Gravitational wave fluxes for a spinning particle in the Hamilton-Jacobi formalism

Work in progress...

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Extreme-mass ratio inspirals (EMRIs)

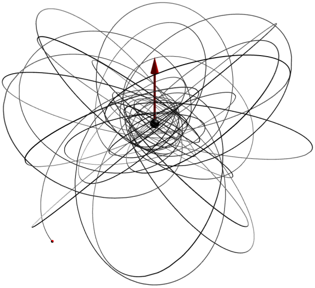
Primary mass M: $10^{5.5} - 10^{7} M_{\odot}$ Secondary mass μ: $1 - 100 M_{\odot}$

$q = \mu/\mathrm{M} \sim 10^{-4} - 10^{-7} \implies$ expand Einstein equations in q

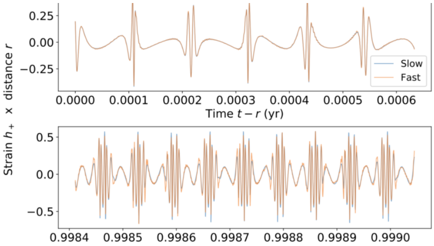
Some features:

mHz GWs $\implies$ targets for LISA

$\sim 1/q$ orbits in 1 year before inspiral in strong gravity regime



Credit: Maarten van de Meent



Credit: Niels Warburton

Why should we consider a spinning secondary in EMRIs?

$$\Phi_{\rm GW} = \underbrace{q^{-1}\mathcal{C}^{(0)}}_{\text{adiabatic}} + \underbrace{q^{-1/2}\mathcal{C}^{(1/2)}}_{\text{resonances}} + \underbrace{q^{0}\mathcal{C}^{(1)}}_{\text{post-1-adiabatic}} + \mathcal{O}(q)$$

$\mathcal{C}^{(1)}$ contains self force plus secondary spin s effects

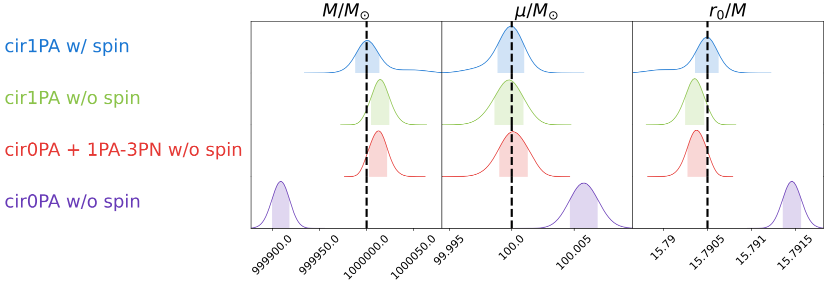


Figure: Biases if you neglect 1PA! ($q = 10^{-4}$). Burke, Piovano+,2024

We propose a new method to produce EMRI waveform for spinning binaries

EoM for a spinning body in first order form

(linearized) MPD equations of motion - 2nd order form

$$S/(\mu M) = qs \qquad s = S/\mu^2 = \sqrt{s_\parallel^2 + s_\perp^2}$$

TD condition: $s^{\mu\nu} v_\nu^{\mathrm{g}} = 0$

$$\begin{cases} \dfrac{D_{\mathrm{g}} v_{\mathrm{g}}^\mu}{\mathrm{d}\tau} = 0 \ , \\ \dfrac{D_{\mathrm{g}} v_{\mathrm{s}}^\mu}{\mathrm{d}\tau} = -\dfrac{1}{2} R^\mu{}_{\nu\rho\sigma} v_{\mathrm{g}}^\nu s^{\rho\sigma} \ , \\ \dfrac{D_{\mathrm{g}} s^{\mu\rho}}{\mathrm{d}\tau} = 0 \ , \end{cases}$$

Spin vector: $s^\mu = s_\perp \left(\tilde{e}^\mu{}_{(1)} \cos\psi_{\mathrm{p}} + \tilde{e}^\mu{}_{(2)} \sin\psi_{\mathrm{p}} \right) + s_\parallel e^\mu{}_{(3)}$

Precession angle: ψ_{p}

For generic orbits, see

Drummond&Hughes (2022), Skuopy+ (2023), Drummond+ (2024), Witzany&Piovano (2024)

