Symmetry reduction of gravitational Lagrangians

based on: G. Frausto, I. Kolář, TM, Ch. Torre, (soon on arXiv)

Tomáš Málek



Institute of Mathematics Academy of Sciences of the Czech Republic

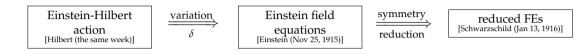
> EREP2024 Coimbra July 23, 2024

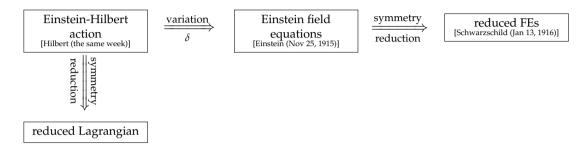
- **1** Motivation: Weyl trick
- 2 Rigorous treatment: Principle of symmetric criticality
- **3** Systematic study
- 4 Examples

Einstein field equations [Einstein (Nov 25, 1915)]

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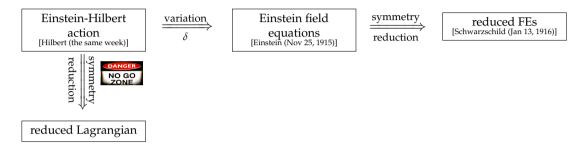






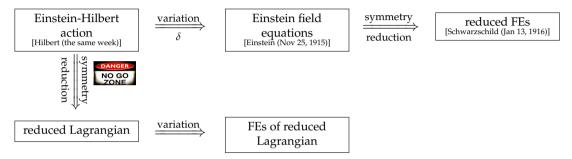
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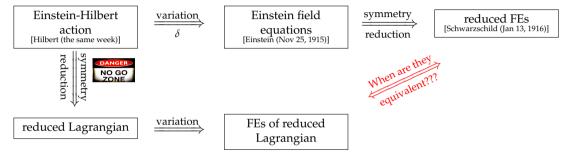
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$$b' = 0$$
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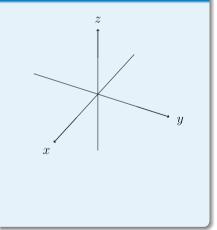
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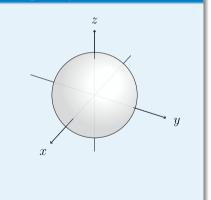
Infinitesimal group action Γ on M

given by *d*-dim Lie algebra of isometry generators $X \in \Gamma$ (Killing vectors)



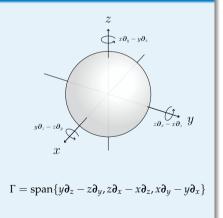
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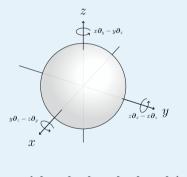


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$$\Gamma = \operatorname{span}\{y\partial_z - z\partial_y, z\partial_x - x\partial_z, x\partial_y - y\partial_x\}$$

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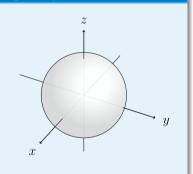
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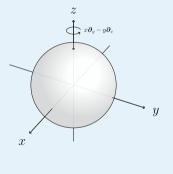
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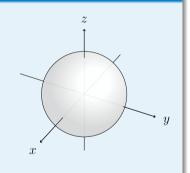
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dim of orbit l =dim of linearly independent KVs at x relation to dim p of Γ_x : l = d - p



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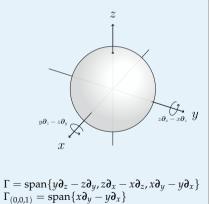
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Example: symmetries of S₂



 $\Gamma_{(0,0,1)} = \operatorname{span} \{ x \partial_y - y \partial_x \}$ orbit of (0, 0, 1) is unit sphere

Purely gravitational theory on a 4-dimensional spacetime

$$S = \int_{\mathbf{M}} \underline{\epsilon}(\mathbf{g}) L[\mathbf{g}]$$

• Levi-Civita tensor $\underline{\epsilon}(g)$ defines the volume element $\sqrt{-g} d^4 x$

Lagrangian L[g] constructed from $g, R, \nabla \cdots \nabla R$ (Lagrangian 4-form $\underline{L}[g] \equiv \underline{\epsilon}(g)L[g]$)

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$$\pounds_{\boldsymbol{X}_i} \boldsymbol{\chi} = 0, \quad i = 1, \dots, d, \qquad \chi^{i_1 \dots i_l} = \chi^{[i_1 \dots i_l]}$$

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• reduced Lagrangian $\underline{\hat{L}} = \chi \bullet \underline{L}[\hat{g}] = \underline{\hat{e}}(\hat{g})L[\hat{g}]$, (where $\underline{\hat{e}}(\hat{g}) = \chi \bullet \underline{e}(\hat{g})$)

Principle of symmetric criticality

Variation of Lagrangian 4-form

$$\delta \underline{L} = \underline{E}(\underline{L}) \cdot \delta g + \underline{d} \underline{\eta}(\delta g)$$

Euler-Lagrange expression <u>E</u>(<u>L</u>) gives the field equations <u>E</u>(<u>L</u>)[g] = 0
 η is boundary 3-form

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Principle of symmetric criticality [Palais (1979), M. E. Fels, C. G. Torre (2002)]

Variation of Lagrangian commutes with symmetry reduction for all possible theories:

$$\forall \underline{L} : \underline{E}(\underline{L})[\hat{g}] = 0 \iff \underline{E}(\underline{\hat{L}})[\hat{g}] = 0$$

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Two conditions imposed solely on Γ are necessary and sufficient for validity of PSC.

PSC1 "Lie algebra condition"

PSC1 ensures that the reduction of the boundary term $\underline{d\eta}$ is a boundary term $\underline{d\hat{\eta}}$ for the reduced Lagrangian

$$\delta \underline{\hat{L}} = \underline{E}(\underline{\hat{L}}) \cdot \delta \hat{g} + \underline{d} \underline{\hat{\eta}}(\delta \hat{g})$$

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■ PSC1 most simply formulated as an extra condition on *l*-chain:

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• if PSC1 satisfied then Euler-Lagrange equations of the reduced Lagrangian always yield at least a subset of the reduced equations

PSC2 "(local) Palais condition"

PSC2 arises from the requirement that this subset contains all reduced equations i.e. all reduced FEs appear in the reduction of Euler-Lagrange term <u>E(L</u>)

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PSC2

Let S_x and S_x^* denote the vector space of Γ_x -invariant $\binom{0}{2}$ and $\binom{2}{0}$ tensors at x, respectively. Denote by V_x^0 the vector space of $\binom{2}{0}$ tensors which have a vanishing scalar contraction with all elements of S_x . Then in the neighborhood of x:

$$S_x^* \cap V_x^0 = \{0\}$$

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PSC2 satisfied iff the isotropy algebra contains no null-rotation subalgebra

Classification of infinitesimal group actions

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Hicks classification [Hicks, Ph.D. thesis (2016)]

based on classifying isometry algebra and isotropy subalgebra pairs (Γ , Γ _x)

- isotropy subalgebras Γ_x can be identified with subalgebras of the Lorentz algebra
- cases denoted by [*d*, *l*, *c*]
 - **1** d is dim of Γ
 - 2 *l* is dim of orbits (l = d p)
 - 3 *c* enumerates possible cases of given dimensions

explicit infinitesimal generators given for each case

Systematic study

1 1

1 1 1

1 1 1

1 1 1

V X X

PSC-compatible infinitesimal group actions

	Law or over	Law or other	Law or out									Hicks #
Hicks $\#$	PSCI	PSC2	PSC	Hicks #	PSC1	PSC2	PSC	Hicks #	PSC1	PSC2	PSC	
[3,2,1]	1	1	1	[4,3,10]	1	\checkmark	\checkmark	4,4,13	×	1	X	[5,4,7]
[3,2,2]	1	1	1	[4,3,11]	1	1	1	4,4,14	×	1	X	[5,4,8]
[3,2,3]	1	1	1	[4,3,12]	×	1	X	4,4,15	×	1	X	[5,4,9]
[3,2,4]	1	1	1	4,3,13	1	×	×	4,4,16	×	1	X	[5,4,10]
[3,2,5]	1	1	1	[4,3,14]	1	×	X	[4,4,17]	×	1	X	[5,4,11]
[3,3,1]	×		X	[4,3,15]	1	×	X	4,4,18	1	1	1	[6,3,1]
[3,3,2]	1	1	1	[4,3,16]	1	×	×	[4,4,19]	×	1	×	[6,3,2]
[3,3,3]	1	1	1	[4,3,17]	1	×	×	[4,4,20]	×	1	×	[6,3,3]
[3,3,4]	×	1	X	[4,3,18]	1	×	X	[4,4,21]	×	1	X	[6,3,4]
3,3,5	×	1	×	[4,3,19]	1	×	×	4,4,22	1	1	1	[6,3,5]
3,3,6	×	1	X	[4,3,20]	1	×	X	4,4,23	×	1	X	[6,3,6]
[3,3,7]	×	1	×	[4,4,1]	1	1	1	[5,4,-1]	1	X	X	[6, 4, -1]
3,3,8	1	1	1	4,4,2	1	1	1	5,4,-2	X	X	X	[6,4,1]
[3,3,9]	1	1	1	4,4,3	X	1	X	5.43	×	X	X	6.4.2
[4,3,1]	1		1	4.4.4	X	1	×	[5,4,-4]	×	X	×	6,4,3
4,3,2	1	1	1	4.4.5	×	1	X	5.45	×	×	X	[6,4,4]
4.3.3	1	1	1	4.4.6	X	1	X	[5,4,-6]	×	X	X	[6,4,5]
4.3.4	1	1	1	4,4,7	×	1	X	[5,4,1]	1	1	1	[6,4,6]
4,3,5	1	1	1	4.4.8	×	1	X	[5,4,2]	1	1	1	[7,4,1]
4.3.6	1	1	1	4,4,9	1	1	1	5,4,3	1	1	1	7.4.2
4.3.7	×	1	×	4.4.10	×	1	×	[5,4,4]	×	1	×	7,4,3
4.3.8	1	1	1	4.4.11	X	1	X	5.4.5	×	1	X	7,4,4
		1		4.4.12	X	1	X	5,4,6	17	1	1	
[4,3,9]	V .	V	· ·	(1,4,12)	· ^			0,4,0				[7,4,5]

