Symmetry reduction of gravitational Lagrangians

based on: G. Frausto, I. Kolář, TM, Ch. Torre, (soon on arXiv)

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Outline

Weyl trick to derive the Schwarzschild solution [Weyl (1917)]

Einstein field equations
[Einstein (Nov 25, 1915)]

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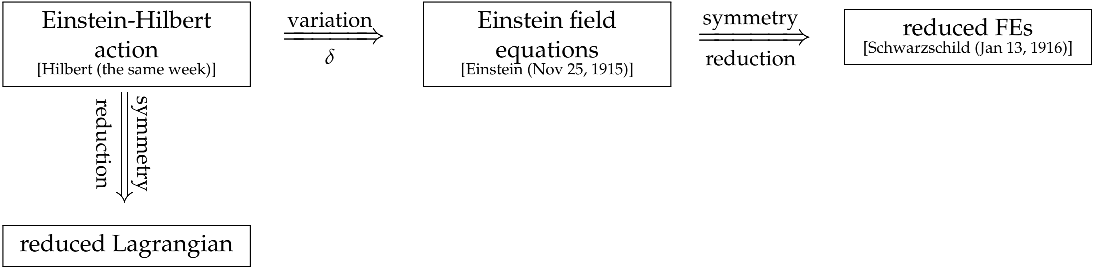

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$$\boldsymbol{g} = -a(r)b(r)^2\,\mathbf{d}t^2 + a(r)^{-1}\mathbf{d}r^2 + r^2\boldsymbol{g}_{S_2}: \qquad \int \mathrm{d}x^4\sqrt{-g}R \implies \int \mathrm{d}r\, r(a-1)b'$$

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When are they equivalent???

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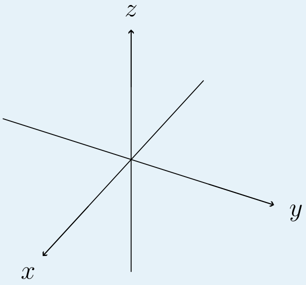
Isometries

Infinitesimal group action Γ on M

given by d-dim Lie algebra of isometry generators $\boldsymbol{X} \in \Gamma$ (Killing vectors)

Example: symmetries of S_2



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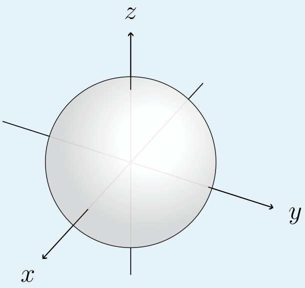

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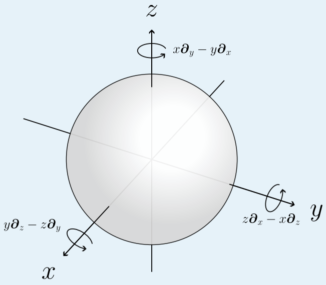

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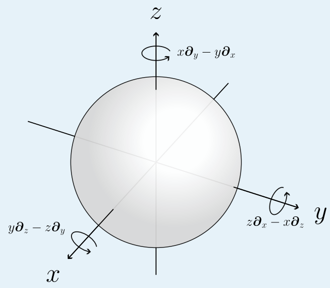

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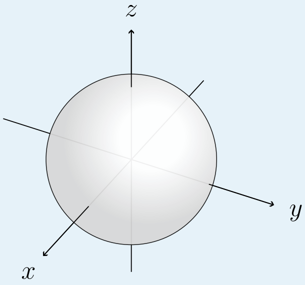

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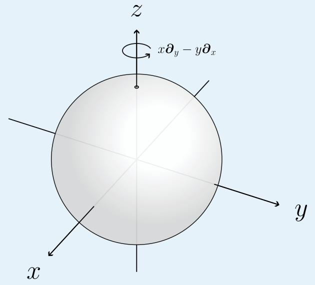

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relation to dim p of Γ_x: $l = d - p$

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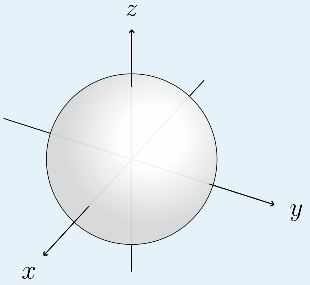

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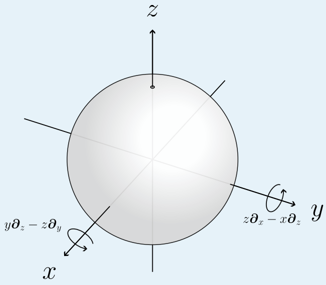

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orbit of $(0, 0, 1)$ is unit sphere

Symmetry reduction of Lagrangian

Purely gravitational theory on a 4-dimensional spacetime

$$S = \int_{\mathrm{M}} \underline{\epsilon}(\boldsymbol{g}) L[\boldsymbol{g}]$$

- Levi-Civita tensor $\underline{\epsilon}(\boldsymbol{g})$ defines the volume element $\sqrt{-g}\mathrm{d}^4x$
- Lagrangian $L[\boldsymbol{g}]$ constructed from $\boldsymbol{g}$, $\boldsymbol{R}$, $\boldsymbol{\nabla} \cdots \boldsymbol{\nabla}\boldsymbol{R}$
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- accomplished by contraction with Γ-invariant antisymmetric l-chain $\chi = \chi^{i_1 \dots i_l} X_{i_1} \cdots X_{i_l}$

$$\pounds_{X_i} \chi = 0, \quad i = 1, \dots, d, \qquad \chi^{i_1 \dots i_l} = \chi^{[i_1 \dots i_l]}$$

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- reduced Lagrangian $\underline{\hat{L}} = \boldsymbol{\chi} \bullet \underline{L}[\hat{\boldsymbol{g}}] = \underline{\hat{\epsilon}}(\hat{\boldsymbol{g}}) L[\hat{\boldsymbol{g}}]$, (where $\underline{\hat{\epsilon}}(\hat{\boldsymbol{g}}) = \boldsymbol{\chi} \bullet \underline{\epsilon}(\hat{\boldsymbol{g}})$)

Principle of symmetric criticality

Variation of Lagrangian 4-form

$$\delta \underline{L} = \underline{\boldsymbol{E}}(\underline{L}) \cdot \delta \boldsymbol{g} + \underline{d\eta}(\delta \boldsymbol{g})$$

- Euler-Lagrange expression $\underline{\boldsymbol{E}}(\underline{L})$ gives the field equations $\underline{\boldsymbol{E}}(\underline{L})[\boldsymbol{g}] = 0$
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Principle of symmetric criticality [Palais (1979), M. E. Fels, C. G. Torre (2002)]

Variation of Lagrangian commutes with symmetry reduction for all possible theories:

$$\forall \underline{L} : \underline{\boldsymbol{E}}(\underline{L})[\hat{\boldsymbol{g}}] = 0 \Longleftrightarrow \underline{\boldsymbol{E}}(\hat{\underline{L}})[\hat{\boldsymbol{g}}] = 0$$

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Two conditions imposed solely on Γ are necessary and sufficient for validity of PSC.

PSC1 “Lie algebra condition”

- PSC1 ensures that the reduction of the boundary term $\underline{d\eta}$ is a boundary term $\underline{d\hat{\eta}}$ for the reduced Lagrangian

$$\delta\underline{\hat{L}} = \underline{\boldsymbol{E}}(\underline{\hat{L}}) \cdot \delta\hat{\boldsymbol{g}} + \underline{d\hat{\eta}}(\delta\hat{\boldsymbol{g}})$$

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$\pounds_v \chi = 0$ for all Γ-invariant vector fields $\boldsymbol{v}$ ($\pounds_{X_i}\boldsymbol{v} = 0$)

- if PSC1 satisfied then Euler-Lagrange equations of the reduced Lagrangian always yield at least a subset of the reduced equations

PSC2 "(local) Palais condition"

- PSC2 arises from the requirement that this subset contains all reduced equations i.e. all reduced FEs appear in the reduction of Euler-Lagrange term $\underline{\boldsymbol{E}}(\underline{\hat{L}})$

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Let S_x and S_x^* denote the vector space of Γ_x-invariant $\binom{0}{2}$ and $\binom{2}{0}$ tensors at x, respectively. Denote by V_x^0 the vector space of $\binom{2}{0}$ tensors which have a vanishing scalar contraction with all elements of S_x. Then in the neighborhood of x:

$$S_x^* \cap V_x^0 = \{0\}$$

i.e. there is no Γ_x-invariant $\binom{2}{0}$ tensor that contracts to zero with all Γ_x-invariant $\binom{0}{2}$ tensors

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- PSC2 satisfied iff the isotropy algebra contains no null-rotation subalgebra

Classification of infinitesimal group actions

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Hicks classification [Hicks, Ph.D. thesis (2016)]

based on classifying isometry algebra and isotropy subalgebra pairs (Γ, Γ_x)

- isotropy subalgebras Γ_x can be identified with subalgebras of the Lorentz algebra
- cases denoted by $[d, l, c]$
  1. d is dim of Γ
  2. l is dim of orbits ($l = d - p$)
  3. c enumerates possible cases of given dimensions
- explicit infinitesimal generators given for each case