We solve MPD equation in 1st form

Geodesics equations of motion

$$\begin{cases} \frac{\mathrm{d}t_{\mathrm{g}}}{\mathrm{d}\lambda} = V^t_{\mathrm{g}}(r_{\mathrm{g}}) + V^t_{\mathrm{g}}(z_{\mathrm{g}}) \\ \frac{\mathrm{d}r_{\mathrm{g}}}{\mathrm{d}\lambda} = \pm\sqrt{R_{\mathrm{g}}(r_{\mathrm{g}})} = \pm\sqrt{(r_{1\mathrm{g}} - r_{\mathrm{g}})(r_{\mathrm{g}} - r_{2\mathrm{g}})Y_{r\mathrm{g}}(r_{\mathrm{g}})} \\ \frac{\mathrm{d}z_{\mathrm{g}}}{\mathrm{d}\lambda} = \pm\sqrt{Z_{\mathrm{g}}(z_{\mathrm{g}})} = \pm\sqrt{(z_{1\mathrm{g}} - z_{\mathrm{g}})^2 Y_{z\mathrm{g}}(z_{\mathrm{g}})} \\ \frac{\mathrm{d}\phi_{\mathrm{g}}}{\mathrm{d}\lambda} = V^\phi_{\mathrm{g}}(r_{\mathrm{g}}) + V^\phi_{\mathrm{g}}(z_{\mathrm{g}}) \end{cases}$$

$z = \cos\theta$ and $\lambda =$ Mino time

EoM fully separable!!!

See

- Carter Phys. Rev. (1968) 174, 1559
- Schmidt CQG (2002) 19 2743
- Fujita and Hikida (2009) CQG 26 135002
- van de Meent (2020) CQG 37 145007

(linearized) MPD equations of motion - 1st order form

Spin-corrections to the 4-velocities in the Hamilton-Jacobi formalism[1]

$$
\begin{cases}
\dfrac{\mathrm{d}t_\mathrm{s}}{\mathrm{d}\lambda} = \dfrac{\partial V^t_\mathrm{rg}}{\partial r_\mathrm{g}}\delta r + \dfrac{\partial V^t_\mathrm{zg}}{\partial z_\mathrm{g}}\delta z + \displaystyle\sum_{i=1}^{2} \dfrac{\partial V^t_\mathrm{g}}{\partial C_i}\delta C_i + V^t_\mathrm{s} \\
\dfrac{\mathrm{d}r_\mathrm{s}}{\mathrm{d}\lambda} = \pm\dfrac{1}{2\sqrt{R_\mathrm{g}}}\left(\dfrac{\partial R_\mathrm{g}}{\partial r_\mathrm{g}}\delta r + \displaystyle\sum_{i=1}^{3} \dfrac{\partial R_\mathrm{g}}{\partial C_i}\delta C_i + R_\mathrm{s}\right) \\
\dfrac{\mathrm{d}z_\mathrm{s}}{\mathrm{d}\lambda} = \pm\dfrac{1}{2\sqrt{Z_\mathrm{g}}}\left(\dfrac{\partial Z_\mathrm{g}}{\partial z_\mathrm{g}}\delta z + \displaystyle\sum_{i=1}^{3} \dfrac{\partial Z_\mathrm{g}}{\partial C_i}\delta C_i + Z_\mathrm{s}\right) \\
\dfrac{\mathrm{d}\phi_\mathrm{s}}{\mathrm{d}\lambda} = \dfrac{\partial V^\phi_\mathrm{rg}}{\partial r_\mathrm{g}}\delta r + \dfrac{\partial V^\phi_\mathrm{zg}}{\partial z_\mathrm{g}}\delta z + \displaystyle\sum_{i=1}^{2} \dfrac{\partial V^\phi_\mathrm{g}}{\partial C_i}\delta C_i + V^\phi_\mathrm{s}
\end{cases}
$$

with $(C_1, C_2, C_3) = (E_\mathrm{g}, L_\mathrm{zg}, K_\mathrm{g})$. EoM valid for any orbits!

