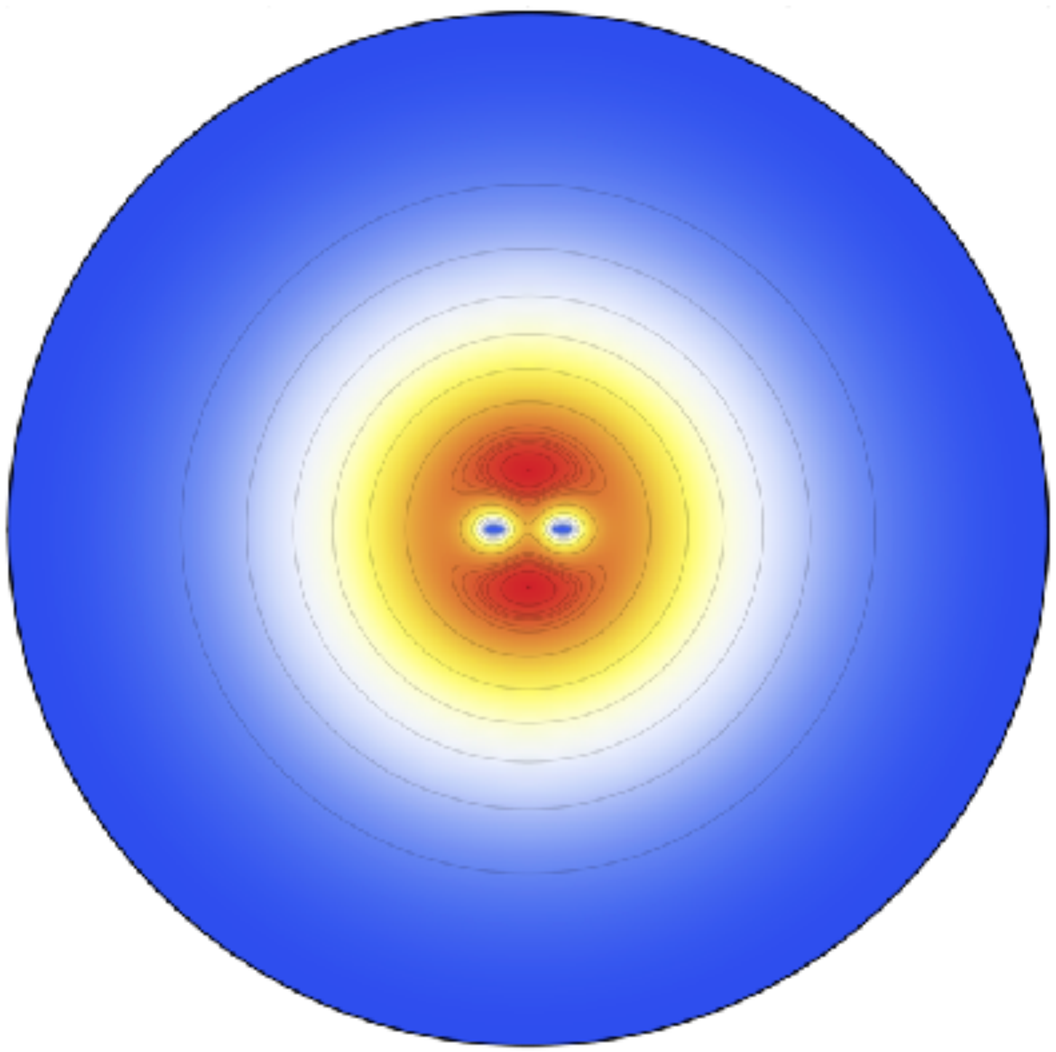
Black hole Binaries in an Expanding Universe

Óscar Dias

STAG RESEARCH CENTRE

UNIVERSITY OF Southampton



OD, Gary Gibbons, Jorge Santos, Benson Way,

2303.07361 (PRL) **&** **2406.10333** (submitted PRL)

→ Introduction & Motivation & short Summary

Our **Universe** appears to be undergoing a **accelerated expansion**
due to the presence of a **positive cosmological constant, Λ>0** ...

(**ΛCDM model:** ~69% **vacuum energy** <—> **Λ** <—> **dark matter** with EoS p=w ρ, w~-1)

So we should ask:

- What is the phase space of **stationary black hole (BH)** solutions of the Einstein equation in **de Sitter**?
- Are there **other solutions** besides **de Sitter-Schwarzschild** & **de Sitter-Kerr** ?
- Can we have **multi-BHs** (eg **BH binaries**) ?

→ **Can we have multi-BHs (eg BH binaries) ?**

- On one hand, **Newton-Hooke analysis:** **cosmological expansion** should be able to **balance gravitational attraction**

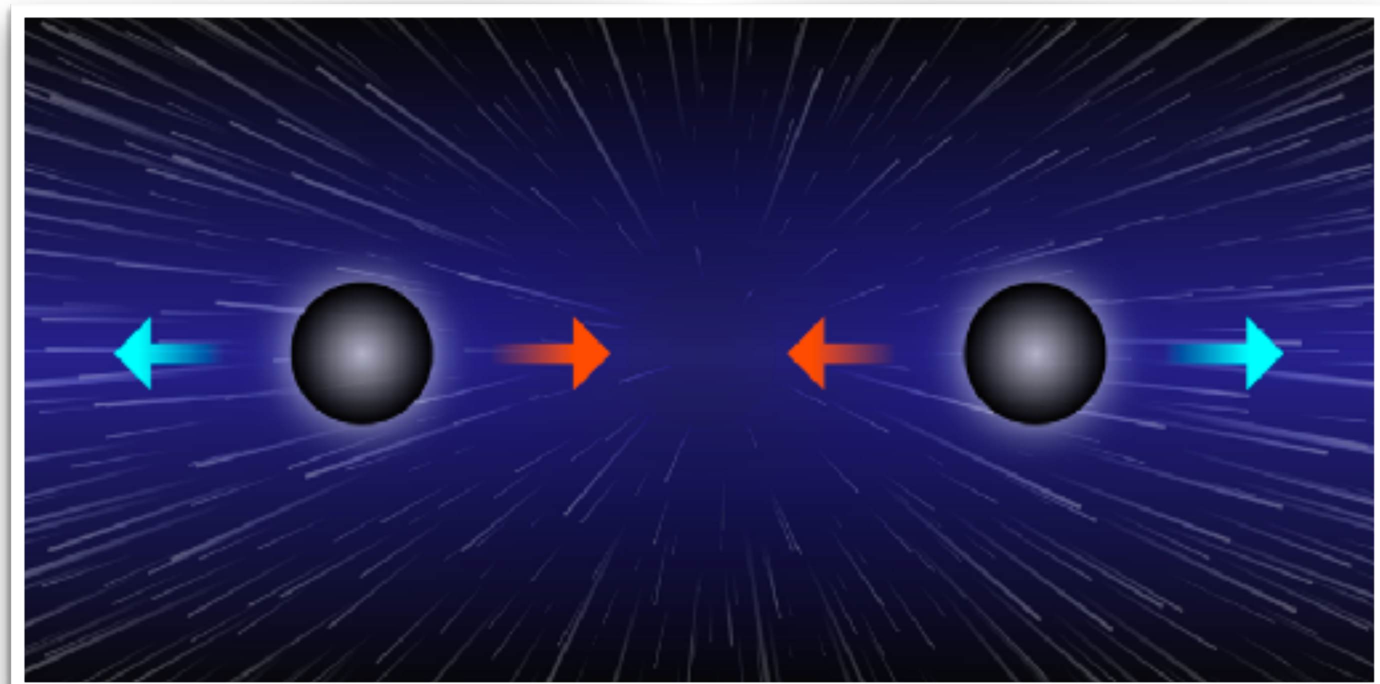
- On the other hand, some **mathematical theorems** in the literature **claim uniqueness** of Schwarzschild/Kerr solutions in de Sitter !!!

[LeFloch, Rozoy '10] [Borghini, Chruściel, Mazzieri '19]
[ul Alam, Yu '14]

→ Solve the Einstein equations to **settle the issue**!

- We find that

regular static/stationary BH binaries do exist in de Sitter.



- Not **in conflict** with available **Uniqueness theorems**:

we have (explicitly identified) **assumptions** of these theorems that can be **evaded**

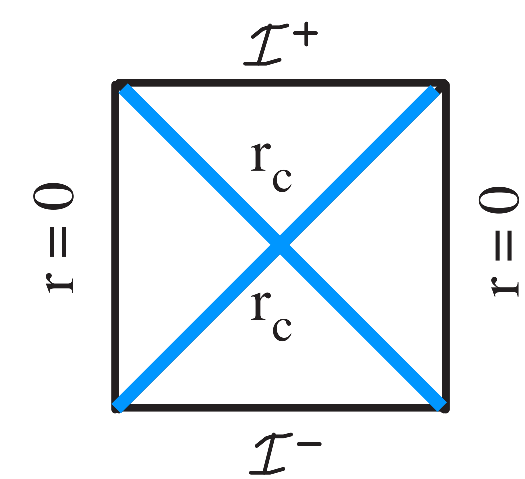
→ Λ=0: Uniqueness, No-hair theorems & multi-BHs

- When Λ=0, **Stationarity** => **axisymmetry** [Hawking '73 and Wald '92, Chrúsciel '23]
  - => BHs are uniquely characterized by their M, J, Q: the **Kerr-Newman BH** family
  - => **No-hair** & **Uniqueness theorems** [Kerr '67, Carter '71, Robinson '75]
- For **static** configurations, mathematical theorems **preclude** the existence of regular asymptotically **flat** **multiple** BHs [Bunting, Masood-Ul-Alam '97].
- Asymptotically **flat** **multi-Kerr BHs**, where their gravitational attraction might be balanced by **spin-spin** interactions, have **been ruled out**. [Neugebauer, Hennig '10-'14, Chrusciel et al '11]
- All Einstein(-Maxwell) **binary (multi-BH)** solutions in **4-dim** found so far have **naked singularities** or **conical singularities** (e.g. **Bach-Weyl** and **Israel-Khan**), except **Majumdar-Papapetrou** solution

→ What about de Sitter (Λ>0) ? … Uniqueness ?

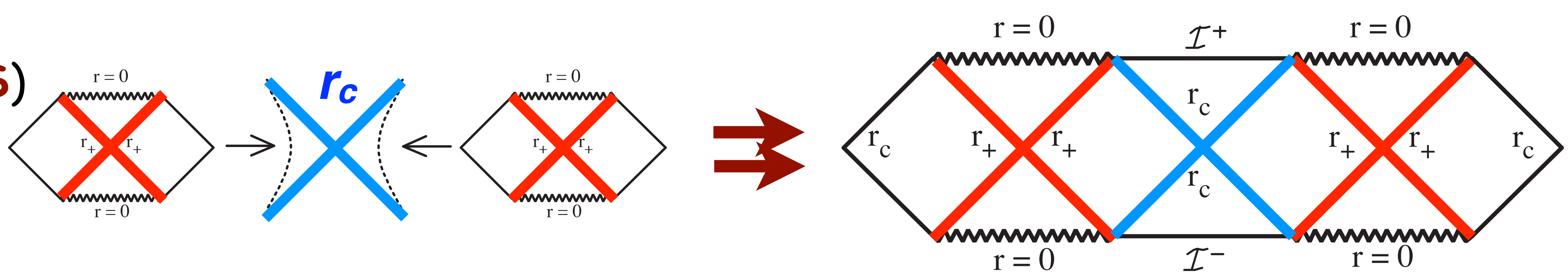
- Our Universe appears to be **expanding & accelerating** due to the presence of a **positive Λ**.
- Einstein equation with a **positive cosmological constant:** $R_{ab} = \frac{3}{\ell^2} g_{ab}$ $\Lambda \equiv 3/\ell^2 > 0$
- We would like to understand the moduli space of **static/stationary BHs** of this theory.
- **For Λ>0 <u>uniqueness</u> of Kerr-dS is <u>not</u> established**
- Spacetimes with a **positive cosmological constant** have **spatial slices that grow exponentially.**
  => at late times, an **inertial observer O** in de Sitter experiences a **cosmological horizon**.
- Region **visible** to O — the **de Sitter static patch** — can be described by a **static** metric:

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad f = 1 - \frac{r^2}{\ell^2}$$



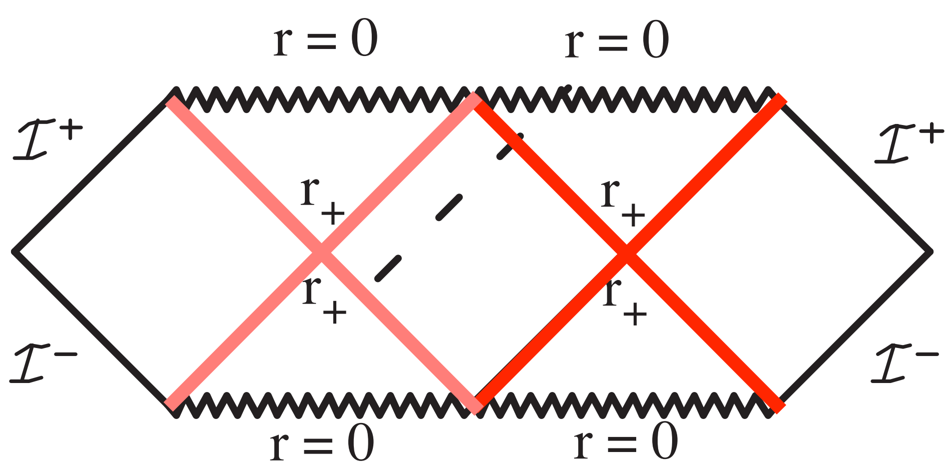
where polar coords are built **around an inertial observer O** placed at r = 0.
Null hypersurface $r = r_c = \ell$, is a **cosmological** horizon:
a surface beyond which nothing influences O

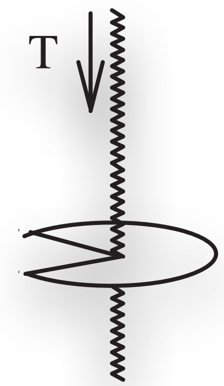
- **Kottler BH (Schw-dS)**
- **Kerr-dS BH**



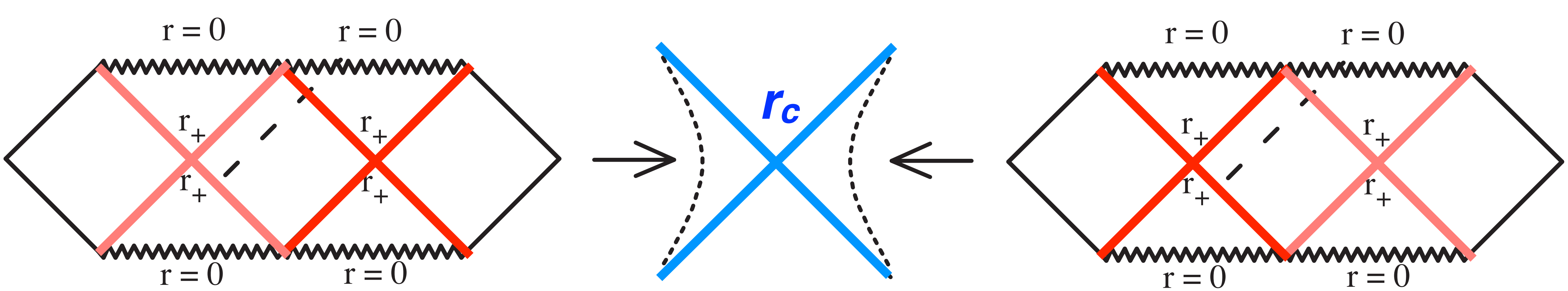
→ Can we have multi-BH, eg BH binaries ?

