Critical gravitational collapse in elastic matter models

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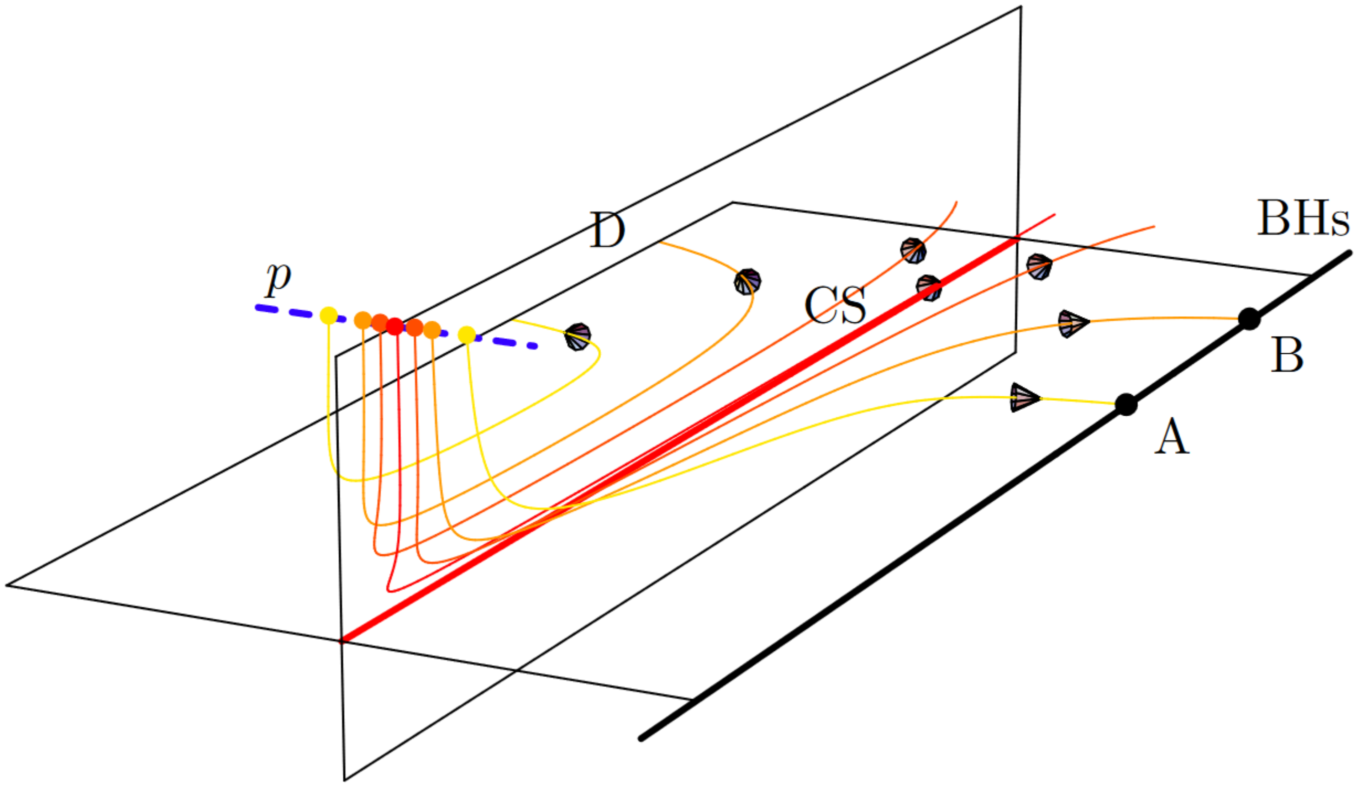
Introduction

- Critical solutions are well established for spherically symmetric distributions;
- Critical solutions separate black hole formation and dissipation;
- In phase space, the critical point verifies a local one-dimensional unstable submanifold



Gundlach and Martín-García(2007)