Systematic study

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Hicks #	PSC1	PSC2	PSC	Hicks #	PSC1	PSC2	nea	TT: d -	# PSC1	PS
			_					Hicks		
[3,2,1]	1	1	1	[4,3,10]	1	1	1	[4,4,1]		•
[3,2,2]	1	1	1	[4,3,11]	1	1	1	[4,4,1]		۰,
[3,2,3]	1	1	1	[4,3,12]	×	1	×	[4,4,1]		•
[3,2,4]	1	1	1	[4,3,13]	1	X	×	[4,4,1]		•
$^{3,2,5]}$	1	1	1	[4,3,14]	1	×	×	[4,4,1]		•
[3,3,1]	×	1	×	[4,3,15]	1	×	×	[4,4,1]		•
3, 3, 2	1	1	1	[4,3,16]	1	×	×	[4,4,1]		•
3,3,3	1	1	1	[4,3,17]	1	×	×	[4,4,2]		•
[3,3,4]	×	1	X	[4,3,18]	1	×	×	[4,4,2]		
3, 3, 5	×	1	X	[4,3,19]	1	X	×	[4,4,2]		•
3, 3, 6	×	1	X	[4,3,20]	1	×	×	[4,4,2]	3 🗡	•
3, 3, 7	×	1	X	[4,4,1]	1	1	1	[5, 4, -1]		1
3, 3, 8	1	1	1	[4,4,2]	1	1	1	5, 4, -2	×	1
3, 3, 9	1	1	1	4,4,3	X	1	×	5, 4, -3	X	1
[4,3,1]	1	1	1	[4,4,4]	×	1	X	[5,4,-4]	X	1
4.3.2	1	1	1	[4,4,5]	X	1	×	5, 4, -5	×	1
4.3.3	1	1	1	[4,4,6]	X	1	×	[5,4,-6]	X	1
4,3,4	1	1	1	[4,4,7]	X	1	×	[5,4,1]	1	
4,3,5	1	1	1	[4,4,8]	×	1	×	[5,4,2]	1	
4,3,6	1	1	1	[4,4,9]	1	1	1	[5,4,3]	1	•
4,3,7	×	1	X	[4,4,10]	X	1	×	[5,4,4]	×	
4,3,8	1	1	1	[4,4,11]	X	1	×	[5,4,5]	×	۰,
4.3.9	1	1	1	[4,4,12]	X	1	×	[5,4,6]	1	

	PSC	Hicks #	PSC1	PSC2	PSC
	×	[5,4,7]	1	1	1
1	×	[5,4,8]	×	1	×
1	X	[5,4,9]	X	1	×
1	X	[5,4,10]	×	×	×
1	X	[5,4,11]	1	×	×
1	1	[6,3,1]	1	1	1
1	X	[6,3,2]	1	1	1
1	X	6,3,3	1	1	1
1	X	[6,3,4]	1	1	1
1	1	[6,3,5]	1	1	1
1	×	[6,3,6]	1	1	1
1	×	[6, 4, -1]	1	×	×
1	X	[6,4,1]	1	1	1
1	X	6, 4, 2	1	1	
1	X	[6,4,3]	1	1	/ /
1	X	[6,4,4]	1	1	1
1	X	[6,4,5]	1	1	1
1	1	[6,4,6]	1	×	×
1	1	[7,4,1]	1	1	1
1	1	[7,4,2]	1	1	1
1	×	7,4,3	1	1	\ \ \ \
1	×	7,4,4	1	1	1
]	1	[7,4,5]	1	×	×

number of cases

total	PSC1	PSC2	PSC
92	57	71	44

Systematic study

X

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PSC-compatible infinitesimal group actions

Hicks

4.4.

Hicks $\#$	PSC1	PSC2	PSC	Hicks #	PSC1	PSC2	PSC
[3,2,1]	1	1	1	[4,3,10]	1	1	1
[3,2,2]	1	1	1	[4,3,11]	1	1	1
[3,2,3]	1	1	1	[4,3,12]	×	1	×
[3,2,4]	1	1	1	[4,3,13]	1	×	×
[3,2,5]	1	1	1	[4,3,14]	1	×	×
[3,3,1]	X	1	X	[4,3,15]	1	×	×
[3,3,2]	1	1	1	[4,3,16]	1	×	×
[3,3,3]	1	1	1	[4,3,17]	1	×	×
[3,3,4]	×	1	×	[4,3,18]	1	×	×
[3,3,5]	×	1	X	[4,3,19]	1	×	×
[3, 3, 6]	×	1	×	[4,3,20]	1	×	×
[3,3,7]	×	1	×	[4,4,1]	1	1	1
[3, 3, 8]	1	1	1	[4,4,2]	1	1	1
[3, 3, 9]	1	1	1	[4,4,3]	×	1	×
[4,3,1]	1	1	1	[4,4,4]	×	1	×
[4,3,2]	1	1	1	[4,4,5]	×	1	×
[4,3,3]	1	1	1	[4,4,6]	×	1	×
[4,3,4]	1	1	1	[4,4,7]	×	1	×
[4,3,5]	1	1	1	[4,4,8]	×	1	×
[4,3,6]	1	1	1	[4,4,9]	1	1	1
[4,3,7]	×	1	×	[4,4,10]	×	1	×
[4,3,8]	1	1	1	[4,4,11]	×	1	×
[4,3,9]	1	1	1	[4,4,12]	×	1	×

						-
#	PSC1	PSC2	PSC	Hicks #	PSC1	
13	X	1	X	[5,4,7]	1	
14	×	1	×	[5,4,8]	×	
15	×	1	X	[5,4,9]	×	
16	×	1	×	[5,4,10]	×	
17	×	1	×	[5,4,11]	1	
18	1	1	1	[6,3,1]	1	Γ
19]	×	1	X	[6,3,2]	1	
20]	×	1	×	[6,3,3]	1	ľ
21]	×	1	×	[6,3,4]	1	
22]	1	1	1	[6,3,5]	1	
23	×	1	×	[6,3,6]	1	
1]	1	×	×	[6,4,-1]	1	
2	×	×	×	[6,4,1]	1	
3	×	×	×	[6,4,2]	1	ľ
4	×	×	×	[6,4,3]	1	ľ
5	×	×	×	[6,4,4]	1	ľ
6	×	×	×	[6,4,5]	1	
L]	1	1	1	[6,4,6]	1	
2	1	1	1 1	[7,4,1]	1	
3	 Image: A start of the start of	1		[7,4,2]	1	ľ
1	×	1	×	[7,4,3]	1	t
5	×	1	×	[7,4,4]	1	
3	1	1	1	[7,4,5]	1	ľ

number of cases

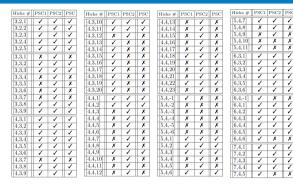
total	PSC1	PSC2	PSC
92	57	71	44

 only 42 qualitatively different cases (the answer to the ultimate question of life, the universe, and everything.)



Systematic study

PSC-compatible infinitesimal group actions



number of cases

to	tal	PSC1	PSC2	PSC
9	92 57		71	44

 only 42 qualitatively different cases (the answer to the ultimate question of life, the universe, and everything.)

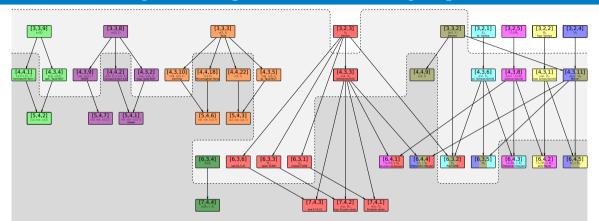


for each PSC-compatible Γ we determined the corresponding *l*-chains *χ* and Γ-invariant metrics *ĝ* in adapted coordinates

$$\hat{g} = \sum_{i=1}^{s} \phi_i q_i, \text{ where } s = \begin{cases} 2, & \text{for } [6,3,\star], [6,4,\star], [7,4,\star] \\ 4, & \text{for } [3,2,\star], [4,3,\star], [5,4,\star] \\ 10, & \text{for } [3,3,\star], [4,4,\star] \end{cases} \text{ and } \phi_i = \phi_i(x_1, x_2, \dots, x_{(4-l)})$$

Systematic study

Relations among PSC-compatible infinitesimal group actions



$$\hat{g} = \sum_{i=1}^{s} \phi_{i} q_{i}, \text{ where } s = \begin{cases} 2, & \text{for } [6,3,\star], [6,4,\star], [7,4,\star] \\ 4, & \text{for } [3,2,\star], [4,3,\star], [5,4,\star] \\ 10, & \text{for } [3,3,\star], [4,4,\star] \end{cases} \text{ and } \phi_{i} = \phi_{i}(x_{1},x_{2},\ldots,x_{(4-l)})$$

Tomáš Málek

Symmetry reduction of gravitational Lagrangians

Weyl trick revisited ([4,3,3]: stationary S_2)

infinitesimal group action

$$\Gamma = \operatorname{span} \{ \cos \varphi \, \partial_{\vartheta} - \cot \vartheta \sin \varphi \, \partial_{\varphi}, \, \sin \varphi \, \partial_{\vartheta} + \cot \vartheta \cos \varphi \, \partial_{\varphi}, \, \partial_{\varphi}, \, \partial_{t} \}$$

Weyl trick revisited ([4,3,3]: stationary S_2)

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$$\Gamma = \operatorname{span} \{ \cos \varphi \, \partial_{\vartheta} - \cot \vartheta \sin \varphi \, \partial_{\varphi}, \ \sin \varphi \, \partial_{\vartheta} + \cot \vartheta \cos \varphi \, \partial_{\varphi}, \ \partial_{\varphi}, \ \partial_{t} \}$$

Γ-invariant metric

$$\hat{g} = -\phi_1(r) \, \mathbf{d}t^2 + \phi_2(r) (\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r) \, \mathbf{d}r^2 + \phi_4(r) (\mathbf{d}\vartheta^2 + \sin^2\vartheta \, \mathbf{d}\varphi^2)$$

infinitesimal group action

$$\Gamma = \operatorname{span} \{ \cos \varphi \, \partial_{\vartheta} - \cot \vartheta \sin \varphi \, \partial_{\varphi}, \ \sin \varphi \, \partial_{\vartheta} + \cot \vartheta \cos \varphi \, \partial_{\varphi}, \ \partial_{\varphi}, \ \partial_{t} \}$$

Γ-invariant metric

$$\hat{\mathbf{g}} = -\phi_1(r)\,\mathbf{d}t^2 + \phi_2(r)(\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r)\,\mathbf{d}r^2 + \phi_4(r)(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2)$$