- Λ=0 **Bach-Weyl** (1922) or **Israel-Khan** (1964) solution. But it has **conical singularities**:





- We want a **BH binary without conical singularities: maybe possible with Λ>0 ?**



→ Start with **Newtonian** analysis: consider a configuration of **N small BHs in de Sitter** space

- **Newton-Hooke** equations of motion: $$m_a \frac{d^2\mathbf{x}_a}{dt^2} - m_a \frac{\mathbf{x}_a}{\ell^2} = -\sum_{b\neq a}^{b=N} \frac{m_a m_b(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3}$$

- **Static** solutions exist when: $$\frac{\mathbf{x}_a}{\ell^2} = \sum_{b\neq a}^{b=N} \frac{m_b(\mathbf{x}_a - \mathbf{x}_b)}{|\mathbf{x}_a - \mathbf{x}_b|^3} \quad (1)$$ $$\Lambda \equiv 3/\ell^2 > 0.$$

- **Two equal mass BHs** aligned along z axis and separated by a **distance d**:

$$N = 2, \quad x_1 = -x_2 = \frac{d}{2}\,\hat{e}_z, \quad m_a = m_b = M$$

- Then (1) yields: $$\frac{d^3}{\ell^3} = \frac{r_+}{\ell} \quad \Rightarrow \quad \frac{d}{\ell} = \frac{1}{(4\pi\ell\, T_+)^{1/3}} \quad (2)$$ $$r_+ = 2M$$ $$T_+ = (4\pi r_+)^{-1}$$

- Require **validity** of **Newton + Hooke** approxs (**BHs inside** a **single** **cosmological** horizon):

$$r_+ \ll d\,, \quad d \ll \ell \quad (\Rightarrow r_+ \ll \ell\,)$$

- These conditions are consistent with Newton-Hooke equilibrium condition (2):

$$\frac{r_+}{\ell} = \frac{d^3}{\ell^3} \ll 1 \Rightarrow \begin{cases} d \ll \ell \\ \frac{r_+}{d} = \frac{d^2}{\ell^2} \ll 1 \end{cases}$$

=> $\exists$ **static de Sitter binaries** with **small BHs** are **consistent** with **Newton-Hooke theory**.

→ Going beyond the Newton-Hooke approximation: General Relativity (GR) solution

We find exact solutions to this 2-body problem in GR with $\Lambda > 0$ using numerics.

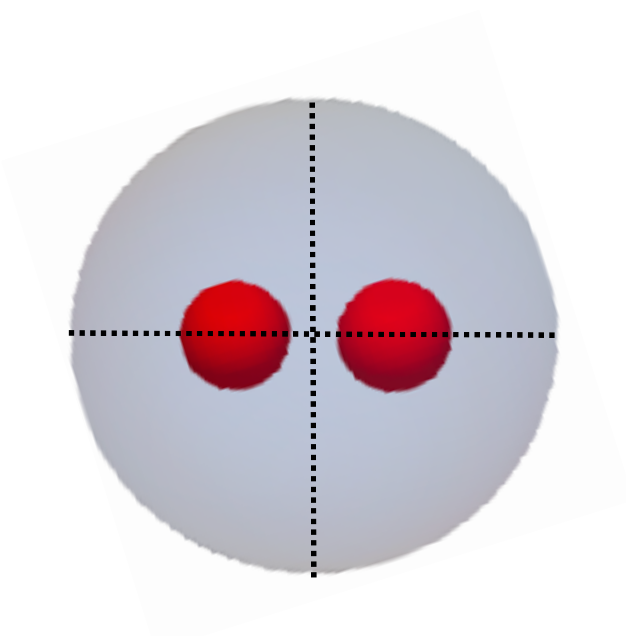
Use **Einstein-deTurck formulation of GR:**

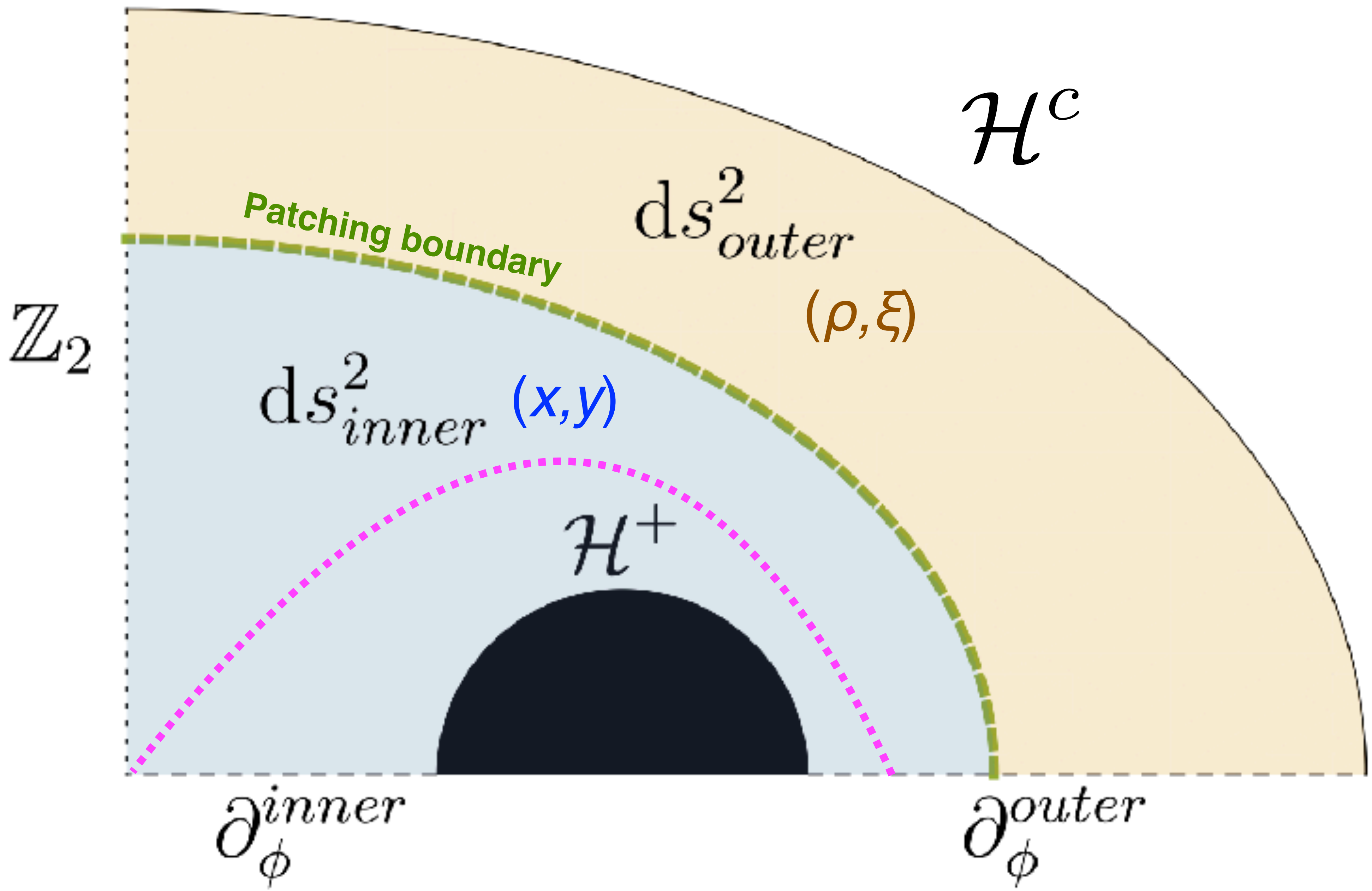
[Headrick, Kitchen, Wiseman '09] [Review: OD, Santos, Way '15]

$\Lambda \equiv 3/\ell^2 > 0$

- Solve instead $G^{\rm H}_{ab} \equiv R_{ab} - \nabla_{(a}\xi_{b)} = \frac{3}{\ell^2} g_{ab}$
- **De Turck vector** ξ can be arbitrary. We choose: $\xi^a \equiv g^{bc}\left[\Gamma^a_{bc}(g) - \Gamma^a_{bc}(\bar{g})\right]$
- $\bar{g}$ is a **reference metric** of choice: it must have the <u>same</u> **asymptotics** & **causal structure** as g.
- <u>**Advantage:**</u> **Principal symbol** of $G^{\rm H}_{ab} = 0$ is simply $\mathcal{P} \sim g^{ab}\partial_a\partial_b$
- For stationary problems, $G^{\rm H} = 0$, together with appropriate **BCs**, yields a set of **Elliptic PDEs!**
- Ultimately, we want to solve $R_{ab} = \frac{3}{\ell^2} g_{ab}$ & thus we **want solutions** of $G^{\rm H} = 0$ that have $\xi = 0$.

- Find a solution, and check that $\xi \to 0$ in the **continuum** limit:
  Ellipticity (local uniqueness) **guarantees** that solutions w/ $\xi \neq 0$ will <u>**not**</u> be **nearby** those w/ $\xi = 0$.





- **Outer region:** near (single) cosmological horizon, solution looks like **de Sitter** space; **(ρ,ξ) coords**
- **Inner region:** solution looks like **warped Israel-Khan** but <u>without</u> conical singularity; **(x,y)** coords
- Inner region is **pentagonal** (5 boundaries) => so split it into 2 squared (4 boundaries) sub-regions

→ Choosing a good reference metric $\bar{g}$ & metric ansatz with patching

1) de Turck reference metric:

→ Israel-Khan <u>without</u> conical singularity; (x,y)

$$ds^2_{\rm ref} = \frac{\ell^2}{g_+^2}\left\{-f g_-^2\, F\, dt^2 + \frac{\lambda^2}{m^2\Delta_{xy}^2}\left[p^2\left(\frac{4dx^2}{(2-x^2)\Delta_x}+\frac{4dy^2}{(2-y^2)\Delta_y}\right)+y^2(2-y^2)(1-y^2)^2\, s\, d\phi^2\right]\right\}$$

$$= \frac{\ell^2}{g_+^2}\left\{-f g_-^2\, F\, dt^2 + \frac{\lambda^2 h}{f}\left[d\rho^2+\rho^2\left(\frac{4d\xi^2}{2-\xi^2}+\frac{(1-\xi^2)^2}{h}\, s\, d\phi^2\right)\right]\right\}.$$

$s = 1-\alpha(1-y^2)^2$

$\alpha = \cdots$

→ de Sitter space: (ρ,ξ) coords

2) metric ansatz with patching:

$$ds^2 = \frac{\ell^2}{g_+^2}\left\{-f g_-^2\, F\,\mathcal{T}\, dt^2 + \frac{\lambda^2}{m^2\Delta_{xy}^2}\left[w^2\left(\frac{4\mathcal{A}\,dx^2}{(2-x^2)\Delta_x}+\frac{4\mathcal{B}}{(2-y^2)\Delta_y}\left(dy - x(1-x^2)\,y\,(2-y^2)(1-y^2)\mathcal{F}\,dx\right)^2\right)\right.\right.$$
$$\left.\left.+\,y^2(2-y^2)(1-y^2)^2\, s\,\mathcal{S}\, d\phi^2\right]\right\}$$

$$= \frac{\ell^2}{g_+^2}\left\{-f g_-^2\, F\,\widetilde{\mathcal{T}}\, dt^2 + \frac{\lambda^2 h}{f}\left[\widetilde{\mathcal{A}}\,d\rho^2+\rho^2\left(\frac{4\widetilde{\mathcal{B}}}{2-\xi^2}\left(d\xi - \xi\,(2-\xi^2)(1-\xi^2)\,\rho\,\widetilde{\mathcal{F}}\,d\rho\right)^2+\frac{(1-\xi^2)^2}{h}\, s\,\widetilde{\mathcal{S}}\, d\phi^2\right)\right]\right\}$$

<u>**Our mission**</u>: find the **unknown functions** $\{\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{F},\mathcal{S}\}_{(x,y)}$ $\{\widetilde{\mathcal{T}},\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{F}},\widetilde{\mathcal{S}}\}_{(\rho,\xi)}$

We know the map: ρ(x,y), ξ(x,y)

by **solving** the **Einstein-de Turck EoM** (**ξ=0**)

subject to the appropriate physical **Boundary Conditions**

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We know the map: $\rho(x,y)$, $\xi(x,y)$

by **solving** the **Einstein-de Turck EoM** (**ξ=0**)

subject to the appropriate (regularity) physical **Boundary Conditions**

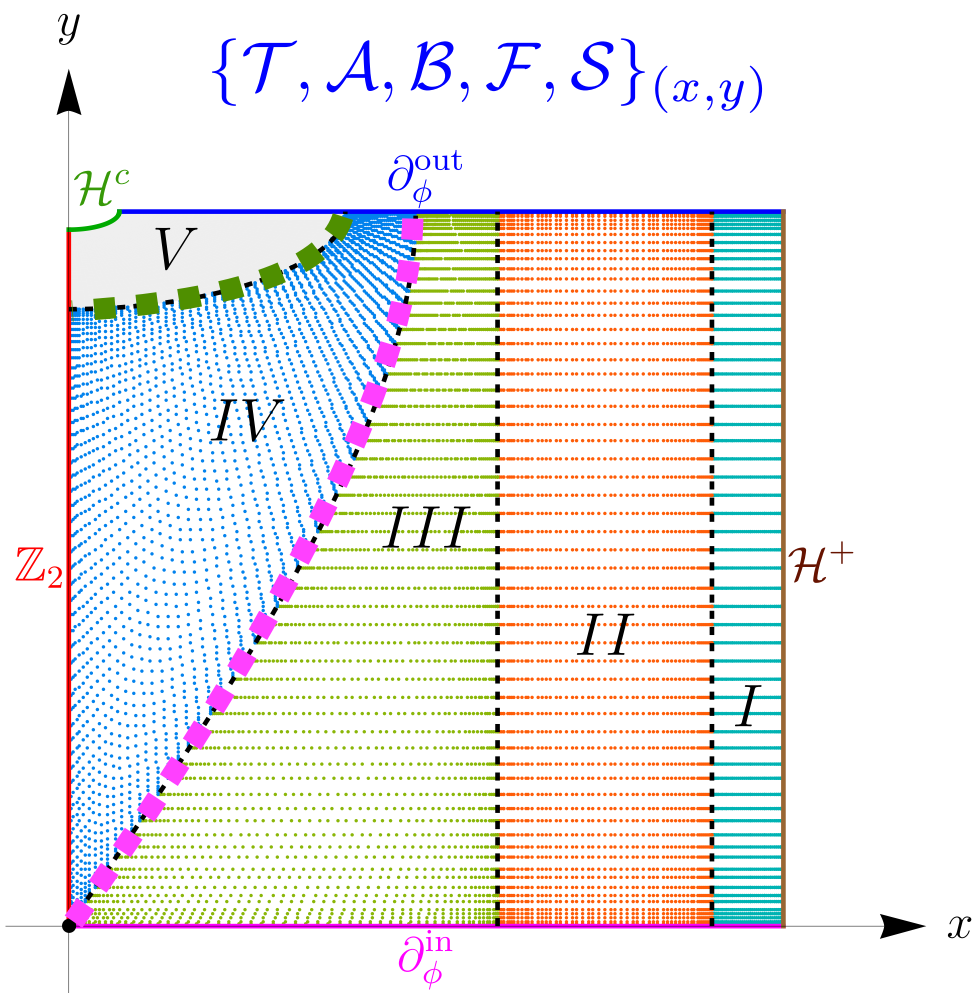
• **Numerical method:** [Review: OD, Santos,Way 1510.02804]