PSC-compatible infinitesimal group actions

Hicks #	PSC1	PSC2	PSC
[3,2,1]	✓	✓	✓
[3,2,2]	✓	✓	✓
[3,2,3]	✓	✓	✓
[3,2,4]	✓	✓	✓
[3,2,5]	✓	✓	✓
[3,3,1]	✗	✓	✗
[3,3,2]	✓	✓	✓
[3,3,3]	✓	✓	✓
[3,3,4]	✗	✓	✗
[3,3,5]	✗	✓	✗
[3,3,6]	✗	✓	✗
[3,3,7]	✗	✓	✗
[3,3,8]	✓	✓	✓
[3,3,9]	✓	✓	✓
[4,3,1]	✓	✓	✓
[4,3,2]	✓	✓	✓
[4,3,3]	✓	✓	✓
[4,3,4]	✓	✓	✓
[4,3,5]	✓	✓	✓
[4,3,6]	✓	✓	✓
[4,3,7]	✗	✓	✗
[4,3,8]	✓	✓	✓
[4,3,9]	✓	✓	✓

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[4,3,10]	✓	✓	✓
[4,3,11]	✓	✓	✓
[4,3,12]	✗	✓	✗
[4,3,13]	✓	✗	✗
[4,3,14]	✓	✗	✗
[4,3,15]	✓	✗	✗
[4,3,16]	✓	✗	✗
[4,3,17]	✓	✗	✗
[4,3,18]	✓	✗	✗
[4,3,19]	✓	✗	✗
[4,3,20]	✓	✗	✗
[4,4,1]	✓	✓	✓
[4,4,2]	✓	✓	✓
[4,4,3]	✗	✓	✗
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[4,4,9]	✓	✓	✓
[4,4,10]	✗	✓	✗
[4,4,11]	✗	✓	✗
[4,4,12]	✗	✓	✗

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[4,4,16]	✗	✓	✗
[4,4,17]	✗	✓	✗
[4,4,18]	✓	✓	✓
[4,4,19]	✗	✓	✗
[4,4,20]	✗	✓	✗
[4,4,21]	✗	✓	✗
[4,4,22]	✓	✓	✓
[4,4,23]	✗	✓	✗
[5,4,-1]	✓	✗	✗
[5,4,-2]	✗	✗	✗
[5,4,-3]	✗	✗	✗
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[5,4,-5]	✗	✗	✗
[5,4,-6]	✗	✗	✗
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[5,4,9]	✗	✓	✗
[5,4,10]	✗	✗	✗
[5,4,11]	✓	✗	✗
[6,3,1]	✓	✓	✓
[6,3,2]	✓	✓	✓
[6,3,3]	✓	✓	✓
[6,3,4]	✓	✓	✓
[6,3,5]	✓	✓	✓
[6,3,6]	✓	✓	✓
[6,4,-1]	✓	✗	✗
[6,4,1]	✓	✓	✓
[6,4,2]	✓	✓	✓
[6,4,3]	✓	✓	✓
[6,4,4]	✓	✓	✓
[6,4,5]	✓	✓	✓
[6,4,6]	✓	✗	✗
[7,4,1]	✓	✓	✓
[7,4,2]	✓	✓	✓
[7,4,3]	✓	✓	✓
[7,4,4]	✓	✓	✓
[7,4,5]	✓	✗	✗

PSC-compatible infinitesimal group actions

Hicks #	PSC1	PSC2	PSC
[3,2,1]	✓	✓	✓
[3,2,2]	✓	✓	✓
[3,2,3]	✓	✓	✓
[3,2,4]	✓	✓	✓
[3,2,5]	✓	✓	✓
[3,3,1]	✗	✓	✗
[3,3,2]	✓	✓	✓
[3,3,3]	✓	✓	✓
[3,3,4]	✗	✓	✗
[3,3,5]	✗	✓	✗
[3,3,6]	✗	✓	✗
[3,3,7]	✗	✓	✗
[3,3,8]	✓	✓	✓
[3,3,9]	✓	✓	✓
[4,3,1]	✓	✓	✓
[4,3,2]	✓	✓	✓
[4,3,3]	✓	✓	✓
[4,3,4]	✓	✓	✓
[4,3,5]	✓	✓	✓
[4,3,6]	✓	✓	✓
[4,3,7]	✗	✓	✗
[4,3,8]	✓	✓	✓
[4,3,9]	✓	✓	✓

Hicks #	PSC1	PSC2	PSC
[4,3,10]	✓	✓	✓
[4,3,11]	✓	✓	✓
[4,3,12]	✗	✓	✗
[4,3,13]	✓	✗	✗
[4,3,14]	✓	✗	✗
[4,3,15]	✓	✗	✗
[4,3,16]	✓	✗	✗
[4,3,17]	✓	✗	✗
[4,3,18]	✓	✗	✗
[4,3,19]	✓	✗	✗
[4,3,20]	✓	✗	✗
[4,4,1]	✓	✓	✓
[4,4,2]	✓	✓	✓
[4,4,3]	✗	✓	✗
[4,4,4]	✗	✓	✗
[4,4,5]	✗	✓	✗
[4,4,6]	✗	✓	✗
[4,4,7]	✗	✓	✗
[4,4,8]	✗	✓	✗
[4,4,9]	✓	✓	✓
[4,4,10]	✗	✓	✗
[4,4,11]	✗	✓	✗
[4,4,12]	✗	✓	✗

Hicks #	PSC1	PSC2	PSC
[4,4,13]	✗	✓	✗
[4,4,14]	✗	✓	✗
[4,4,15]	✗	✓	✗
[4,4,16]	✗	✓	✗
[4,4,17]	✗	✓	✗
[4,4,18]	✓	✓	✓
[4,4,19]	✗	✓	✗
[4,4,20]	✗	✓	✗
[4,4,21]	✗	✓	✗
[4,4,22]	✓	✓	✓
[4,4,23]	✗	✓	✗
[5,4,-1]	✓	✗	✗
[5,4,-2]	✗	✗	✗
[5,4,-3]	✗	✗	✗
[5,4,-4]	✗	✗	✗
[5,4,-5]	✗	✗	✗
[5,4,-6]	✗	✗	✗
[5,4,1]	✓	✓	✓
[5,4,2]	✓	✓	✓
[5,4,3]	✓	✓	✓
[5,4,4]	✗	✓	✗
[5,4,5]	✗	✓	✗
[5,4,6]	✓	✓	✓

Hicks #	PSC1	PSC2	PSC
[5,4,7]	✓	✓	✓
[5,4,8]	✗	✓	✗
[5,4,9]	✗	✓	✗
[5,4,10]	✗	✗	✗
[5,4,11]	✓	✗	✗
[6,3,1]	✓	✓	✓
[6,3,2]	✓	✓	✓
[6,3,3]	✓	✓	✓
[6,3,4]	✓	✓	✓
[6,3,5]	✓	✓	✓
[6,3,6]	✓	✓	✓
[6,4,-1]	✓	✗	✗
[6,4,1]	✓	✓	✓
[6,4,2]	✓	✓	✓
[6,4,3]	✓	✓	✓
[6,4,4]	✓	✓	✓
[6,4,5]	✓	✓	✓
[6,4,6]	✓	✗	✗
[7,4,1]	✓	✓	✓
[7,4,2]	✓	✓	✓
[7,4,3]	✓	✓	✓
[7,4,4]	✓	✓	✓
[7,4,5]	✓	✗	✗

- number of cases

total	PSC1	PSC2	PSC
92	57	71	44

PSC-compatible infinitesimal group actions

Hicks #	PSC1	PSC2	PSC
[3,2,1]	✓	✓	✓
[3,2,2]	✓	✓	✓
[3,2,3]	✓	✓	✓
[3,2,4]	✓	✓	✓
[3,2,5]	✓	✓	✓
[3,3,1]	✗	✓	✗
[3,3,2]	✓	✓	✓
[3,3,3]	✓	✓	✓
[3,3,4]	✗	✓	✗
[3,3,5]	✗	✓	✗
[3,3,6]	✗	✓	✗
[3,3,7]	✗	✓	✗
[3,3,8]	✓	✓	✓
[3,3,9]	✓	✓	✓
[4,3,1]	✓	✓	✓
[4,3,2]	✓	✓	✓
[4,3,3]	✓	✓	✓
[4,3,4]	✓	✓	✓
[4,3,5]	✓	✓	✓
[4,3,6]	✓	✓	✓
[4,3,7]	✗	✓	✗
[4,3,8]	✓	✓	✓
[4,3,9]	✓	✓	✓

Hicks #	PSC1	PSC2	PSC
[4,3,10]	✓	✓	✓
[4,3,11]	✓	✓	✓
[4,3,12]	✗	✓	✗
[4,3,13]	✓	✗	✗
[4,3,14]	✓	✗	✗
[4,3,15]	✓	✗	✗
[4,3,16]	✓	✗	✗
[4,3,17]	✓	✗	✗
[4,3,18]	✓	✗	✗
[4,3,19]	✓	✗	✗
[4,3,20]	✓	✗	✗
[4,4,1]	✓	✓	✓
[4,4,2]	✓	✓	✓
[4,4,3]	✗	✓	✗
[4,4,4]	✗	✓	✗
[4,4,5]	✗	✓	✗
[4,4,6]	✗	✓	✗
[4,4,7]	✗	✓	✗
[4,4,8]	✗	✓	✗
[4,4,9]	✓	✓	✓
[4,4,10]	✗	✓	✗
[4,4,11]	✗	✓	✗
[4,4,12]	✗	✓	✗

