Accelerating cosmological evolution and dark energy from the osculating Barthel-Kropina geometry

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Overview

Finsler spaces

Finsler spaces (Finsler, 1918): metric spaces, with the distance ds between $x=(x^I)$ and $x+dx=(x^I+dx^I)$

$$d\hat{s}=F(x,dx).$$

Finsler metric function F: positively homogeneous of degree one function in dx, satisfying

$$F(x,\lambda dx)=\lambda F(x,dx),\mathrm{for}\lambda>0.$$

Canonical coordinates $(x,y)=(x^I,y^I)$ of the tangent bundle TM, where $y=y^I\dfrac{\partial}{\partial x^I}$, is a tangent vector at x.

Finsler metric tensor $\hat{g}_{IJ}$

$$\hat{g}_{IJ}(x,y)=\frac{1}{2}\frac{\partial^2F^2(x,y)}{\partial y^I\partial y^J}.$$

Metric in Finsler geometry:

$$d\hat{s}^2=\hat{g}_{IJ}(x,y)\,y^Iy^J$$

Finsler geometry

$$\hat{g}_{IJ}(x,y) = g_{IJ}(x),\ y^I = dx^I\text{-Riemann geometry}$$

$$ds^2 = g_{IJ}(x) dx^I dx^J$$

Cartan tensor $\hat{C}(x,y)$:

$$\hat{C}_{IJK} = \frac{1}{2}\frac{\partial \hat{g}_{IJ}(x,y)}{\partial y^K}.$$

Randers spaces (Randers, 1941):

$$F = \left[g_{IJ}(x) dx^I dx^J\right]^{1/2} + A_I(x) dx^I = \alpha + \beta,$$

Kropina spaces (Kropina, 1959):

$$F(x,y) = \frac{g_{IJ}(x) y^I y^J}{A_I(x) y^I} = \frac{\alpha^2}{\beta}.$$

(α, β) metrics (Matsumoto, 1972): $F(\alpha, \beta)$,
$\alpha(x,y) = \left[g_{IJ}(x) dx^I dx^J\right]^{1/2}$ and $\beta(x,y) = A_I(x) y^I$

General (α, β) metrics:

$$F(\alpha, \beta) = \alpha\phi\left(\beta/\alpha\right) = \alpha\phi\left(s\right), s = \beta/\alpha, \phi = \phi(s)$$

Metric tensor of the (α, β) metric

$$\hat{g}_{IJ}(x, y) = \frac{L_{\alpha}}{\alpha} h_{IJ} + \frac{L_{\alpha\alpha}}{\alpha^2} y_I y_J + \frac{L_{\alpha\beta}}{\alpha}\left(y_I A_J + y_J A_I\right) + L_{\beta\beta} A_I A_J,$$

where $L = F^2/2$, and

$$h_{IJ} = \alpha \frac{\partial^2 \alpha(x, y)}{\partial y^I \partial y^J} = g_{IJ} - \frac{y_I y_J}{\alpha^2}.$$

The Barthel connection

Let (M^n, F) be a Finsler space, defined on a base manifold M^n. On M^n a vector field $Y(x) \neq 0$ is also defined.
We define a particular mathematical structure $(M^n, F(x,y), Y(x))$, a Finsler space $(M^n, F(x,y))$ *with a tangent vector field* $Y(x)$.
If the vector Y *that does not vanish in any point on* M, $\hat{g}(x,y)$ generates the Y-Riemann metric

$$\hat{g}_Y(x) = \hat{g}(x, Y)$$

Let's assume that a point vector field $Y^I(x)$ and a Finsler metric tensor $\hat{g}(x,y)$ are given.
The absolute differential of the vector Y:

$$DY^I = dY^I + Y^K b^I_{KH}(x, Y) dx^H,$$

$b^I_{KH}(x, Y)$ are the coefficients of the Barthel connection (Barthel, 1953) are obtained with the help of the generalized Christoffel symbols $\hat{\gamma}_{JIH}$

The Barthel connection

$$\hat{\gamma}_{IJH} := \frac{1}{2}\left(\frac{\partial \hat{g}_{JI}}{\partial x^H} + \frac{\partial \hat{g}_{IH}}{\partial x^J} - \frac{\partial \hat{g}_{HJ}}{\partial x^I}\right).$$

$$b^I_{KH} = \hat{\gamma}^I_{KH} - \hat{\gamma}^R_{KS} Y^S \hat{C}^I_{RH}.$$

Interesting properties of the Barthel connection:

-It depends on the vector field on which it acts - very different to the connections in Riemann geometry

- The dependence is only on the direction of the vector field, and not on its magnitude
- Keeps the metric function unchanged by the parallel transport
- Permits a transition to the Cartan geometry of the Finsler spaces
- The Barthel connections do not live on the base manifold M, but on the total space of the tangent bundle
- Major differences between the geometrical Riemann and Finsler gravitational theories

The Y osculating Riemann geometry

The osculating approach (Nazim, 1936) associates to a complex geometric structure, like, for example, a Finsler geometry, and a Finsler connection, a simpler mathematical format, like a Riemann metric, or an affine or a linear connection. Thus the simpler, osculating structure, approximates, in some sense, the most complicated one.

Hence, one can obtain mathematical results that allow the understanding of the properties of the mathematically more complicated geometries.

Let's consider now a local section Y of $\pi_M : TM \to M$. By taking into account that $\hat{g}_{IJ} \circ Y$ is a function defined on U, we can introduce a new metric

$$\hat{g}_{IJ}(x) := \hat{g}_{IJ}(x, y)|_{y=Y(x)}, \quad x \in U. \tag{1}$$

The pair $(U, \hat{g}_{IJ})$ *correspond to a Riemannian manifold*, while $\hat{g}_{IJ}(x)$ represents the Y*-osculating Riemannian metric* corresponding to (M, F).

The Christoffel symbols of the first kind for the metric (1)

$$\hat{\gamma}_{IJK}(x) := \frac{1}{2}\left\{\frac{\partial}{\partial x^J}\left[\hat{g}_{IK}(x, Y(x))\right] + \frac{\partial}{\partial x^K}\left[\hat{g}_{IJ}(x, Y(x))\right] - \frac{\partial}{\partial x^I}\left[\hat{g}_{JK}(x, Y(x))\right]\right\}.$$

$$\hat{\gamma}_{IJK}(x) = \hat{\gamma}_{IJK}(x, y)\Big|_{y=Y(x)} + 2\left(\hat{C}_{IJL}\frac{\partial Y^L}{\partial x^K} + \hat{C}_{IKL}\frac{\partial Y^L}{\partial x^J} - \hat{C}_{JKL}\frac{\partial Y^L}{\partial x^I}\right)\Bigg|_{y=Y(x)}$$

If a non-vanishing global section Y of TM does exist, with the property $Y(x) \neq 0$, $\forall x \in M$, the osculating Riemannian manifold $(M, \hat{g}_{ij})$ can always be defined.

The case of the (α, β) metrics

For the (α, β) metrics we take $Y^I = A^I$, with $A^I = g^{IJ}A_J$.
We define the A osculating Riemannian manifold $(M, \hat{g}_{IJ})$, with

$$\hat{g}_{IJ}(x) := \hat{g}_{IJ}(x, A), \tilde{a}^2 = A_I A^I = \alpha^2 (x, A), Y_I (x, A) = A_I$$

$$\hat{g}_{IJ}(x) = \left.\frac{L_\alpha}{\tilde{a}}\right|_{y=A(x)} g_{IJ} + \left.\left(\frac{L_{\alpha\alpha}}{\tilde{a}^2} + 2\frac{L_{\alpha\beta}}{\tilde{a}} + L_{\beta\beta} - \frac{L_\alpha}{\tilde{a}^3}\right)\right|_{y=A(x)} A_I A_J.$$

$$\hat{C}_{IJK}(x, A) = 0.$$

$$\hat{\gamma}_{IJK}(x) = \hat{\gamma}_{IJK}(x, y)\big|_{y=A(x)}. \tag{2}$$

For *an (α, β)-metric, the Barthel connection is the Levi-Civita connection of the A -Riemannian metric*.

After evaluating $g_{ij}(x, y)$ of (M, F) at $(x, A(x))$, one obtains a Riemannian metric g_A on M, with its own Levi-Civita connection.