Motivation

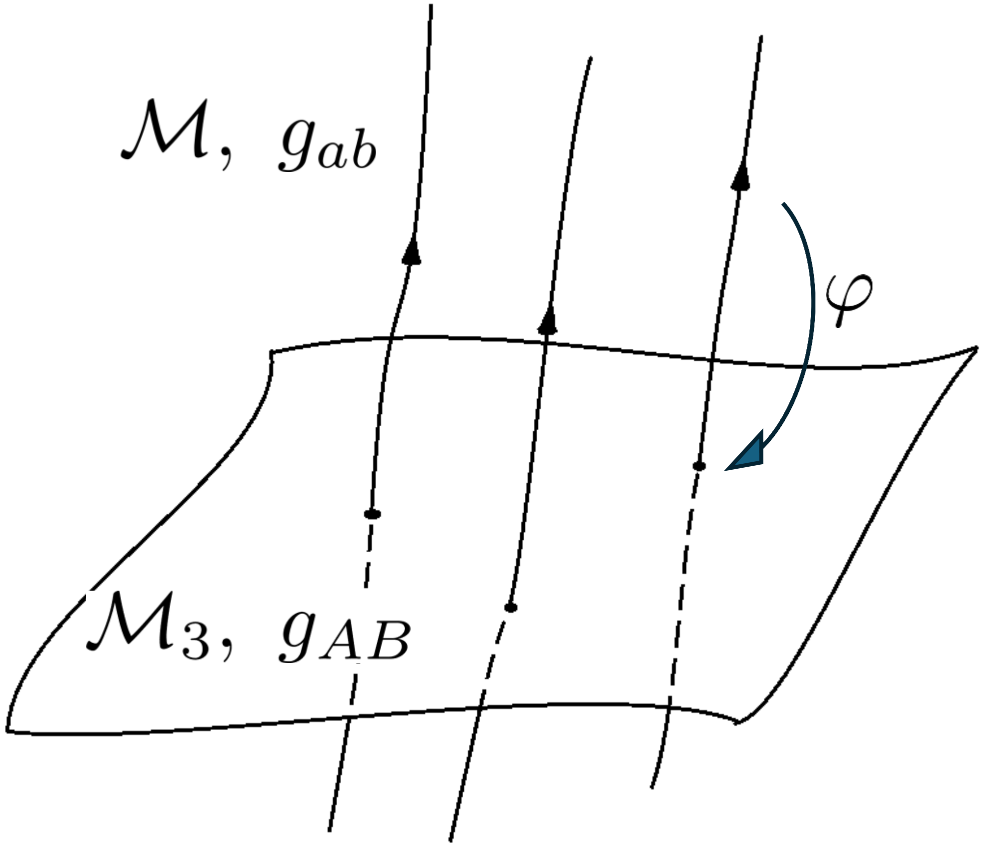
Critical black holes are the smallest possible obtainable. As such they:

- May denote regions of high spacetime curvature;

- May give rise to naked singularities.

Relativistic Elasticity

Consider the 4-manifold, spacetime, and a projection into a 3-submanifold, “material” space.

