Hyperboloidal evolution in Numerical Relativity: the case of Generalized Harmonic Gauge

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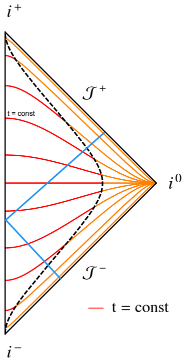
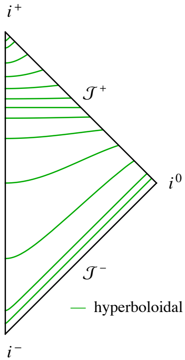
We want to extract the gravitational wave signal from the code at $\mathscr{J}^+$.





*Courtesy of A. Vaño-Viñuales

Hyperboloidal coordinates: $\tau = T - H(R)\,, \;\; R = \frac{r}{\Omega(r)}$

General Relativity in GHG vs. GBU(F) model

Gauge choice is important!

We want hyperboloidal evolution for GR in Generalized Harmonic ($\mathcal{C}^\mu \equiv \Gamma^\mu + F^\mu = 0$)

$$R_{\mu\nu} - \nabla_{(\mu}\mathcal{C}_{\nu)} + W_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T_g$$

Equations of motion are

$$\Box\Psi = \mathcal{N}_\Psi$$

$$\Box\mathcal{C}^R_- = -\tfrac{K}{2}(\nabla_T\Psi)^2 - 2\nabla_T\check{F} + \mathcal{N}_{\mathcal{C}^R_-}$$

$$\Box\mathcal{C}^R_+ = \frac{2}{R}\nabla_T\mathcal{C}^R_+ + \mathcal{N}_{\mathcal{C}^R_+}$$

$$\Box\mathcal{C}^\pm_A = \frac{2}{R}\nabla_T\mathcal{C}^\pm_A + \mathcal{N}_{\mathcal{C}^\pm_A}$$

$$\Box\epsilon = \frac{2}{R}\nabla_T\epsilon + \mathcal{N}_\epsilon$$

$$\Box\delta = \nabla_T\check{F}^\sigma + \mathcal{N}_\delta$$

They decay asymptotically as the *Good-Bad-Ugly (-F)* model

$$\Box g \simeq 0$$

$$\Box b \simeq (\nabla_T g)^2 + \tfrac{1}{R}\nabla_T f$$

$$\Box u \simeq \tfrac{2}{R}\nabla_T u$$

$$\Box f \simeq \tfrac{2}{R}\nabla_T f + 2(\nabla_T g)^2$$

Asymptotic decay of the fields
GBU(F) model

The good, bad and ugly fields have different asymptotic decay:

$$\Box g \simeq 0 \qquad \qquad g \sim \frac{G_1(T-R)}{R}$$

$$\Box b \simeq (\nabla_T g)^2 \qquad \Rightarrow \qquad b \sim \frac{B_1(T-R) + \log R \; B_2(T-R)}{R}$$

$$\Box u \simeq \frac{2}{R}\nabla_T u \qquad \qquad u \sim \frac{m_u}{R}$$

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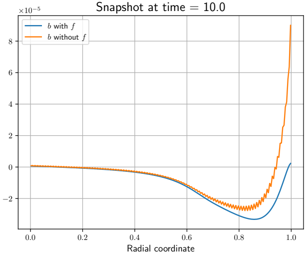
$\Rightarrow$

$$g \sim \frac{G_1(T-R)}{R}$$
$$b \sim \frac{B_1(T-R) + \cancel{\log R \, B_2(T-R)}}{R}$$
$$u \sim \frac{m_u}{R}$$

The quesion is: can we capture this asymptotic cancellation numerically?