The Barthel connection $\left(b^A_{BC}(x)\right)$, *is an affine connection*.
Curvature tensor with local coefficients $\left(\Gamma^A_{BC}(x)\right)$:

$$R^A_{BCD} = \frac{\partial \Gamma^A_{BD}}{\partial x^C} - \frac{\partial \Gamma^A_{BC}}{\partial x^D} + \Gamma^E_{BD}\Gamma^A_{EC} - \Gamma^E_{BC}\Gamma^A_{ED}.$$

Kropina metric $F = \alpha^2/\beta$, Barthel connection = Levi-Civita connection of the osculating metric $\hat{g}_{AB}(x) = g_{AB}\left(x, A(x)\right)$, $A_I(x)$ - components of the one-form β, g_{AB} - the fundamental tensor of F.

$$b^A_{BC} = \hat{\gamma}^A_{BC}$$

$$\hat{R}^A_{BCD} = \frac{\partial \hat{\gamma}^A_{BD}}{\partial x^C} - \frac{\partial \hat{\gamma}^A_{BC}}{\partial x^D} + \hat{\gamma}^E_{BD}\hat{\gamma}^A_{EC} - \hat{\gamma}^E_{BC}\hat{\gamma}^A_{ED},$$

$$\hat{R}_{BD} = \sum_A \left[\frac{\partial \hat{\gamma}^A_{BD}}{\partial x^A} - \frac{\partial \hat{\gamma}^A_{BA}}{\partial x^D} + \sum_E \left(\hat{\gamma}^E_{BD}\hat{\gamma}^A_{EA} - \hat{\gamma}^E_{BA}\hat{\gamma}^A_{ED}\right)\right],$$

Ricci scalar $\hat{R} = \hat{R}^B_B$.

The Barthel-Kropina cosmological model (Hama, Harko and Sabau, EPJC 82, 385, 2022; EPJC 83 1030, 2023): based on a Finsler type (α, β) geometry, with

$$F = \frac{\alpha^2}{\beta} = \frac{g_{ij}(x)dx^i dx^j}{A_I y^I}, \quad \hat{g}_{ij}(x) = \hat{g}_{ij}(x, A)$$

α is a positive non-degenerate Riemann metric, β is an one form.

- α is the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = \left(dx^0\right)^2 - a^2\left(x^0\right)\left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right],$$

-Validity of the Cosmological Principle - $A_I = A_I\left(x^0\right)$

-The vector A has only one time-like independent component $A_0\left(x^0\right)$

$$A_I = \left(a\left(x^0\right)\eta\left(x^0\right), 0, 0, 0\right)$$

- Matter comoves with the cosmological expansion
- Non-vanishing components of the matter energy-momentum tensor $\hat{T}_{AB}$

$$\hat{T}_0^0 = \rho c^2, \, \hat{T}_{00} = \hat{g}_{00}\hat{T}_0^0, \, \hat{T}_k^k = -p, \, \hat{T}_{ii} = -\hat{g}_{ik}\hat{T}_i^k.$$

The Barthel-Kropina cosmological model

- The Einstein gravitational field equations are given in the Barthel-Kropina geometry by (Hama, Harko and Sabau, EPJC 82, 385, 2022; EPJC 83 1030, 2023)

$$\hat{R}_{BD} - \frac{1}{2}\hat{g}_{BD}\hat{R} = \kappa^2 \hat{T}_{BD}, \tag{3}$$

1. $(A_I) = \left(a\left(x^0\right)\eta\left(x^0\right), 0, 0, 0\right) = \left(A^I\right);$
2. $(g_{IJ}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(x^0) & 0 & 0 \\ 0 & 0 & -a^2(x^0) & 0 \\ 0 & 0 & 0 & -a^2(x^0) \end{pmatrix};$
3. $\alpha|_{y=A(x)} = a(x^0)\eta(x^0);$
4. $\beta|_{y=A(x)} = [a(x^0)\eta(x^0)]^2.$

The generalized Friedmann equations

$$\frac{3(\eta')^2}{\eta^2} = \frac{8\pi G}{c^2}\frac{1}{a^2\eta^2}\rho,$$

$$2\frac{\eta''}{\eta} + 2H\frac{\eta'}{\eta} - 3\frac{(\eta')^2}{\eta^2} = \frac{8\pi G}{c^4}\frac{p}{a^2\eta^2},$$

$$a\eta\frac{d}{dx^0}\left(\eta' a\right) = \frac{4\pi G}{c^4}\left(\rho c^2 + p\right).$$

$$\frac{4\pi G}{c^4}\left[\frac{d}{dx^0}\left(\rho c^2 a^3\right) + p\frac{d}{dx^0}a^3\right] = 3a^5\left[\frac{8\pi G}{2c^4}\left(\frac{5}{3}\rho c^2 + p\right)\frac{H}{a^2} + \eta'\eta''\right].$$