Hicks #	PSC1	PSC2	PSC
[4,4,13]	✗	✓	✗
[4,4,14]	✗	✓	✗
[4,4,15]	✗	✓	✗
[4,4,16]	✗	✓	✗
[4,4,17]	✗	✓	✗
[4,4,18]	✓	✓	✓
[4,4,19]	✗	✓	✗
[4,4,20]	✗	✓	✗
[4,4,21]	✗	✓	✗
[4,4,22]	✓	✓	✓
[4,4,23]	✗	✓	✗
[5,4,-1]	✓	✗	✗
[5,4,-2]	✗	✗	✗
[5,4,-3]	✗	✗	✗
[5,4,-4]	✗	✗	✗
[5,4,-5]	✗	✗	✗
[5,4,-6]	✗	✗	✗
[5,4,1]	✓	✓	✓
[5,4,2]	✓	✓	✓
[5,4,3]	✓	✓	✓
[5,4,4]	✗	✓	✗
[5,4,5]	✗	✓	✗
[5,4,6]	✓	✓	✓

Hicks #	PSC1	PSC2	PSC
[5,4,7]	✓	✓	✓
[5,4,8]	✗	✓	✗
[5,4,9]	✗	✓	✗
[5,4,10]	✗	✗	✗
[5,4,11]	✓	✗	✗
[6,3,1]	✓	✓	✓
[6,3,2]	✓	✓	✓
[6,3,3]	✓	✓	✓
[6,3,4]	✓	✓	✓
[6,3,5]	✓	✓	✓
[6,3,6]	✓	✓	✓
[6,4,-1]	✓	✗	✗
[6,4,1]	✓	✓	✓
[6,4,2]	✓	✓	✓
[6,4,3]	✓	✓	✓
[6,4,4]	✓	✓	✓
[6,4,5]	✓	✓	✓
[6,4,6]	✓	✗	✗
[7,4,1]	✓	✓	✓
[7,4,2]	✓	✓	✓
[7,4,3]	✓	✓	✓
[7,4,4]	✓	✓	✓
[7,4,5]	✓	✗	✗

- number of cases

total	PSC1	PSC2	PSC
92	57	71	44

- only 42 qualitatively different cases
  (the answer to the ultimate question of life, the universe, and everything.)

PSC-compatible infinitesimal group actions

Hicks #	PSC1	PSC2	PSC
[3,2,1]	✓	✓	✓
[3,2,2]	✓	✓	✓
[3,2,3]	✓	✓	✓
[3,2,4]	✓	✓	✓
[3,2,5]	✓	✓	✓
[3,3,1]	✗	✓	✗
[3,3,2]	✓	✓	✓
[3,3,3]	✓	✓	✓
[3,3,4]	✗	✓	✗
[3,3,5]	✗	✓	✗
[3,3,6]	✗	✓	✗
[3,3,7]	✗	✓	✗
[3,3,8]	✓	✓	✓
[3,3,9]	✓	✓	✓
[4,3,1]	✓	✓	✓
[4,3,2]	✓	✓	✓
[4,3,3]	✓	✓	✓
[4,3,4]	✓	✓	✓
[4,3,5]	✓	✓	✓
[4,3,6]	✓	✓	✓
[4,3,7]	✗	✓	✗
[4,3,8]	✓	✓	✓
[4,3,9]	✓	✓	✓
[4,3,10]	✓	✓	✓
[4,3,11]	✓	✓	✓
[4,3,12]	✗	✓	✗
[4,3,13]	✓	✗	✗
[4,3,14]	✓	✗	✗
[4,3,15]	✓	✗	✗
[4,3,16]	✓	✗	✗
[4,3,17]	✓	✗	✗
[4,3,18]	✓	✗	✗
[4,3,19]	✓	✗	✗
[4,3,20]	✓	✗	✗
[4,4,1]	✓	✓	✓
[4,4,2]	✓	✓	✓
[4,4,3]	✗	✓	✗
[4,4,4]	✗	✓	✗
[4,4,5]	✗	✓	✗
[4,4,6]	✗	✓	✗
[4,4,7]	✗	✓	✗
[4,4,8]	✗	✓	✗
[4,4,9]	✓	✓	✓
[4,4,10]	✗	✓	✗
[4,4,11]	✗	✓	✗
[4,4,12]	✗	✓	✗
[4,4,13]	✗	✓	✗
[4,4,14]	✗	✓	✗
[4,4,15]	✗	✓	✗
[4,4,16]	✗	✓	✗
[4,4,17]	✗	✓	✗
[4,4,18]	✓	✓	✓
[4,4,19]	✗	✓	✗
[4,4,20]	✗	✓	✗
[4,4,21]	✗	✓	✗
[4,4,22]	✓	✓	✓
[4,4,23]	✗	✓	✗
[5,4,-1]	✓	✗	✗
[5,4,-2]	✗	✗	✗
[5,4,-3]	✗	✗	✗
[5,4,-4]	✗	✗	✗
[5,4,-5]	✗	✗	✗
[5,4,-6]	✗	✗	✗
[5,4,1]	✓	✓	✓
[5,4,2]	✓	✓	✓
[5,4,3]	✓	✓	✓
[5,4,4]	✗	✓	✗
[5,4,5]	✗	✓	✗
[5,4,6]	✓	✓	✓
[5,4,7]	✓	✓	✓
[5,4,8]	✗	✓	✗
[5,4,9]	✗	✓	✗
[5,4,10]	✗	✗	✗
[5,4,11]	✓	✗	✗
[6,3,1]	✓	✓	✓
[6,3,2]	✓	✓	✓
[6,3,3]	✓	✓	✓
[6,3,4]	✓	✓	✓
[6,3,5]	✓	✓	✓
[6,3,6]	✓	✓	✓
[6,4,-1]	✓	✗	✗
[6,4,1]	✓	✓	✓
[6,4,2]	✓	✓	✓
[6,4,3]	✓	✓	✓
[6,4,4]	✓	✓	✓
[6,4,5]	✓	✓	✓
[6,4,6]	✓	✗	✗
[7,4,1]	✓	✓	✓
[7,4,2]	✓	✓	✓
[7,4,3]	✓	✓	✓
[7,4,4]	✓	✓	✓
[7,4,5]	✓	✗	✗

- number of cases

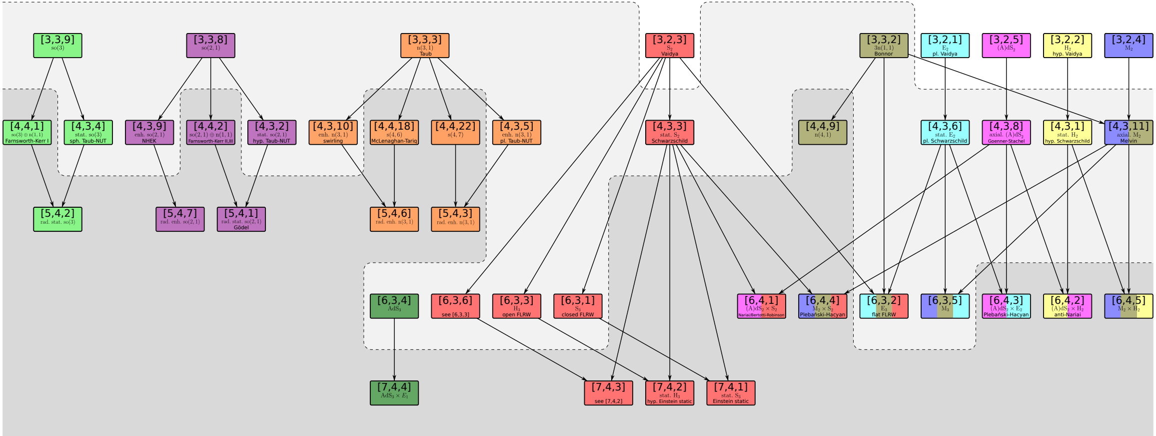
total	PSC1	PSC2	PSC
92	57	71	44

- only 42 qualitatively different cases
  (the answer to the ultimate question of life, the universe, and everything.)