Take (T, R, θ, ϕ) coordinates that asymptote spherical polars (GHG coords)

The functions $C_\pm(T, R)$ are defined by requiring that the following vectors are null

$$\xi^a = \partial_T^a + C_+\partial_R^a\,, \quad \underline{\xi}^a = \partial_T^a + C_-\partial_R^a$$

The function $\delta(T, R)$ is defined by the covectors

$$\sigma_a = e^{-\delta}\xi_a,\ \underline{\sigma}_a = e^{-\delta}\underline{\xi}_a$$

satisfying

$$\sigma_a\partial_R^a = -\underline{\sigma}_a\partial_R^a = 1.$$

Defining aereal radius $\mathring{R} = Re^{\epsilon/2}$ the metric takes the form

$$g = \begin{pmatrix} \frac{2e^\delta C_+ C_-}{C_+ - C_-} & \frac{e^\delta\left(C_- + C_+\right)}{C_- - C_+} & 0 & 0 \\ \frac{e^\delta\left(C_- + C_+\right)}{C_- - C_+} & \frac{2e^\delta}{C_+ - C_-} & 0 & 0 \\ 0 & 0 & e^\epsilon R^2 & 0 \\ 0 & 0 & 0 & e^\epsilon R^2 \sin^2\theta \end{pmatrix}.$$

Equations of motion

Reduced Einstein Field equations

Recall GHG is to impose $\mathcal{C}^\mu \equiv \Gamma^\mu + F^\mu = 0.$

Spherical symmetry $\Rightarrow$ $F^\theta = \mathring{R}^{-2}\cot\theta\,,\;\; F^\phi = 0\,.$

Reduced Einstein Field Equations + massless scalar field

$$D_\sigma\left(\frac{2}{\kappa}\mathring{R}^2 D_{\underline{\sigma}} C_+\right) + \mathring{R} D_\sigma\left(\mathring{R} F^\sigma\right) - D_\sigma \mathring{R}^2 \frac{D_\sigma C_+}{\kappa} + W_{\sigma\sigma} = -8\pi \mathring{R}^2 (D_\sigma\psi)^2\,,$$

$$D_{\underline{\sigma}}\left(\frac{2}{\kappa}\mathring{R}^2 D_\sigma C_-\right) - \mathring{R} D_{\underline{\sigma}}\left(\mathring{R} F^{\underline{\sigma}}\right) - D_{\underline{\sigma}} \mathring{R}^2 \frac{D_{\underline{\sigma}} C_-}{\kappa} + W_{\underline{\sigma\sigma}} = 8\pi \mathring{R}^2 (D_{\underline{\sigma}}\psi)^2\,,$$

$$\Box_2 \mathring{R}^2 - 2 + W_{\theta\theta} = 0\,,$$

$$\Box_2\delta + D_\nu F^\nu + 2e^\delta\left(\frac{D_{\underline{\sigma}} C_+ D_\sigma C_- - D_\sigma C_+ D_{\underline{\sigma}} C_-}{\kappa^3}\right) + \frac{2}{\mathring{R}^2}\left(1 - \frac{2M_{\rm MS}}{\mathring{R}}\right) = 16\pi\frac{e^\delta}{\kappa} D_\sigma\psi D_{\underline{\sigma}}\psi\,,$$

$$\Box_4\psi = 0$$

where $\kappa \equiv C_+ - C_-$ and $M_{\rm MS} \equiv \frac{1}{2}\mathring{R}\left(2\frac{e^\delta}{\kappa}(D_\sigma\mathring{R})(D_{\underline{\sigma}}\mathring{R}) + 1\right)$

According to our classification:

- ψ is a good wave equation
- C_+ and ϵ are uglies (by constraint addition)
- δ is good/ugly depending on gauge choice
- ✱ C_- has $O(R^{-2})$ source $\Rightarrow$ **bad** wave equation

To regularize C_- we take

$$F^{\sigma} = \frac{2}{\mathring{R}} + \frac{2p}{\mathring{R}}(e^{-\delta} - 1)\,, \qquad F^{\underline{\sigma}} = -\frac{2}{\mathring{R}} - \frac{p}{\mathring{R}}(1 + C_-) + \frac{1}{\mathring{R}}F_{GD}$$

where F_{GD} satisfies

$$\Box F_{GD} \simeq \frac{2}{R} D_{\underline{\sigma}} F_1 + 16\pi (D_{\underline{\sigma}}\psi)^2$$

