New Dark-Energy parameterisations and applications to Quintessence (arXiv:2407.14378 [gr-qc])

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Outline

- Exact Relations for Dark-Energy (DE)
- ΛCDM and wCDM models
- New DE parameterisations
- Chevallier-Polarski-Linder (CPL) parameterisation
- Quintessence: Thawing and Tracking models

Preliminaries

- **Spatially flat homogeneous and isotropic cosmology:**

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \\ &= (1+z)^{-2}\left[H^{-2}(z)dz^2 + a_0^2\delta_{ij}dx^i dx^j\right] \end{aligned}$$

- Scale factor: $a(t) > 0$;
- Hubble variable: $H = \frac{d}{dt}\ln(a)$;
- Cosmological redshift $z = \frac{a_0}{a} - 1$

- **Observables:** $H(z)$; $q(z) = -1 + (1+z)\frac{d\ln(H)}{dz}$; Distance measures:

$$\text{Hubble distance:} \quad D_H = \frac{1}{H(z)},$$

$$\text{Coordinate distance:} \quad D_M = \int_0^z H^{-1}(\tilde{z})d\tilde{z},$$

$$\text{Luminosity distance:} \quad D_L = (1+z)\int_0^z H^{-1}(\tilde{z})d\tilde{z},$$

$$\text{Angular diameter distance:} \quad D_A = (1+z)^{-1}\int_0^z H^{-1}(\tilde{z})d\tilde{z}.$$

- Parameterisations of $H(z)$, or $D_L(z)$, etc.. More common is to start from the equation of state of DE

$$w_{\rm DE} = \frac{p_{DE}}{\rho_{DE}}$$

- However w_{DE} is not an observable!
- Computing H from w_{DE} requires EFEs + Conservation Equations.
- Computational purposes use e-fold time: $e^{-N} = 1 + z$
- Conservation equations:

$$\begin{aligned} \rho'_{\rm m} &= -3\rho_{\rm m} & \Rightarrow \quad \rho_{\rm m} &= \rho_{\rm m0}\exp(-3N), \\ \rho'_{\rm DE} &= -3(1+w_{\rm DE})\rho_{\rm DE} & \Rightarrow \quad \rho_{\rm DE} &= \rho_{\rm DE0}\exp(-3N)f(N), \end{aligned}$$

where

$$f(N) = \exp\left(-3\int_0^N w_{\rm DE}(\tilde{N})d\tilde{N}\right).$$

- Dimensionless bounded *Hubble normalized* variables

$$\Omega_{\rm m} := \frac{\rho_{\rm m}}{3H^2}, \qquad \Omega_{\rm DE} := \frac{\rho_{\rm DE}}{3H^2}$$

- Gauss constraint:

$$\Omega_{\rm m} + \Omega_{\rm DE} = 1 \quad \Rightarrow \quad \Omega_{\rm m0} + \Omega_{\rm DE0} = 1.$$

- The present time normalized Hubble parameter

$$E = \frac{H}{H_0}.$$

is given by

$$E^2_{\rm DE} = \Omega_{\rm m0}\exp(-3N) + \Omega_{\rm DE0}\exp(-3N)f(N).$$

- For the Hubble normalised variables:

$$\Omega_{\rm DE} = \frac{\Omega_{\rm DE0}}{[\Omega_{\rm m0}/f(N)] + \Omega_{\rm DE0}}, \qquad \Omega_{\rm m} = \frac{\Omega_{\rm m0}}{\Omega_{\rm m0} + \Omega_{\rm DE0}f(N)}.$$

- The above expressions involve $f(N)$, which is computed from the integral of $w_{\rm DE}(N)$.