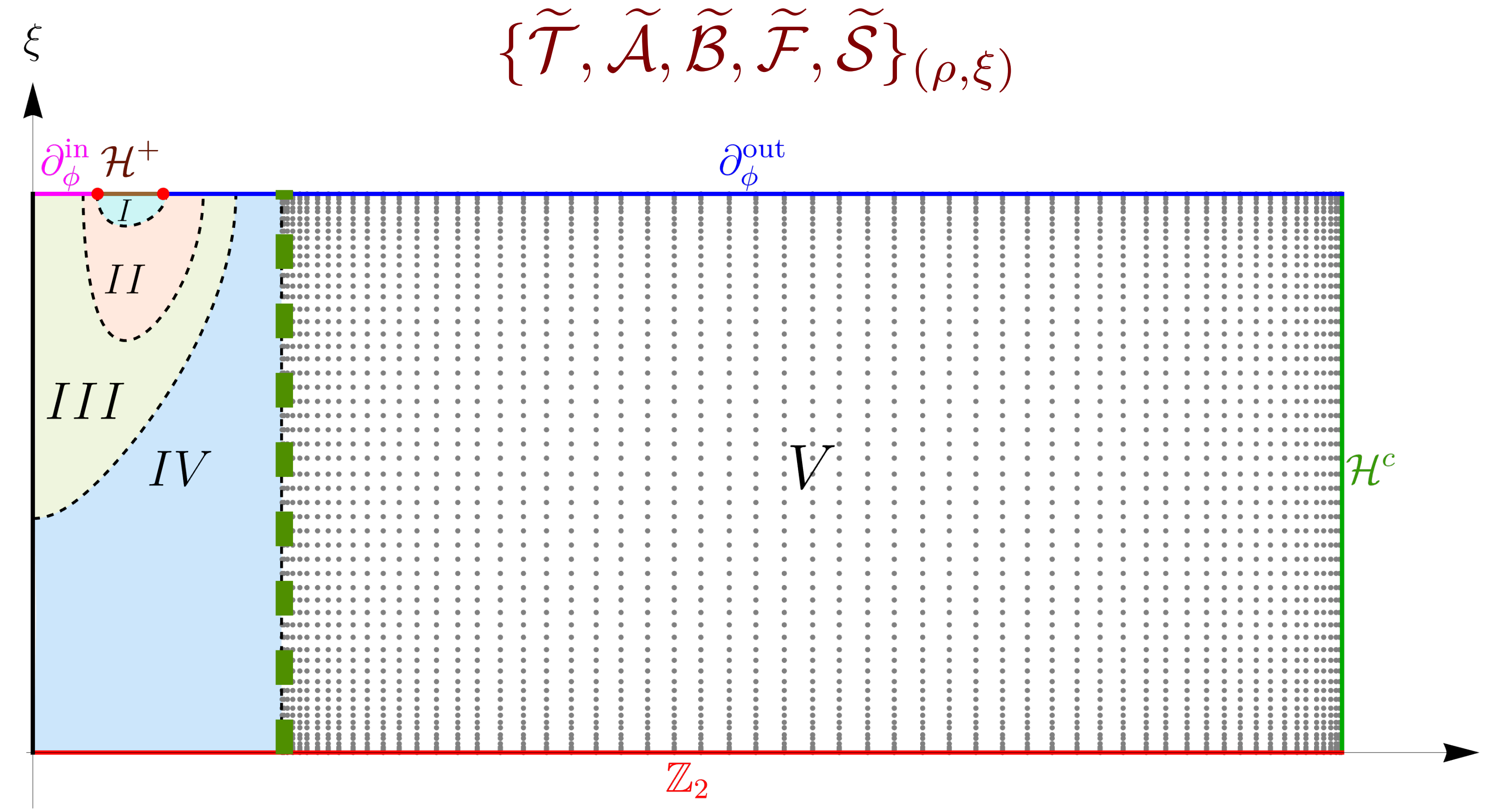
Use a **Newton-Raphson algorithm** with **pseudospectral grid**.

Also use **transfinite interpolation** to complete the **patching**.

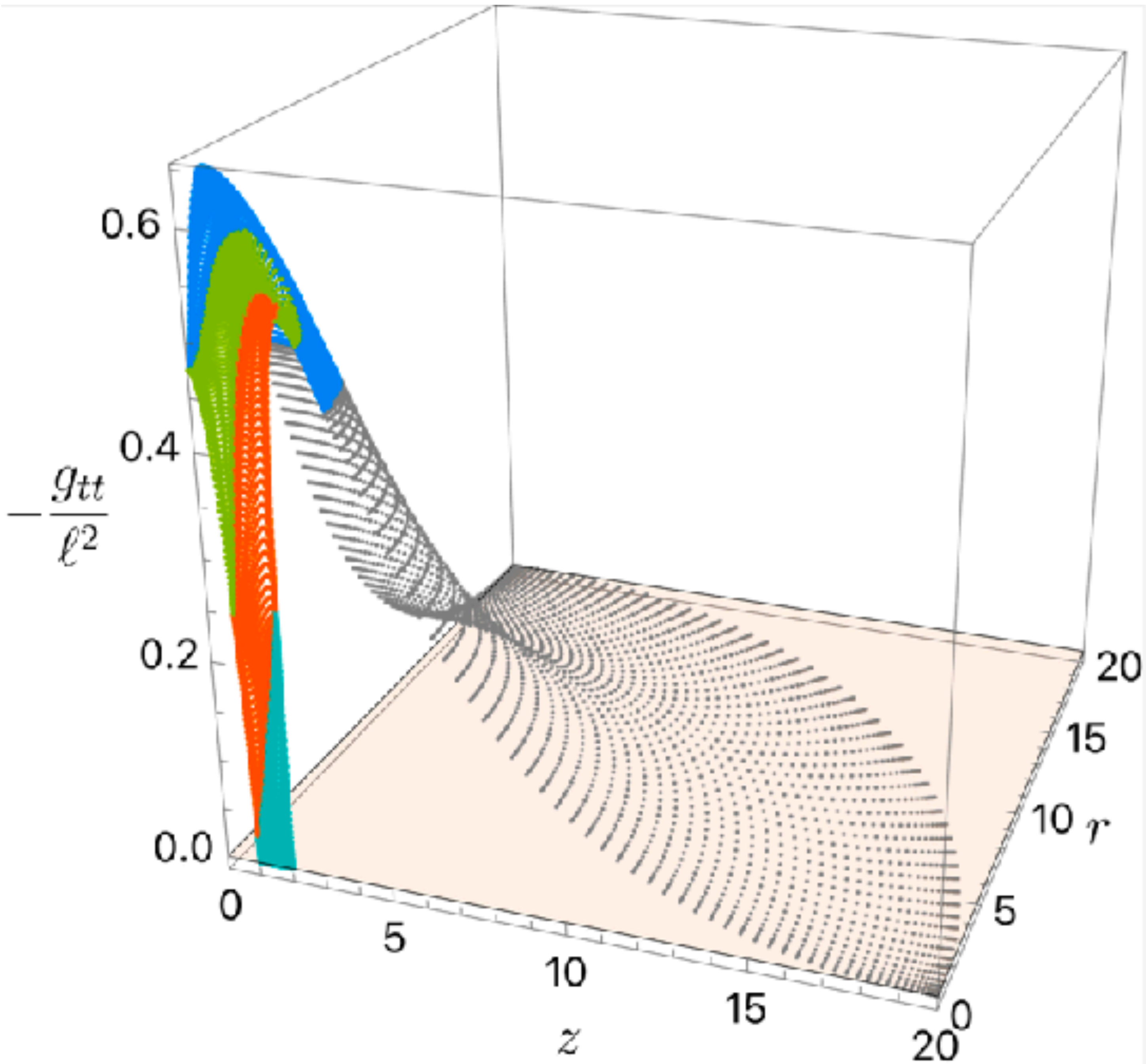
@ patching boundary, require:

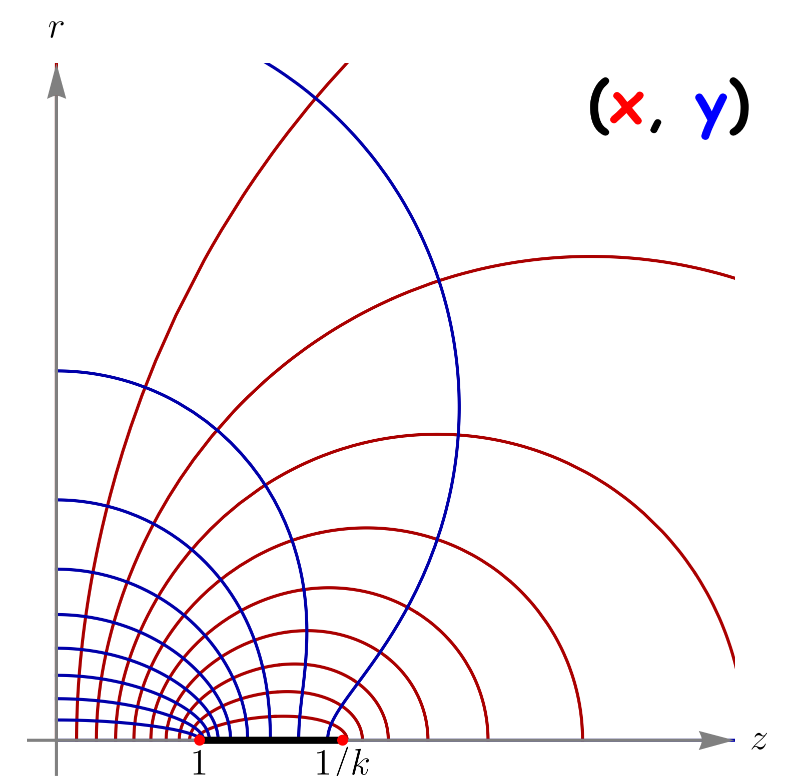
1) **matching** of **two line elements**, & **2)** **matching** of the **normal derivative** across patch bdry





Testing patching: g_{tt} & $g_{\phi\phi}$ are gauge invariant since ∂_t and ∂_ϕ are KVFs





(x, y)

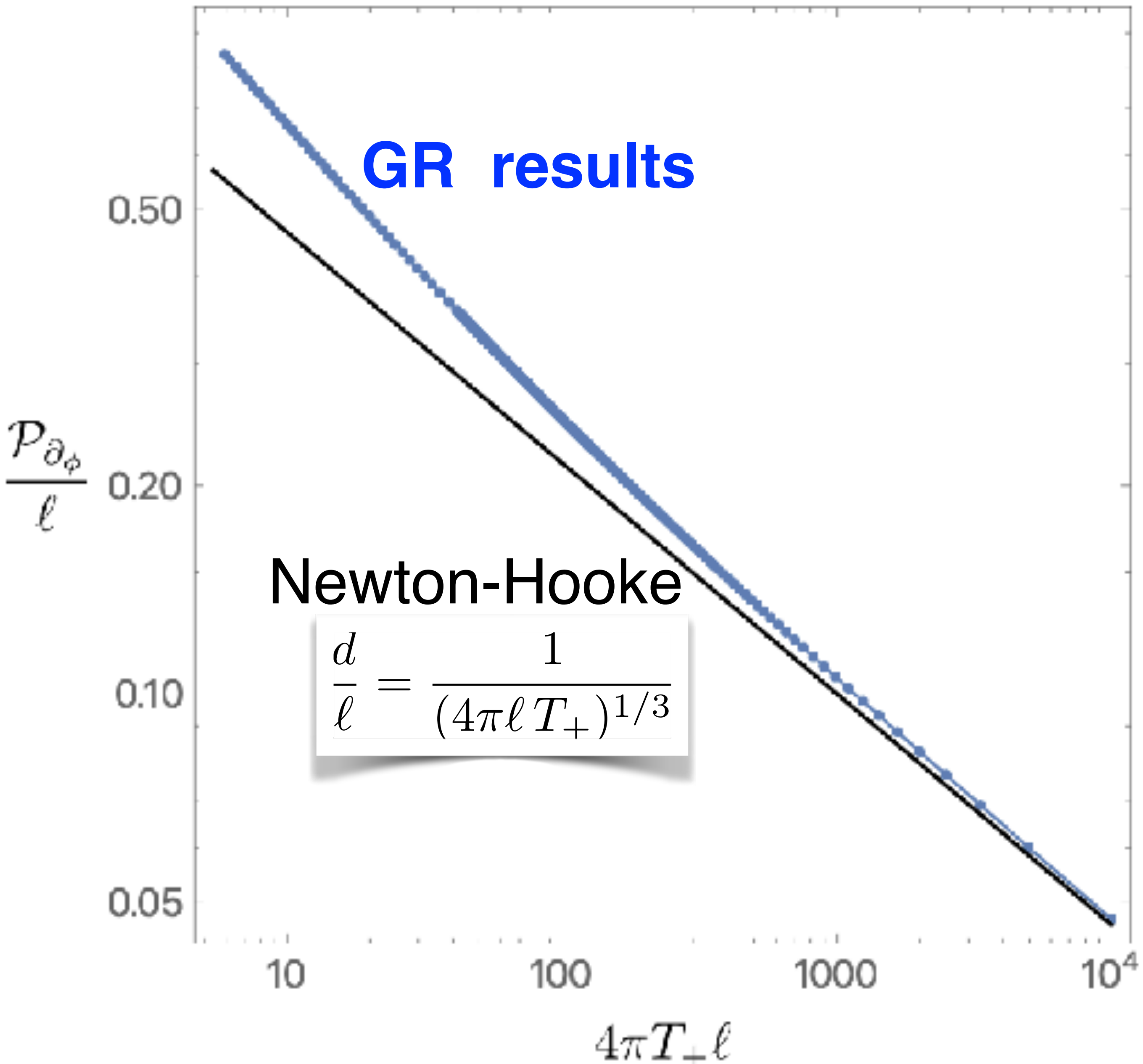
Recall **Bach-Weyl (Israel-Khan) cylindrical-Weyl coord {r,z}** and its **rod-structure** where:

1) the rotation axis and the BH horizons are all located at r = 0
2) there is a Z_2 symmetry

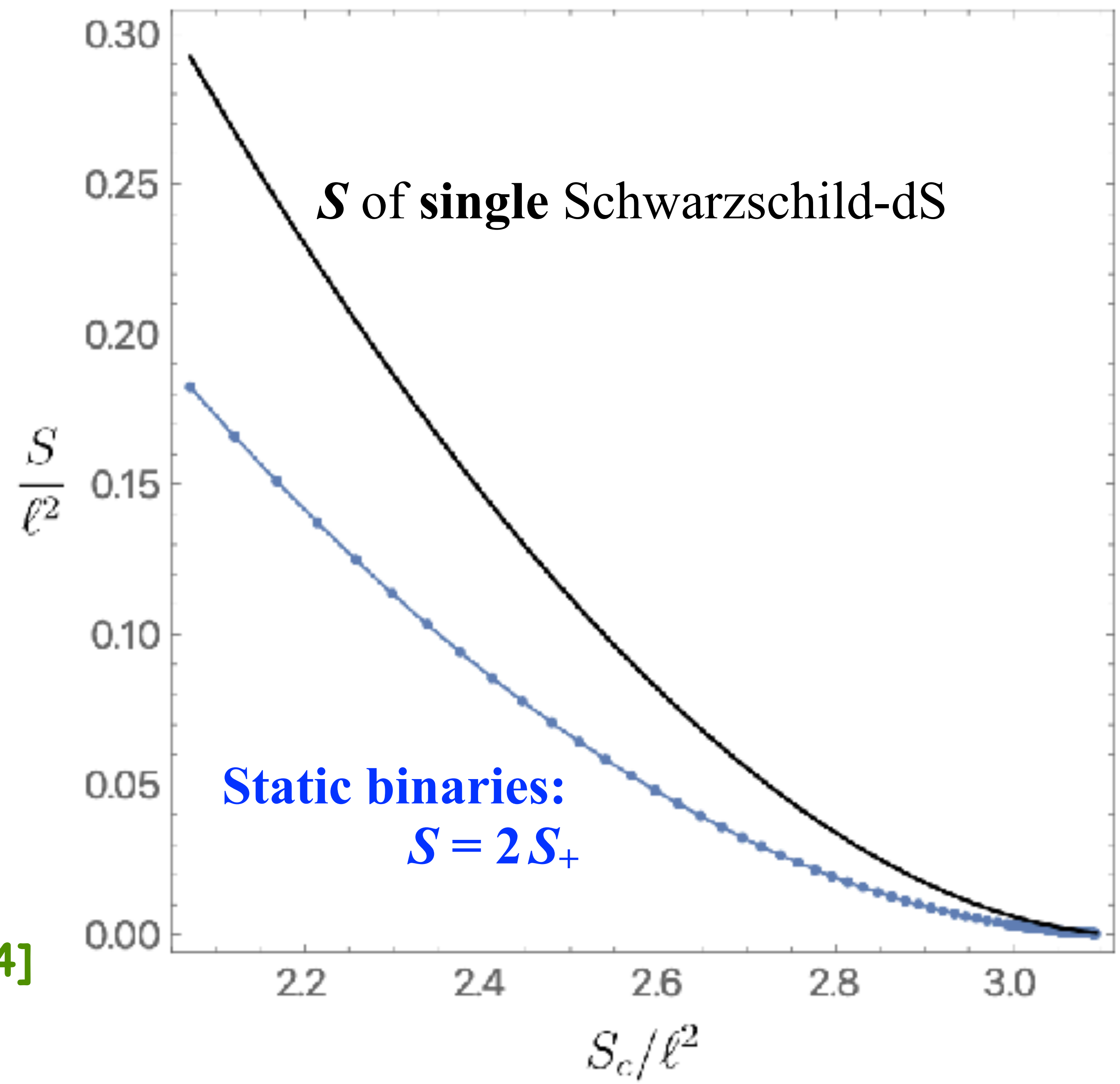
→ Properties of static de Sitter BH binaries

Proper distance between the **BH horizons** versus the BH **temperature**:

NON-Uniqueness in 4D!



Total BH entropy versus the **cosmological horizon entropy**:



First law of thermodynamics:

$$-T_c\,\mathrm{d}S_c = 2\,T_+\,\mathrm{d}S_+$$

[Hawking, Gibbons '74]

Our data satisfies it up to 0.01%

→ Properties of static de Sitter BH binaries

• **Not in conflict** with available Uniqueness theorems:

we have (explicitly identified) **assumptions** of these theorems that can be **evaded**

[LeFloch, Rozoy '10] [Borghini, Chruściel, Mazzieri '19]
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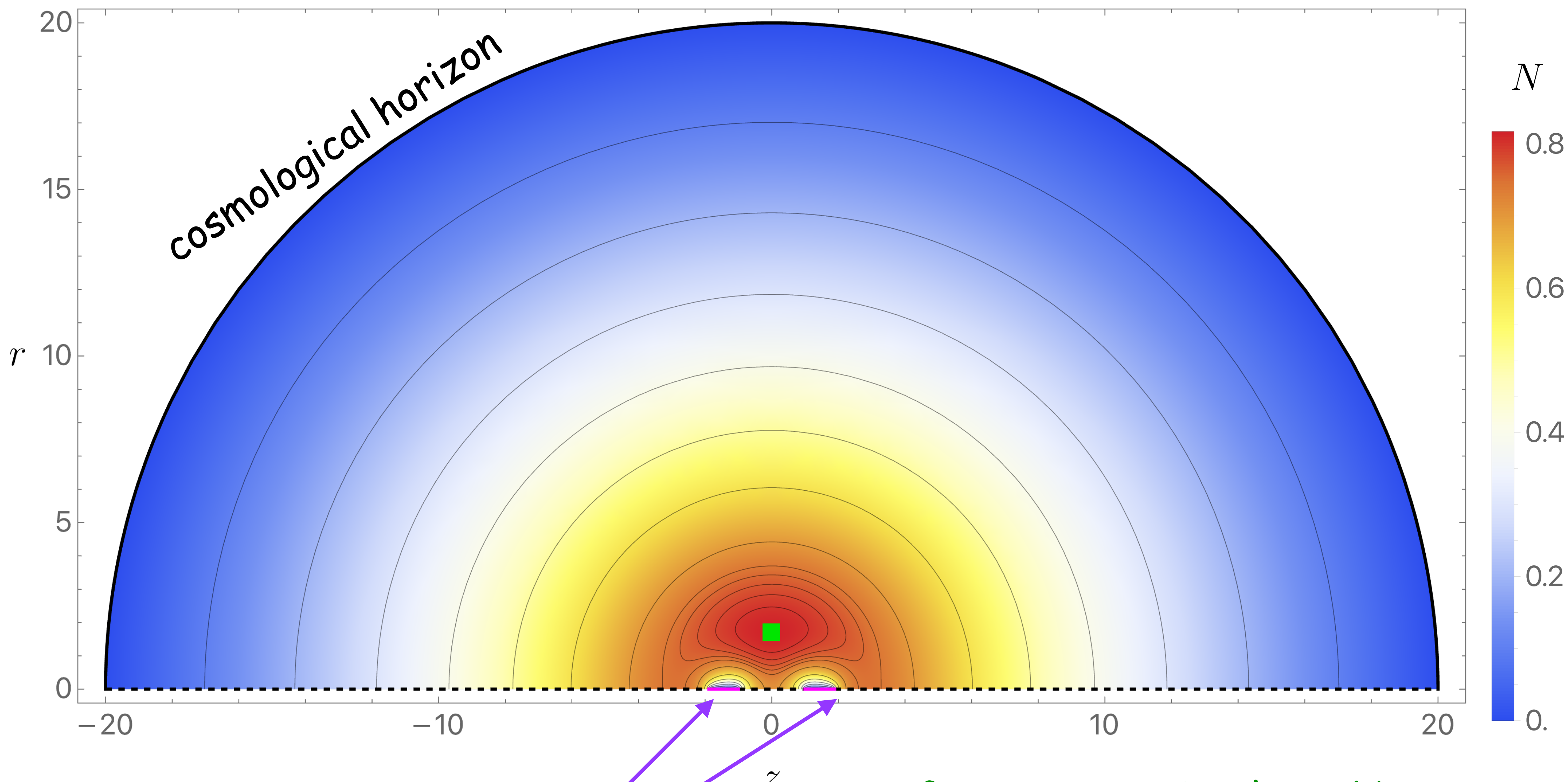
Contour plot showing the level sets of the lapse function $N = \sqrt{-g_{tt}}$



Two BH horizons

Green square is where N takes its maximum value.

Spinning Black hole binaries

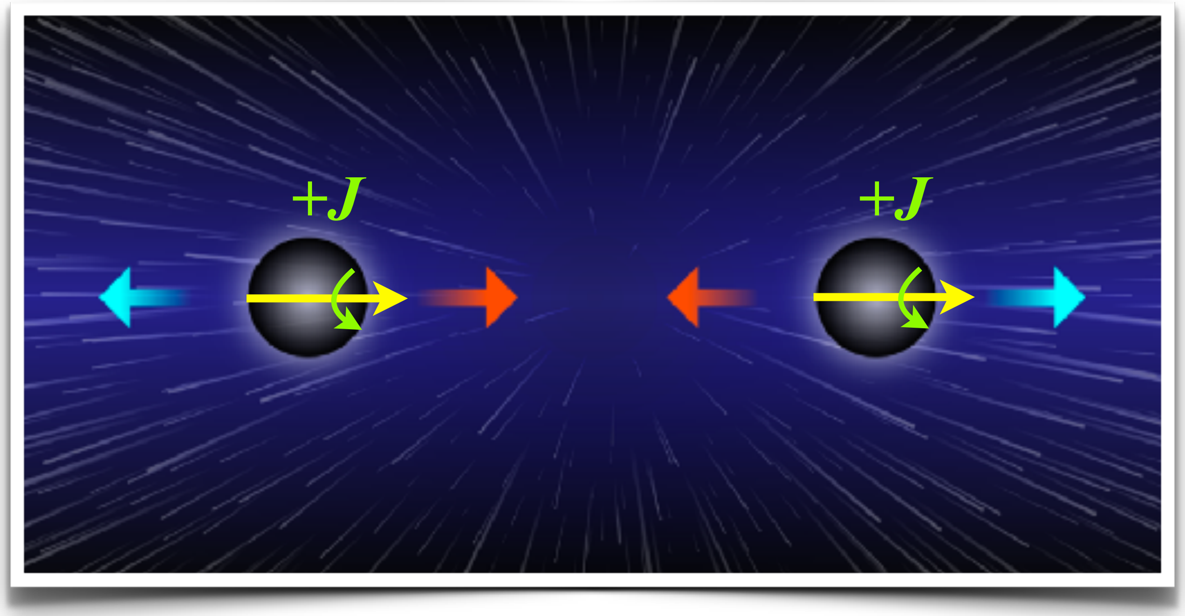
- Add **spin** (not orbital angular momentum) to BHs:

  Cosmological expansion + Grav attraction + Spin-Spin interaction [Wald '72]

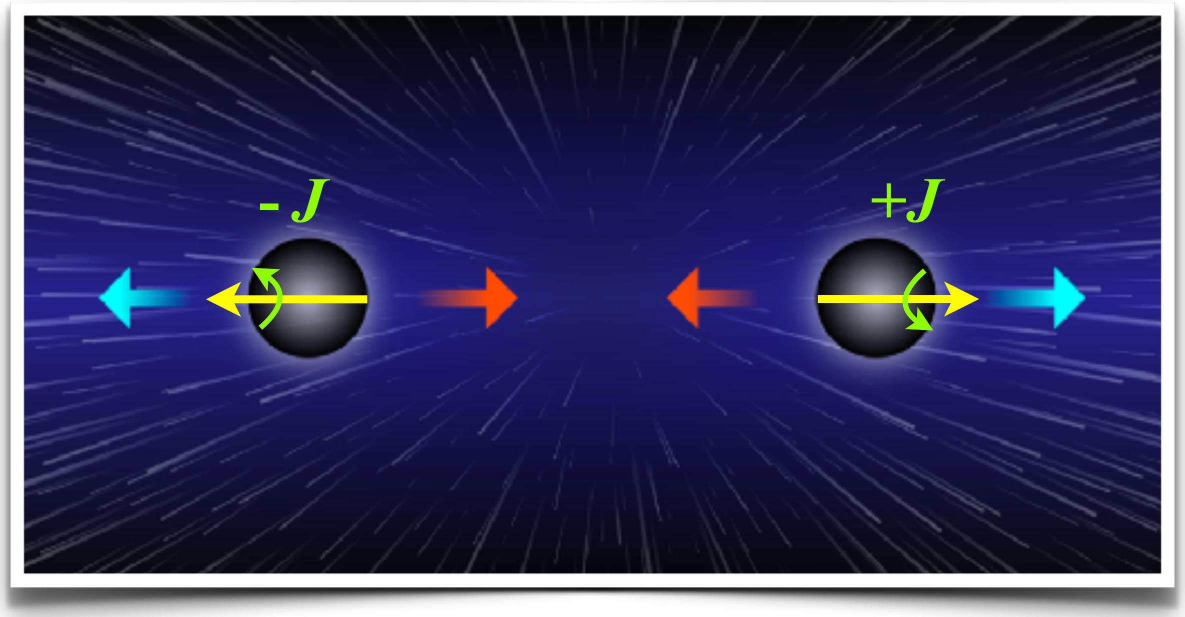
- Can we have stationary (no quadrupole momentum, no radiation) **spinning** BH binaries ?

- Can Spin-Spin interaction **stabilise** the binaries (alike in molecular systems) ?