The Barthel-Kropina cosmological Friedmann equations reduce to the Friedmann equations of general relativity *in the limit* $\eta \to \pm 1/a$, $\beta = (1,0,0,0)$,

The generalized Friedmann equations

$$\eta\left(x^0\right)=\frac{1}{a\left(x^0\right)}\left[1+\psi\left(x^0\right)\right],$$

$$3\frac{\left(a'\right)^2}{a^2} = \frac{8\pi G}{c^2}\rho+6(1+\psi)\psi' H-3\left(\psi'\right)^2-3(2+\psi)\psi H^2=\frac{8\pi G}{c^2}\rho+\rho_{DE},$$

$$\frac{2a''}{a}+\frac{\left(a'\right)^2}{a^2} = -\frac{8\pi G}{c^4}\frac{p}{(1+\psi)^2}+4\frac{\psi'}{1+\psi}H-3\frac{\left(\psi'\right)^2}{(1+\psi)^2} +2\frac{\psi''}{1+\psi}=-\frac{8\pi G}{c^4}\frac{p}{(1+\psi)^2}+p_{DE},$$

$$\rho_{DE}=6(1+\psi)\psi' H-3\left(\psi'\right)^2-3(2+\psi)\psi H^2,$$

$$p_{DE}=4\frac{\psi'}{1+\psi}H-3\frac{\left(\psi'\right)^2}{(1+\psi)^2}+2\frac{\psi''}{1+\psi}$$

For $\psi\to 0$, and $\eta\to 1/a$, we reobtain the Friedmann equations of GR

Dark energy from the Barthel-Kropina cosmology

$$p_{DE} = \omega \rho_{DE}, \omega = \text{constant},$$

Redshift variable $1+z=1/a$

$$-(1+z)h\frac{d\psi}{dz} = \sigma,$$

$$-2(1+z)h\frac{dh}{dz} + 3h^2 = 4h\frac{\sigma}{1+\psi} - 3\frac{\sigma^2}{(1+\psi)^2} - 2(1+z)\frac{h}{1+\psi}\frac{d\sigma}{dz},$$

$$-2(1+z)\frac{h}{1+\psi}\frac{d\sigma}{dz} + 2h\left[\frac{2}{1+\psi} - 3\omega(1+\psi)\right]\sigma$$
$$+3\left[\omega - \frac{1}{(1+\psi)^2}\right]\sigma^2 + 3\omega\psi(2+\psi)h^2 = 0.$$

Matter density parameter

$$\Omega_m = h^2\left[1 + (1+z)^2\left(\frac{d\psi}{dz}\right)^2 + (2+\psi)\psi + 2(1+z)(1+\psi)\frac{d\psi}{dz}\right].$$

Datasets and methodology

We use the $H(z)$ measurements, and the Pantheon sample (Bouali et al., EPJC 83, 121, 2023).

Estimation of the model parameters ω, σ_0, ψ_0, h: chi-square function

$$\chi^2_H(\omega, \sigma_0, \psi_0, h) = \sum_{i=1}^{57} \frac{\left[H_{th}(z_i, \omega, \sigma_0, \psi_0, h) - H_{obs}(z_i)\right]^2}{\sigma^2_{H(z_i)}}$$

The Akaike's Information Criterion (AIC_c)

$$AIC_c = \chi^2_{min} + 2\mathcal{N}_{\rm f} + \frac{2\mathcal{N}_{\rm f}(\mathcal{N}_{\rm f}+1)}{\mathcal{N}_{\rm Tot} - \mathcal{N}_{\rm f} - 1},$$

$\mathcal{N}_{\rm f}$ and $\mathcal{N}_{\rm Tot}$ are the number of free parameters, and the total data points.

The model with the minimal AIC_c is taken as reference

$$\Delta AIC_c = AIC_{c,model} - AIC_{c,reference},$$

If $0 < \Delta \mathrm{AIC}_c < 2$, the model is substantially supported by the data

If $4 < \Delta \mathrm{AIC}_c < 7$, the model has less observational support

If $\Delta \mathrm{AIC}_c > 10$, the model is not supported by the data

Results

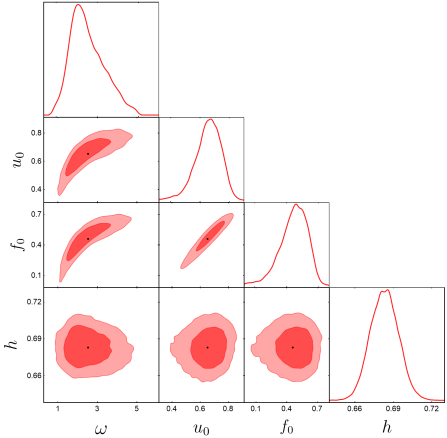

Figure: MCMC confidence contours at 1σ and 2σ, obtained after constraining the Barthel-Kropina dark energy model with SNIa+$\mathrm{H}(z)$ data.

