# Numerical study of the Hawking and dynamical Casimir effects

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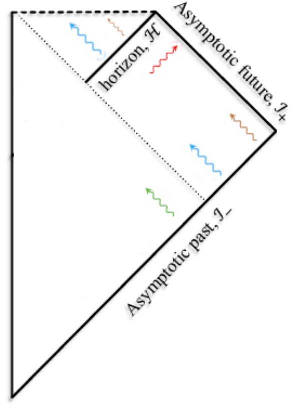
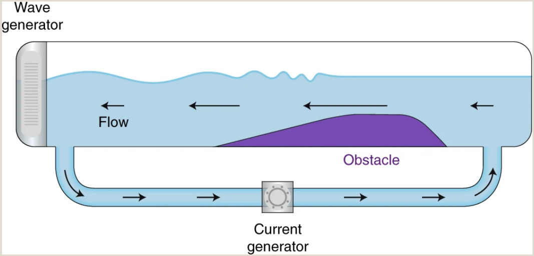
# Introduction

- Casimir effect and Hawking radiation are among the most important phenomena in Quantum Field Theory in Curved Spacetimes.
- The latter has deep roots in BH physics and is at the heart of one of the most important open problems in fundamental physics: information loss problem (quantum gravity at the rescue).
- It is very difficult to closely observe realistic black holes, even more Hawking radiation.
- This fact has led to propose several experiments to verify Hawking radiation. One of the earliest theoretical proposals are moving (accelerating) mirrors. However, a realistic implementation is very hard ... but not impossible.



# Analogue gravity

- Bose-Einstein condensates: detection of spontaneous Hawking effect reported by Steinhauer (2016) and criticized by Leonhardt (2018).
- Surface waves in water flows allow for the study of stimulated effect.
- Nonlinear optics systems (electromagnetic waveguides, optical fibres, quantum fluids of light). Promising experiments.
- Moving mirrors (AnaBHEL, CPW ended in SQUIDs).

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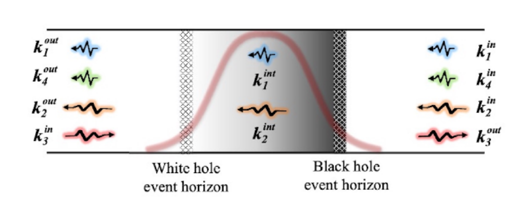


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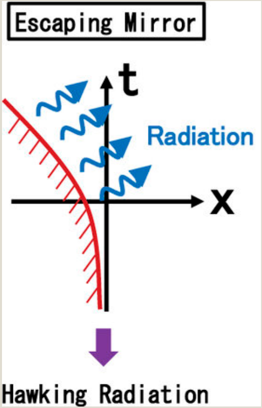


# The original model (Fulling & Davies - 76)

- The field theory we consider is

$$S = -\frac{1}{2}\int d^2x\sqrt{-\eta}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

  where $\eta_{\mu\nu}$ is the 1+1 flat spacetime metric.

- The Klein-Gordon equation

$$\frac{1}{\sqrt{-\eta}}\partial_\mu\left(\sqrt{-\eta}\eta^{\mu\nu}\partial_\nu\right)\phi = 0.$$

- A boundary condition $\phi(t, x = g(t)) = 0$ with $t \in (-\infty, \infty)$ has non-trivial consequences on the field evolution.
- For instance, consider a mirror initially at rest at $g(t) = L_0$ for $t < 0$, and at $t \geq 0$ following the trajectory

$$g(t) = L_0 - B + t + Be^{-2\kappa t}.$$

- At very late times $t \to +\infty$, it radiates with a Planckian spectrum

$$|\beta_{\omega\omega'}|^2 = \frac{1}{2\pi\kappa\omega'}\frac{1}{(e^{2\pi\omega/\kappa} - 1)}, \quad \text{where} \quad \kappa/(2\pi) = k_B T/\hbar.$$