- for each PSC-compatible Γ we determined the corresponding l-chains χ and Γ-invariant metrics $\hat{g}$ in adapted coordinates

$$\hat{g} = \sum_{i=1}^{s} \phi_i \boldsymbol{q}_i, \text{ where } s = \begin{cases} 2, & \text{for } [6,3,\star],\ [6,4,\star],\ [7,4,\star] \\ 4, & \text{for } [3,2,\star],\ [4,3,\star],\ [5,4,\star] \\ 10, & \text{for } [3,3,\star],\ [4,4,\star] \end{cases} \text{ and } \phi_i = \phi_i(x_1, x_2, \ldots, x_{(4-l)})$$

Relations among PSC-compatible infinitesimal group actions



$$\hat{g} = \sum_{i=1}^{s} \phi_i \boldsymbol{q}_i, \text{ where } s = \begin{cases} 2, & \text{for } [6,3,\star], [6,4,\star], [7,4,\star] \\ 4, & \text{for } [3,2,\star], [4,3,\star], [5,4,\star] \\ 10, & \text{for } [3,3,\star], [4,4,\star] \end{cases} \text{ and } \phi_i = \phi_i(x_1, x_2, \ldots, x_{(4-l)})$$

Weyl trick revisited ([4,3,3]: stationary S_2)

- infinitesimal group action

$$\Gamma = \operatorname{span}\{\cos\varphi\,\boldsymbol{\partial}_\vartheta - \cot\vartheta\sin\varphi\,\boldsymbol{\partial}_\varphi,\ \sin\varphi\,\boldsymbol{\partial}_\vartheta + \cot\vartheta\cos\varphi\,\boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_t\}$$

Weyl trick revisited ([4,3,3]: stationary S_2)

- infinitesimal group action

$$\Gamma = \operatorname{span}\{\cos\varphi\,\boldsymbol{\partial}_\vartheta - \cot\vartheta\sin\varphi\,\boldsymbol{\partial}_\varphi,\ \sin\varphi\,\boldsymbol{\partial}_\vartheta + \cot\vartheta\cos\varphi\,\boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_t\}$$

- Γ-invariant metric

$$\hat{g} = -\phi_1(r)\,\mathbf{d}t^2 + \phi_2(r)(\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r)\,\mathbf{d}r^2 + \phi_4(r)(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2)$$

Weyl trick revisited ([4,3,3]: stationary S_2)

- infinitesimal group action

$$\Gamma = \operatorname{span}\{\cos\varphi\,\boldsymbol{\partial}_\vartheta - \cot\vartheta\sin\varphi\,\boldsymbol{\partial}_\varphi,\ \sin\varphi\,\boldsymbol{\partial}_\vartheta + \cot\vartheta\cos\varphi\,\boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_t\}$$

- Γ-invariant metric

$$\hat{g} = -\phi_1(r)\,\mathbf{d}t^2 + \phi_2(r)(\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r)\,\mathbf{d}r^2 + \phi_4(r)(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2)$$

- residual gauge freedom $t \to t + A(r)$, $r \to B(r)$ do not invalidate PSC and allows us to fix $\phi_2 = 0$ and $\phi_4 = r^2$

Weyl trick revisited ([4,3,3]: stationary S_2)

- infinitesimal group action

$$\Gamma = \mathrm{span}\{\cos\varphi\,\boldsymbol{\partial}_\vartheta - \cot\vartheta\sin\varphi\,\boldsymbol{\partial}_\varphi,\ \sin\varphi\,\boldsymbol{\partial}_\vartheta + \cot\vartheta\cos\varphi\,\boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_t\}$$

- Γ-invariant metric

$$\hat{g} = -\phi_1(r)\,\mathbf{d}t^2 + \phi_2(r)(\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r)\,\mathbf{d}r^2 + \phi_4(r)(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2)$$

- residual gauge freedom $t \to t + A(r)$, $r \to B(r)$ do not invalidate PSC and allows us to fix $\phi_2 = 0$ and $\phi_4 = r^2$
- l-chain

$$\chi = \csc\vartheta\,\boldsymbol{\partial}_t \wedge \boldsymbol{\partial}_\vartheta \wedge \boldsymbol{\partial}_\varphi$$

Weyl trick revisited ([4,3,3]: stationary S_2)

- infinitesimal group action

$$\Gamma = \operatorname{span}\{\cos\varphi\,\boldsymbol{\partial}_\vartheta - \cot\vartheta\sin\varphi\,\boldsymbol{\partial}_\varphi,\ \sin\varphi\,\boldsymbol{\partial}_\vartheta + \cot\vartheta\cos\varphi\,\boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_t\}$$

- Γ-invariant metric

$$\hat{g} = -\phi_1(r)\,\mathbf{d}t^2 + \phi_2(r)(\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r)\,\mathbf{d}r^2 + \phi_4(r)(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2)$$

- residual gauge freedom $t \to t + A(r)$, $r \to B(r)$ do not invalidate PSC and allows us to fix $\phi_2 = 0$ and $\phi_4 = r^2$
- l-chain

$$\chi = \csc\vartheta\,\boldsymbol{\partial}_t \wedge \boldsymbol{\partial}_\vartheta \wedge \boldsymbol{\partial}_\varphi$$

- Levi-Civita tensor

$$\underline{\epsilon}(\hat{g}) = r^2\sin\vartheta\sqrt{\phi_1\phi_3}\,\mathbf{d}t \wedge \mathbf{d}r \wedge \mathbf{d}\vartheta \wedge \mathbf{d}\varphi$$

Weyl trick revisited ([4,3,3]: stationary S_2)

- infinitesimal group action

$$\Gamma = \mathrm{span}\{\cos\varphi\,\boldsymbol{\partial}_\vartheta - \cot\vartheta\sin\varphi\,\boldsymbol{\partial}_\varphi,\ \sin\varphi\,\boldsymbol{\partial}_\vartheta + \cot\vartheta\cos\varphi\,\boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_\varphi,\ \boldsymbol{\partial}_t\}$$

- Γ-invariant metric

$$\hat{g} = -\phi_1(r)\,\mathbf{d}t^2 + \phi_2(r)(\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r)\,\mathbf{d}r^2 + \phi_4(r)(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2)$$

- residual gauge freedom $t \to t + A(r)$, $r \to B(r)$ do not invalidate PSC and allows us to fix $\phi_2 = 0$ and $\phi_4 = r^2$
- l-chain

$$\chi = \csc\vartheta\,\boldsymbol{\partial}_t \wedge \boldsymbol{\partial}_\vartheta \wedge \boldsymbol{\partial}_\varphi$$

- Levi-Civita tensor

$$\underline{\epsilon}(\hat{g}) = r^2\sin\vartheta\sqrt{\phi_1\phi_3}\,\mathbf{d}t \wedge \mathbf{d}r \wedge \mathbf{d}\vartheta \wedge \mathbf{d}\varphi$$

- reduced Lagrangian 1-form

$$\underline{\hat{L}} = \chi \bullet \underline{\epsilon}(\hat{g})L[\hat{g}] = r^2\sqrt{\phi_1\phi_3}\,L[\hat{g}]\,\mathbf{d}r$$

Symmetry reduction for flat FLRW cosmologies ([6,3,2]: E_3)

- infinitesimal group action

$$\Gamma = \mathrm{span}\{\partial_x,\ \partial_y,\ \partial_z,\ x\,\partial_y - y\,\partial_x,\ y\,\partial_z - z\,\partial_y,\ z\,\partial_x - x\,\partial_z\}$$

Symmetry reduction for flat FLRW cosmologies ([6,3,2]: E_3)

- infinitesimal group action

$$\Gamma = \mathrm{span}\{\boldsymbol{\partial}_x,\ \boldsymbol{\partial}_y,\ \boldsymbol{\partial}_z,\ x\,\boldsymbol{\partial}_y - y\,\boldsymbol{\partial}_x,\ y\,\boldsymbol{\partial}_z - z\,\boldsymbol{\partial}_y,\ z\,\boldsymbol{\partial}_x - x\,\boldsymbol{\partial}_z\}$$

- Γ-invariant metric

$$\hat{g} = -\phi_1(t)\,\mathbf{d}t^2 + \phi_2(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2)$$

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- l-chain

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$$\chi = \boldsymbol{\partial}_x \wedge \boldsymbol{\partial}_y \wedge \boldsymbol{\partial}_z$$

- Levi-Civita tensor

$$\sqrt{\phi_1\phi_2^3}\,\mathbf{d}t \wedge \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z$$

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- Γ-invariant metric

$$\hat{g} = -\phi_1(t)\,\mathbf{d}t^2 + \phi_2(t)\left(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2\right)$$

- residual gauge freedom $t \to A(t)$ which would allow us to set $\phi_1 = 1$ breaks PSC
- l-chain

$$\chi = \boldsymbol{\partial}_x \wedge \boldsymbol{\partial}_y \wedge \boldsymbol{\partial}_z$$

- Levi-Civita tensor

$$\sqrt{\phi_1\phi_2^3}\,\mathbf{d}t \wedge \mathbf{d}x \wedge \mathbf{d}y \wedge \mathbf{d}z$$

- reduced Lagrangian 1-form

$$\underline{\hat{L}} = \chi \bullet \underline{\epsilon}(\hat{g})L[\hat{g}] = \sqrt{\phi_1\phi_3^2}\,L[\hat{g}]\,\mathbf{d}t$$

Conclusion

- We established the essential ingredients for a successful symmetry reduction:
  1. identified all possible PSC-compatible infinitesimal group actions Γ
  2. determined corresponding Γ-invariant metrics
     and l-chains in adapted coordinates
  3. minimized the amount of unknown functions
     employing residual gauge freedom compliant with PSC
- As a by-product, we implemeted the symmetry reduction of Lagrangians in MATHEMATICA employing the `xAct` package.

Conclusion

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  3. minimized the amount of unknown functions employing residual gauge freedom compliant with PSC
- As a by-product, we implemeted the symmetry reduction of Lagrangians in MATHEMATICA employing the `xAct` package.

Thank you! Obrigado!