■ residual gauge freedom $t \to t + A(r)$, $r \to B(r)$ do not invalidate PSC and allows us to fix $\phi_2 = 0$ and $\phi_4 = r^2$

infinitesimal group action

$$\Gamma = \operatorname{span} \{ \cos \varphi \, \partial_{\vartheta} - \cot \vartheta \sin \varphi \, \partial_{\varphi}, \ \sin \varphi \, \partial_{\vartheta} + \cot \vartheta \cos \varphi \, \partial_{\varphi}, \ \partial_{\varphi}, \ \partial_{t} \}$$

Γ-invariant metric

$$\hat{\mathbf{g}} = -\phi_1(r)\,\mathbf{d}t^2 + \phi_2(r)(\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r)\,\mathbf{d}r^2 + \phi_4(r)(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2)$$

■ residual gauge freedom $t \to t + A(r), r \to B(r)$ do not invalidate PSC and allows us to fix $\phi_2 = 0$ and $\phi_4 = r^2$

*l-*chain

$$\boldsymbol{\chi} = \csc \, \vartheta \, \boldsymbol{\partial}_t \wedge \boldsymbol{\partial}_\vartheta \wedge \boldsymbol{\partial}_\varphi$$

infinitesimal group action

$$\Gamma = \operatorname{span} \{ \cos \varphi \, \partial_{\vartheta} - \cot \vartheta \sin \varphi \, \partial_{\varphi}, \ \sin \varphi \, \partial_{\vartheta} + \cot \vartheta \cos \varphi \, \partial_{\varphi}, \ \partial_{\varphi}, \ \partial_{t} \}$$

Γ-invariant metric

$$\hat{\mathbf{g}} = -\phi_1(r)\,\mathbf{d}t^2 + \phi_2(r)(\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r)\,\mathbf{d}r^2 + \phi_4(r)(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2)$$

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*l-*chain

$$\boldsymbol{\chi} = \csc \, \vartheta \, \boldsymbol{\partial}_t \wedge \boldsymbol{\partial}_\vartheta \wedge \boldsymbol{\partial}_\varphi$$

Levi-Civita tensor

$$\underline{\epsilon}(\hat{g}) = r^2 \sin \vartheta \sqrt{\phi_1 \phi_3} \, \mathbf{d} t \wedge \mathbf{d} r \wedge \mathbf{d} \vartheta \wedge \mathbf{d} \varphi$$

infinitesimal group action

$$\Gamma = \operatorname{span} \{ \cos \varphi \, \partial_{\vartheta} - \cot \vartheta \sin \varphi \, \partial_{\varphi}, \ \sin \varphi \, \partial_{\vartheta} + \cot \vartheta \cos \varphi \, \partial_{\varphi}, \ \partial_{\varphi}, \ \partial_{t} \}$$

Γ-invariant metric

$$\hat{\mathbf{g}} = -\phi_1(r)\,\mathbf{d}t^2 + \phi_2(r)(\mathbf{d}t \vee \mathbf{d}r) + \phi_3(r)\,\mathbf{d}r^2 + \phi_4(r)(\mathbf{d}\vartheta^2 + \sin^2\vartheta\,\mathbf{d}\varphi^2)$$

- residual gauge freedom $t \to t + A(r)$, $r \to B(r)$ do not invalidate PSC and allows us to fix $\phi_2 = 0$ and $\phi_4 = r^2$
- *l-*chain

$$oldsymbol{\chi} = \cscartheta \,oldsymbol{\partial}_t \wedge oldsymbol{\partial}_artheta \wedge oldsymbol{\partial}_arphi$$

Levi-Civita tensor

$$\underline{\epsilon}(\hat{g}) = r^2 \sin artheta \sqrt{\phi_1 \phi_3} \, \mathbf{d} t \wedge \mathbf{d} r \wedge \mathbf{d} artheta \wedge \mathbf{d} arphi$$

reduced Lagrangian 1-form

$$\underline{\hat{L}} = \boldsymbol{\chi} \bullet \underline{\boldsymbol{\epsilon}}(\hat{\boldsymbol{g}}) L[\hat{\boldsymbol{g}}] = r^2 \sqrt{\phi_1 \phi_3} L[\hat{\boldsymbol{g}}] \, \mathbf{d}r$$

infinitesimal group action

$$\Gamma = \operatorname{span}\{\partial_x, \partial_y, \partial_z, x \partial_y - y \partial_x, y \partial_z - z \partial_y, z \partial_x - x \partial_z\}$$

infinitesimal group action

$$\Gamma = \operatorname{span}\{\partial_x, \partial_y, \partial_z, x \partial_y - y \partial_x, y \partial_z - z \partial_y, z \partial_x - x \partial_z\}$$

Γ-invariant metric

$$\hat{g} = -\phi_1(t) \, \mathbf{d}t^2 + \phi_2(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2)$$

infinitesimal group action

$$\Gamma = \operatorname{span}\{\partial_x, \partial_y, \partial_z, x \partial_y - y \partial_x, y \partial_z - z \partial_y, z \partial_x - x \partial_z\}$$

Γ-invariant metric

$$\hat{g} = -\phi_1(t) \, \mathbf{d}t^2 + \phi_2(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2)$$

• residual gauge freedom $t \rightarrow A(t)$ which would allow us to set $\phi_1 = 1$ breaks PSC

infinitesimal group action

$$\Gamma = \operatorname{span}\{\partial_x, \partial_y, \partial_z, x \partial_y - y \partial_x, y \partial_z - z \partial_y, z \partial_x - x \partial_z\}$$

Γ-invariant metric

$$\hat{g} = -\phi_1(t) \, \mathbf{d}t^2 + \phi_2(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2)$$

■ residual gauge freedom $t \rightarrow A(t)$ which would allow us to set $\phi_1 = 1$ breaks PSC ■ *l*-chain

$$\boldsymbol{\chi} = \boldsymbol{\partial}_x \wedge \, \boldsymbol{\partial}_y \wedge \boldsymbol{\partial}_z$$

infinitesimal group action

$$\Gamma = \operatorname{span}\{\partial_x, \partial_y, \partial_z, x \partial_y - y \partial_x, y \partial_z - z \partial_y, z \partial_x - x \partial_z\}$$

Γ-invariant metric

$$\hat{g} = -\phi_1(t) \, \mathbf{d}t^2 + \phi_2(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2)$$

■ residual gauge freedom $t \rightarrow A(t)$ which would allow us to set $\phi_1 = 1$ breaks PSC ■ *l*-chain

$$\boldsymbol{\chi} = \boldsymbol{\partial}_x \wedge \, \boldsymbol{\partial}_y \wedge \boldsymbol{\partial}_z$$

Levi-Civita tensor

$$\sqrt{\phi_1\phi_2^3}\,\mathbf{d}t\wedge\mathbf{d}x\wedge\mathbf{d}y\wedge\mathbf{d}z$$

infinitesimal group action

$$\Gamma = \operatorname{span}\{\partial_x, \partial_y, \partial_z, x \partial_y - y \partial_x, y \partial_z - z \partial_y, z \partial_x - x \partial_z\}$$

Γ-invariant metric

$$\hat{g} = -\phi_1(t) \, \mathbf{d}t^2 + \phi_2(t)(\mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2)$$

■ residual gauge freedom $t \rightarrow A(t)$ which would allow us to set $\phi_1 = 1$ breaks PSC ■ *l*-chain

$$\boldsymbol{\chi} = \boldsymbol{\partial}_x \wedge \, \boldsymbol{\partial}_y \wedge \boldsymbol{\partial}_z$$

Levi-Civita tensor

$$\sqrt{\phi_1\phi_2^3}\,\mathbf{d}t\wedge\mathbf{d}x\wedge\mathbf{d}y\wedge\mathbf{d}z$$

reduced Lagrangian 1-form

$$\underline{\hat{L}} = \chi \bullet \underline{\epsilon}(\hat{g}) L[\hat{g}] = \sqrt{\phi_1 \phi_3^2} L[\hat{g}] \, \mathrm{d}t$$

Conclusion

• We established the essential ingredients for a successful symmetry reduction:

- 1 identified all possible PSC-compatible infinitesimal group actions Γ
- 2 determined corresponding Γ-invariant metrics and *l*-chains in adapted coordinates
- 3 minimized the amount of unknown functions employing residual gauge freedom compliant with PSC
- As a by-product, we implemeted the symmetry reduction of Lagrangians in MATHEMATICA employing the xAct package.

Conclusion

• We established the essential ingredients for a successful symmetry reduction:

- 1 identified all possible PSC-compatible infinitesimal group actions Γ
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- 3 minimized the amount of unknown functions employing residual gauge freedom compliant with PSC
- As a by-product, we implemeted the symmetry reduction of Lagrangians in MATHEMATICA employing the xAct package.

Thank you! Obrigado!