# The Fourier space & equations of motion

- We are interested in two boundary conditions $\phi(t, x = f(t)) = 0 = \phi(t, x = g(t))$.
- For our purposes it is useful to introduce the following Fourier decomposition

$$\phi(\tau,\xi) = \sum_{n=1}^{\infty} \phi_n(\tau)\sin(n\pi\xi), \quad t = \tau, \quad \xi = L_0 \frac{x - f(t)}{L(t)}, \quad L(t) = g(t) - f(t) > 0.$$

which guaranties that the field fulfills $\phi(\tau, \xi = 0) = 0 = \phi(\tau, \xi = 1)$, namely, (stationary) Dirichlet boundary conditions.

- The Hamilton's equations of motion are

$$\dot{\phi}_n = \frac{1}{L}\pi_n - \frac{\dot{L}}{2L}\phi_n + 2\sum_{m\neq n} \frac{mn}{m^2 - n^2}\left[\frac{\dot{f}}{L}\left((-1)^{m+n} - 1\right) + \frac{\dot{L}}{L}(-1)^{m+n}\right]\phi_m$$

$$\dot{\pi}_n = -\frac{1}{L}(n\pi)^2\phi_n + \frac{\dot{L}}{2L}\pi_n + 2\sum_{m\neq n} \frac{nm}{m^2 - n^2}\left[\frac{\dot{f}}{L}\left((-1)^{n+m} - 1\right) + \frac{\dot{L}}{L}(-1)^{n+m}\right]\pi_m.$$

We infer that if $\dot{L}(\tau) \neq 0$ and/or $\dot{f}(\tau) \neq 0$ there will be dynamical mode mixing.

# KG product & complex basis of solutions

- A complex solution $\mathbf{U}(\tau)$

$$\mathbf{U}(\tau) = \Big(\phi_1(\tau), \pi_1(\tau), \phi_2(\tau), \pi_2(\tau), \cdots\Big),$$

is an element of a (complexified) space of solutions, i.e. $\mathbf{U}(\tau) \in \mathcal{S}^{\mathbb{C}}$, endowed with a natural Klein-Gordon product

$$\langle \mathbf{U}^{(1)}(\tau), \mathbf{U}^{(2)}(\tau)\rangle = \frac{i}{2}\sum_{n=1}^{\infty} \bar{\phi}_n^{(1)}(\tau)\pi_n^{(2)}(\tau) - \bar{\pi}_n^{(1)}(\tau)\phi_n^{(2)}(\tau),$$

which is preserved under the evolution and is not positive definite.

- We now choose a basis of (orthonormal) complex solutions $\big(\mathbf{u}^{(I)}, \bar{\mathbf{u}}^{(I)}\big)$, with $I = 1, 2, \ldots$ This basis satisfies

$$\langle \mathbf{u}^{(I)}(\tau), \mathbf{u}^{(J)}(\tau)\rangle = \delta^{IJ},\ \langle \mathbf{u}^{(I)}(\tau), \bar{\mathbf{u}}^{(J)}(\tau)\rangle = 0,\ \langle \bar{\mathbf{u}}^{(I)}(\tau), \bar{\mathbf{u}}^{(J)}(\tau)\rangle = -\delta^{IJ},$$

- Any real solution can be written as $\underline{\mathbf{U}}(\tau) = \sum_I a_I \mathbf{u}^{(I)}(\tau) + \bar{a}_I \bar{\mathbf{u}}^{(I)}(\tau)$, with $a_I$ and $\bar{a}_I$ annihilation and creation variables.