List of infinitesimal group actions Γ

Hicks #	Isometry algebra Γ
[3,2,1]	$\partial_{y_3},\ \partial_{y_4},\ -y_4\,\partial_{y_3} + y_3\,\partial_{y_4}$
[3,2,2]	$\cos y_4\,\partial_{y_3} - \operatorname{cth} y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \operatorname{cth} y_3\,\partial_{y_4},\ \partial_{y_4}$
[3,2,3]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4}$
[3,2,4]	$\partial_{y_1},\ \partial_{y_2},\ y_2\,\partial_{y_1} + y_1\,\partial_{y_2}$
[3,2,5]	$-\sin y_1 \operatorname{th} y_2\,\partial_{y_1} + \cos y_1\,\partial_{y_2},\ -\cos y_1 \operatorname{th} y_2\,\partial_{y_1} - \sin y_1\,\partial_{y_2},\ \partial_{y_1}$
[3,3,2]	$\partial_{y_1},\ \partial_{y_2},\ \partial_{y_4}$
[3,3,3]	$\partial_{y_1},\ \partial_{y_4},\ y_4\,\partial_{y_1} + \partial_{y_3}$
[3,3,8]	$-\operatorname{csch} y_3 \sin y_4\,\partial_{y_1} + \cos y_4\,\partial_{y_3} - \operatorname{cth} y_3 \sin y_4\,\partial_{y_4},\ -\cos y_4 \operatorname{csch} y_3\,\partial_{y_1} - \sin y_4\,\partial_{y_3} - \cos y_4 \operatorname{cth} y_3\,\partial_{y_4},\ \partial_{y_4}$
[3,3,9]	$-\csc y_3 \sin y_4\,\partial_{y_1} + \cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\cos y_4 \csc y_3\,\partial_{y_1} - \sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4}$
[4,3,1]	$\cos y_4\,\partial_{y_3} - \operatorname{cth} y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \operatorname{cth} y_3\,\partial_{y_4},\ \partial_{y_4},\ \partial_{y_1}$
[4,3,2]	$-\operatorname{csch} y_3 \sin y_4\,\partial_{y_1} + \cos y_4\,\partial_{y_3} - \operatorname{cth} y_3 \sin y_4\,\partial_{y_4},\ -\cos y_4 \operatorname{csch} y_3\,\partial_{y_1} - \sin y_4\,\partial_{y_3} - \cos y_4 \operatorname{cth} y_3\,\partial_{y_4},\ \partial_{y_4},\ \partial_{y_1}$
[4,3,3]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \partial_{y_1}$
[4,3,4]	$-\csc y_3 \sin y_4\,\partial_{y_1} + \cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\cos y_4 \csc y_3\,\partial_{y_1} - \sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \partial_{y_1}$
[4,3,5]	$\partial_{y_1},\ \partial_{y_4},\ y_4\,\partial_{y_1} + \partial_{y_3},\ \frac{1}{2}(-y_3^2 + y_4^2)\,\partial_{y_1} + y_4\,\partial_{y_3} - y_3\,\partial_{y_4}$
[4,3,6]	$\partial_{y_3},\ \partial_{y_4},\ -y_4\,\partial_{y_3} + y_3\,\partial_{y_4},\ \partial_{y_1}$
[4,3,8]	$-\sin y_1 \operatorname{th} y_2\,\partial_{y_1} + \cos y_1\,\partial_{y_2},\ -\cos y_1 \operatorname{th} y_2\,\partial_{y_1} - \sin y_1\,\partial_{y_2},\ \partial_{y_1},\ \partial_{y_4}$
[4,3,9]	$-\sin y_1 \operatorname{th} y_2\,\partial_{y_1} + \cos y_1\,\partial_{y_2} - \operatorname{sech} y_2 \sin y_1\,\partial_{y_3},\ -\cos y_1 \operatorname{th} y_2\,\partial_{y_1} - \sin y_1\,\partial_{y_2} - \cos y_1 \operatorname{sech} y_2\,\partial_{y_3},\ \partial_{y_1},\ \partial_{y_3}$
[4,3,10]	$\partial_{y_1},\ \partial_{y_4},\ y_4\,\partial_{y_1} + \partial_{y_3},\ y_3\,\partial_{y_3} - y_4\,\partial_{y_4}$
[4,3,11]	$\partial_{y_1},\ \partial_{y_2},\ \partial_{y_4},\ y_2\,\partial_{y_1} + y_1\,\partial_{y_2}$
[4,4,1]	$-\csc y_3 \sin y_4\,\partial_{y_1} + \cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\cos y_4 \csc y_3\,\partial_{y_1} - \sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \partial_{y_2}$
[4,4,2]	$-\operatorname{csch} y_3 \sin y_4\,\partial_{y_1} + \cos y_4\,\partial_{y_3} - \operatorname{cth} y_3 \sin y_4\,\partial_{y_4},\ -\cos y_4 \operatorname{csch} y_3\,\partial_{y_1} - \sin y_4\,\partial_{y_3} - \cos y_4 \operatorname{cth} y_3\,\partial_{y_4},\ \partial_{y_4},\ \partial_{y_2}$
[4,4,9]	$\partial_{y_1},\ \partial_{y_2},\ \partial_{y_4},\ -y_4\,\partial_{y_1} + \partial_{y_3} + y_2\,\partial_{y_4}$
[4,4,18]	$\partial_{y_1},\ \partial_{y_4},\ y_4\,\partial_{y_1} + \partial_{y_3},\ \partial_{y_2} - y_3\,\partial_{y_3} + y_4\,\partial_{y_4}$
[4,4,22]	$\partial_{y_1},\ \partial_{y_4},\ y_4\,\partial_{y_1} + \partial_{y_3},\ \frac{1}{2}(-y_3^2 + y_4^2)\,\partial_{y_1} + \partial_{y_2} + y_4\,\partial_{y_3} - y_3\,\partial_{y_4}$
[5,4,1]	$-\operatorname{csch} y_3 \sin y_4\,\partial_{y_1} + \cos y_4\,\partial_{y_3} - \operatorname{cth} y_3 \sin y_4\,\partial_{y_4},\ -\cos y_4 \operatorname{csch} y_3\,\partial_{y_1} - \sin y_4\,\partial_{y_3} - \cos y_4 \operatorname{cth} y_3\,\partial_{y_4},\ \partial_{y_4},\ \partial_{y_1},\ \partial_{y_2}$
[5,4,2]	$-\csc y_3 \sin y_4\,\partial_{y_1} + \cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\cos y_4 \csc y_3\,\partial_{y_1} - \sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \partial_{y_1},\ \partial_{y_2}$
[5,4,3]	$\partial_{y_1},\ \partial_{y_4},\ y_4\,\partial_{y_1} + \partial_{y_3},\ \frac{1}{2}(-y_3^2 + y_4^2)\,\partial_{y_1} + y_4\,\partial_{y_3} - y_3\,\partial_{y_4},\ \partial_{y_2}$
[5,4,6]	$\partial_{y_1},\ \partial_{y_4},\ y_4\,\partial_{y_1} + \partial_{y_3},\ y_3\,\partial_{y_3} - y_4\,\partial_{y_4},\ \partial_{y_2}$
[5,4,7]	$-\sin y_1 \operatorname{th} y_2\,\partial_{y_1} + \cos y_1\,\partial_{y_2} - \operatorname{sech} y_2 \sin y_1\,\partial_{y_3},\ -\cos y_1 \operatorname{th} y_2\,\partial_{y_1} - \sin y_1\,\partial_{y_2} - \cos y_1 \operatorname{sech} y_2\,\partial_{y_3},\ \partial_{y_1},\ \partial_{y_3},\ \partial_{y_4}$
[6,3,1]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \cos y_4 \sin y_3\,\partial_{y_2} + \cos y_3 \cos y_4 \cot y_2\,\partial_{y_3} - \cot y_2 \csc y_3 \sin y_4\,\partial_{y_4},\ \sin y_3 \sin y_4\,\partial_{y_2} + \cos y_3 \cot y_2 \sin y_4\,\partial_{y_3} + \cos y_4 \cot y_2 \csc y_3\,\partial_{y_4},\ \cos y_3\,\partial_{y_2} - \cot y_2 \sin y_3\,\partial_{y_3}$
[6,3,2]	$\partial_{y_3},\ \partial_{y_4},\ -y_4\,\partial_{y_3} + y_3\,\partial_{y_4},\ y_4\,\partial_{y_2} - y_2\,\partial_{y_4},\ y_3\,\partial_{y_2} - y_2\,\partial_{y_3},\ -\partial_{y_2}$
[6,3,3]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \cos y_4 \sin y_3\,\partial_{y_2} + \cos y_3 \cos y_4 \operatorname{cth} y_2\,\partial_{y_3} - \operatorname{cth} y_2 \csc y_3 \sin y_4\,\partial_{y_4},\ \sin y_3 \sin y_4\,\partial_{y_2} + \cos y_3 \operatorname{cth} y_2 \sin y_4\,\partial_{y_3} + \cos y_4 \operatorname{cth} y_2 \csc y_3\,\partial_{y_4},\ \cos y_3\,\partial_{y_2} - \operatorname{cth} y_2 \sin y_3\,\partial_{y_3}$
[6,3,4]	$-\sin(y_1 - y_3) \operatorname{th} y_2\,\partial_{y_1} + \cos(y_1 - y_3)\,\partial_{y_2} + \operatorname{cth} y_2 \sin(y_1 - y_3)\,\partial_{y_3},\ \cos(y_1 - y_3) \operatorname{th} y_2\,\partial_{y_1} + \sin(y_1 - y_3)\,\partial_{y_2} - \cos(y_1 - y_3) \operatorname{cth} y_2\,\partial_{y_3},\ -\partial_{y_1} + \partial_{y_3},\ \sin(y_1 + y_3) \operatorname{th} y_2\,\partial_{y_1} - \cos(y_1 + y_3)\,\partial_{y_2} + \operatorname{cth} y_2 \sin(y_1 + y_3)\,\partial_{y_3},\ \cos(y_1 + y_3) \operatorname{th} y_2\,\partial_{y_1} + \sin(y_1 + y_3)\,\partial_{y_2} + \cos(y_1 + y_3) \operatorname{cth} y_2\,\partial_{y_3},\ \partial_{y_1} + \partial_{y_3}$
[6,3,5]	$\partial_{y_1},\ \partial_{y_2},\ \partial_{y_4},\ y_2\,\partial_{y_1} + y_1\,\partial_{y_2},\ y_4\,\partial_{y_1} + y_1\,\partial_{y_4},\ -y_4\,\partial_{y_2} + y_2\,\partial_{y_4}$
[6,3,6]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \cos y_4 \sin y_3\,\partial_{y_2} + \cos y_3 \cos y_4 \operatorname{cth} y_2\,\partial_{y_3} - \operatorname{cth} y_2 \csc y_3 \sin y_4\,\partial_{y_4},\ \sin y_3 \sin y_4\,\partial_{y_2} + \cos y_3 \operatorname{cth} y_2 \sin y_4\,\partial_{y_3} + \cos y_4 \operatorname{cth} y_2 \csc y_3\,\partial_{y_4},\ \cos y_3\,\partial_{y_2} - \operatorname{cth} y_2 \sin y_3\,\partial_{y_3}$
[6,4,1]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ -\sin y_1 \operatorname{th} y_2\,\partial_{y_1} + \cos y_1\,\partial_{y_2},\ -\cos y_1 \operatorname{th} y_2\,\partial_{y_1} - \sin y_1\,\partial_{y_2},\ \partial_{y_1}$
[6,4,2]	$\cos y_4\,\partial_{y_3} - \operatorname{cth} y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \operatorname{cth} y_3\,\partial_{y_4},\ \partial_{y_4},\ -\sin y_1 \operatorname{th} y_2\,\partial_{y_1} + \cos y_1\,\partial_{y_2},\ -\cos y_1 \operatorname{th} y_2\,\partial_{y_1} - \sin y_1\,\partial_{y_2},\ \partial_{y_1}$
[6,4,3]	$-\sin y_1 \operatorname{th} y_2\,\partial_{y_1} + \cos y_1\,\partial_{y_2},\ -\cos y_1 \operatorname{th} y_2\,\partial_{y_1} - \sin y_1\,\partial_{y_2},\ \partial_{y_1},\ \partial_{y_3},\ \partial_{y_4},\ -y_4\,\partial_{y_3} + y_3\,\partial_{y_4}$
[6,4,4]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \partial_{y_1},\ \partial_{y_2},\ y_2\,\partial_{y_1} + y_1\,\partial_{y_2}$
[6,4,5]	$\cos y_4\,\partial_{y_3} - \operatorname{cth} y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \operatorname{cth} y_3\,\partial_{y_4},\ \partial_{y_4},\ y_2\,\partial_{y_1} + y_1\,\partial_{y_2},\ \partial_{y_1},\ \partial_{y_2}$
[7,4,1]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \cos y_4 \sin y_3\,\partial_{y_2} + \cos y_3 \cos y_4 \cot y_2\,\partial_{y_3} - \cot y_2 \csc y_3 \sin y_4\,\partial_{y_4},\ \sin y_3 \sin y_4\,\partial_{y_2} + \cos y_3 \cot y_2 \sin y_4\,\partial_{y_3} + \cos y_4 \cot y_2 \csc y_3\,\partial_{y_4},\ \cos y_3\,\partial_{y_2} - \cot y_2 \sin y_3\,\partial_{y_3},\ \partial_{y_1}$
[7,4,2]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \cos y_4 \sin y_3\,\partial_{y_2} + \cos y_3 \cos y_4 \operatorname{cth} y_2\,\partial_{y_3} - \operatorname{cth} y_2 \csc y_3 \sin y_4\,\partial_{y_4},\ \sin y_3 \sin y_4\,\partial_{y_2} + \cos y_3 \operatorname{cth} y_2 \sin y_4\,\partial_{y_3} + \cos y_4 \operatorname{cth} y_2 \csc y_3\,\partial_{y_4},\ \cos y_3\,\partial_{y_2} - \operatorname{cth} y_2 \sin y_3\,\partial_{y_3},\ \partial_{y_1}$
[7,4,3]	$\cos y_4\,\partial_{y_3} - \cot y_3 \sin y_4\,\partial_{y_4},\ -\sin y_4\,\partial_{y_3} - \cos y_4 \cot y_3\,\partial_{y_4},\ \partial_{y_4},\ \cos y_4 \sin y_3\,\partial_{y_2} + \cos y_3 \cos y_4 \operatorname{cth} y_2\,\partial_{y_3} - \operatorname{cth} y_2 \csc y_3 \sin y_4\,\partial_{y_4},\ \sin y_3 \sin y_4\,\partial_{y_2} + \cos y_3 \operatorname{cth} y_2 \sin y_4\,\partial_{y_3} + \cos y_4 \operatorname{cth} y_2 \csc y_3\,\partial_{y_4},\ \cos y_3\,\partial_{y_2} - \operatorname{cth} y_2 \sin y_3\,\partial_{y_3},\ \partial_{y_1}$
[7,4,4]	$-\sin(y_1 - y_3) \operatorname{th} y_2\,\partial_{y_1} + \cos(y_1 - y_3)\,\partial_{y_2} + \operatorname{cth} y_2 \sin(y_1 - y_3)\,\partial_{y_3},\ \cos(y_1 - y_3) \operatorname{th} y_2\,\partial_{y_1} + \sin(y_1 - y_3)\,\partial_{y_2} - \cos(y_1 - y_3) \operatorname{cth} y_2\,\partial_{y_3},\ -\partial_{y_1} + \partial_{y_3},\ \sin(y_1 + y_3) \operatorname{th} y_2\,\partial_{y_1} - \cos(y_1 + y_3)\,\partial_{y_2} + \operatorname{cth} y_2 \sin(y_1 + y_3)\,\partial_{y_3},\ \cos(y_1 + y_3) \operatorname{th} y_2\,\partial_{y_1} + \sin(y_1 + y_3)\,\partial_{y_2} + \cos(y_1 + y_3) \operatorname{cth} y_2\,\partial_{y_3},\ \partial_{y_1} + \partial_{y_3},\ \partial_{y_4}$