List of infinitesimal group actions Γ

	[4, 4, 1]			
		$-\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}, \partial_{y_4}, \partial_{y_2}$		
	[4,4,2]	$-\operatorname{csch} y_3 \sin y_4 \partial_{y_1} + \cos y_4 \partial_{y_3} - \operatorname{cth} y_3 \sin y_4 \partial_{y_4}, -\cos y_4 \operatorname{csch} y_3 \partial_{y_1}$		
		$-\sin y_4 \partial_{y_3} - \cos y_4 \operatorname{cth} y_3 \partial_{y_4}, \partial_{y_4}, \partial_{y_2}$		
	[4.4.9]	$\partial_{y_1}, \partial_{y_2}, \partial_{y_4}, -y_4 \partial_{y_1} + \partial_{y_3} + y_2 \partial_{y_4}$	1	
	4,4,18			
		$\frac{\partial_{y_1}}{\partial_{y_1}}, \frac{\partial_{y_4}}{\partial_{y_4}}, \frac{\partial_{y_1}}{\partial_{y_1}} + \frac{\partial_{y_3}}{\partial_{y_3}}, \frac{\partial_{y_2}}{\partial_{y_2}} + \frac{\partial_{y_3}}{\partial_{y_1}} + \frac{\partial_{y_2}}{\partial_{y_2}} + \frac{\partial_{y_3}}{\partial_{y_4}} + \frac{\partial_{y_3}}{\partial_{y_4}} - \frac{\partial_{y_3}}{\partial_{y_4}} + \frac{\partial_{y_3}}{\partial_{y_4}} + \frac{\partial_{y_3}}{\partial_{y_4}} + \frac{\partial_{y_3}}{\partial_{y_4}} + \frac{\partial_{y_3}}{\partial_{y_4}} + \frac{\partial_{y_3}}{\partial_{y_4}} + \frac{\partial_{y_4}}{\partial_{y_4}} + \frac$		
Hicks # Isometry algebra Γ	[5,4,1]	$-\operatorname{csch} y_3 \sin y_4 \partial_{y_1} + \cos y_4 \partial_{y_3} - \operatorname{cth} y_3 \sin y_4 \partial_{y_4}, -\cos y_4 \operatorname{csch} y_3 \partial_{y_1}$	[6,4,1]	$\cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}$, $-\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}$, ∂_{y_4} ,
$[3,2,1]$ $\ \partial_{y_3}, \partial_{y_4}, -y_4 \partial_{y_3} + y_3 \partial_{y_4}$		$-\sin y_4 \partial_{y_3} - \cos y_4 \operatorname{cth} y_3 \partial_{y_4}, \partial_{y_4}, \partial_{y_1}, \partial_{y_2}$		$-\sin y_1 \operatorname{th} y_2 \partial_{y_1} + \cos y_1 \partial_{y_2}, -\cos y_1 \operatorname{th} y_2 \partial_{y_1} - \sin y_1 \partial_{y_2}, \partial_{y_1}$
$3,2,2$ $\cos y_4 \partial_{y_3} - \operatorname{cth} y_3 \sin y_4 \partial_{y_4}, -\sin y_4 \partial_{y_3} - \cos y_4 \operatorname{cth} y_3 \partial_{y_4}, \partial_{y_4}$	[5,4,2]		[6,4,2]	$\cos y_4 \partial_{y_3} - \operatorname{cth} y_3 \sin y_4 \partial_{y_4}, - \sin y_4 \partial_{y_3} - \cos y_4 \operatorname{cth} y_3 \partial_{y_4}, \partial_{y_4},$
[3,2,3] $\cos y_1 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4} - \sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4} \partial_{y_4}$ [3,2,3] $\cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4} - \sin y_4 \partial_{y_1} - \cos y_4 \cot y_3 \partial_{y_4} \partial_{y_4}$		$-\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}, \partial_{y_4}, \partial_{y_1}, \partial_{y_2}$		$-\sin y_1 \operatorname{th} y_2 \partial_{y_1} + \cos y_1 \partial_{y_2}, -\cos y_1 \operatorname{th} y_2 \partial_{y_1} - \sin y_1 \partial_{y_2}, \partial_{y_1}$
$\begin{array}{c} (52.6) & (\cos y_1 O_{y_1} - \cos y_2 O_{y_1} + y_1 O_{y_2} \\ (3.2.4) & (\partial_{y_1}, \partial_{y_2}, y_2 \partial_{y_1} + y_1 \partial_{y_2} \end{array}$	[5,4,3]	$ \partial_{y_1}, \partial_{y_4}, y_4 \partial_{y_1} + \partial_{y_3}, \frac{1}{2}(-y_3^2 + y_4^2) \partial_{y_1} + y_4 \partial_{y_3} - y_3 \partial_{y_4}, \partial_{y_2}$	[6,4,3]	$-\sin y_1 \operatorname{th} y_2 \partial_{y_1} + \cos y_1 \partial_{y_2}, -\cos y_1 \operatorname{th} y_2 \partial_{y_1} - \sin y_1 \partial_{y_2}, \partial_{y_1}, \partial_{y_2}, \partial_{y_3}, \partial_{y_4}, \partial_{y_4}$
	5,4,6	$\partial_{y_1}, \partial_{y_4}, y_4 \partial_{y_1} + \partial_{y_3}, y_3 \partial_{y_3} - y_4 \partial_{y_4}, \partial_{y_2}$	1 1	$-y_4 \partial_{y_1} + y_3 \partial_{y_4}$
$[3,2,5] = -\sin y_1 \text{ th } y_2 \partial_{y_1} + \cos y_1 \partial_{y_2}, -\cos y_1 \text{ th } y_2 \partial_{y_1} - \sin y_1 \partial_{y_2}, \partial_{y_1}$	5,4,7	$-\sin y_1 \operatorname{th} y_2 \partial_{y_1} + \cos y_1 \partial_{y_2} - \operatorname{sech} y_2 \sin y_1 \partial_{y_3}, -\cos y_1 \operatorname{th} y_2 \partial_{y_1}$	[6.4.4]	$\cos y_4 \partial_{y_5} - \cot y_1 \sin y_4 \partial_{y_4} - \sin y_4 \partial_{y_5} - \cos y_4 \cot y_3 \partial_{y_4} \partial_{y_4} \partial_{y_5} \partial_{y_5}$
$[3,3,2] = [\partial_{y_1}, \partial_{y_2}, \partial_{y_4}]$		$-\sin y_1 \partial_{y_2} - \cos y_1 \operatorname{sech} y_2 \partial_{y_3}, \partial_{y_4}, \partial_{y_5}, \partial_{y_4}$		$y_2 \partial_{y_1} + y_1 \partial_{y_2}$
$[3,3,3]$ $[\partial_{y_1}, \partial_{y_4}, y_4, \partial_{y_1} + \partial_{y_3}]$	[6,3,1]	$ \cos y_4 \partial_{y_1} - \cot y_3 \sin y_4 \partial_{y_1} - \sin y_4 \partial_{x_2} - \cos y_4 \cot y_3 \partial_{y_1} \partial_{y_2}$	[6.4.5]	$\cos y_4 \partial_{y_1} - \operatorname{cth} y_3 \sin y_4 \partial_{y_4}, - \sin y_4 \partial_{y_5} - \cos y_4 \operatorname{cth} y_3 \partial_{y_4}, \partial_{y_4}, y_2 \partial_{y_5}$
$[3,3,8]$ $-\operatorname{csch} y_3 \sin y_4 \partial_{y_1} + \cos y_4 \partial_{y_3} - \operatorname{cth} y_3 \sin y_4 \partial_{y_4}, -\cos y_4 \operatorname{csch} y_3 \partial_{y_4}$	[0,0,1]	$\cos y_4 \ \delta y_3 - \cos y_3 \sin y_4 \ \delta y_4, -\sin y_4 \ \delta y_3 - \cos y_4 \cot y_3 \ \delta y_4, \delta y_4, \cos y_4 \sin y_4 \ \delta y_4, \cos y_4 \sin y_4 \ \delta y_4, \cos y_4 \cos y_4 \cos y_4 \ \delta y_4, \sin y_4 \ \delta y_5, \sin y_5,$	Contraction	$+y_1\partial_{y_2}, \partial_{y_1}, \partial_{y_2}$
$-\sin y_4 \partial_{y_3} - \cos y_4 \operatorname{cth} y_3 \partial_{y_4}, \partial_{y_4}$		$\sin y_3 \sin y_4 \partial_{y_1} + \cos y_3 \cos y_4 \cos y_2 \partial_{y_3} - \cos y_2 \cos y_3 \sin y_4 \partial_{y_4},$ $\sin y_3 \sin y_4 \partial_{y_1} + \cos y_3 \cot y_2 \sin y_4 \partial_{y_3} + \cos y_4 \cot y_2 \csc y_3 \partial_{y_4}, \cos y_3 \partial_{y_5}$	177 4 13	
$[3,3,9] = -\csc y_3 \sin y_4 \partial_{y_1} + \cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}, -\cos y_4 \csc y_3 \partial_{y_1}$		$\sin y_3 \sin y_4 \partial_{y_2} + \cos y_3 \cos y_2 \sin y_4 \partial_{y_3} + \cos y_4 \cot y_2 \csc y_3 \partial_{y_4}, \cos y_3 \partial_{y_2}$ - $\cot y_2 \sin y_3 \partial_{y_3}$	[7,4,1]	$\cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}, -\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}, \partial_{y_4},$
$-\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}, \partial_{y_4}$	[6,3,2]			$\cos y_4 \sin y_3 \partial_{y_2} + \cos y_3 \cos y_4 \cot y_2 \partial_{y_3} - \cot y_2 \csc y_3 \sin y_4 \partial_{y_4}$