Aligned [Neuman BC W(-x)=+W(x)]
Repulsive Spin-Spin interaction



Anti-aligned [Dirichlet BC W(-x)=-W(x)]
Attractive Spin-Spin interaction



OD, Jorge Santos, Benson Way, 2406.10333

→ Spinning binaries within Newton-Hooke + Spin-Spin interaction

- **Newton-Hooke + Spin-Spin interaction [Wald '72]** equations of motion:

$$m_a \frac{\mathrm{d}^2 \mathbf{x}_a}{\mathrm{d}t^2} = m_a \frac{\mathbf{x}_a}{\ell^2} + \nabla \left(\frac{m_a \, m_b}{|\mathbf{r}_{ab}|} \right) + \nabla \left[\frac{\mathcal{S}_a \cdot \mathcal{S}_b}{|\mathbf{r}_{ab}|^3} - \frac{3\,(\mathcal{S}_a \cdot \mathbf{r}_{ab})\,(\mathcal{S}_b \cdot \mathbf{r}_{ab})}{|\mathbf{r}_{ab}|^5} \right]$$

- **Stationary** solutions exist when: $\frac{\mathrm{d}^2 \mathbf{x}_a}{\mathrm{d}t^2} = 0$ **(1)**

$\Lambda \equiv 3/\ell^2 > 0.$

- **Two equal mass BHs** aligned along z axis and separated by a **distance d**:

$$N = 2, \quad x_1 = -x_2 = \tfrac{d}{2}\,\hat{e}_z, \quad m_a = m_b = M$$

$$\mathcal{S}_{1,2} = m\,\sigma_{1,2}\,\mathbf{e}_z \qquad \sigma_2 = \gamma\sigma_1 \equiv \gamma\sigma$$

$\gamma = -1$ (attractive SS)
$\gamma = +1$ (repulsive SS)

- Then (1) yields:

$$\frac{d^3}{\ell^3} = \frac{2m}{\ell}\left(1 - 6\,\gamma\,\frac{\sigma^2}{d^2}\right) \quad \Rightarrow \quad \frac{d}{\ell} \simeq \frac{1}{(4\pi T_+ \ell)^{\frac{1}{3}}}\left\{1 - \left[\frac{1}{3} + \frac{2\gamma}{(4\pi T_+ \ell)^{4/3}}\right]\frac{\Omega_+^2}{(4\pi T_+)^2}\right\} \quad (2)$$

$$2\,m = \frac{1}{2\pi T_+ + \sqrt{4\pi^2 T_+^2 + \Omega_+^2}}$$

$$\sigma = \frac{m\,\Omega_+}{\sqrt{4\pi^2 T_+^2 + \Omega_+^2}}\,.$$

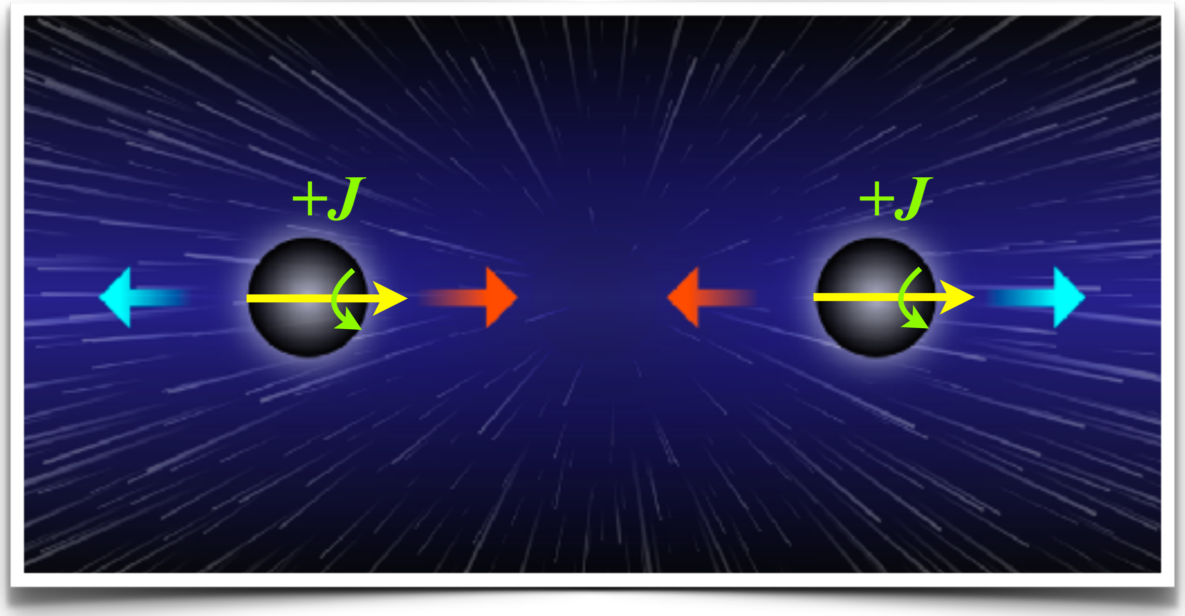
- **Equilibrium condition (2)** can fall within the regime of **validity of Newton-Hook theory**:

$$r_+ \ll d \ll \ell \;\; (\text{i.e. large } T_+\ell)$$

→ Choosing a good reference metric $\bar{g}$ & metric ansatz with BCs

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{\ell^2}{g_+^2}\Bigg\{ - f g_-^2\, F\, \mathcal{T}\, \mathrm{d}t^2 + \frac{\lambda^2}{m^2\Delta_{xy}^2}\Bigg[w^2\left(\frac{4\mathcal{A}\,\mathrm{d}x^2}{(2-x^2)\Delta_x} + \frac{4\mathcal{B}}{(2-y^2)\Delta_y}\left(\mathrm{d}y - x\,(1-x^2)\,y\,(2-y^2)(1-y^2)\mathcal{F}\,\mathrm{d}x\right)^2\right) \\
&\qquad + y^2(2-y^2)(1-y^2)^2\, s\,\mathcal{S}\left(\mathrm{d}\phi - g_-^2\, w\,\mathcal{W}\,\mathrm{d}t\right)^2 \Bigg]\Bigg\} \\
&= \frac{\ell^2}{g_+^2}\Bigg\{ - f g_-^2\, F\, \widetilde{\mathcal{T}}\, \mathrm{d}t^2 + \frac{\lambda^2 h}{f}\Bigg[\widetilde{\mathcal{A}}\,\mathrm{d}\rho^2 + \rho^2\left(\frac{4\widetilde{\mathcal{B}}}{2-\xi^2}\left(\mathrm{d}\xi - \xi\,(2-\xi^2)(1-\xi^2)\,\rho\,\widetilde{\mathcal{F}}\,\mathrm{d}\rho\right)^2 + \frac{(1-\xi^2)^2}{h}\, s\,\widetilde{\mathcal{S}}\left(\mathrm{d}\phi - g_-^2\, w\,\widetilde{\mathcal{W}}\,\mathrm{d}t\right)^2\right)\Bigg]\Bigg\}
\end{aligned}
$$

Aligned [Neuman BC W(-x)=+W(x)]
Repulsive Spin-Spin interaction



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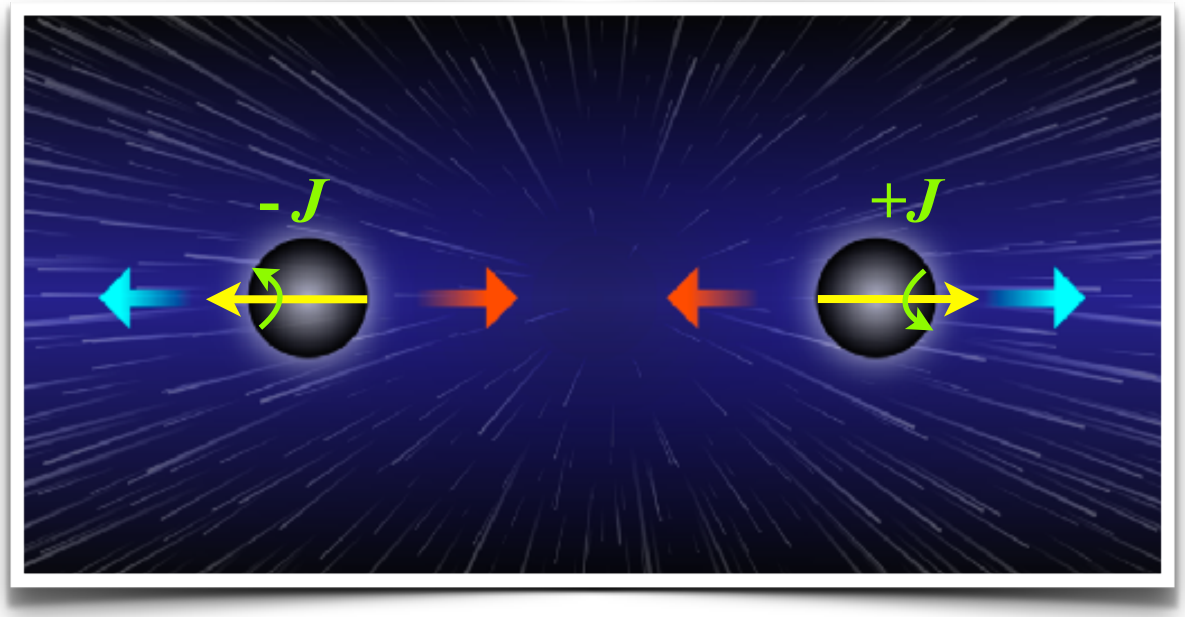
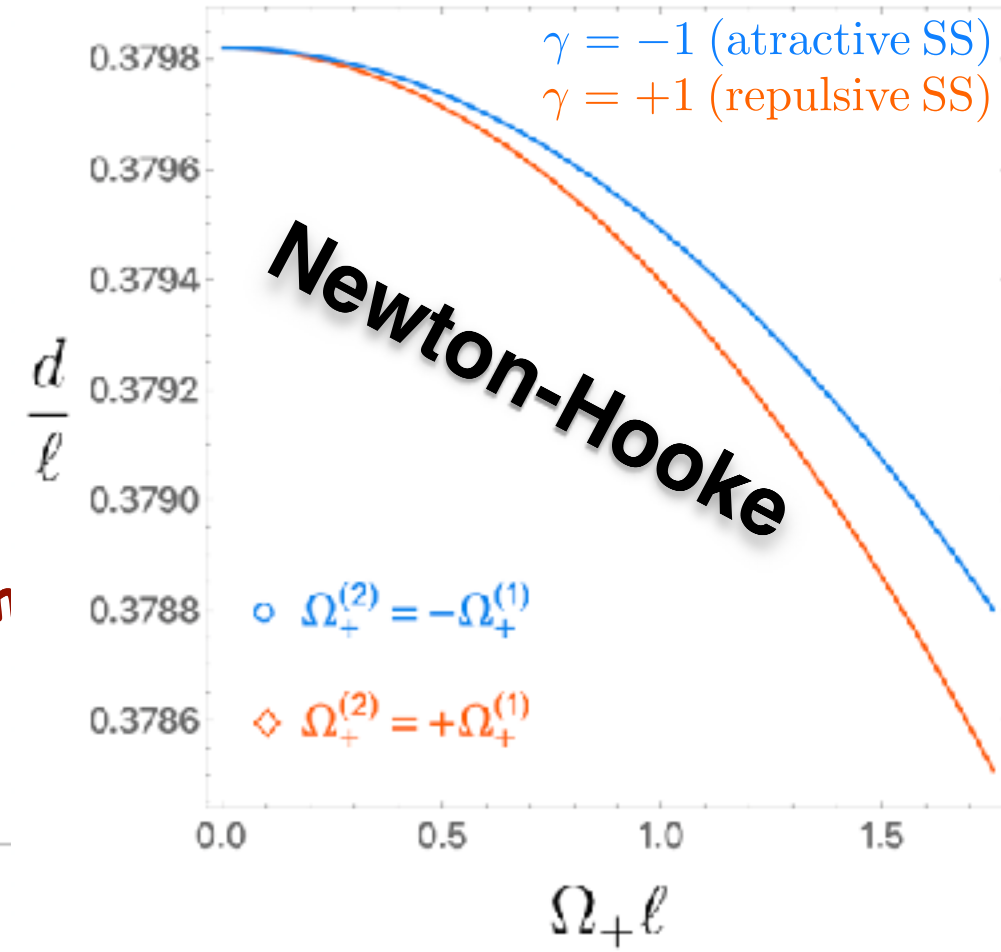
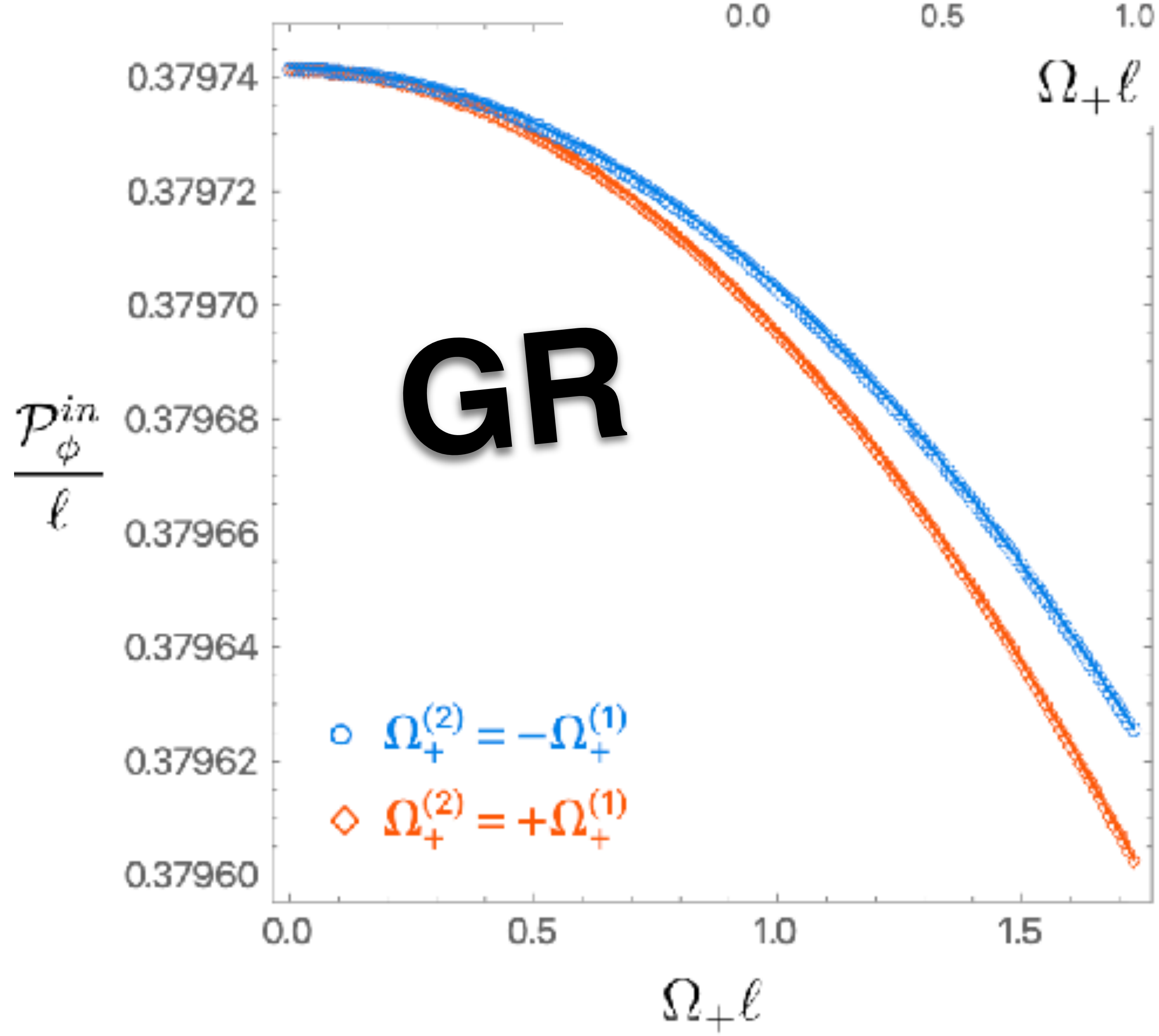


→ Properties of SPINNING de Sitter BH binaries

- $d(\Omega_+)$ curve for the aligned binary ($\gamma = 1$) is always **below** the curve for the anti-aligned case ($\gamma = -1$).