ΛCDM and wCDM models

- **ΛCDM model:** Simplest hypothesis compatible with observations DE is a positive cosmological constant $\Lambda > 0$,

$$\rho_{\rm DE} = \rho_\Lambda = \Lambda, \quad p_{\rm DE} = p_\Lambda = -\Lambda, \quad \Omega_{\rm DE} = \Omega_\Lambda := \frac{\Lambda}{3H^2}$$

,

- Equation of state

$$w_{\rm DE} = w_\Lambda = -1,$$

- This leads to

$$f_\Lambda = e^{3N}$$

and

$$\Omega_\Lambda = \frac{\Omega_{\Lambda 0}}{\Omega_{\Lambda 0} + (1-\Omega_{\Lambda 0})\, e^{-3N}}, \quad E_\Lambda^2 = (1-\Omega_{\Lambda 0})e^{-3N} + \Omega_{\Lambda 0}.$$

- ***w*CDM model:**

$$w_{\rm DE} = w = \text{const.}, \quad -1 < w < 0.$$

:

- This leads to

$$f_\Lambda(N) = e^{-3wN}$$

and

$$\Omega_w = \frac{\Omega_{w0}}{\Omega_{w0} + (1-\Omega_{w0})\, e^{3wN}}, \quad E_w^2 = (1-\Omega_{w0})e^{3wN} + \Omega_{w0} e^{-3(1+w)N}.$$

Past DE series expansions

- Letting $w_{\rm DE} \to w_\infty \in [-1, 0)$ and $\Omega_{\rm DE} \to 0$ when $N \to -\infty$ results in

$$\Omega'_{\rm DE} \approx -3 w_\infty \Omega_{\rm DE}$$

  and hence

$$\Omega_{\rm DE} \approx C e^{-3 w_\infty N}.$$

- Improve approximation by series expansion in

$$T = T_0 e^{-3 w_\infty N},$$

  where $T \to 0$ when $N \to -\infty$.

$$w_{\rm DE} = w_\infty \left[1 - (\gamma - 1) T + (\gamma - 1)\beta T^2 + \dots\right]$$

$$\Omega_{\rm DE} = T \left(1 - \gamma T + \left[1 + \frac{1}{2}(\gamma - 1)(3 + \gamma + \beta)\right] T^2 + \dots\right).$$

DE Padé approximants

- Padé for w_{DE}:

$$w_{\rm DE} \approx [0/1]_{w_{\rm DE}}(T) = \frac{w_\infty}{1+(\gamma-1)T},$$

$$w_{\rm DE} \approx [1/1]_{w_{\rm DE}}(T) = w_\infty \left(1 - \frac{(\gamma-1)T}{1+\beta T}\right).$$

- Connect with the present time $N=0$: solve for $T=T_0$ in terms of $w_{\rm DE0}$. However, w_{DE} is not an observable!

- To connect with present time we take the Padé approximant

$$\Omega_{\rm DE} \approx [1/1]_{\Omega_{\rm DE}}(T) = \frac{T}{1+\gamma T},$$

- At $N=0$: $\Omega_{\rm DE0} = T_0/[1+\gamma T_0]$, and hence

$$T_0 = \frac{\Omega_{\rm DE0}}{1-\gamma\Omega_{\rm DE0}}.$$

- Parameterisations of $w_{\rm DE}$

$$w_{\rm DE} \approx w_\infty \left(1 - \frac{(\gamma-1)\Omega_{\rm DE0}}{(\gamma-1)\Omega_{\rm DE0} + (1-\gamma\Omega_{\rm DE0})e^{3w_\infty N}}\right),$$

$$w_{\rm DE} \approx w_\infty \left(1 - \frac{(\gamma-1)\Omega_{\rm DE0}}{\beta\Omega_{\rm DE0} + (1-\gamma\Omega_{\rm DE0})e^{3w_\infty N}}\right).$$

- Present time values

$$w_{\rm DE0} \approx w_\infty \left(\frac{1-\gamma\Omega_{\rm DE0}}{\Omega_{\rm m0}}\right),$$

$$w_{\rm DE0} \approx w_\infty \left(1 - \frac{(\gamma-1)\Omega_{\rm DE0}}{1+(\beta-\gamma)\Omega_{\rm DE0}}\right).$$