[4,3,1] $\cos y_4 \partial_{y_3} - \operatorname{cth} y_3 \sin y_4 \partial_{y_4}, -\sin y_4 \partial_{y_3} - \cos y_4 \operatorname{cth} y_3 \partial_{y_4}, \partial_{y_5}$		$\partial_{y_3}, \partial_{y_4}, -y_4 \partial_{y_3} + y_5 \partial_{y_4}, y_4 \partial_{y_2} - y_2 \partial_{y_4}, y_3 \partial_{y_2} - y_2 \partial_{y_3}, -\partial_{y_2}$		$\sin y_3 \sin y_4 \partial_{y_2} + \cos y_3 \cot y_2 \sin y_4 \partial_{y_3} + \cos y_4 \cot y_2 \csc y_3 \partial_{y_4}, \cos y_3 \partial_{y_2}$
$\begin{array}{c} 4,3,1 \\ \hline 4,3,2 \\ \hline -\cosh y_1 \ \delta_{y_3} - \cosh y_3 \ \sin y_4 \ \delta_{y_3} - \cosh y_1 \ \delta_{y_3} - \cos y_4 \ \cosh y_1 \ \delta_{y_1}, \ \delta_{y_1}, \ \delta_{y_1} \\ \hline 4,3,2 \\ \hline -\cosh y_1 \ \sin y_4 \ \delta_{y_1} + \cos y_4 \ \delta_{y_2} - \cosh y_1 \ \sin y_4 \ \delta_{y_1}, \ -\cos y_4 \ \cosh y_1 \ \delta_{y_1} \\ \hline \delta_{y_1} \ \delta_{y_1} \ \delta_{y_1} \ \delta_{y_1} \\ \hline \delta_{y_1} \ \delta_{y_1} \ \delta_{y_1} \ \delta_{y_1} \ \delta_{y_1} \ \delta_{y_1} \\ \hline \delta_{y_1} \ \delta_{y_1} \$	[6,3,3]	$\cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}$, $-\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}$, ∂_{y_4} ,	100 1 03	$-\cot y_2 \sin y_3 \partial_{y_3}, \partial_{y_1}$
$\begin{bmatrix} q, 3, 2 \end{bmatrix} = \frac{-\cos y_3 \sin y_4 \partial_{y_1} + \cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}, -\cos y_4 \cosh y_3 \partial_{y_1}}{-\sin y_4 \partial_{y_3} - \cos y_4 \coth y_3 \partial_{y_4}, \partial_{y_4}, \partial_{y_1}}$		$\cos y_4 \sin y_3 \partial_{y_2} + \cos y_3 \cos y_4 \operatorname{cth} y_2 \partial_{y_3} - \operatorname{cth} y_2 \csc y_3 \sin y_4 \partial_{y_4}$	[7,4,2]	$\cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}$, $-\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}$, ∂_{y_4} ,
		$\sin y_3 \sin y_4 \partial_{y_2} + \cos y_3 \operatorname{cth} y_2 \sin y_4 \partial_{y_3} + \cos y_4 \operatorname{cth} y_2 \csc y_3 \partial_{y_4}, \cos y_3 \partial_{y_2}$		$\cos y_4 \sin y_3 \partial_{y_2} + \cos y_3 \cos y_4 \operatorname{cth} y_2 \partial_{y_3} - \operatorname{cth} y_2 \operatorname{csc} y_3 \sin y_4 \partial_{y_4}$
$[4,3,3] \cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}, -\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}, \partial_{y_4}, \partial_{y_1}$		$-\operatorname{cth} y_2 \sin y_3 \partial_{y_3}$		$\sin y_3 \sin y_4 \partial_{y_2} + \cos y_3 \operatorname{cth} y_2 \sin y_4 \partial_{y_3} + \cos y_4 \operatorname{cth} y_2 \csc y_3 \partial_{y_4}, \cos y_3 \partial_{y_2}$
$[4,3,4] = -\csc y_3 \sin y_4 \partial_{y_1} + \cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}, -\cos y_4 \csc y_3 \partial_{y_1}$	[6,3,4]	$-\sin(y_1 - y_3)$ th $y_2 \partial_{y_1} + \cos(y_1 - y_3) \partial_{y_2} + \operatorname{cth} y_2 \sin(y_1 - y_3) \partial_{y_3}$, $\cos(y_1 - y_3) \partial_{y_3}$		$-\operatorname{cth} y_2 \sin y_3 \partial_{y_3}, \partial_{y_1}$
$-\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}, \partial_{y_4}, \partial_{y_1}$		$(-y_3)$ th $y_2 \partial_{y_1} + \sin(y_1 - y_3) \partial_{y_2} - \cos(y_1 - y_3)$ cth $y_2 \partial_{y_3}, -\partial_{y_1} + \partial_{y_3},$	[7,4,3]	$\cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}$, $-\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}$, ∂_{y_4} ,
[4,3,5] $\partial_{y_1}, \partial_{y_4}, y_4 \partial_{y_1} + \partial_{y_3}, \frac{1}{2}(-y_3^2 + y_4^2) \partial_{y_1} + y_4 \partial_{y_3} - y_3 \partial_{y_4}$		$\sin(y_1 + y_3)$ th $y_2 \partial_{y_1} - \cos(y_1 + y_3) \partial_{y_2} + \operatorname{cth} y_2 \sin(y_1 + y_3) \partial_{y_3}$, $\cos(y_1 + y_3) \partial_{y_3}$		$\cos y_4 \sin y_3 \partial_{y_2} + \cos y_3 \cos y_4 \operatorname{cth} y_2 \partial_{y_3} - \operatorname{cth} y_2 \operatorname{csc} y_3 \sin y_4 \partial_{y_4}$
$[4,3,6]$ $[\partial_{y_3}, \partial_{y_4}, -y_4 \partial_{y_3} + y_3 \partial_{y_4}, \partial_{y_1}$		$(+ y_3) \operatorname{th} y_2 \partial_{y_1} + \sin(y_1 + y_3) \partial_{y_2} + \cos(y_1 + y_3) \operatorname{cth} y_2 \partial_{y_3}, \partial_{y_1} + \partial_{y_3}$		$\sin y_3 \sin y_4 \partial_{y_2} + \cos y_3 \operatorname{cth} y_2 \sin y_4 \partial_{y_3} + \cos y_4 \operatorname{cth} y_2 \csc y_3 \partial_{y_4}, \cos y_3 \partial_{y_2}$
$[4,3,8]$ $-\sin y_1 \operatorname{th} y_2 \partial_{y_1} + \cos y_1 \partial_{y_2}, -\cos y_1 \operatorname{th} y_2 \partial_{y_1} - \sin y_1 \partial_{y_2}, \partial_{y_1}, \partial_{y_4}$	6,3,5	$\partial_{y_1}, \partial_{y_2}, \partial_{y_4}, y_2 \partial_{y_1} + y_1 \partial_{y_2}, y_4 \partial_{y_1} + y_1 \partial_{y_4}, -y_4 \partial_{y_2} + y_2 \partial_{y_4}$		$-\operatorname{cth} y_2 \sin y_3 \partial_{y_3}, \partial_{y_1}$
$[4,3,9]$ $-\sin y_1 \operatorname{th} y_2 \partial_{y_1} + \cos y_1 \partial_{y_2} - \operatorname{sech} y_2 \sin y_1 \partial_{y_3}, -\cos y_1 \operatorname{th} y_2 \partial_{y_1}$	[6,3,6]	$\cos y_4 \partial_{y_3} - \cot y_3 \sin y_4 \partial_{y_4}$, $-\sin y_4 \partial_{y_3} - \cos y_4 \cot y_3 \partial_{y_4}$, ∂_{y_4} ,	[7,4,4]	$-\sin(y_1 - y_3)$ th $y_2 \partial_{y_1} + \cos(y_1 - y_3) \partial_{y_2} + \text{cth } y_2 \sin(y_1 - y_3) \partial_{y_3}$, $\cos(y_1 - y_3) \partial_{y_3}$
$-\sin y_1 \partial_{y_2} - \cos y_1 \operatorname{sech} y_2 \partial_{y_3}, \partial_{y_1}, \partial_{y_3}$	1	$\cos y_4 \sin y_3 \partial_{y_2} + \cos y_3 \cos y_4 \operatorname{cth} y_2 \partial_{y_3} - \operatorname{cth} y_2 \operatorname{csc} y_3 \sin y_4 \partial_{y_4}$		$(-y_3)$ th $y_2 \partial_{y_1}$ + sin $(y_1 - y_3) \partial_{y_2}$ - cos $(y_1 - y_3)$ cth $y_2 \partial_{y_3}$, $-\partial_{y_1} + \partial_{y_3}$,
$[4,3,10]$ ∂_{y_1} , ∂_{y_4} , $y_4 \partial_{y_1} + \partial_{y_3}$, $y_3 \partial_{y_3} - y_4 \partial_{y_4}$		$\sin y_3 \sin y_4 \partial_{y_1} + \cos y_3 \operatorname{cth} y_2 \sin y_4 \partial_{y_1} + \cos y_4 \operatorname{cth} y_2 \csc y_3 \partial_{y_4}, \cos y_3 \partial_{y_1}$		$\sin(y_1 + y_3)$ th $y_2 \partial_{y_1} - \cos(y_1 + y_3) \partial_{y_2} + \operatorname{cth} y_2 \sin(y_1 + y_3) \partial_{y_3}$, $\cos(y_1 + y_3) \partial_{y_3}$
$[4,3,11]$ ∂_{y_1} , ∂_{y_2} , ∂_{y_4} , $y_2 \partial_{y_1} + y_1 \partial_{y_2}$		$-\operatorname{cth} y_2 \sin y_3 \partial_{y_3}$		$(+y_3)$ th $y_2 \partial_{y_1} + \sin(y_1 + y_3) \partial_{y_2} + \cos(y_1 + y_3)$ cth $y_2 \partial_{y_3}, \partial_{y_1} + \partial_{y_3}, \partial_{y_4}$
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List of Γ -invariant metrics