List of Γ-invariant metrics

Hicks #	Γ-invariant metric $\hat{\boldsymbol{g}}$
[3,2,1]	$-\phi_1(y_1,y_2)\,\mathbf{d}y_1{}^2 + \phi_2(y_1,y_2)(\mathbf{d}y_1 \vee \mathbf{d}y_2) + \phi_3(y_1,y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_1,y_2)(\mathbf{d}y_3{}^2 + \mathbf{d}y_4{}^2)$
[3,2,2]	$-\phi_1(y_1,y_2)\,\mathbf{d}y_1{}^2 + \phi_2(y_1,y_2)(\mathbf{d}y_1 \vee \mathbf{d}y_2) + \phi_3(y_1,y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_1,y_2)(\mathbf{d}y_3{}^2 + \operatorname{sh}^2 y_3\,\mathbf{d}y_4{}^2)$
[3,2,3]	$-\phi_1(y_1,y_2)\,\mathbf{d}y_1{}^2 + \phi_2(y_1,y_2)(\mathbf{d}y_1 \vee \mathbf{d}y_2) + \phi_3(y_1,y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_1,y_2)(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)$
[3,2,4]	$\phi_1(y_3,y_4)(-\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(y_3,y_4)\,\mathbf{d}y_3{}^2 + \phi_3(y_3,y_4)(\mathbf{d}y_3 \vee \mathbf{d}y_4) + \phi_4(y_3,y_4)\,\mathbf{d}y_4{}^2$
[3,2,5]	$\phi_1(y_3,y_4)(-\operatorname{ch}^2 y_2\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(y_3,y_4)\,\mathbf{d}y_3{}^2 + \phi_3(y_3,y_4)(\mathbf{d}y_3 \vee \mathbf{d}y_4) + \phi_4(y_3,y_4)\,\mathbf{d}y_4{}^2$
[3,3,2]	$-\phi_1(y_3)\,\mathbf{d}y_1{}^2 + \phi_2(y_3)\,\mathbf{d}y_3{}^2 + \phi_3(y_3)(\mathbf{d}y_3 \vee \mathbf{d}y_4) + \phi_4(y_3)\,\mathbf{d}y_4{}^2 + \phi_5(y_3)\,\mathbf{d}y_2{}^2 + \phi_6(y_3)(\mathbf{d}y_1 \vee \mathbf{d}y_2)$ $+ \phi_7(y_3)(\mathbf{d}y_1 \vee \mathbf{d}y_3) + \phi_8(y_3)(\mathbf{d}y_1 \vee \mathbf{d}y_4) + \phi_9(y_3)(\mathbf{d}y_2 \vee \mathbf{d}y_3) + \phi_{10}(y_3)(\mathbf{d}y_2 \vee \mathbf{d}y_4)$
[3,3,3]	$-\phi_1(y_2)(\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)^2 + \phi_2(y_2)\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)\,\mathbf{d}y_3{}^2 + \phi_5(y_2)\big(\mathbf{d}y_3 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big)$ $+ \phi_6(y_2)\big(\mathbf{d}y_4 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_7(y_2)(\mathbf{d}y_2 \vee \mathbf{d}y_3) + \phi_8(y_2)(\mathbf{d}y_2 \vee \mathbf{d}y_4) + \phi_9(y_2)(\mathbf{d}y_3 \vee \mathbf{d}y_4) + \phi_{10}(y_2)\,\mathbf{d}y_4{}^2$
[3,3,8]	$-\phi_1(y_2)(\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4)^2 + \phi_2(y_2)\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4)\big) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)(\sin y_1\,\mathbf{d}y_3 + \cos y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)^2 +$ $\phi_5(y_2)\big(\mathbf{d}y_2 \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) + \phi_6(y_2)\big((\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4) \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) +$ $\phi_7(y_2)\big((\sin y_1\,\mathbf{d}y_3 + \cos y_1 \operatorname{sh} y_3\,\mathbf{d}y_4) \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) + \phi_8(y_2)\big((\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4) \vee (\sin y_1\,\mathbf{d}y_3 +$ $\cos y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) + \phi_9(y_2)\big(\mathbf{d}y_2 \vee (\sin y_1\,\mathbf{d}y_3 + \cos y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) + \phi_{10}(y_2)(\cos y_1\,\mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)^2$
[3,3,9]	$-\phi_1(y_2)(\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4)^2 + \phi_2(y_2)\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4)\big) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)(\sin y_1\,\mathbf{d}y_3 + \cos y_1 \sin y_3\,\mathbf{d}y_4)^2 +$ $\phi_5(y_2)\big(\mathbf{d}y_2 \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_3 \sin y_1\,\mathbf{d}y_4)\big) + \phi_6(y_2)\big((\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4) \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_3 \sin y_1\,\mathbf{d}y_4)\big) +$ $\phi_7(y_2)\big((\sin y_1\,\mathbf{d}y_3 + \cos y_1 \sin y_3\,\mathbf{d}y_4) \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_3 \sin y_1\,\mathbf{d}y_4)\big) + \phi_8(y_2)\big((\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4) \vee (\sin y_1\,\mathbf{d}y_3 +$ $\cos y_1 \sin y_3\,\mathbf{d}y_4)\big) + \phi_9(y_2)\big(\mathbf{d}y_2 \vee (\sin y_1\,\mathbf{d}y_3 + \cos y_1 \sin y_3\,\mathbf{d}y_4)\big) + \phi_{10}(y_2)(\cos y_1\,\mathbf{d}y_3 - \sin y_3 \sin y_1\,\mathbf{d}y_4)^2$
[4,3,1]	$-\phi_1(y_2)\,\mathbf{d}y_1{}^2 + \phi_2(y_2)(\mathbf{d}y_1 \vee \mathbf{d}y_2) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)(\mathbf{d}y_3{}^2 + \operatorname{sh}^2 y_3\,\mathbf{d}y_4{}^2)$
[4,3,2]	$-\phi_1(y_2)(\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4)^2 + \phi_2(y_2)\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4)\big) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)(\mathbf{d}y_3{}^2 + \operatorname{sh}^2 y_3\,\mathbf{d}y_4{}^2)$
[4,3,3]	$-\phi_1(y_2)\,\mathbf{d}y_1{}^2 + \phi_2(y_2)(\mathbf{d}y_1 \vee \mathbf{d}y_2) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)$
[4,3,4]	$-\phi_1(y_2)(\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4)^2 + \phi_2(y_2)\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4)\big) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)$
[4,3,5]	$-\phi_1(y_2)(\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)^2 + \phi_2(y_2)\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)(\mathbf{d}y_3{}^2 + \mathbf{d}y_4{}^2)$
[4,3,6]	$-\phi_1(y_2)\,\mathbf{d}y_1{}^2 + \phi_2(y_2)(\mathbf{d}y_1 \vee \mathbf{d}y_2) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)(\mathbf{d}y_3{}^2 + \mathbf{d}y_4{}^2)$
[4,3,8]	$\phi_1(y_3)(-\operatorname{ch}^2 y_2\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(y_3)\,\mathbf{d}y_3{}^2 + \phi_3(y_3)(\mathbf{d}y_3 \vee \mathbf{d}y_4) + \phi_4(y_3)\,\mathbf{d}y_4{}^2$
[4,3,9]	$\phi_1(y_4)(-\operatorname{ch}^2 y_2\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(y_4)(\operatorname{sh} y_2\,\mathbf{d}y_1 + \mathbf{d}y_3)^2 + \phi_3(y_4)\big(\mathbf{d}y_4 \vee (\operatorname{sh} y_2\,\mathbf{d}y_1 + \mathbf{d}y_3)\big) + \phi_4(y_4)\,\mathbf{d}y_4{}^2$
[4,3,10]	$-\phi_1(y_2)(\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)^2 + \phi_2(y_2)\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_3(y_2)\,\mathbf{d}y_2{}^2 + \phi_4(y_2)(\mathbf{d}y_3 \vee \mathbf{d}y_4)$
[4,3,11]	$\phi_1(y_3)(-\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(y_3)\,\mathbf{d}y_3{}^2 + \phi_3(y_3)(\mathbf{d}y_3 \vee \mathbf{d}y_4) + \phi_4(y_3)\,\mathbf{d}y_4{}^2$
[4,4,1]	$-\phi_1(\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4)^2 + \phi_2\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4)\big) + \phi_3\mathbf{d}y_2{}^2 + \phi_4(\sin y_1\,\mathbf{d}y_3 + \cos y_1 \sin y_3\,\mathbf{d}y_4)^2 + \phi_5\big(\mathbf{d}y_2 \vee$ $(\cos y_1\,\mathbf{d}y_3 - \sin y_3 \sin y_1\,\mathbf{d}y_4)\big) + \phi_6\big((\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4) \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_3 \sin y_1\,\mathbf{d}y_4)\big) + \phi_7\big((\sin y_1\,\mathbf{d}y_3 +$ $\cos y_1 \sin y_3\,\mathbf{d}y_4) \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_3 \sin y_1\,\mathbf{d}y_4)\big) + \phi_8\big((\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4) \vee (\sin y_1\,\mathbf{d}y_3 + \cos y_1 \sin y_3\,\mathbf{d}y_4)\big) +$ $\phi_9\big(\mathbf{d}y_2 \vee (\sin y_1\,\mathbf{d}y_3 + \cos y_1 \sin y_3\,\mathbf{d}y_4)\big) + \phi_{10}(\cos y_1\,\mathbf{d}y_3 - \sin y_3 \sin y_1\,\mathbf{d}y_4)^2$