Take a fixed compactification $R = \dfrac{r}{(1-r^2)^{\frac{1}{n-1}}}$

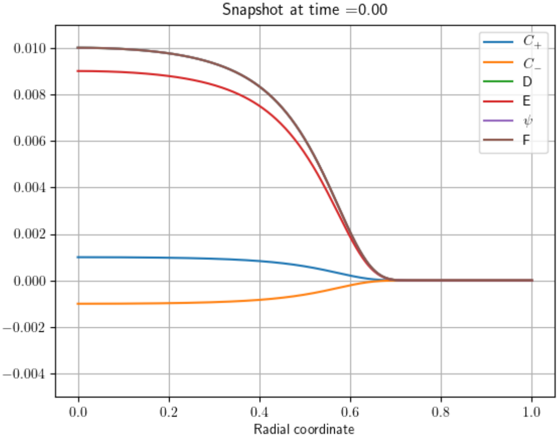
We can construct hyperboloidal coordintes two ways:

- With height function: $\tau = T - H(R)$ with $H = R + m_{C_+} \log R - r$
- Through the eikonal equation: $\tau = u + r$ where $g^{ab}\nabla_a u \nabla_b u = 0$

Demanding $\nabla^a u \propto \sigma^a$ implies outgoing coordinate lightspeed $c_+^r \equiv 1$

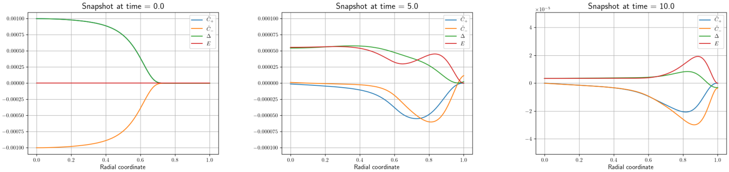
Dual Foliation: (T, R, θ, ϕ) rEFEs components evolved in (τ, r) coordinates

- This approach allows us to keep hyperbolicity properties while reaching $\mathscr{J}^+$.

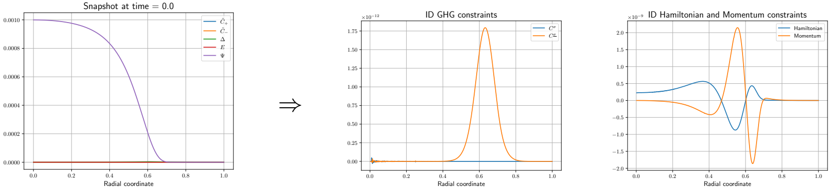

Snapshot at time =0.00
0.010
0.008
0.006
0.004
0.002
0.000
−0.002
−0.004
0.0
0.2
0.4
0.6
0.8
1.0
Radial coordinate
C_+
C_-
D
E
ψ
F

Spherical GR numerical evolution
Regular center case

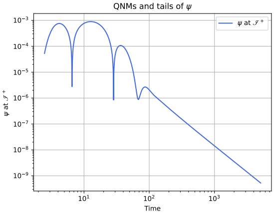
Gauge perturbations



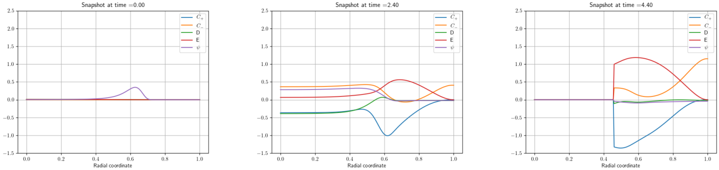
Constraint satisfying ID



Schwarzschild perturbations



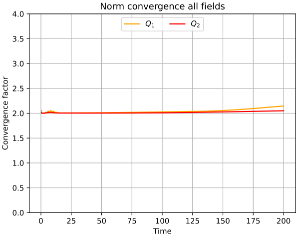
Scalar field collapse to a BH

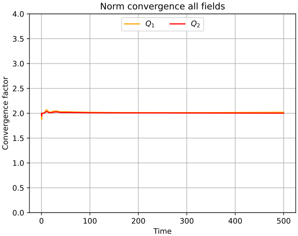


Spherical GR numerical evolution

Convergence and summary

For $n \leq 1.5$ we get very good convergence





Regular center (left) and Schwarzschild perturbations (right)

Summary:

- We have an evolution code working and converging for every possible case
- Constraint satisfying ID in spherical symmetry

What we aim for:

- **Full 3D GR**