Hicks #	Γ -invariant metric \hat{g}	
	$ \ -\phi_1(y_1, y_2) \mathbf{d}y_1^2 + \phi_2(y_1, y_2) (\mathbf{d}y_1 \vee \mathbf{d}y_2) + \phi_3(y_1, y_2) \mathbf{d}y_2^2 + \phi_4(y_1, y_2) (\mathbf{d}y_3^2 + \mathbf{d}y_4^2) $	
	$\frac{-\phi_1(y_1, y_2)(\mathbf{d}y_1 + \phi_2(y_1, y_2)(\mathbf{d}y_1 \vee \mathbf{d}y_2) + \phi_3(y_1, y_2)(\mathbf{d}y_2 + \phi_4(y_1, y_2)(\mathbf{d}y_3 + \mathbf{d}y_4))}{-\phi_1(y_1, y_2)(\mathbf{d}y_1^2 + \phi_2(y_1, y_2)(\mathbf{d}y_1 \vee \mathbf{d}y_2) + \phi_3(y_1, y_2)(\mathbf{d}y_2^2 + \phi_4(y_1, y_2)(\mathbf{d}y_3^2 + \mathbf{sh}^2 y_3)(\mathbf{d}y_4^2)}$	
	$\frac{-\phi_1(y_1, y_2) dy_1^2 + \phi_2(y_1, y_2)(dy_1 \vee dy_2) + \phi_3(y_1, y_2) dy_2^2 + \phi_4(y_1, y_2)(dy_3^2 + \sin^2 y_3 dy_4)}{-\phi_1(y_1, y_2) dy_1^2 + \phi_2(y_1, y_2)(dy_1 \vee dy_2) + \phi_3(y_1, y_2) dy_2^2 + \phi_4(y_1, y_2)(dy_3^2 + \sin^2 y_3 dy_4^2)}$	
	$ \begin{array}{c} +(g_1,g_2) dg_1 + (g_1,g_2) (dg_1 + dg_2) + (g_1g_2) dg_2 + (g_1g_2) (dg_3 + dg_3) g_3 dg_4 \\ \phi_1(y_3,y_4) (-dy_1^2 + dy_2^2) + \phi_2(y_3,y_4) dy_3^2 + \phi_3(y_3,y_4) (dy_3 \vee dy_4) + \phi_4(y_3,y_4) dy_4^2 \end{array} $	
	$ \phi_1(y_3, y_4)(-\operatorname{ch}^2 y_2 \mathbf{d} y_1^2 + \mathbf{d} y_2^2) + \phi_2(y_3, y_4) \mathbf{d} y_3^2 + \phi_3(y_3, y_4) (\mathbf{d} y_3 \vee \mathbf{d} y_4) + \phi_4(y_3, y_4) \mathbf{d} y_4^2 $	
	$\ -\phi_1(y_3) \mathbf{d}{y_1}^2 + \phi_2(y_3) \mathbf{d}{y_3}^2 + \phi_3(y_3)(\mathbf{d}{y_3} \lor \mathbf{d}{y_4}) + \phi_4(y_3) \mathbf{d}{y_4}^2 + \phi_5(y_3) \mathbf{d}{y_2}^2 + \phi_6(y_3)(\mathbf{d}{y_1} \lor \mathbf{d}{y_2})$	
[0,0,2]	$ \begin{array}{l} + \phi_7(y_3)(\mathbf{d}y_1 + \phi_2(y_3)(\mathbf{d}y_3 + \phi_3(y_3)(\mathbf{d}y_3 + \mathbf{d}y_4) + \phi_4(y_3)(\mathbf{d}y_4 + \phi_5(y_3)(\mathbf{d}y_2 + \phi_6(y_3)(\mathbf{d}y_1 + \mathbf{d}y_2)) \\ + \phi_7(y_3)(\mathbf{d}y_1 + \mathbf{d}y_3) + \phi_8(y_3)(\mathbf{d}y_1 + \mathbf{d}y_4) + \phi_9(y_3)(\mathbf{d}y_2 + \mathbf{d}y_4) + \phi_{10}(y_3)(\mathbf{d}y_2 + \mathbf{d}y_4) \end{array} $	
[3,3,3]	$\frac{ -\phi_1(y_2)(\mathbf{d}y_1 - y_3 \mathbf{d}y_4)^2 + \phi_2(y_2)(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(y_2) \mathbf{d}y_2^2 + \phi_4(y_2) \mathbf{d}y_3^2 + \phi_5(y_2)(\mathbf{d}y_3 \vee (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) }{ \phi_1(y_1) - \phi_1(y_2)(\mathbf{d}y_1 - y_3 \mathbf{d}y_4) + \phi_2(y_2) \mathbf{d}y_2^2 + \phi_5(y_2)(\mathbf{d}y_3 \vee (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) }$	$\left[[4,4,9] \ \left\ -\phi_1 (\mathbf{d}y_1 - \frac{1}{3}y_3^2 \mathbf{d}y_2 + y_3 \mathbf{d}y_4)^2 + \phi_2 \mathbf{d}y_3^2 + \phi_3 (\mathbf{d}y_3 \vee (-y_3 \mathbf{d}y_2 + \mathbf{d}y_4)) + \phi_4 (-y_3 \mathbf{d}y_2 + \mathbf{d}y_4)^2 + \phi_5 \mathbf{d}y_2^2 + \phi_6 (\mathbf{d}y_2 \vee (-y_3 \mathbf{d}y_3 + \mathbf{d}y_4)) + \phi_4 (-y_3 \mathbf{d}y_2 + \mathbf{d}y_4)^2 + \phi_5 \mathbf{d}y_2^2 + \phi_6 (\mathbf{d}y_2 \vee (-y_3 \mathbf{d}y_3 + \mathbf{d}y_4)) + \phi_4 (-y_3 \mathbf{d}y_2 + \mathbf{d}y_4)^2 + \phi_5 \mathbf{d}y_2^2 + \phi_6 (\mathbf{d}y_2 \vee (-y_3 \mathbf{d}y_3 + \mathbf{d}y_4)) + \phi_4 (-y_3 \mathbf{d}y_2 + \mathbf{d}y_4)^2 + \phi_5 \mathbf{d}y_2^2 + \phi_6 (\mathbf{d}y_2 \vee (-y_3 \mathbf{d}y_3 + \mathbf{d}y_4)) + \phi_4 (-y_3 \mathbf{d}y_2 + \mathbf{d}y_4)^2 + \phi_5 \mathbf{d}y_2^2 + \phi_6 (\mathbf{d}y_2 \vee (-y_3 \mathbf{d}y_3 + \mathbf{d}y_4)) + \phi_4 (-y_3 \mathbf{d}y_2 + \mathbf{d}y_4)^2 + \phi_5 \mathbf{d}y_2^2 + \phi_6 (\mathbf{d}y_2 \vee (-y_3 \mathbf{d}y_3 + \mathbf{d}y_4)) + \phi_4 (-y_3 \mathbf{d}y_2 + \mathbf{d}y_4)^2 + \phi_6 (\mathbf{d}y_2 \vee (-y_3 \mathbf{d}y_3 + \mathbf{d}y_4)) + \phi_6 (-y_3 \mathbf{d}y_2 + \mathbf{d}y_4)^2 + \phi_6 (\mathbf{d}y_2 \vee (-y_3 \mathbf{d}y_4 + \mathbf{d}y_4)) + \phi_6 (\mathbf{d}y_4 \vee (-y_3 \mathbf{d}y_4 + \mathbf{d}y_4)) + \phi_6 (-y_4 \mathbf{d}y_4) + \phi_6 (\mathbf{d}y_4 \vee (-y_4 \mathbf{d}y_4)) + \phi_6 (\mathbf{d}y_4 \vee (-y_4 \mathbf{d}y_4) + \phi_6 \mathbf{d}y_4 \vee (-y_4 \mathbf{d}y_4) + \phi_6 $
	$+\phi_{6}(y_{2})(\mathbf{d}y_{4}\vee(\mathbf{d}y_{1}-y_{3}\mathbf{d}y_{4}))+\phi_{7}(y_{2})(\mathbf{d}y_{2}\vee\mathbf{d}y_{3})+\phi_{8}(y_{2})(\mathbf{d}y_{2}\vee\mathbf{d}y_{4})+\phi_{9}(y_{2})(\mathbf{d}y_{3}\vee\mathbf{d}y_{4})+\phi_{10}(y_{2})\mathbf{d}y_{4}^{2}$	$(\mathbf{d}y_1 - \frac{1}{3}y_3^2 \mathbf{d}y_2 + y_3 \mathbf{d}y_4)) + \phi_7(\mathbf{d}y_3 \vee (\mathbf{d}y_1 - \frac{1}{3}y_3^2 \mathbf{d}y_2 + y_3 \mathbf{d}y_4)) + \phi_8((-y_3 \mathbf{d}y_2 + \mathbf{d}y_4) \vee (\mathbf{d}y_1 - \frac{1}{3}y_3^2 \mathbf{d}y_4) \vee (\mathbf{d}y_1 - \frac{1}{3}y_4 + \mathbf{d}y$
[3,3,8]	$-\phi_1(y_2)(\mathbf{d}y_1 - \mathbf{ch} y_3 \mathbf{d}y_4)^2 + \phi_2(y_2)(\mathbf{d}y_2 \lor (\mathbf{d}y_1 - \mathbf{ch} y_3 \mathbf{d}y_4)) + \phi_3(y_2) \mathbf{d}y_2^2 + \phi_4(y_2)(\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_4(y_2)(\mathbf{d}y_1 - \mathbf{ch} y_3 \mathbf{d}y_4) + \phi_4(y_2)(\mathbf{d}y_1 - \mathbf{ch} \mathbf{d}y_3 \mathbf{d}y_4) + \phi_4(y_2)(\mathbf{d}y_1 - \mathbf{ch} \mathbf{d}y_4) + \phi_4(y_2)(\mathbf{d}y_1 -$	$(y_3 \mathbf{d} y_4) + \phi_9(\mathbf{d} y_2 \lor \mathbf{d} y_3) + \phi_{10}(\mathbf{d} y_2 \lor (-y_3 \mathbf{d} y_2 + \mathbf{d} y_4))$
	$\phi_5(y_2)(\mathbf{d}y_2 \lor (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4)) + \phi_6(y_2)((\mathbf{d}y_1 - \operatorname{ch} y_3 \mathbf{d}y_4) \lor (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4)) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos y_1 \mathbf{d}y_3 - \sin y_1 \operatorname{sh} y_3 \mathbf{d}y_4) + (\cos$	$[4,4,18] -\phi_1(\mathbf{d}y_1 - y_3 \mathbf{d}y_4)^2 + \phi_2(\mathbf{d}y_2 \lor (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(\mathbf{d}y_2^2 + e^{2y_2} \phi_4(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(\mathbf{d}y_2^2 + e^{2y_2} \phi_4(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_1 - y_3 \mathbf{d}y_4))) + \phi_3(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_3 \lor (\mathbf{d}y_3 - \mathbf{d}y_3))) + \phi_3(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_3 \lor (\mathbf{d}y_3 - \mathbf{d}y_3))) + \phi_3(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_3 \lor (\mathbf{d}y_3 - \mathbf{d}y_3))) + \phi_3(\mathbf{d}y_3^2 + e^{y_2} \phi_5(\mathbf{d}y_3 \lor (\mathbf{d}y_3 \lor (\mathbf{d}$
	$\phi_7(y_2)((\sin y_1 \mathbf{d} y_3 + \cos y_1 \sin y_3 \mathbf{d} y_4) \vee (\cos y_1 \mathbf{d} y_3 - \sin y_1 \sin y_3 \mathbf{d} y_4)) + \phi_8(y_2)((\mathbf{d} y_1 - ch y_3 \mathbf{d} y_4) \vee (\sin y_1 \mathbf{d} y_3 + ch \mathbf{d} y_4))$	$+e^{-y_2}\phi_0(\mathbf{d}y_4\vee(\mathbf{d}y_1-y_3\mathbf{d}y_4))+e^{y_2}\phi_7(\mathbf{d}y_2\vee\mathbf{d}y_3)+e^{-y_2}\phi_8(\mathbf{d}y_2\vee\mathbf{d}y_4)+\phi_0(\mathbf{d}y_3\vee\mathbf{d}y_4)+e^{-2y_2}\phi_{10}\mathbf{d}y_4^2$
	$\cos y_1 \text{sh} y_3 \mathbf{d} y_4) + \phi_9(y_2) (\mathbf{d} y_2 \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \text{sh} y_3 \mathbf{d} y_4)) + \phi_{10}(y_2) (\cos y_1 \mathbf{d} y_3 - \sin y_1 \text{sh} y_3 \mathbf{d} y_4)^2$	$\boxed{[4,4,22]} - \phi_1(\mathbf{d}y_1 - y_3 \mathbf{d}y_4)^2 + \phi_2(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(\mathbf{d}y_2)^2 + \phi_4(\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)^2 + \phi_5((\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4) \vee [(\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)]^2 + \phi_5((\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4) \vee [(\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)]^2 + \phi_5((\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) \vee [(\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)]^2 + \phi_5((\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)]^2 + \phi_5((\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) \vee [(\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)]^2 + \phi_5((\cos y_2 \mathbf{d}y_3 - \sin y$
[3,3,9]	$-\phi_1(y_2)(\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4)^2 + \phi_2(y_2)(\mathbf{d}y_2 \lor (\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4)) + \phi_3(y_2) \mathbf{d}y_2^2 + \phi_4(y_2)(\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_4(y_2)(\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_4(y_2)(\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) + \phi_4(y_2)(\mathbf{d}y_2 - \cos y_3 \mathbf{d}y_4) + \phi_4(y_3)(\mathbf{d}y_3 - \cos y_3 \mathbf{d}y_4) $	$(\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_6((\sin y_2 \mathbf{d}y_3 + \cos y_2 \mathbf{d}y_4) \lor (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_7(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_2 \lor (\cos y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_4 \lor (\cos y_2 \mathbf{d}y_4)) + \phi_8(\mathbf{d}y_4 \lor (\cos y_4 $
	$\phi_{5}(y_{2})(\mathbf{d}y_{2} \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})((\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{3} \mathbf{d}y_{4}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{2} \mathbf{d}y_{3}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{2} \mathbf{d}y_{3}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) + \phi_{6}(y_{2})(\mathbf{d}y_{1} - \cos y_{2} \mathbf{d}y_{3}) \lor (\cos y_{1} \mathbf{d}y_{3} - \sin y_{3} \sin y_{1} \mathbf{d}y_{4})) $	$(\sin y_2 \mathbf{d}y_3 + \cos y_2 \mathbf{d}y_4)) + \phi_9((\sin y_2 \mathbf{d}y_3 + \cos y_2 \mathbf{d}y_4) \vee (\cos y_2 \mathbf{d}y_3 - \sin y_2 \mathbf{d}y_4)) + \phi_{10}(\sin y_2 \mathbf{d}y_3 + \cos y_2 \mathbf{d}y_4)^2$
	$\phi_7(y_2)((\sin y_1 \mathbf{d} y_3 + \cos y_1 \sin y_3 \mathbf{d} y_4) \lor (\cos y_1 \mathbf{d} y_3 - \sin y_3 \sin y_1 \mathbf{d} y_4)) + \phi_8(y_2)((\mathbf{d} y_1 - \cos y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \sin y_2 \mathbf{d} y_4))$	$[[5,4,1] -\phi_1(\mathbf{d}y_1 - \operatorname{ch} y_3 \mathbf{d}y_4)^2 + \phi_2(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \operatorname{ch} y_3 \mathbf{d}y_4)) + \phi_3(\mathbf{d}y_2^2 + \phi_4(\mathbf{d}y_3^2 + \operatorname{sh}^2 y_3 \mathbf{d}y_4^2)$