- The integral $f(N)$:

$$f = \frac{\Omega_{\rm m0}e^{-3w_\infty N}}{1-\gamma\Omega_{\rm DE0} + (\gamma-1)\Omega_{\rm DE0}e^{-3w_\infty N}},$$

$$f = e^{-3w_\infty N}\left(\frac{1+(\beta-\gamma)\Omega_{\rm DE0}}{1-\gamma\Omega_{\rm DE0}+\beta\Omega_{\rm DE0}e^{-3w_\infty N}}\right)^{\frac{\gamma-1}{\beta}},$$

CPL parameterisation

- Chevallier-Polarski-Linder (CPL) parameterisation:

$$w_{\rm DE} = w_{\rm CPL} = w_0 + w_a \left(\frac{z}{1+z} \right) = w_\infty - w_a e^N,$$

  where

$$w_\infty = w_0 + w_a.$$

- The integral $f(N)$

$$f = e^{-3w_\infty N} \cdot e^{-3w_a(1-e^N)} = (1+z)^{3w_\infty} \cdot e^{-3w_a\left(\frac{z}{1+z}\right)},$$

- The CPL can be regarded as a series expansion in e^N, but this is inconsistent with the past matter dominant requirement, with a natural expansion in $e^{-3w_\infty N}$, which is particularly pertinent as observations probe increasingly large redshift.

Quintessence

- **Quintessence models:** minimally coupled scalar field, φ, with a potential $V(\varphi) > 0$:

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \qquad p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{8}$$

- The dark-energy conservation equation

$$\ddot{\varphi} = -3H\dot{\varphi} - V_{,\varphi}. \tag{9}$$

- $\Omega_{\rm DE} = \Omega_\varphi$, $w_{\rm DE} = w_\varphi$, with $-1 \leq w_\varphi \leq 1$.

- **Thawing models:** Thawing quintessence exist for all potential where

$$\lambda(\varphi) = -\frac{V_{,\varphi}}{V}.$$

is bounded. It corresponds to a value of w_φ beginning near -1 increasing with time, i.e.

$$w'_\varphi > 0$$

- **Freezing models:** freezing quintessence corresponds a decreasing value of w_φ, i.e.

$$w'_\varphi < 0$$

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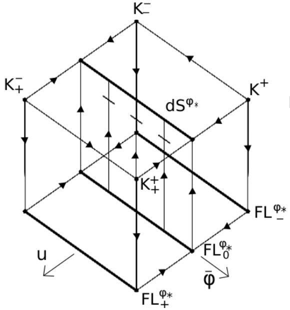
Thawing Quintessence

- A. Alho & C. Uggla: Scalar field deformations of ΛCDM cosmology. Phys. Rev. D 92, 103502 (2015).
- A. Alho & C. Uggla, and John Wainwright : Quintessence from a state space perspective, Phys. Dark Universe, 39 (2023) 101146.



(a) $\lambda = 0$ (b) $\lambda_- = 1$, $\lambda_+ = 0$

- $\lambda(\varphi)$ bounded; line of matter dominated 'Friedmann-Lemaître' fixed points, FL_{φ_*}, with $w_\infty = -1$, parameterized by the constant values φ_*.
- Thawing quintessence is associated with the unstable manifold of FL_{φ_*}.

Thawing Approximations

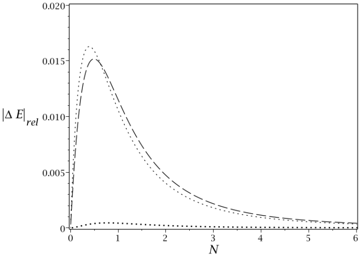
- Higher-order expansions on the unstable manifold leads to the identifications

$$\gamma := 1 + \left(\frac{2}{3}\right)^3 \epsilon_*, \qquad \beta := \frac{4}{5}\left(1 + \frac{\eta_*}{6}\right)$$

and $\epsilon_* \equiv \epsilon(\varphi_*), \eta_* \equiv \eta(\varphi_*)$ are the potential slow-roll parameters

$$\epsilon(\varphi) = \frac{1}{2}\left(\frac{V_{,\varphi}}{V}\right)^2 = \frac{\lambda^2}{2}, \qquad \eta(\varphi) = \frac{V_{,\varphi\varphi}}{V} = \lambda^2 - \lambda_{,\varphi}.$$