Results

Model	Parameter	Prior	Best fit	Mean
ΛCDM	$\Omega_{\rm m}$	$[0.001, 1]$	$0.27859^{+0.0139588}_{-0.0139588}$	$0.279249^{+0.0139493}_{-0.0139493}$
	h	$[0.4, 1]$	$0.691892^{+0.008881}_{-0.0088881}$	$0.691857^{+0.0088838}_{-0.00888348}$
Barthel-Kropina	ω	$[0, 6]$	$2.02382^{+0.819655}_{-0.819655}$	$2.53314^{+0.817497}_{-0.817497}$
	σ_0	$[-3, 3]$	$0.619373^{+0.0939505}_{-0.0939505}$	$0.652573^{+0.0936745}_{-0.0936745}$
	ψ_0	$[-3, 3]$	$0.40882^{+0.141627}_{-0.141627}$	$0.458615^{+0.141176}_{-0.141176}$
	h	$[0.4, 1]$	$0.684579^{+0.0108078}_{-0.0108078}$	$0.683052^{+0.0108017}_{-0.0108017}$

Table: Summary of the best fit and of the mean values of the free cosmological parameters of the Barthel-Kropina dark energy model.

Results

Model	${\chi^2_{tot}}^{min}$	χ^2_{red}	AIC_c	ΔAIC_c
ΛCDM	1081.5479	0.978776	1085.56	0
Barthel-Kropina	1078.0028	0.975568	1086.04	0.48

Table: Summary of the ${\chi^2_{tot}}^{min}$, χ^2_{red} , AIC_c and ΔAIC_c.

Results

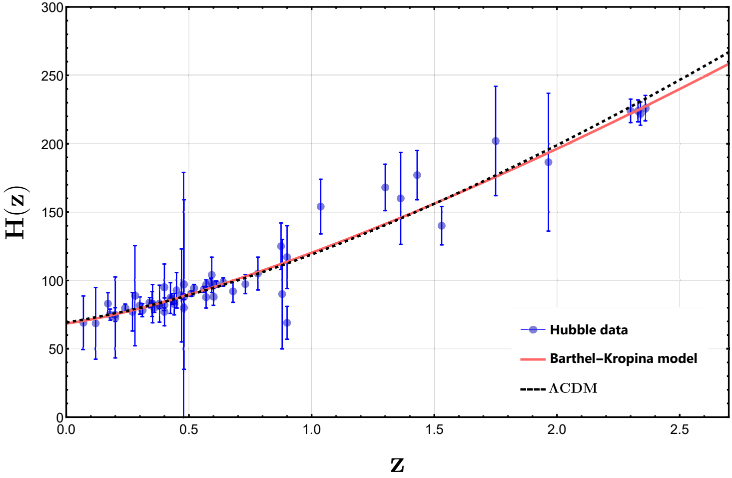

Figure: The evolution of the Hubble parameter $\mathrm{H}(\mathrm{z})$ of the Barthel-Kropina and ΛCDM models as a function of the redshift z against the Hubble measurements.

Results

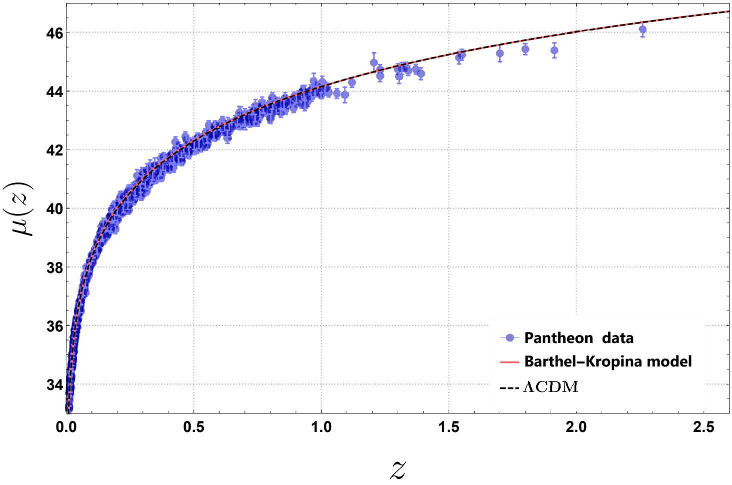

Figure: The evolution of the distance modulus $\mu(z)$ of the Barthel-Kropina dark energy model, and of the ΛCDM model in terms of the redshift z against the Pantheon data.