	$\left[\cos y_{1}\sin y_{3}\mathbf{d}y_{4})\right) + \phi_{9}(y_{2})\left(\mathbf{d}y_{2} \lor (\sin y_{1}\mathbf{d}y_{3} + \cos y_{1}\sin y_{3}\mathbf{d}y_{4})\right) + \phi_{10}(y_{2})(\cos y_{1}\mathbf{d}y_{3} - \sin y_{3}\sin y_{1}\mathbf{d}y_{4})^{2}$	$[5,4,2] -\phi_1(\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4)^2 + \phi_2(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4)) + \phi_3(\mathbf{d}y_2^2 + \phi_4(\mathbf{d}y_3^2 + \sin^2 y_3 \mathbf{d}y_4^2))$
[4,3,1]	$\ -\phi_1(y_2) \mathbf{d}y_1^2 + \phi_2(y_2)(\mathbf{d}y_1 \lor \mathbf{d}y_2) + \phi_3(y_2) \mathbf{d}y_2^2 + \phi_4(y_2)(\mathbf{d}y_3^2 + \mathrm{sh}^2 y_3 \mathbf{d}y_4^2)$	$[5,4,3] = -\phi_1(\mathbf{d}y_1 - y_3 \mathbf{d}y_4)^2 + \phi_2(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3 \mathbf{d}y_2^2 + \phi_4(\mathbf{d}y_3^2 + \mathbf{d}y_4^2)$
[4,3,2]	$-\phi_1(y_2)(\mathbf{d}y_1 - ch y_3 \mathbf{d}y_4)^2 + \phi_2(y_2)(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - ch y_3 \mathbf{d}y_4)) + \phi_3(y_2) \mathbf{d}y_2^2 + \phi_4(y_2)(\mathbf{d}y_3^2 + sh^2 y_3 \mathbf{d}y_4^2)$	$[5,4,6] = -\phi_1(\mathbf{d}y_1 - y_3 \mathbf{d}y_4)^2 + \phi_2(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3 \mathbf{d}y_2^2 + \phi_4(\mathbf{d}y_3 \vee \mathbf{d}y_4)$
	$-\phi_1(y_2) dy_1^2 + \phi_2(y_2) (dy_1 \vee dy_2) + \phi_3(y_2) dy_2^2 + \phi_4(y_2) (dy_3^2 + \sin^2 y_3 dy_4^2)$	$[5,4,7] \phi_1(-\operatorname{ch}^2 y_2 \mathbf{d} y_1^2 + \mathbf{d} y_2^2) + \phi_2(\operatorname{sh} y_2 \mathbf{d} y_1 + \mathbf{d} y_3)^2 + \phi_3 \left(\mathbf{d} y_4 \vee (\operatorname{sh} y_2 \mathbf{d} y_1 + \mathbf{d} y_3) \right) + \phi_4 \mathbf{d} y_4^2$
	$-\phi_1(y_2)(\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4)^2 + \phi_2(y_2)(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4)) + \phi_3(y_2) \mathbf{d}y_2^2 + \phi_4(y_2)(\mathbf{d}y_3^2 + \sin^2 y_3 \mathbf{d}y_4^2)$	$[6,3,1] = -\phi_1(y_1) dy_1^2 + \phi_2(y_1) (dy_2^2 + \sin^2 y_2 (dy_3^2 + \sin^2 y_3 dy_4^2))$
[4,3,5]	$-\phi_1(y_2)(\mathbf{d}y_1 - y_3 \mathbf{d}y_4)^2 + \phi_2(y_2)(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(y_2) \mathbf{d}y_2^2 + \phi_4(y_2)(\mathbf{d}y_3^2 + \mathbf{d}y_4^2)$	$[6,3,2] = -\phi_1(y_1) dy_1^2 + \phi_2(y_1)(dy_2^2 + dy_3^2 + dy_4^2)$
	$-\phi_1(y_2) \mathbf{d}{y_1}^2 + \phi_2(y_2) (\mathbf{d}{y_1} \lor \mathbf{d}{y_2}) + \phi_3(y_2) \mathbf{d}{y_2}^2 + \phi_4(y_2) (\mathbf{d}{y_3}^2 + \mathbf{d}{y_4}^2)$	$[6,3,3] = -\phi_1(y_1) dy_1^2 + \phi_2(y_1) (dy_2^2 + sh^2 y_2 (dy_3^2 + sin^2 y_3 dy_4^2))$
	$\phi_1(y_3)(-\operatorname{ch}^2 y_2 \mathbf{d} y_1^2 + \mathbf{d} y_2^2) + \phi_2(y_3) \mathbf{d} y_3^2 + \phi_3(y_3)(\mathbf{d} y_3 \vee \mathbf{d} y_4) + \phi_4(y_3) \mathbf{d} y_4^2$	$\begin{bmatrix} 6,3,4 \end{bmatrix} = \begin{bmatrix} \phi_1(y_4)(-\operatorname{ch}^2 y_2 \mathbf{d} y_1^2 + \mathbf{d} y_2^2 + \operatorname{sh}^2 y_2 \mathbf{d} y_3^2) + \phi_2(y_4) \mathbf{d} y_4^2 \end{bmatrix}$
	$\phi_1(y_4)(-\operatorname{ch}^2 y_2 \mathbf{d}y_1^2 + \mathbf{d}y_2^2) + \phi_2(y_4)(\operatorname{sh} y_2 \mathbf{d}y_1 + \mathbf{d}y_3)^2 + \phi_3(y_4)(\mathbf{d}y_4 \lor (\operatorname{sh} y_2 \mathbf{d}y_1 + \mathbf{d}y_3)) + \phi_4(y_4) \mathbf{d}y_4^2$	$[6.3.5] \phi_1(y_3)(-\mathbf{d}y_1^2 + \mathbf{d}y_2^2 + \mathbf{d}y_4^2) + \phi_2(y_3) \mathbf{d}y_3^2$
[4,3,10]	$-\phi_1(y_2)(\mathbf{d}y_1 - y_3 \mathbf{d}y_4)^2 + \phi_2(y_2)(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - y_3 \mathbf{d}y_4)) + \phi_3(y_2) \mathbf{d}y_2^2 + \phi_4(y_2)(\mathbf{d}y_3 \vee \mathbf{d}y_4)$	$\begin{bmatrix} [6,3,6] \\ -\phi_1(y_1) dy_1^2 + \phi_2(y_1) (dy_2^2 + sh^2 y_2 (dy_3^2 + sin^2 y_3 dy_4^2)) \end{bmatrix}$
	$\phi_1(y_3)(-\mathbf{d}y_1^2+\mathbf{d}y_2^2)+\phi_2(y_3)\mathbf{d}y_3^2+\phi_3(y_3)(\mathbf{d}y_3\vee\mathbf{d}y_4)+\phi_4(y_3)\mathbf{d}y_4^2$	$[6,4,1] = [\phi_1(-ch^2 y_2 dy_1^2 + dy_2^2) + \phi_2(dy_3^2 + sin^2 y_3 dy_4^2)$
[4, 4, 1]	$-\phi_1(\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4)^2 + \phi_2(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4)) + \phi_3 \mathbf{d}y_2^2 + \phi_4(\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_2 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee \mathbf{d}y_3 + \cos y_1 \sin y_2 $	$[6,4,2] \left \phi_1(-ch^2 y_2 dy_1^2 + dy_2^2) + \phi_2(dy_3^2 + sh^2 y_3 dy_4^2) \right $
	$(\cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_6 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 + \cos y_3 \mathbf{d}y_4) \lor (\cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 + \cos y_3 \mathbf{d}y_4) \lor (\cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 + \cos y_3 \mathbf{d}y_4) \lor (\cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 + \cos y_3 \mathbf{d}y_4) \lor (\cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 + \cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 + \cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 + \cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 + \sin y_1 \mathbf{d}y_3 + \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 - \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 - \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 - \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 - \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 - \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 - \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 - \sin y_1 \mathbf{d}y_4)) + \phi_7 ((\sin y_1 \mathbf{d}y_3 - \sin y_1 \mathbf{d}y_4))$	$\begin{bmatrix} [6,4,3] \\ \phi_1(-ch^2 y_2 dy_1^2 + dy_2^2) + \phi_2(dy_3^2 + dy_4^2) \end{bmatrix}$
	$\cos y_1 \sin y_3 \mathbf{d}y_4) \lor (\cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)) + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \cdots + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \cdots + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \cdots + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \cdots + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \cdots + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \cdots + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \cdots + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \cdots + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \cdots + \phi_8 ((\mathbf{d}y_1 - \cos y_3 \mathbf{d}y_4) \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4))$	$\begin{bmatrix} 6,4,4 \end{bmatrix} \phi_1(-\mathbf{d}y_1^2 + \mathbf{d}y_2^2) + \phi_2(\mathbf{d}y_3^2 + \sin^2 y_3 \mathbf{d}y_4^2)$
	$\phi_9 (\mathbf{d}y_2 \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \phi_{10} (\cos y_1 \mathbf{d}y_3 - \sin y_3 \sin y_1 \mathbf{d}y_4)^2$	$\begin{bmatrix} 6,4,5 \end{bmatrix} \phi_1(-\mathbf{d}y_1^2 + \mathbf{d}y_2^2) + \phi_2(\mathbf{d}y_3^2 + \mathrm{sh}^2 y_3 \mathbf{d}y_4^2)$
[4,4,2]	$-\phi_1(\mathbf{d}y_1 - \mathbf{ch}y_3\mathbf{d}y_4)^2 + \phi_2(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \mathbf{ch}y_3\mathbf{d}y_4)) + \phi_3\mathbf{d}y_2^2 + \phi_4(\sin y_1\mathbf{d}y_3 + \cos y_1\mathbf{sh}y_3\mathbf{d}y_4)^2 + \phi_5(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \mathbf{ch}y_3\mathbf{d}y_4)) + \phi_5(\mathbf{d}y_2 \vee (\mathbf{d}y_1 - \mathbf{ch}y_3,\mathbf{d}y_4)) + \phi_5(\mathbf{d}y_2 \vee ($	$[7,4,1] = -\phi_1 dy_1^2 + \phi_2 (dy_2^2 + \sin^2 y_2 (dy_3^2 + \sin^2 y_3 dy_4^2))$
	$\left(\cos y_{1} \mathbf{d} y_{3} - \sin y_{1} \mathrm{sh} y_{3} \mathbf{d} y_{4}\right)\right) + \phi_{6}\left(\left(\mathbf{d} y_{1} - \mathrm{ch} y_{3} \mathbf{d} y_{4}\right) \lor \left(\cos y_{1} \mathbf{d} y_{3} - \sin y_{1} \mathrm{sh} y_{3} \mathbf{d} y_{4}\right)\right) + \phi_{7}\left(\left(\sin y_{1} \mathbf{d} y_{3} + \frac{1}{2}\right) + \frac{1}{2}\right)$	$[7,4,2] - \phi_1 dy_1^2 + \phi_2 (dy_2^2 + sh^2 y_2 (dy_3^2 + sin^2 y_3 dy_4^2))$
	$\cos y_1 \text{sh} y_3 \mathbf{d} y_4) \lor (\cos y_1 \mathbf{d} y_3 - \sin y_1 \text{sh} y_3 \mathbf{d} y_4)) + \phi_8 \left((\mathbf{d} y_1 - \mathbf{ch} y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \text{sh} y_3 \mathbf{d} y_4) \right) + \phi_8 \left((\mathbf{d} y_1 - \mathbf{ch} y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \text{sh} y_3 \mathbf{d} y_4) \right) + \phi_8 \left((\mathbf{d} y_1 - \mathbf{ch} y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \text{sh} y_3 \mathbf{d} y_4) \right) + \phi_8 \left((\mathbf{d} y_1 - \mathbf{ch} y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \mathbf{sh} y_3 \mathbf{d} y_4) \right) + \phi_8 \left((\mathbf{d} y_1 - \mathbf{ch} y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \mathbf{sh} y_3 \mathbf{d} y_4) \right) + \phi_8 \left((\mathbf{d} y_1 - \mathbf{ch} y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \mathbf{sh} y_3 \mathbf{d} y_4) \right) + \phi_8 \left((\mathbf{d} y_1 - \mathbf{ch} y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \mathbf{sh} y_3 \mathbf{d} y_4) \right) + \phi_8 \left((\mathbf{d} y_1 - \mathbf{ch} y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \mathbf{sh} y_3 \mathbf{d} y_4) \right) + \phi_8 \left((\mathbf{d} y_1 - \mathbf{ch} y_3 \mathbf{d} y_4) \lor (\sin y_1 \mathbf{d} y_3 + \cos y_1 \mathbf{sh} y_3 \mathbf{d} y_4) \right)$	$\begin{bmatrix} 7,4,3 \\ -\phi_1 \mathbf{d}y_1^2 + \phi_2 (\mathbf{d}y_2^2 + \operatorname{sh}^2 y_2 (\mathbf{d}y_3^2 + \operatorname{sin}^2 y_3 \mathbf{d}y_4^2)) \end{bmatrix}$
	$\phi_9(\mathbf{d}y_2 \lor (\sin y_1 \mathbf{d}y_3 + \cos y_1 \sin y_3 \mathbf{d}y_4)) + \phi_{10}(\cos y_1 \mathbf{d}y_3 - \sin y_1 \sin y_3 \mathbf{d}y_4)^2$	$[7,4,4]$ $\phi_1(-ch^2 y_2 dy_1^2 + dy_2^2 + sh^2 y_2 dy_3^2) + \phi_2 dy_4^2$