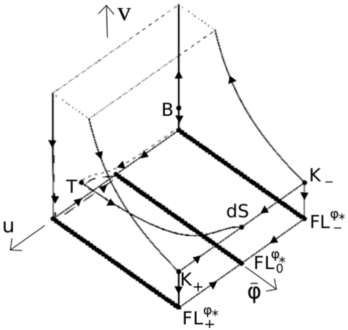
- Example: Quintessential α-attractor *EC* potential with $\lambda_* \approx 0.8$, $\gamma \approx 1.1$



(a) $E(z)$

Tracking Quintessence

- Prototype model: inverse power-law potential $V(\varphi) = V_0\varphi^{-p}, \qquad p > 0.$
- $\lambda(\varphi)$ becomes unbounded when $\varphi \to 0^+$.
- A. Alho, C. Uggla, J. Wainwright. Tracking Quintessence. Phys. Dark Universe, 44 (2024) 101433



- Tracker fixed point T: $\tilde{w}_\varphi := w_\varphi|_{\mathrm{T}} = -\frac{2}{2+p}$.
- Tracking quintessence is associated with the unstable manifold of T (Tracker orbit).

Tracker Approximations

- Introduce

$$\Gamma := \frac{V\, V_{,\varphi\varphi}}{V_{,\varphi}^2} = 1 + (\lambda^{-1})_{,\varphi}.$$

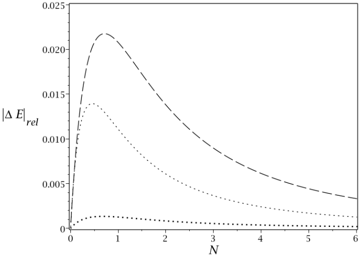
- Higher-order expansions for the unstable manifold leads to the identifications

$$\gamma := 1 - \tilde{w}_\varphi^{-1}(1 - \tilde{w}_\varphi^2)k,$$

$$\beta := \frac{2\tilde{w}_\varphi^2(3\tilde{w}_\varphi - 1) + k\left(12\tilde{w}_\varphi^4 - \tilde{w}_\varphi^3 - 3\tilde{w}_\varphi^2 + 2\tilde{w}_\varphi - 1\right) + k^{(2)}}{\tilde{w}_\varphi(12\tilde{w}_\varphi^2 - 3\tilde{w}_\varphi + 1)},$$

and

$$k := \frac{\tilde{w}_\varphi - \frac{2}{3}\Gamma^{(1)}}{4\tilde{w}_\varphi^2 - 2\tilde{w}_\varphi + 1}, \qquad k^{(2)} := \frac{\tilde{w}_\varphi \Gamma^{(2)}}{9(\tilde{w}_\varphi + 1)k}, \qquad \Gamma^{(0)} = 1 + p^{-1}.$$



- We have present a new consistent and unified approximation scheme for Quintessence models having a continuous ΛCDM limit. These are simpler and accurate as previous approximations.

- Contextualize in a broader DE context

- General procedure which can be applied to a plethora of models: modified gravity, etc..
  1. Identify fixed points whose unstable manifolds are associated with physical "viable" solutions:
  2. Produce expansions and improve range of convergence with Padés. Connect with present time initial data.

- **Linear perturbations:**
  - A. Alho, C. Uggla & J. Wainwright: Perturbations of the Lambda-CDM model in a dynamical systems perspective. J. Cosmol. Astropart. Phys. 2019 (09), 045 .
  - A. Alho, C. Uggla & J. Wainwright: Dynamical systems in perturbative scalar field cosmology. Class. Quantum Grav. 37 225011 (2020).