- **Spin-spin forces are repulsive for aligned spins**
  => BHs need to be closer apart to remain in equilibrium
  (for fixed gravitational and cosmological forces).
  The opposite is true for anti-aligned spins.

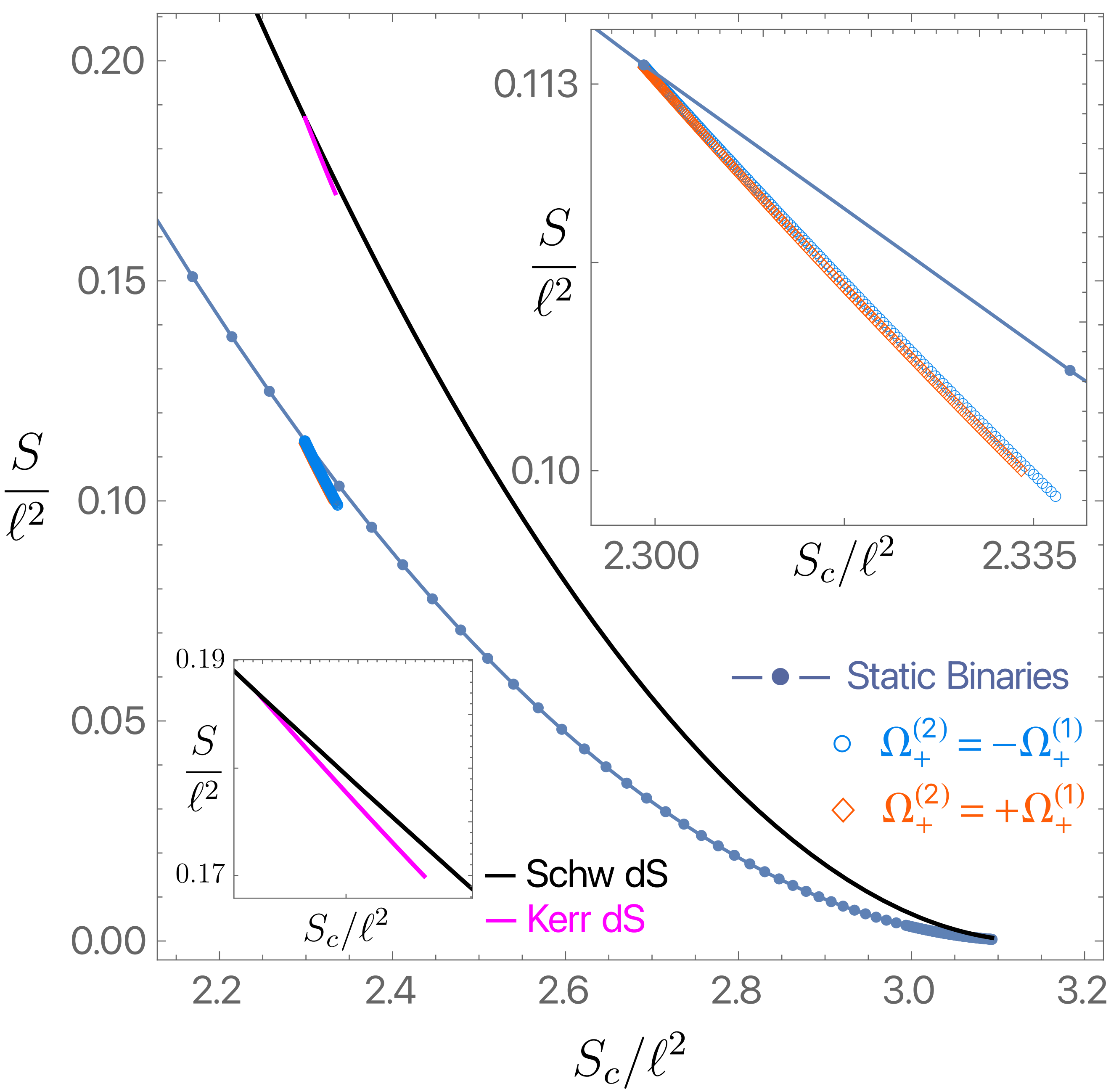


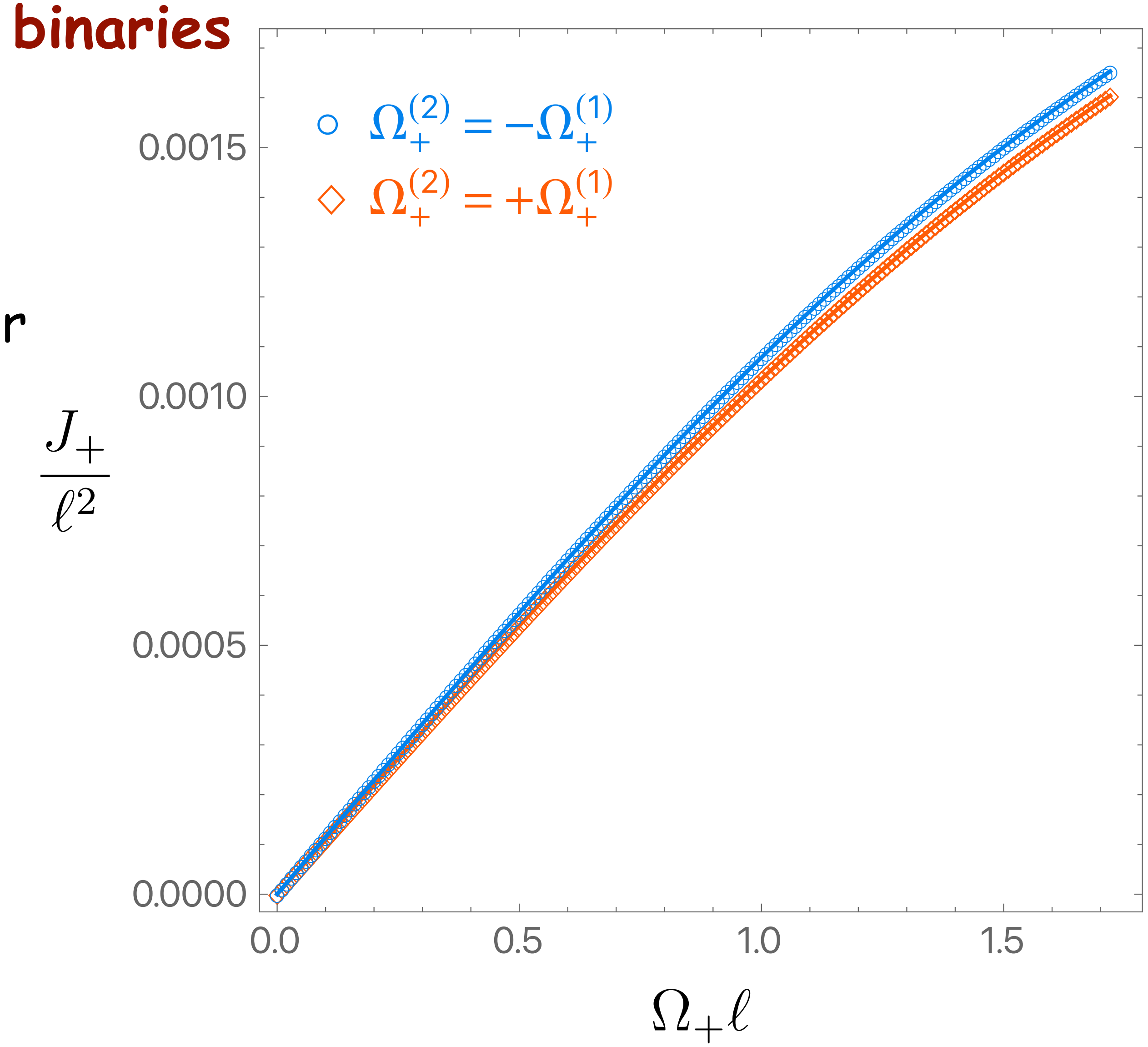


→ Properties of SPINNING de Sitter BH binaries

- Binaries are **thermodynamically unstable**
  For a given S_c and J_{tot}, spinning binaries have lower total event horizon S than dS Schw/Kerr

- **Continuous Non-Uniqueness**





First law of thermodynamics:

$$-T_c\,\mathrm{d}S_c = 2\,T_+\,\mathrm{d}S_+ + 2\,\Omega_+\,\mathrm{d}J_+$$

[Hawking,Gibbons '74]

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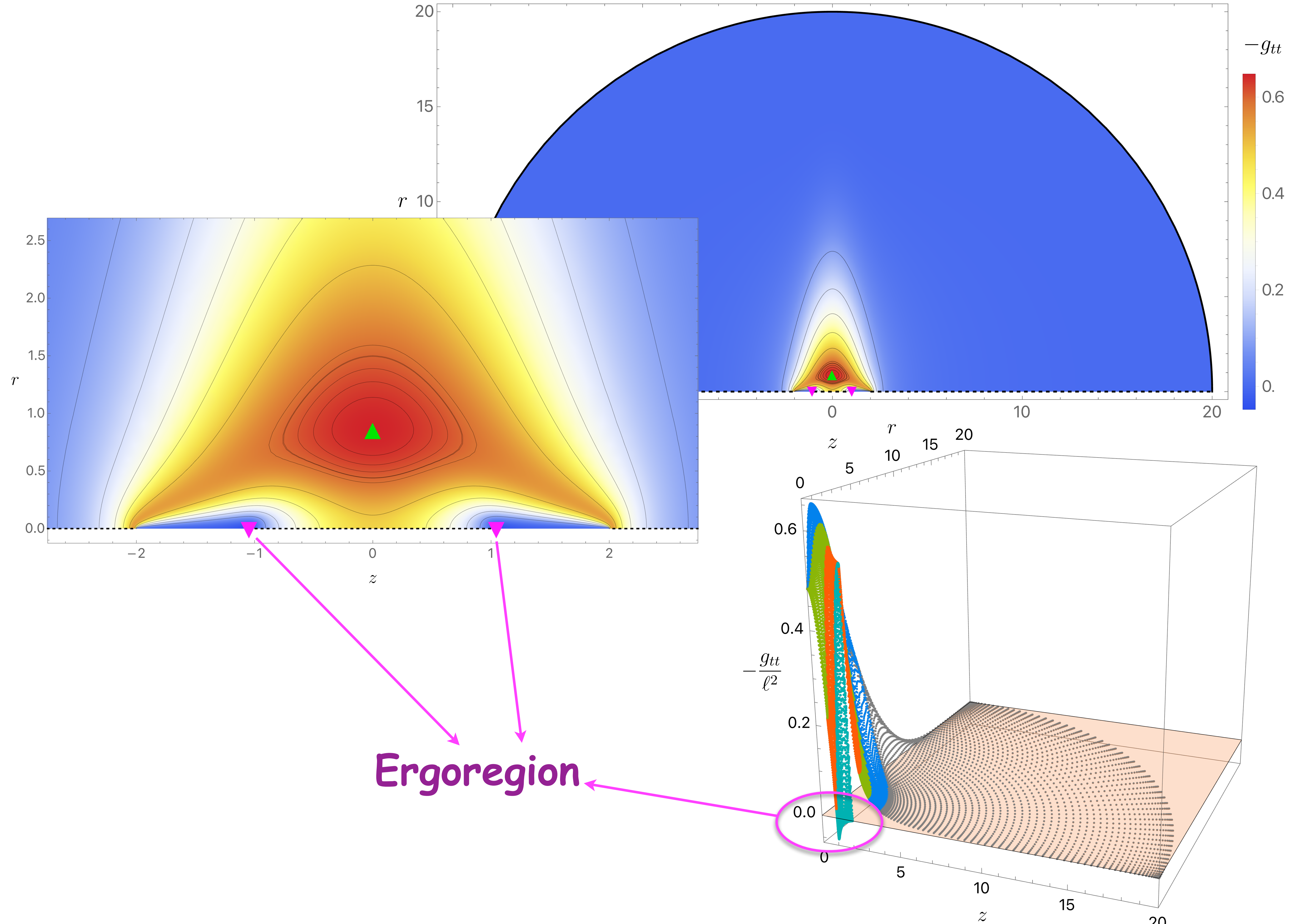
→ Properties of SPINNING de Sitter BH binaries



→ Outlook

- Our binaries are **thermodynamically unstable**: but, under small perturbations, the BH pair necessarily needs to merge into a single BH or fly apart, ie it can be **dynamically stable (?)**.

- **Future**: study the **dynamical stability** of spinning binaries by perturbing our stationary solutions.