# The *in* basis

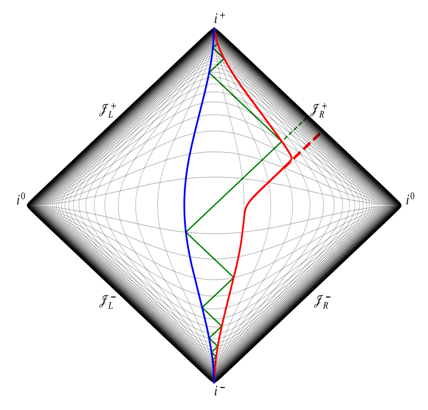
- We assume that the plates are stationary in the past. The *in* basis is given by the initial conditions

$$\mathbf{u}^{(1)}(\tau_0) = \left(\frac{1}{\sqrt{\omega_1}}, -i\sqrt{\omega_1}, 0, 0, \cdots\right),$$
$$\mathbf{u}^{(2)}(\tau_0) = \left(0, 0, \frac{1}{\sqrt{\omega_2}}, -i\sqrt{\omega_2}, 0, 0, \cdots\right),$$
$$\vdots$$
$$\mathbf{u}^{(I)}(\tau_0) = \left(0, 0, \ldots, \frac{1}{\sqrt{\omega_I}}, -i\sqrt{\omega_I}, 0, 0, \ldots\right)$$
$$\vdots$$

and the complex conjugate, where $\omega_1 = \pi$, $\omega_2 = 2\pi$, $\ldots$ are frequencies of the modes $n = 1, 2, \ldots$, respectively. These modes correspond to standard "plane waves" in a cavity.

- We solve the dynamics numerically with an explicit embedded Runge-Kutta-Prince-Dormand (8,9). It can integrate a finite (but large) number of modes. Limit of infinite modes adopting a Richardson extrapolation. Error controlled by inner products and closure relations of $\mathbf{u}^{(I)}(\tau)$.

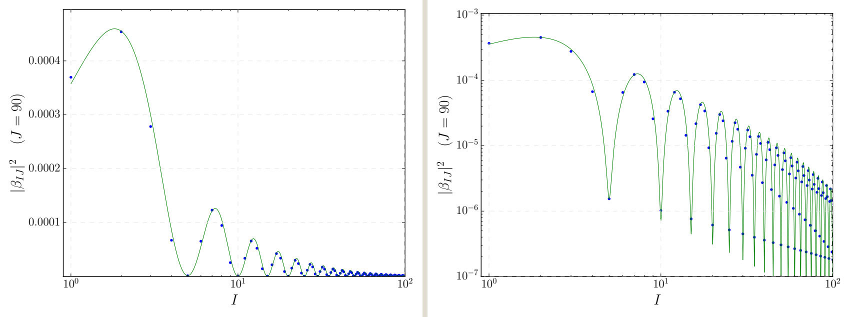
# The Hawking effect: accelerating mirrors




The explicit trajectories for a light ray (green line), and for the boundaries are (blue line) $f(t) = 0$ and (red line)

$$g(t) = 1 + \frac{s}{2\kappa} + \frac{1}{2\kappa}\left[\log\Big(\cosh\big(\kappa(t-t_0)\big)\Big) - \log\Big(\cosh\big(s-\kappa(t-t_0)\big)\Big)\right].$$

# Particle production & Hawking radiation



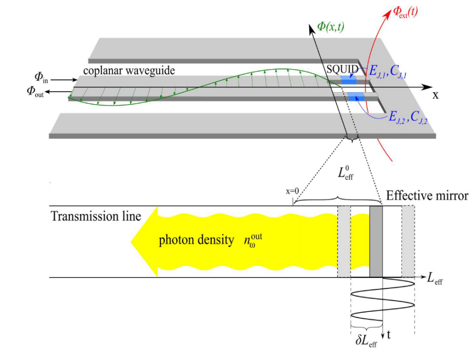

We show here the results (blue dots) for $\kappa = 1200$ and $s = 150$. It amounts to $g(t = +\infty) = (1+\epsilon)$ (with $\epsilon = s/\kappa = 0.25$). The numerical results fit very well the spectrum (green line)

$$|\beta_{IJ}^{(\text{fit})}|^2 = \frac{2\Delta\omega_I \Delta\omega_J}{\pi\kappa\omega_J} \frac{\Gamma(\kappa,\epsilon)}{(e^{2\pi\omega_I/\kappa} - 1)}, \quad \Gamma(\kappa,\epsilon) \simeq \left[A + B\sin^2(\epsilon\,\omega_I)\right],$$