EoM **non-separable** because of R_s, Z_s, and V^t_s, V^ϕ_s

[1]Original form derived in Witzany, PRD 100 (2019) 10, 104030

Radial and polar spin corrections

$$\frac{\mathrm{d}r_{\mathrm{s}}}{\mathrm{d}\lambda} = \pm\frac{1}{2\sqrt{R_{\mathrm{g}}}}\left(\frac{\partial R_{\mathrm{g}}}{\partial r_{\mathrm{g}}}\delta r + \sum_{i=1}^{3}\frac{\partial R_{\mathrm{g}}}{\partial C_i}\delta C_i + R_{\mathrm{s}}\right)$$

$$\frac{\mathrm{d}z_{\mathrm{s}}}{\mathrm{d}\lambda} = \pm\frac{1}{2\sqrt{Z_{\mathrm{g}}}}\left(\frac{\partial Z_{\mathrm{g}}}{\partial z_{\mathrm{g}}}\delta z + \sum_{i=1}^{3}\frac{\partial Z_{\mathrm{g}}}{\partial C_i}\delta C_i + Z_{\mathrm{s}}\right)$$

$(C_1, C_2, C_3) = (E_{\mathrm{g}}, L_{\mathrm{zg}}, K_{\mathrm{g}})$.

$$G_{\mathrm{s}} = s_{\parallel} G_{\mathrm{s}\parallel}(r_{\mathrm{g}}, z_{\mathrm{g}}) + s_{\perp}\sin(\psi_{\mathrm{p}}) G_{\mathrm{s}\perp}(r_{\mathrm{g}}, z_{\mathrm{g}}) + s_{\perp}\cos(\psi_{\mathrm{p}}) G_{s\perp}(r_{\mathrm{g}}, z_{\mathrm{g}})$$

for $G_{\mathrm{s}} = R_{\mathrm{s}}, Z_{\mathrm{s}}$.

EoM **non-separable** because of $R_{\mathrm{s}}, Z_{\mathrm{s}}$!

$$\delta y = s_{\parallel}\delta y_{\parallel}(r_{\mathrm{g}}, z_{\mathrm{g}}) + s_{\perp}\sin(\psi_{\mathrm{p}})\delta y_{\perp}^{sn}(r_{\mathrm{g}}, z_{\mathrm{g}}) + s_{\perp}\cos(\psi_{\mathrm{p}})\delta y_{\perp}^{cn}(r_{\mathrm{g}}, z_{\mathrm{g}})$$

for $\mathrm{y} = \mathrm{r}, \mathrm{z}$. Removable singularities at the geodesic turning points

Azimuthal and coordinate time spin corrections

$$\frac{\mathrm{d}t_{\mathrm{s}}}{\mathrm{d}\lambda} = \frac{\partial V^t_{\mathrm{rg}}}{\partial r_{\mathrm{g}}}\delta r + \frac{\partial V^t_{\mathrm{zg}}}{\partial z_{\mathrm{g}}}\delta z + \sum_{i=1}^{2} \frac{\partial V^t_{\mathrm{g}}}{\partial C_i}\delta C_i + V^t_{\mathrm{s}}$$

$$\frac{\mathrm{d}\phi_{\mathrm{s}}}{\mathrm{d}\lambda} = \frac{\partial V^\phi_{\mathrm{rg}}}{\partial r_{\mathrm{g}}}\delta r + \frac{\partial V^\phi_{\mathrm{zg}}}{\partial z_{\mathrm{g}}}\delta z + \sum_{i=1}^{2} \frac{\partial V^\phi_{\mathrm{g}}}{\partial C_i}\delta C_i + V^\phi_{\mathrm{s}}$$

$(C_1, C_2, C_3) = (E_{\mathrm{g}}, L_{\mathrm{zg}}, K_{\mathrm{g}})$

$$V^{t,\phi}_{\mathrm{s}} = s_{\parallel} V^{t,\phi}_{s\parallel}(r_{\mathrm{g}}, z_{\mathrm{g}}) + s_{\perp} \sin(\psi_{\mathrm{p}}) V^{t,\phi}_{\mathrm{s}\perp}(r_{\mathrm{g}}, z_{\mathrm{g}}) + s_{\perp} \cos(\psi_{\mathrm{p}}) V^{t,\phi}_{\mathrm{s}\perp}(r_{\mathrm{g}}, z_{\mathrm{g}})$$

EoM **non-separable** because of $V^t_{\mathrm{s}}, V^\phi_{\mathrm{s}}$!
But easy to solve with Fourier series once you have δr and δz.