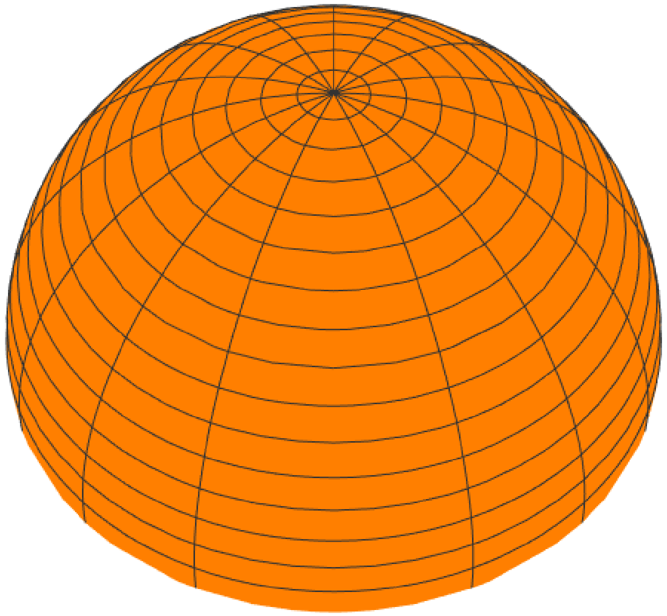


$$\varphi : \mathcal{M} \to \mathcal{M}_3$$

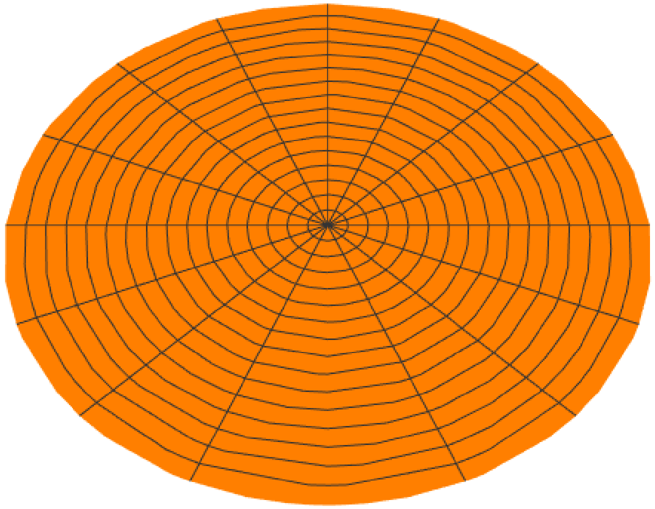
$$\varphi_* g^{ab} = g^{AB}$$

Relativistic Elasticity

Initial Physical State

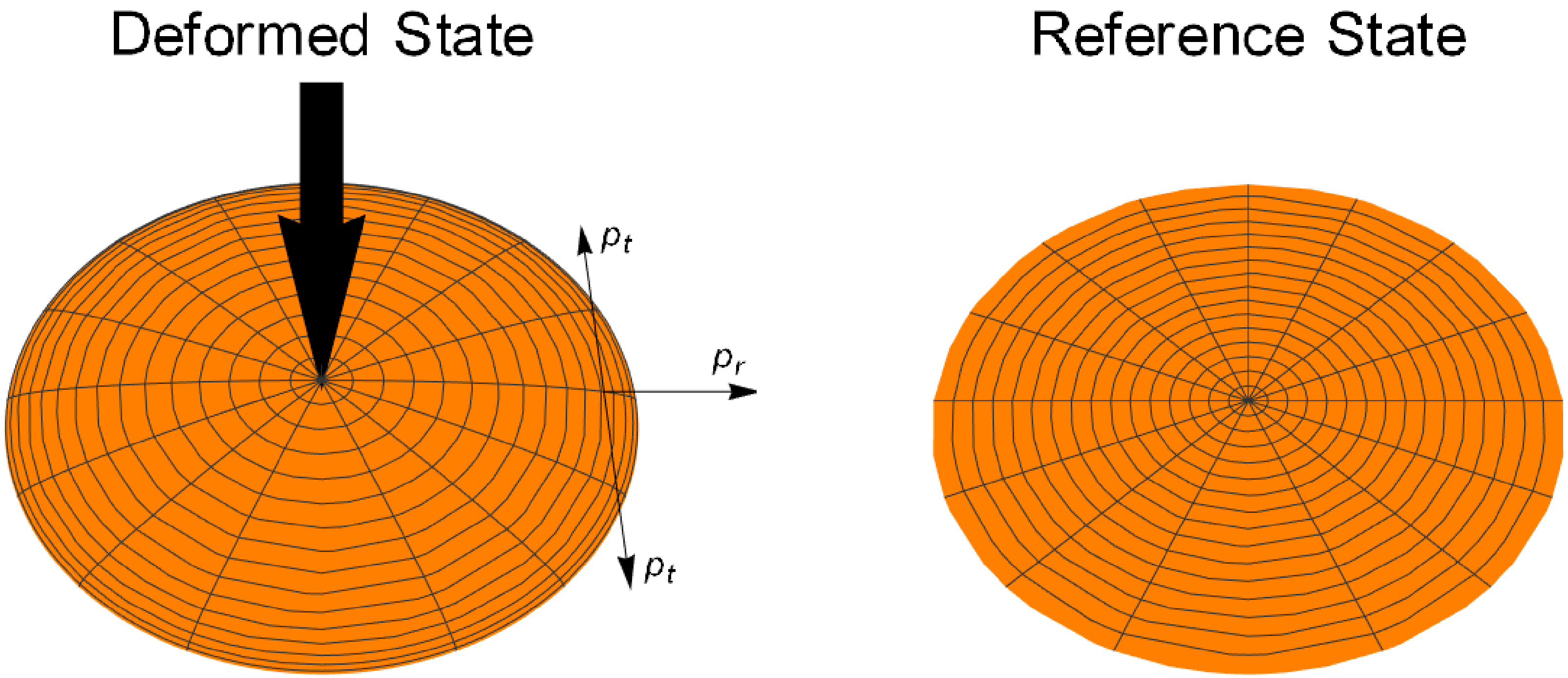


Reference State



- Comparison between physical and reference manifolds yields relations between geometry and physical invariants