where $\omega_I = \pi I/(L_0 + \epsilon)$, $\omega_J = \pi J/L_0$, $\Delta\omega_I = \pi/(L_0 + \epsilon)$, $\Delta\omega_J = \pi/L_0$, $\big($in the continuum $\Delta\omega_I \Delta\omega_J \to d\omega d\omega'\big)$. Here $A = 7.3 \cdot 10^{-3}$ and $B(\kappa) = 0.91$.

# The dynamical Casimir effect: Introduction

- The (dynamical) Casimir effect is one of the most well-known phenomena in quantum field theory.
- It involves the conversion of virtual particles into real particles (photons) due to changes in the boundary conditions of the field.
- This effect was first predicted by Gerald T. Moore 50 years ago.
- Its experimental verification has been claimed by J. R. Johansson, G. Johansson, C. M. Wilson, and F. Nori in 2010 using a coplanar waveguided ended in superconducting interference device.[a]



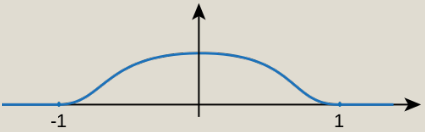

[a] Also by P. Lähteenmäki, G. S. Paraoanu, J. Hassel, and P. J. Hakonen in 2013.

# Trajectory for the dynamical Casimir effect

- We will assume that the boundaries follow simple damped oscillatory trajectories

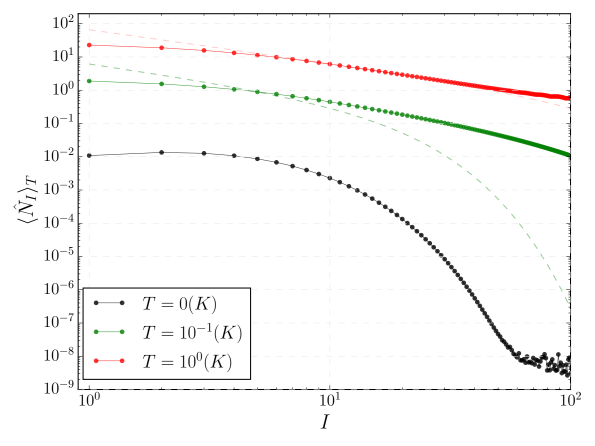
$$
\begin{aligned}
g(t) &= 1 + \epsilon_1 B(t)(\sin(q_1 \pi t + \phi) - \sin(\phi)), \\
f(t) &= \epsilon_2 B(t) \sin(q_2 \pi t),
\end{aligned}
$$

in coordinates $(t, x)$, where $\epsilon_1$ and $\epsilon_2$ control the amplitudes of the oscillations of the boundaries, $q_1 = q_2 = q$ is an integer that controls their frequency, $\phi$ is a relative phase, and $B(t)$ is the bump function



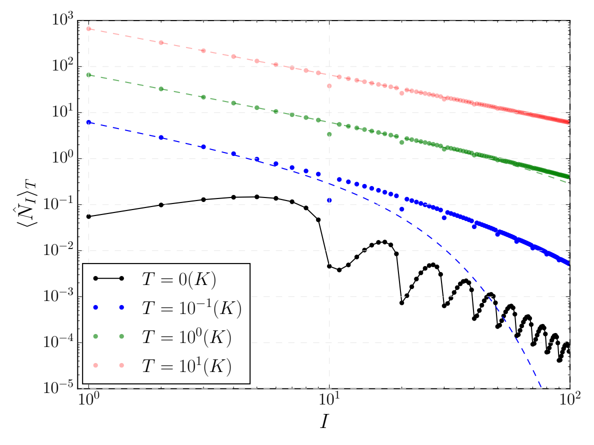

- Moreover, we have studied the trajectories with several values of $q$, and separately the cases i) $\phi = \pi$ and $\epsilon_1 = 1/40$ and $\epsilon_2 = 0$ (one moving boundary), ii) $\phi = \pi$ and $\epsilon_1 = 1/40 = \epsilon_2$ (breathing configuration), and iii) $\phi = 0$ and $\epsilon_1 = 1/40 = \epsilon_2$ (translational configuration).