What is fixed and what is shifted? - part 1

We solved EoM in two parametrizations:

- fixed constants of motion (FC): $\delta E = \delta L_z = \delta K = 0$ ($\delta C_i = 0$)
- fixed (on average) turning points (DH): $\left\langle r_{1\mathrm{s}}^{\mathsf{DH}} \right\rangle = \left\langle r_{2\mathrm{s}}^{\mathsf{DH}} \right\rangle = \left\langle z_{1\mathrm{s}}^{\mathsf{DH}} \right\rangle = 0$

Easier to compute frequencies in the FC way:

$$\Upsilon_{\mathrm{ys}}^{\mathsf{FC}} = \frac{\Upsilon_{\mathrm{yg}}}{2\pi} \int_0^{2\pi} \mathrm{d}\chi_{\mathrm{y}} f(y_{\mathrm{g}}(\chi_{\mathrm{y}}), \mathsf{K}, \mathsf{E}, \Pi) \qquad \mathrm{y} = \mathrm{r}, \mathrm{z}$$

Similar expressions for $\Upsilon_{\mathrm{ts}}^{\mathsf{FC}}$ and $\Upsilon_{\phi\mathrm{s}}^{\mathsf{FC}}$

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Now it's plotting time!

Spin flags! - fixed constants of motion

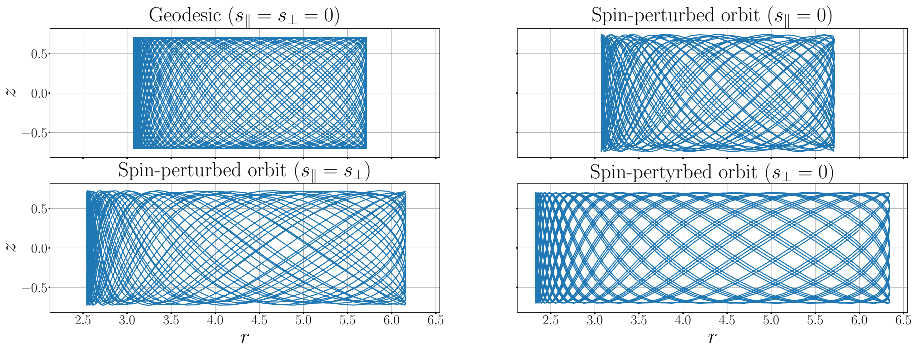
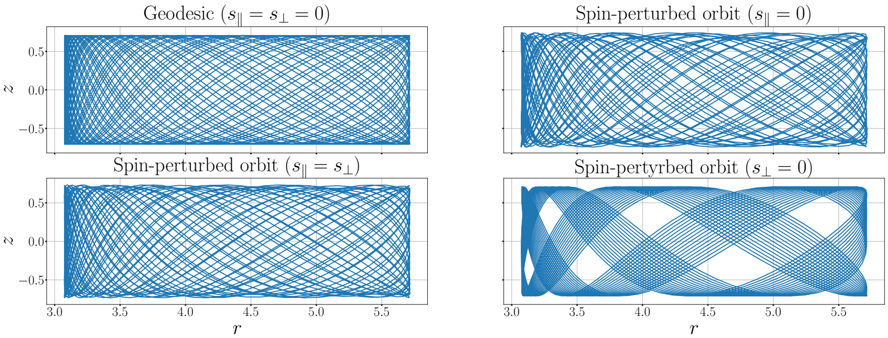


Figure: r vs z plots, $q = 1/20$

$a = 0.9, p = 4, e = 0.3, z_{1\mathrm{g}} = 1/\sqrt{2}$

Spin flags! - fixed turning points (on average)



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Projection over the xy plane

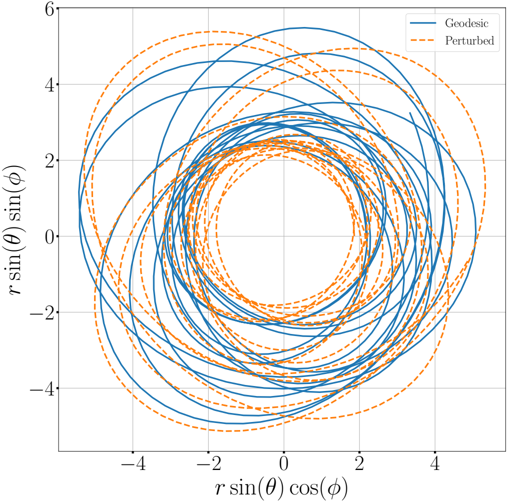

Figure: Projection over the xy plane. Fixed constants of motion, q = 1/20
$a = 0.9, p = 4, e = 0.3, z_{1\mathrm{g}} = 1/\sqrt{2}, s_{\parallel} = s_{\perp} = 1/\sqrt{2}$