[4,4,2]	$-\phi_1(\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4)^2 + \phi_2\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4)\big) + \phi_3\mathbf{d}y_2{}^2 + \phi_4(\sin y_1\,\mathbf{d}y_3 + \cos y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)^2 + \phi_5\big(\mathbf{d}y_2 \vee$ $(\cos y_1\,\mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) + \phi_6\big((\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4) \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) + \phi_7\big((\sin y_1\,\mathbf{d}y_3 +$ $\cos y_1 \operatorname{sh} y_3\,\mathbf{d}y_4) \vee (\cos y_1\,\mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) + \phi_8\big((\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4) \vee (\sin y_1\,\mathbf{d}y_3 + \cos y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) +$ $\phi_9\big(\mathbf{d}y_2 \vee (\sin y_1\,\mathbf{d}y_3 + \cos y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)\big) + \phi_{10}(\cos y_1\,\mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3\,\mathbf{d}y_4)^2$
[4,4,9]	$-\phi_1(\mathbf{d}y_1 - \tfrac{1}{2}y_3^2\,\mathbf{d}y_2 + y_3\,\mathbf{d}y_4)^2 + \phi_2\,\mathbf{d}y_3{}^2 + \phi_3\big(\mathbf{d}y_3 \vee (-y_3\,\mathbf{d}y_2 + \mathbf{d}y_4)\big) + \phi_4(-y_3\,\mathbf{d}y_2 + \mathbf{d}y_4)^2 + \phi_5\,\mathbf{d}y_2{}^2 + \phi_6\big(\mathbf{d}y_2 \vee$ $(\mathbf{d}y_1 - \tfrac{1}{2}y_3^2\,\mathbf{d}y_2 + y_3\,\mathbf{d}y_4)\big) + \phi_7\big(\mathbf{d}y_3 \vee (\mathbf{d}y_1 - \tfrac{1}{2}y_3^2\,\mathbf{d}y_2 + y_3\,\mathbf{d}y_4)\big) + \phi_8\big((-y_3\,\mathbf{d}y_2 + \mathbf{d}y_4) \vee (\mathbf{d}y_1 - \tfrac{1}{2}y_3^2\,\mathbf{d}y_2 +$ $y_3\,\mathbf{d}y_4)\big) + \phi_9(\mathbf{d}y_2 \vee \mathbf{d}y_3) + \phi_{10}\big(\mathbf{d}y_2 \vee (-y_3\,\mathbf{d}y_2 + \mathbf{d}y_4)\big)$
[4,4,18]	$-\phi_1(\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)^2 + \phi_2\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_3\,\mathbf{d}y_2{}^2 + e^{2y_2}\phi_4\,\mathbf{d}y_3{}^2 + e^{y_2}\phi_5\big(\mathbf{d}y_3 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big)$ $+ e^{-y_2}\phi_6\big(\mathbf{d}y_4 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + e^{y_2}\phi_7(\mathbf{d}y_2 \vee \mathbf{d}y_3) + e^{-y_2}\phi_8(\mathbf{d}y_2 \vee \mathbf{d}y_4) + \phi_9(\mathbf{d}y_3 \vee \mathbf{d}y_4) + e^{-2y_2}\phi_{10}\,\mathbf{d}y_4{}^2$
[4,4,22]	$-\phi_1(\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)^2 + \phi_2\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_3\,\mathbf{d}y_2{}^2 + \phi_4(\cos y_2\,\mathbf{d}y_3 - \sin y_2\,\mathbf{d}y_4)^2 + \phi_5\big((\cos y_2\,\mathbf{d}y_3 - \sin y_2\,\mathbf{d}y_4) \vee$ $(\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_6\big((\sin y_2\,\mathbf{d}y_3 + \cos y_2\,\mathbf{d}y_4) \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_7\big(\mathbf{d}y_2 \vee (\cos y_2\,\mathbf{d}y_3 - \sin y_2\,\mathbf{d}y_4)\big) + \phi_8\big(\mathbf{d}y_2 \vee$ $(\sin y_2\,\mathbf{d}y_3 + \cos y_2\,\mathbf{d}y_4)\big) + \phi_9\big((\sin y_2\,\mathbf{d}y_3 + \cos y_2\,\mathbf{d}y_4) \vee (\cos y_2\,\mathbf{d}y_3 - \sin y_2\,\mathbf{d}y_4)\big) + \phi_{10}(\sin y_2\,\mathbf{d}y_3 + \cos y_2\,\mathbf{d}y_4)^2$
[5,4,1]	$-\phi_1(\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4)^2 + \phi_2\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \operatorname{ch} y_3\,\mathbf{d}y_4)\big) + \phi_3\,\mathbf{d}y_2{}^2 + \phi_4(\mathbf{d}y_3{}^2 + \operatorname{sh}^2 y_3\,\mathbf{d}y_4{}^2)$
[5,4,2]	$-\phi_1(\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4)^2 + \phi_2\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \cos y_3\,\mathbf{d}y_4)\big) + \phi_3\,\mathbf{d}y_2{}^2 + \phi_4(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)$
[5,4,3]	$-\phi_1(\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)^2 + \phi_2\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_3\,\mathbf{d}y_2{}^2 + \phi_4(\mathbf{d}y_3{}^2 + \mathbf{d}y_4{}^2)$
[5,4,6]	$-\phi_1(\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)^2 + \phi_2\big(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3\,\mathbf{d}y_4)\big) + \phi_3\,\mathbf{d}y_2{}^2 + \phi_4(\mathbf{d}y_3 \vee \mathbf{d}y_4)$
[5,4,7]	$\phi_1(-\operatorname{ch}^2 y_2\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(\operatorname{sh} y_2\,\mathbf{d}y_1 + \mathbf{d}y_3)^2 + \phi_3\big(\mathbf{d}y_4 \vee (\operatorname{sh} y_2\,\mathbf{d}y_1 + \mathbf{d}y_3)\big) + \phi_4\,\mathbf{d}y_4{}^2$
[6,3,1]	$-\phi_1(y_1)\,\mathbf{d}y_1{}^2 + \phi_2(y_1)\big(\mathbf{d}y_2{}^2 + \sin^2 y_2(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)\big)$
[6,3,2]	$-\phi_1(y_1)\,\mathbf{d}y_1{}^2 + \phi_2(y_1)(\mathbf{d}y_2{}^2 + \mathbf{d}y_3{}^2 + \mathbf{d}y_4{}^2)$
[6,3,3]	$-\phi_1(y_1)\,\mathbf{d}y_1{}^2 + \phi_2(y_1)\big(\mathbf{d}y_2{}^2 + \operatorname{sh}^2 y_2(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)\big)$
[6,3,4]	$\phi_1(y_4)(-\operatorname{ch}^2 y_2\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2 + \operatorname{sh}^2 y_2\,\mathbf{d}y_3{}^2) + \phi_2(y_4)\,\mathbf{d}y_4{}^2$
[6,3,5]	$\phi_1(y_3)(-\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2 + \mathbf{d}y_4{}^2) + \phi_2(y_3)\,\mathbf{d}y_3{}^2$
[6,3,6]	$-\phi_1(y_1)\,\mathbf{d}y_1{}^2 + \phi_2(y_1)\big(\mathbf{d}y_2{}^2 + \operatorname{sh}^2 y_2(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)\big)$
[6,4,1]	$\phi_1(-\operatorname{ch}^2 y_2\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)$
[6,4,2]	$\phi_1(-\operatorname{ch}^2 y_2\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(\mathbf{d}y_3{}^2 + \operatorname{sh}^2 y_3\,\mathbf{d}y_4{}^2)$
[6,4,3]	$\phi_1(-\operatorname{ch}^2 y_2\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(\mathbf{d}y_3{}^2 + \mathbf{d}y_4{}^2)$
[6,4,4]	$\phi_1(-\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)$
[6,4,5]	$\phi_1(-\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2) + \phi_2(\mathbf{d}y_3{}^2 + \operatorname{sh}^2 y_3\,\mathbf{d}y_4{}^2)$
[7,4,1]	$-\phi_1\,\mathbf{d}y_1{}^2 + \phi_2\big(\mathbf{d}y_2{}^2 + \sin^2 y_2(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)\big)$
[7,4,2]	$-\phi_1\,\mathbf{d}y_1{}^2 + \phi_2\big(\mathbf{d}y_2{}^2 + \operatorname{sh}^2 y_2(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)\big)$
[7,4,3]	$-\phi_1\,\mathbf{d}y_1{}^2 + \phi_2\big(\mathbf{d}y_2{}^2 + \operatorname{sh}^2 y_2(\mathbf{d}y_3{}^2 + \sin^2 y_3\,\mathbf{d}y_4{}^2)\big)$
[7,4,4]	$\phi_1(-\operatorname{ch}^2 y_2\,\mathbf{d}y_1{}^2 + \mathbf{d}y_2{}^2 + \operatorname{sh}^2 y_2\,\mathbf{d}y_3{}^2) + \phi_2\,\mathbf{d}y_4{}^2$