List of Γ -invariant *l*-chains

Hicks $\#$	<i>l</i> -chain χ		
[3,2,1]	$oldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$		
[3,2,2]	$\operatorname{csch} y_3 \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$		
[3,2,3]	$\csc y_3 \partial_{y_3} \wedge \partial_{y_4}$		
[3,2,4]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_2}$		
[3,2,5]	$\operatorname{sech} y_2 \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2}$		
[3,3,2]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_2}\wedgeoldsymbol{\partial}_{y_4}$		
[3,3,3]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$	[5,4,2]	$\operatorname{csc} y_3 {oldsymbol \partial}_{y_1} \wedge {oldsymbol \partial}_{y_2} \wedge {oldsymbol \partial}_{y_3} \wedge {oldsymbol \partial}_{y_4}$
[3,3,8]	$\operatorname{csch} y_3 \partial_{y_1} \wedge \partial_{y_3} \wedge \partial_{y_4}$	[5,4,3]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_2}\wedgeoldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$
[3,3,9]	$\csc y_3 \partial_{y_1} \wedge \partial_{y_3} \wedge \partial_{y_4}$	[5,4,6]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_2}\wedgeoldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$
[4,3,1]	$\operatorname{csch} y_3 \partial_{y_1} \wedge \partial_{y_3} \wedge \partial_{y_4}$	[5,4,7]	$\operatorname{csch} y_3 oldsymbol{\partial}_{y_1} \wedge oldsymbol{\partial}_{y_2} \wedge oldsymbol{\partial}_{y_3} \wedge oldsymbol{\partial}_{y_4}$
[4,3,2]	$\operatorname{csch} y_3 \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$	[6,3,1]	$(\csc^2 y_2 \csc y_3 oldsymbol{\partial}_{y_2} \wedge oldsymbol{\partial}_{y_3} \wedge oldsymbol{\partial}_{y_4})$
[4,3,3]	$\csc y_3 \partial_{y_1} \wedge \partial_{y_3} \wedge \partial_{y_4}$	[6,3,2]	$oldsymbol{\partial}_{y_2} \wedge oldsymbol{\partial}_{y_3} \wedge oldsymbol{\partial}_{y_4}$
[4,3,4]	$\csc y_3 \partial_{y_1} \wedge \partial_{y_3} \wedge \partial_{y_4}$	[6,3,3]	$\operatorname{csc} y_3 \operatorname{csch}^2 y_2 {oldsymbol \partial}_{y_2} \wedge {oldsymbol \partial}_{y_3} \wedge {oldsymbol \partial}_{y_4}$
[4,3,5]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$	[6,3,4]	$\operatorname{csch} y_2 \operatorname{sech} y_2 {\boldsymbol \partial}_{y_1} \wedge {\boldsymbol \partial}_{y_2} \wedge {\boldsymbol \partial}_{y_3}$
[4,3,6]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$	[6,3,5]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_2}\wedgeoldsymbol{\partial}_{y_4}$
[4,3,8]	$\operatorname{sech} y_2 \partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_4}$	[6,3,6]	$\operatorname{csc} y_3 \operatorname{csch}^2 y_2 {\boldsymbol \partial}_{y_2} \wedge {\boldsymbol \partial}_{y_3} \wedge {\boldsymbol \partial}_{y_4}$
[4,3,9]	$\operatorname{sech} y_2 {\boldsymbol \partial}_{y_1} \wedge {\boldsymbol \partial}_{y_2} \wedge {\boldsymbol \partial}_{y_3}$	[6,4,1]	$ \operatorname{csc} y_3 \operatorname{sech} y_2 \partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3} \wedge \partial_{y_4} $
[4,3,10]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$	[6,4,2]	$\operatorname{csch} y_3 \operatorname{sech} y_2 \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,3,11]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_2}\wedgeoldsymbol{\partial}_{y_4}$	[6,4,3]	$\operatorname{sech} y_2 \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$
[4,4,1]	$\csc y_3 \partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3} \wedge \partial_{y_4}$	[6,4,4]	$\csc y_3 oldsymbol{\partial}_{y_1} \wedge oldsymbol{\partial}_{y_2} \wedge oldsymbol{\partial}_{y_3} \wedge oldsymbol{\partial}_{y_4}$
[4,4,2]	$\operatorname{csch} y_3 {\boldsymbol \partial}_{y_1} \wedge {\boldsymbol \partial}_{y_2} \wedge {\boldsymbol \partial}_{y_3} \wedge {\boldsymbol \partial}_{y_4}$	[6,4,5]	$\operatorname{csch} y_3 {oldsymbol \partial}_{y_1} \wedge {oldsymbol \partial}_{y_2} \wedge {oldsymbol \partial}_{y_3} \wedge {oldsymbol \partial}_{y_4}$
[4,4,9]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_2}\wedgeoldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$	[7,4,1]	$ \csc^2 y_2 \csc y_3 \partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3} \wedge \partial_{y_4} $
[4,4,18]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_2}\wedgeoldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$	[7,4,2]	$\operatorname{csc} y_3 \operatorname{csch}^2 y_2 \partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3} \wedge \partial_{y_4}$
[4,4,22]	$oldsymbol{\partial}_{y_1}\wedgeoldsymbol{\partial}_{y_2}\wedgeoldsymbol{\partial}_{y_3}\wedgeoldsymbol{\partial}_{y_4}$	[7,4,3]	$\operatorname{csc} y_3 \operatorname{csch}^2 y_2 \partial_{y_1} \wedge \partial_{y_2} \wedge \partial_{y_3} \wedge \partial_{y_4}$
[5,4,1]	$\operatorname{csch} y_3 \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$	[7,4,4]	$\operatorname{csch} y_2 \operatorname{sech} y_2 \boldsymbol{\partial}_{y_1} \wedge \boldsymbol{\partial}_{y_2} \wedge \boldsymbol{\partial}_{y_3} \wedge \boldsymbol{\partial}_{y_4}$