Gravitational waveforms for a spin-precessing secondary

Teukolsky waveforms for spin precessing secondary

$$h = -\frac{2\mu}{r}\sum_{\ell m\vec{\kappa}}\left(\mathcal{A}^{\rm g}_{\ell m\vec{k}} + q\delta\mathcal{A}^{\rm s}_{\ell m\vec{k}}\right)e^{im\varphi}e^{-i(\omega^{\rm g}_{m\vec{\kappa}}+qs_{\parallel}\omega^{\rm s}_{m\vec{\kappa}})}$$

where $u = t - r^*$, $\vec{\kappa} = (n, k, j)$, $j = -1, 0, 1$

$$\omega^{\rm g}_{m\vec{\kappa}} = m\Omega_{\phi{\rm g}} + n\Omega_{\rm rg} + k\Omega_{\rm zg} + j\Omega_{\rm pg}$$
$$\omega^{\rm s}_{m\vec{\kappa}} = m\Omega_{\phi{\rm s}} + n\Omega_{\rm rs} + k\Omega_{\rm zs} + j\Omega_{\rm ps}$$

$\mathcal{A}^{\rm g}_{\ell mnk1} = \mathcal{A}^{\rm g}_{\ell mnk-1} = 0$ (no dependence on $\psi_{\rm p}$)

$s_{\parallel}$ terms: $\delta\mathcal{A}^{\rm s}_{\ell mnk0}$ $s_{\perp}$ terms: $\delta\mathcal{A}^{\rm s}_{\ell mnk\pm1}$

Write precessing amplitudes as

$$\delta\mathcal{A}^{\rm s}_{\ell mnk\pm1} = |\delta\mathcal{A}^{s}_{\ell mnk\pm1}|e^{i\arg(\delta\mathcal{A}^{\rm s}_{\ell mnk\pm1})}$$

- $\arg(\delta\mathcal{A}^{s}_{\ell mnk\pm1})$ is a 2PA term
- Detecting $|\delta\mathcal{A}^{\rm s}_{\ell mnk\pm1}|$ requires SNR $\sim 1/q$...[2]

[2] See Burke, Piovano+, PRD 109 (2024) 12, 124048

Computation asymptotic amplitudes

- Computed dominant GW fluxes $\mathcal{F}^{\rm g}$ and correction $\mathcal{F}^{\rm s}$ (both for energy and angular momentum)

$$\mathcal{F}^{\rm g} \propto |\mathcal{A}^{\rm g}_{\ell mnk0}| \qquad \mathcal{F}^{\rm s} \propto q\mathcal{A}^{\rm g*}_{\ell mnk0}\delta\mathcal{A}^{\rm s}_{\ell mnk0} + \text{c.c.}$$

- $\mathcal{F}^{\rm s}$ independent on $s_\perp$ because $\mathcal{A}^{\rm g*}_{\ell mnk\pm1} = 0$
- **Excellent agreement** with Skuopy+, PRD 108 (2023) 4, 044041
- Still need to compute amplitude correction to $s_\perp$: $\delta\mathcal{A}^{\rm s}_{\ell mnk\pm1}$

Conclusions and future perspective

Conclusions

We solved MPD equations in 1st form for generic orbits and spin orientation

- excellent agreement with previous results with MPD equations in 2nd order form
- maps between fixed turning points (on average) and fixed constant of motion parametrizations
- ready-to-use expressions for spin corrections to the frequencies

TO DO list

- compute Teukolsky waveform amplitudes with spin-precession...

Paper on arXiv ~~before mid-August~~ ~~before XMas~~ soon!

Final notes and acknowledgments

- Special thanks to Viktor Skuopy and Lisa Drummond for kindly share their codes and data!
- A huge thanks to my collaborators Christiana, Jake and Vojtěch

- this work is proudly powered by the mighty BHPToolkit

https://bhptoolkit.org/

Check it out! It is free!

Recommended by 9 out of 10 self-force researchers!*

*Based on pure speculations and not a real survey

- Feel free to contact me at gabriel.piovano@ucd.ie

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Thank you for you attention!