Relativistic Elasticity



- Comparison between physical and reference manifolds yields relations between geometry and physical invariants

Matter Model

For the elastic matter model, Alho et al. (2024) found the scale invariant elastic model

$$\rho(\xi,\eta) = \frac{k}{\gamma\,(\gamma-1)}\,\eta^{\gamma}\,\left[1-\gamma\,\left(1-3\,\frac{\gamma-1}{\beta-1}\,\left(\frac{1-\nu}{1+\nu}\right)\right)\,(1-\xi)-3\,\frac{\gamma\,(\gamma-1)}{\beta\,(\beta-1)}\,\left(\frac{1-\nu}{1+\nu}\right)\,(1-\xi^{\beta})\right]$$

Where:

$$\gamma = \text{polytropic index} \quad , \quad \beta = \text{shear index} \quad , \quad \nu = \text{Poisson's ratio}$$

This generalizes the perfect fluid, recovered for

$$\begin{cases} \beta &= \gamma \\ \nu &= \frac{1}{2} \end{cases} \quad \Longrightarrow \quad \rho = \frac{k}{\gamma\,(\gamma-1)}\,(\eta\,\xi)^{\gamma}$$

Matter Model

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$$p_r = \xi\,\frac{\partial\rho}{\partial\xi}-\rho \quad , \quad p_t = p_r+\frac{3}{2}\,\eta\,\frac{\partial\rho}{\partial\eta}$$

$$\Downarrow$$

$$p_r = \frac{k}{\gamma}\,\eta^{\gamma}\left[1-3\,\frac{\gamma}{\beta}\left(\frac{1-\nu}{1+\nu}\right)(1-\xi^{\beta})\right]$$

$$p_t = \frac{k}{\gamma}\,\eta^{\gamma}\left[1-\frac{3}{2}\,\gamma\left(1-3\,\frac{\gamma-1}{\beta-1}\left(\frac{1-\nu}{1+\nu}\right)\right)(1-\xi)+\frac{3}{2}\,\frac{\gamma}{\beta}\left(1-3\,\frac{\gamma-1}{\beta-1}\right)\left(\frac{1-\nu}{1+\nu}\right)(1-\xi^{\beta})\right]$$

Metric ansatz

- We consider a spherically symmetric metric given by

$$\mathrm{d}s^2 = -\alpha^2(t,r)\,\mathrm{d}t^2 + a^2(t,r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2$$

- A spacetime is continuously self-similar if there is a homothetic vector field, Z:

$$\mathscr{L}_Z g_{ab} = 2\,g_{ab}$$

$$Z = t\,\partial_t + r\,\partial_r$$

$$x = \log\left(\frac{r}{-t}\right) \quad , \quad \tau = -\log(-t)$$

$$\Longrightarrow \quad a(t,r) \equiv a(x) \quad , \quad \alpha(t,r) \equiv \alpha(x)$$

Metric transformation

The self-similarity of spacetime can be brought out by the choice of new metric functions

$$\alpha = N\sqrt{a}\,\mathrm{e}^{x} \quad , \quad a = \sqrt{A}$$

$$\Downarrow$$

$$\mathrm{d}s^2 = \underbrace{\mathrm{e}^{-2\tau}}_{\tau\ \text{dependency}}\,\mathrm{e}^{2x}\left[-\left(N^2-1\right)A\,\mathrm{d}\tau^2 - 2A\,\mathrm{d}\tau\,\mathrm{d}x + A\,\mathrm{d}x^2 + \mathrm{d}\Omega^2\right]$$

The choice of metric function then reflects on the EFE, showing the correct self-similar form of the matter functions

$$\eta^\gamma(t,r) = \frac{\mathrm{e}^{2\tau}\,\mathrm{e}^{-2x}}{4\pi A}\,\tilde{\eta}^\gamma(x) \quad , \quad \xi(t,r) = \tilde{\xi}(x) \quad \Longrightarrow \quad \rho(t,r) = \frac{\mathrm{e}^{2\tau}\,\mathrm{e}^{-2x}}{4\pi A}\,\tilde{\rho}(x)$$