- The **spin-spin interactions** act on shorter length scales, & **might provide a mechanism for stabilizing binaries** in some windows of parameters, **alike it stabilizes molecules**

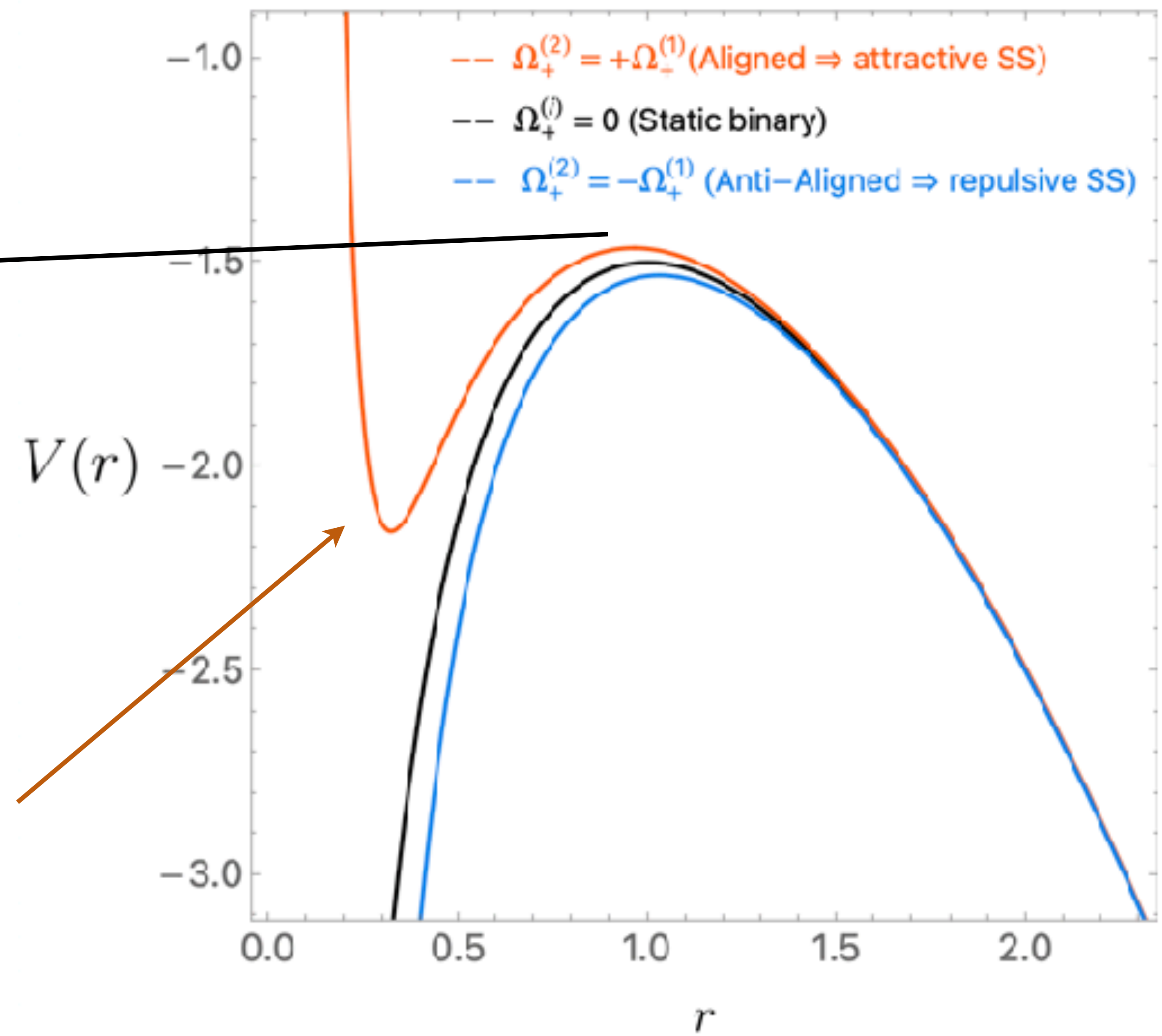
→ Back-of-the-envelop analysis within Newton-Hooke approximation:

- **Static binary:**

$V(r) = -1/r - r^2/2$ <—Schrödinger potential

=> single **maximum** (r=1)

=> **unstable equilibrium**



- **Spinning binary:** Add a $1/r^3$ spin-spin term,

$V_{spin}(r) = V(r) - \gamma/r^3$

For $\gamma > 1/30$ (spin-spin repulsive interaction)

$V_{spin}(r)$ has **local minimum**

=> a **STABLE equilibrium point**

➞ Appendix: technical details of the method employed

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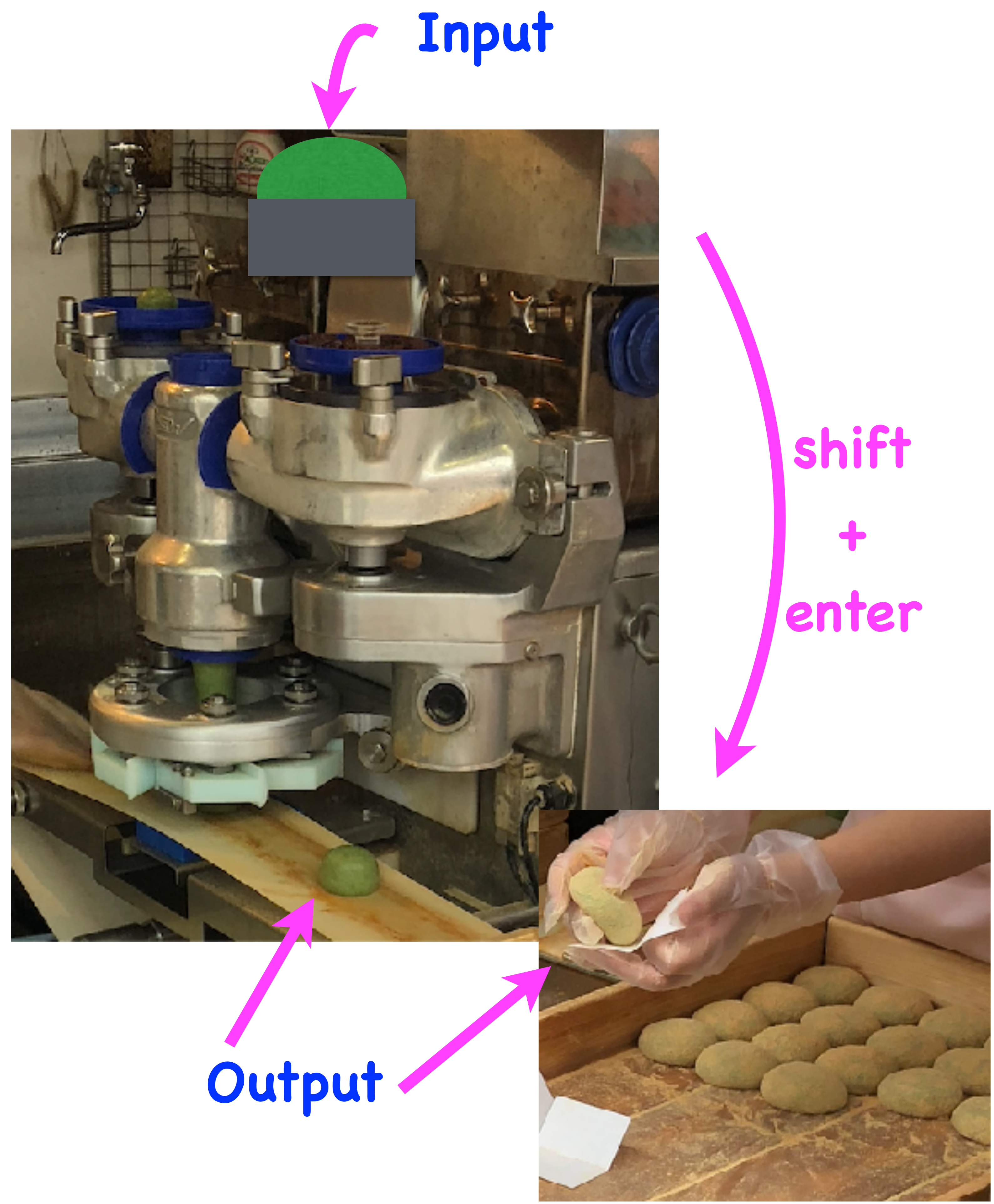


→ Going beyond the Newton-Hooke approximation: General Relativity (GR) solution

We find exact solutions to this 2-body problem in GR with $\Lambda > 0$ using numerics.

Use **Einstein-deTurck formulation of GR:**

[Headrick, Kitchen, Wiseman '09]   [Review: OD, Santos, Way '15]

$\Lambda \equiv 3/\ell^2 > 0$

- Solve instead $G^{\rm H}_{ab} \equiv R_{ab} - \nabla_{(a}\xi_{b)} = \frac{3}{\ell^2} g_{ab}$
- **De Turck vector** ξ can be arbitrary. We choose: $\xi^a \equiv g^{bc}\left[\Gamma^a_{bc}(g) - \Gamma^a_{bc}(\bar{g})\right]$
- $\bar{g}$ is a **reference metric** of choice: it must have the <u>same</u> **asymptotics** & **causal structure** as g.
- <u>**Advantage:**</u> **Principal symbol** of $G^{\rm H}_{ab} = 0$ is simply $\mathcal{P} \sim g^{ab}\partial_a\partial_b$
- For stationary problems, $G^H = 0$, together with appropriate **BCs**, yields a set of Elliptic PDEs!
- Ultimately, we want to solve $R_{ab} = \frac{3}{\ell^2} g_{ab}$ & thus we **want solutions** of $G^H = 0$ that have $\xi=0$.

- Find a solution, and check that $\xi \to 0$ in the **continuum** limit:
  Ellipticity (local uniqueness) **guarantees** that solutions w/ $\xi \neq 0$ will <u>**not**</u> be **nearby** those w/ $\xi=0$.

→ Choosing a good reference metric $\bar{g}$

A) near the event horizons

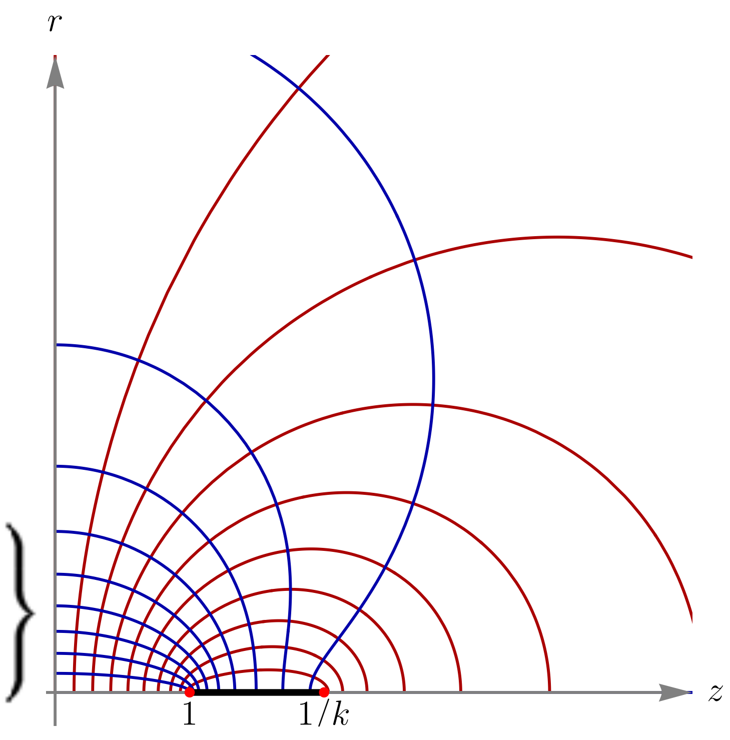
- For binaries well within the **cosmological horizon**, ie **near the event horizons**, the solution should be well approximated by a **Bach-Weyl 1922** (**Israel-Khan 1964**) but **without conical singularity**:

$$\mathrm{d}s^2 = \ell^2\left[-f\mathrm{d}t^2 + \frac{\lambda^2}{f}[h(\mathrm{d}r^2+\mathrm{d}z^2) + s\,r^2\mathrm{d}\phi^2]\right]$$

- Introduce a **Schwarz-Christoffel map** from **cylindrical-Weyl** to **ring-like coordinates** (**x**, **y**):

$$r = \frac{(1-x^2)\sqrt{1-k^2x^2(2-x^2)}\,y\sqrt{2-y^2}(1-y^2)}{(1-y^2)^2+k^2x^2(2-x^2)y^2(2-y^2)}$$

$$z = \frac{x\sqrt{2-x^2}\sqrt{(1-y^2)^2+k^2y^2(2-y^2)}}{(1-y^2)^2+k^2x^2(2-x^2)y^2(2-y^2)}$$



Lines of constant **x** & **y**, along with the **rod structure** of Bach-Weyl

$$\mathrm{d}s^2 = \ell^2\left\{-f\mathrm{d}t^2 + \frac{\lambda^2}{m^2\Delta_{xy}^2}\left[p^2\left(\frac{4\mathrm{d}x^2}{(2-x^2)\Delta_x}+\frac{4\mathrm{d}y^2}{(2-y^2)\Delta_y}\right)+s\,y^2(2-y^2)(1-y^2)^2\mathrm{d}\phi^2\right]\right\}$$

$$s = 1-\alpha(1-y^2)^2$$

- Wish to **join the Bach-Weyl** solution **with** a **de Sitter horizon**. In anticipation, we **write the Bach-Weyl** solution in **polar-Weyl coordinates** (**ρ, ξ**)

$$z = \rho\,\xi\sqrt{2-\xi^2}$$
$$r = \rho(1-\xi^2)$$

$$\mathrm{d}s^2 = \ell^2\left\{-f\mathrm{d}t^2 + \frac{\lambda^2 h}{f}\left[\mathrm{d}\rho^2 + \rho^2\left(\frac{4\mathrm{d}\xi^2}{2-\xi^2} + s\,\frac{(1-\xi^2)^2}{h}\mathrm{d}\phi^2\right)\right]\right\}$$

B) near the cosmological horizon

- **Closer** to the **cosmological horizon**, we would like the metric to **look like pure de Sitter:**

$$\mathrm{d}s^2 = -\left(1-\frac{R^2}{\ell^2}\right)\mathrm{d}\tau^2 + \frac{\mathrm{d}R^2}{1-\frac{R^2}{\ell^2}} + r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)$$

- Introduce **isotropic coordinates:** $\frac{R}{\ell} = \frac{\lambda\rho}{1+\frac{\lambda^2\rho^2}{4}}\,, \qquad \sin\theta = 1-\xi^2\,, \qquad \tau = \ell\, t$

$$\mathrm{d}s^2 = \frac{\ell^2}{g_+^2}\left\{-g_-^2\mathrm{d}t^2 + \lambda^2\left[\mathrm{d}\rho^2+\rho^2\left(\frac{4\mathrm{d}\xi^2}{2-\xi^2}+(1-\xi^2)^2\mathrm{d}\phi^2\right)\right]\right\} \qquad g_\pm = 1\pm\frac{\lambda^2\rho^2}{4}$$

- In these coords, the **de Sitter horizon** is at **$\rho = 2/\lambda$** (where $g_-^2 = 0$)
  & has a constant temperature of **$T_C = 1/(2\pi)$.**

- **de Sitter space** in isotropic coords **<u>resembles</u>** **Bach-Weyl solution** in polar-Weyl coords:

$$\mathrm{d}s^2 = \ell^2\left\{-f\mathrm{d}t^2 + \frac{\lambda^2 h}{f}\left[\mathrm{d}\rho^2+\rho^2\left(\frac{4\mathrm{d}\xi^2}{2-\xi^2} + s\,\frac{(1-\xi^2)^2}{h}\mathrm{d}\phi^2\right)\right]\right\} \qquad f,h\big|_{\rho\gg1}\to 1$$

→ Choosing a good reference metric $\bar{g}$ & metric ansatz with patching

1) de Turck reference metric:

→ Israel-Khan without conical singularity: (x,y)

$$ds^2_{\rm ref} = \frac{\ell^2}{g_+^2}\left\{-f g_-^2\, F\, dt^2 + \frac{\lambda^2}{m^2\Delta_{xy}^2}\left[p^2\left(\frac{4dx^2}{(2-x^2)\Delta_x} + \frac{4dy^2}{(2-y^2)\Delta_y}\right) + y^2(2-y^2)(1-y^2)^2\, s\, d\phi^2\right]\right\}$$

$$= \frac{\ell^2}{g_+^2}\left\{-f g_-^2\, F\, dt^2 + \frac{\lambda^2 h}{f}\left[d\rho^2 + \rho^2\left(\frac{4d\xi^2}{2-\xi^2} + \frac{(1-\xi^2)^2}{h}\, s\, d\phi^2\right)\right]\right\}.$$

$s = 1 - \alpha(1-y^2)^2$

$\alpha = \cdots$

→ de Sitter space: (ρ,ξ) coords

2) metric ansatz with patching:

$$ds^2 = \frac{\ell^2}{g_+^2}\Bigg\{-f g_-^2\, F\, \mathcal{T}\, dt^2 + \frac{\lambda^2}{m^2\Delta_{xy}^2}\Bigg[w^2\left(\frac{4\mathcal{A}\, dx^2}{(2-x^2)\Delta_x} + \frac{4\mathcal{B}}{(2-y^2)\Delta_y}\left(dy - x(1-x^2)\, y\,(2-y^2)(1-y^2)\mathcal{F}\, dx\right)^2\right)$$
$$+ y^2(2-y^2)(1-y^2)^2\, s\, \mathcal{S}\, d\phi^2\Bigg]\Bigg\}$$

$$= \frac{\ell^2}{g_+^2}\left\{-f g_-^2\, F\, \widetilde{\mathcal{T}}\, dt^2 + \frac{\lambda^2 h}{f}\left[\widetilde{\mathcal{A}}\, d\rho^2 + \rho^2\left(\frac{4\widetilde{\mathcal{B}}}{2-\xi^2}\left(d\xi - \xi\,(2-\xi^2)(1-\xi^2)\,\rho\,\widetilde{\mathcal{F}}\, d\rho\right)^2 + \frac{(1-\xi^2)^2}{h}\, s\, \widetilde{\mathcal{S}}\, d\phi^2\right)\right]\right\}$$

Our mission: find the **unknown functions** $\{\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{F},\mathcal{S}\}_{(x,y)}$ $\{\widetilde{\mathcal{T}},\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{F}},\widetilde{\mathcal{S}}\}_{(\rho,\xi)}$

We know the map: *ρ(x,y)*, *ξ(x,y)*

by **solving** the **Einstein-de Turck EoM** (**ξ=0**)

subject to the appropriate physical **Boundary Conditions**

• Our mission: find the **unknown functions** $\{\mathcal{T},\mathcal{A},\mathcal{B},\mathcal{F},\mathcal{S}\}_{(x,y)}$ $\{\widetilde{\mathcal{T}},\widetilde{\mathcal{A}},\widetilde{\mathcal{B}},\widetilde{\mathcal{F}},\widetilde{\mathcal{S}}\}_{(\rho,\xi)}$

We know the map: $\rho(x,y)$, $\xi(x,y)$

by **solving** the **Einstein-de Turck EoM** (**ξ=0**)

subject to the appropriate (regularity) physical **Boundary Conditions**

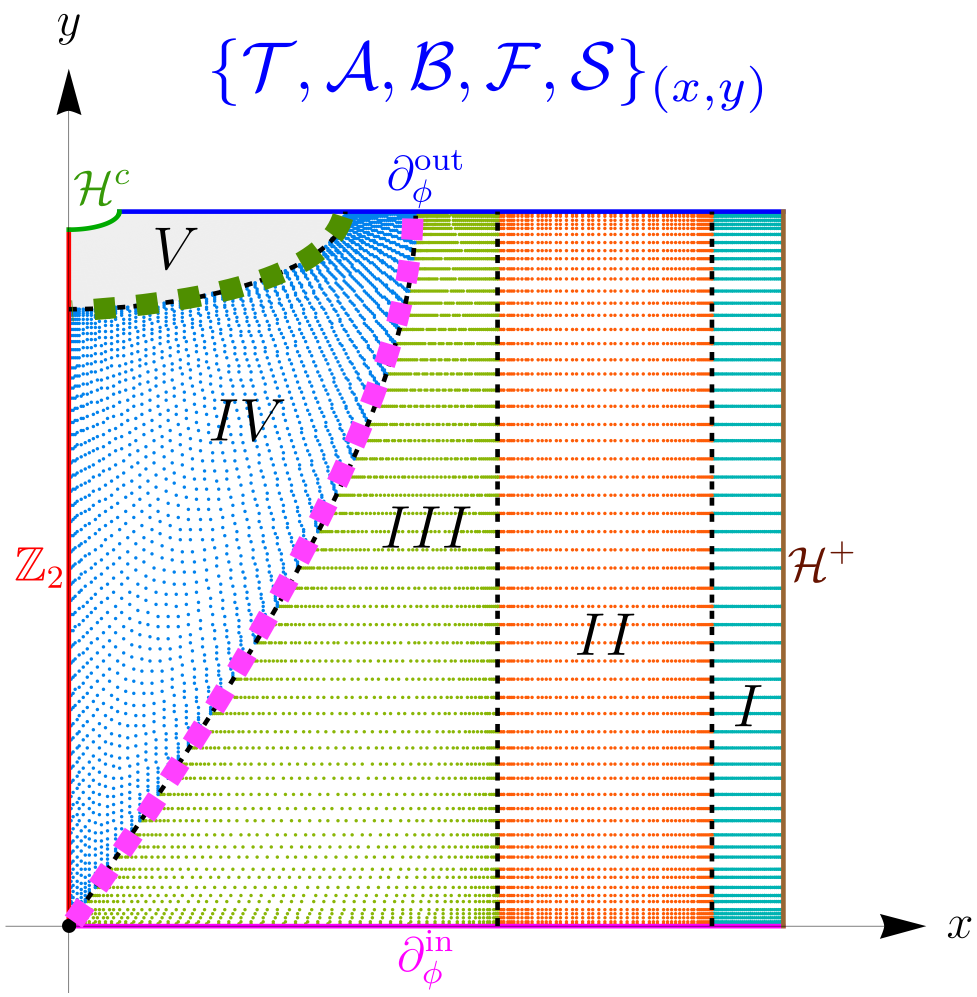
• **Numerical method:** [Review: OD, Santos,Way 1510.02804]

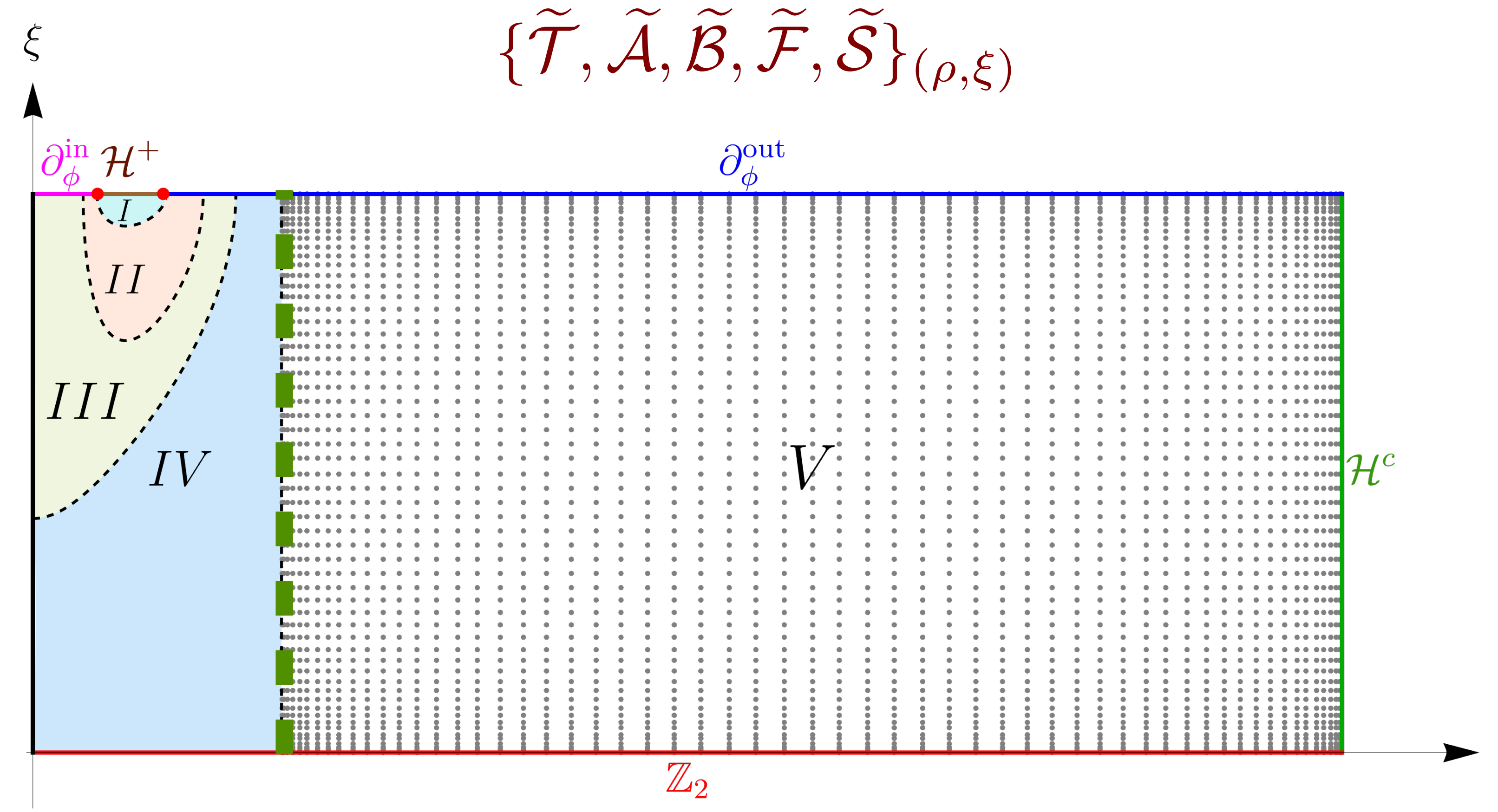
Use a **Newton-Raphson algorithm** with **pseudospectral grid**.

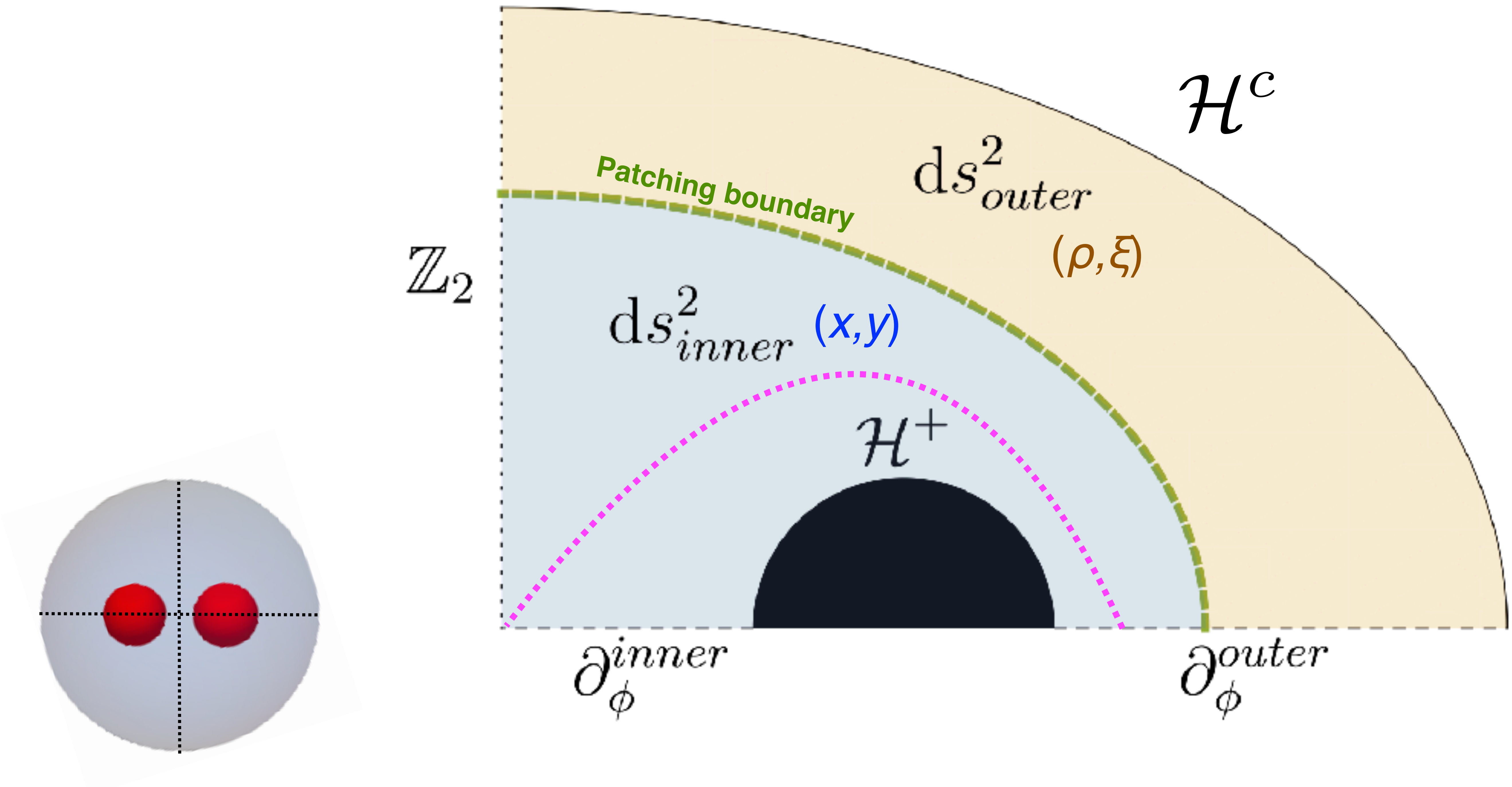
Also use **transfinite interpolation** to complete the **patching**.

@ patching boundary, require:

1) **matching** of **two line elements**, & **2)** **matching** of the **normal derivative** across patch bdry







- **Outer region:** near (single) cosmological horizon, solution looks like **de Sitter** space; **(ρ,ξ) coords**
- **Inner region:** solution looks like **warped Israel-Khan** but <u>without</u> conical singularity; **(x,y)** coords
- Inner region is **pentagonal** (5 boundaries) => so split it into 2 squared (4 boundaries) sub-regions