Backup slides

What is fixed and what is shifted? - part 2

Shifts to the constants of motion and frequencies in the DH way

$$\left\langle r_{1\mathrm{s}}^{\mathsf{DH}}\right\rangle = \left\langle r_{2\mathrm{s}}^{\mathsf{DH}}\right\rangle = \left\langle z_{1\mathrm{s}}^{\mathsf{DH}}\right\rangle = 0$$

$$\begin{pmatrix} \delta E^{\mathsf{DH}} \\ \delta L_z^{\mathsf{DH}} \\ \delta K^{\mathsf{DH}} \end{pmatrix} = -\begin{pmatrix} \frac{\partial E_{\mathrm{g}}}{\partial r_{1\mathrm{g}}} & \frac{\partial E_{\mathrm{g}}}{\partial r_{2\mathrm{g}}} & \frac{\partial E_{\mathrm{g}}}{\partial z_{1\mathrm{g}}} \\ \frac{\partial L_{\mathrm{zg}}}{\partial r_{1\mathrm{g}}} & \frac{\partial L_{\mathrm{zg}}}{\partial r_{2\mathrm{g}}} & \frac{\partial L_{\mathrm{zg}}}{\partial z_{1\mathrm{g}}} \\ \frac{\partial K_{\mathrm{g}}}{\partial r_{1\mathrm{g}}} & \frac{\partial K_{\mathrm{g}}}{\partial r_{2\mathrm{g}}} & \frac{\partial K_{\mathrm{g}}}{\partial z_{1\mathrm{g}}} \end{pmatrix} \cdot \begin{pmatrix} \left\langle r_{1\mathrm{s}}^{\mathsf{FC}}\right\rangle \\ \left\langle r_{2\mathrm{s}}^{\mathsf{FC}}\right\rangle \\ \left\langle z_{1\mathrm{s}}^{\mathsf{FC}}\right\rangle \end{pmatrix}$$

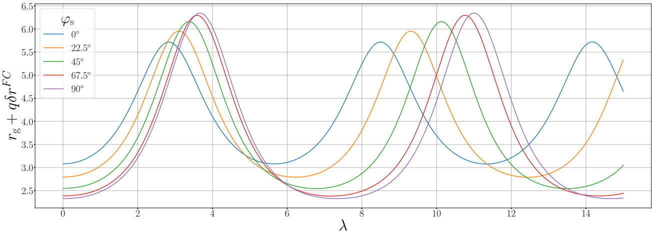
$$\Upsilon_{\mathrm{xs}}^{\mathsf{DH}} = \Upsilon_{\mathrm{xs}}^{\mathsf{FC}} - \frac{\partial \Upsilon_{\mathrm{xg}}}{\partial r_{1\mathrm{g}}}\langle r_{1\mathrm{s}}^{\mathsf{FC}}\rangle - \frac{\partial \Upsilon_{\mathrm{xg}}}{\partial r_{2g}}\langle r_{2\mathrm{s}}^{\mathsf{FC}}\rangle - \frac{\partial \Upsilon_{xg}}{\partial z_{1\mathrm{g}}}\langle z_{1\mathrm{s}}^{\mathsf{FC}}\rangle \qquad \mathrm{x} = \mathrm{t}, \mathrm{r}, \mathrm{z}, \phi$$

Amazing agreement with

Drummond&Hughes, PRD 105 (2022) 12, 124041 and PRD 105 (2022) 12, 124040!

Perturbed radial motion

$$s_{\parallel} = \sin\varphi_s \qquad s_{\perp} = \cos\varphi_s$$



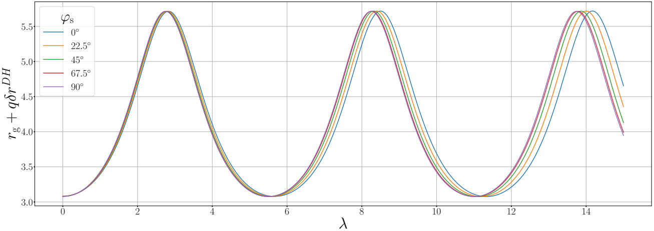


Figure: Top: fixed constants of motion. Bottom: fixed turning points (on average)

$$q = 1/20,\ a = 0.9, p = 4, e = 0.3, z_{1\mathrm{g}} = 1/\sqrt{2}$$