Elastic Equations of Motion

$$G^0_0: \quad \frac{A'}{A} = 1 - A + \frac{2}{1-V^2}\,(\tilde{\rho} + V^2\,\tilde{p}_r)$$

$$G^1_1: \quad \frac{N'}{N} = -2 + A - (\tilde{\rho} - \tilde{p}_r)$$

$$G^1_0: \quad \frac{A'}{A} = -\frac{2\,N\,V}{1-V^2}\,(\tilde{\rho} + \tilde{p}_r)$$

$$T_0{}^{\mu}{}_{,\mu} = 0: \quad a(\ldots)\,\tilde{\xi}' + b(\ldots)\,V' = e(\ldots)$$

$$T_1{}^{\mu}{}_{,\mu} = 0: \quad c(\ldots)\,\tilde{\xi}' + d(\ldots)\,V' = f(\ldots)$$

$$\Longrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \tilde{\xi}' \\ V' \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\text{Elasticity:} \quad \frac{\tilde{\eta}'}{\tilde{\eta}} = -3\,\frac{N\,\sqrt{A}\,V}{1-V^2}\,\tilde{\xi} - \frac{2\,N\,V}{\gamma\,(1-V^2)}\,(\tilde{\rho} + \tilde{p}_r)$$

Elastic Equations of Motion

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Origin requires special care!

$$T_0{}^{\mu}{}_{,\mu} = 0: \quad a(\ldots)\,\tilde\xi' + b(\ldots)\,V' = e(\ldots)$$

$$T_1{}^{\mu}{}_{,\mu} = 0: \quad c(\ldots)\,\tilde\xi' + d(\ldots)\,V' = f(\ldots)$$

$$\Longrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \tilde\xi' \\ V' \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Becomes singular at sonic point!

$$\text{Elasticity:} \quad \frac{\tilde\eta'}{\tilde\eta} = -3\,\frac{N\sqrt{A}\,V}{1-V^2}\,\tilde\xi - \frac{2\,N\,V}{\gamma\,(1-V^2)}\,(\tilde\rho + \tilde p_r)$$

Asymptotic behavior at the center (x→-∞)

- Assuming regularity, the origin can be used to determine the asymptotic behavior of physical solutions.

$$\lim_{r\to 0} a = 1 \quad , \quad \lim_{r\to 0} \alpha = const. \quad \Longrightarrow \quad \lim_{r\to 0} A = 1 \quad , \quad \lim_{r\to 0} N = \infty$$

$$N = \frac{\alpha}{\sqrt{A}} \mathrm{e}^{-x} \quad , \quad \mathrm{e}^{x} = \frac{r}{-t}$$

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- The choice M=N·V shows the origin is a fixed point with

$$A^* = 1 \quad , \quad M^* = -\frac{2}{3\,\gamma} \quad , \quad \tilde{\xi}^* = 1 \quad , \quad \tilde{\eta}^* = 0 \quad , \quad V^* = 0$$

$$A \sim 1 + \delta A(x) \quad , \quad M \sim -\frac{2}{3\,\gamma} + \delta M(x) \quad , \quad \tilde{\xi} \sim 1 + \delta\tilde{\xi}(x) \quad , \quad \tilde{\eta} \sim \delta\tilde{\eta}(x) \quad , \quad V \sim \delta V(x)$$

- And replace them in the EFE

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- Thus we obtain

$$A \sim 1 + A_{-\infty}\, e^{2\,x} \quad , \quad N \sim N_{-\infty}\, e^{-x} \quad , \quad \tilde{\xi} \sim 1 + \tilde{\xi}_{-\infty}\, e^{2\,x} \quad , \quad \tilde{\eta} \sim \tilde{\eta}_{-\infty}\, e^{\frac{2}{\gamma}\,x} \quad , \quad V \sim V_{-\infty}\, e^{x}$$

- The asymptotic rate factors related by

$$A_{-\infty} = \frac{2}{3}\left(\frac{k}{\gamma\,(\gamma-1)}\right)\tilde{\eta}_{-\infty}^{\gamma} \quad , \quad N_{-\infty}\, V_{-\infty} = -\frac{2}{3\,\gamma} \quad , \quad \tilde{\xi}_{-\infty} = \sqrt{\frac{1 - V_{-\infty}^{2}}{A_{-\infty}}}$$

Conclusion and Future Work

We’ve found that:

- Elastic self-similar models show several instances of similar behavior to perfect fluids.

However, to clear the uniqueness of this model we still need to:

- Obtain numerical solutions around the sonic point, validating them with the required asymptotic behavior;
- Develop simulations around the regular center using both Schwarzschild and comoving coordinates;
- Obtain the critical exponent and compare with other matter models.

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