List of Γ-invariant l-chains

Hicks #	l-chain $\boldsymbol{\chi}$
[3,2,1]	$\boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[3,2,2]	$\operatorname{csch} y_3\, \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[3,2,3]	$\csc y_3\, \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[3,2,4]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2}$
[3,2,5]	$\operatorname{sech} y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2}$
[3,3,2]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_4}$
[3,3,3]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[3,3,8]	$\operatorname{csch} y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[3,3,9]	$\csc y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,1]	$\operatorname{csch} y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,2]	$\operatorname{csch} y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,3]	$\csc y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,4]	$\csc y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,5]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,6]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,8]	$\operatorname{sech} y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,9]	$\operatorname{sech} y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3}$
[4,3,10]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,11]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_4}$
[4,4,1]	$\csc y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,4,2]	$\operatorname{csch} y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,4,9]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,4,18]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,4,22]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[5,4,1]	$\operatorname{csch} y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[5,4,2]	$\csc y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[5,4,3]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[5,4,6]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[5,4,7]	$\operatorname{csch} y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[6,3,1]	$\csc^2 y_2 \csc y_3\, \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[6,3,2]	$\boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[6,3,3]	$\csc y_3 \operatorname{csch}^2 y_2\, \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[6,3,4]	$\operatorname{csch} y_2 \operatorname{sech} y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3}$
[6,3,5]	$\boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_4}$
[6,3,6]	$\csc y_3 \operatorname{csch}^2 y_2\, \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[6,4,1]	$\csc y_3 \operatorname{sech} y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[6,4,2]	$\operatorname{csch} y_3 \operatorname{sech} y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[6,4,3]	$\operatorname{sech} y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[6,4,4]	$\csc y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[6,4,5]	$\operatorname{csch} y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[7,4,1]	$\csc^2 y_2 \csc y_3\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[7,4,2]	$\csc y_3 \operatorname{csch}^2 y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[7,4,3]	$\csc y_3 \operatorname{csch}^2 y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[7,4,4]	$\operatorname{csch} y_2 \operatorname{sech} y_2\, \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$