# Cooling down of a thermal state for traj. i) and $q = 1$




We show here $\langle \hat{N}_I \rangle_T = \sum_J |\beta_{JI}|^2 + \sum_J (|\alpha_{JI}|^2 + |\beta_{JI}|^2) \frac{1}{e^{E_J/T} - 1}$ for $q = 1$. The occupation number of IR modes is smaller (cooling down effect). This happens at the expenses of warming up the UV modes (stimulated particle production), since there is a relatively small particle production while we are pumping energy in the system.

# Cooling down of a thermal state for traj. i) and $q = 10$




We now consider the plates oscillating with $q = 10$. The occupation number of only resonant modes is smaller (selective cooling down effect).

# SQUID-terminated CPW

- For the experimental proposals based on a coplanar wave guide (CPW) ended in superconducting interference devices (SQUIDs) acting as effective boundary conditions that oscillate with frequencies smaller than $\omega_p = 37.3$ GHz.

- For a CPW as large as $L_0 = 10.0$ cm, a change in size as large as $\delta L = 0.25$ cm, and a speed of propagation of the phase field (the time integral of the electric field) around $v \simeq 10^{10}$ (cm/s), we get $\omega_d = 3.14$ GHz $\sim \omega_p/10$ for $q = 1$ and $\omega_d = 31.4$ GHz $\sim \omega_p$ for $q = 10$. The configurations we have explored are within the experimental capabilities since $\omega_d \lesssim \omega_p$ for the study of the cooling down within the DCE.

- Regarding the Hawking effect, for $\omega_d = \omega_p/2 = 18.6$ GHz, $\delta L \sim 0.25$ cm $\sim L_{\text{cav}}/4$. If $v \sim 10^{10}$ (cm/s), the maximum acceleration of the boundary $\kappa \simeq \omega_p^2 \delta L \simeq 3.5 \cdot 10^{20}$ (cm/s$^2$), and the maximum Hawking temperature can be as large as $T \simeq 0.5$ K. Current CPW experimental temperatures are $T \simeq 0.025$ K.

- The inductance and capacitance of the CPW can be modified, either by suitable changes in their geometry or by considering substrates with large dielectric permittivities (there exist materials like perovskite or polymers with permittivities higher than $10^4$). Large permittivities imply lower $v$ and hence higher $T$. They also induce a strong polarization field and likely a modified dispersion relation for the phase field.

- As a final remark, the trajectory shown here corresponds to $\kappa \simeq 6 \cdot 10^{18}$ (cm/s$^2$) and a Hawking temperature $T_H \simeq 0.005$ K.

# Summary

- We study a field theory enclosed in a cavity with moving boundaries in the non-perturbative regime with some novel numerical tools.
- Concretely, we apply them to study, on one hand the dynamical Casimir effect, like the cooling down (upconversion) of some modes when the field is initially in a thermal state. Within the Hawking effect, we show that some band frequencies can be excited into a thermal state with the temperature given by the acceleration of the boundary (and a graybody factor).
- We suggest to use a known setup to studying both the cooling down effect discovered by Dodonov and Hawking radiation, experimentally. It involves a CPW with two SQUIDs attached to its endings, where the phase of the electromagnetic field propagating along the CPW can be well-described by a real, scalar quantum field $\phi(t, x)$ and with (tunable) time-dependent boundary conditions determined by the magnetic field threading the SQUIDs.
- We are currently studying the entanglement structure of the out state (log neg for partitions $1 \times (N-1)$) and its robustness against thermal noise and measurement imperfections with an eye on future experiments.