Results

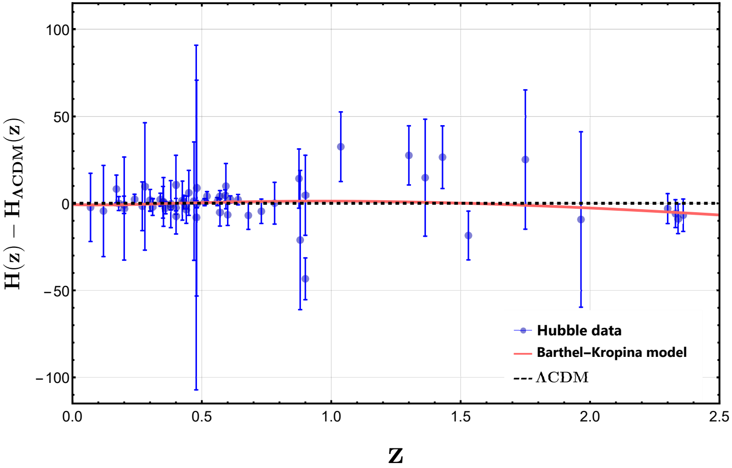

Figure: The variation of the difference between the Barthel-Kropina dark energy model, and the ΛCDM model as a function of the redshift z against the Hubble measurements.

Results

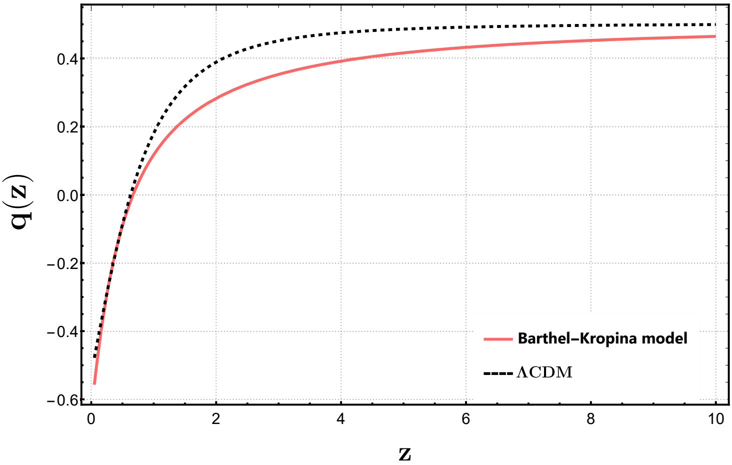

Figure: Evolution of the deceleration parameter as a function of the redshift z for the Barthel-Kropina and ΛCDM cosmologies.

Results

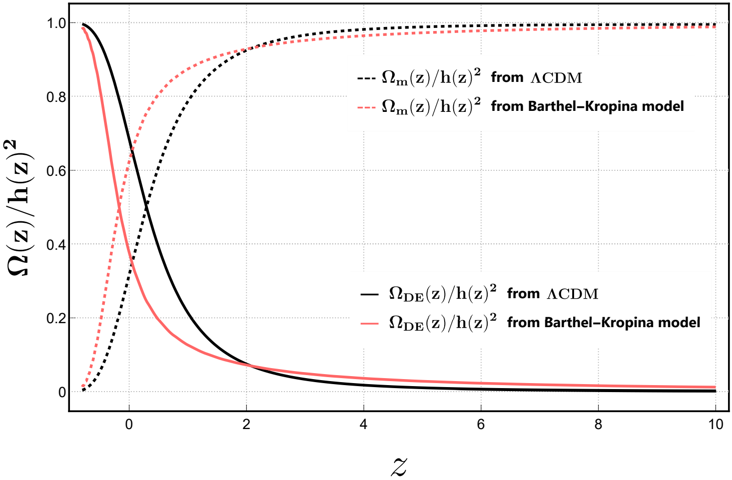

Figure: The reduced matter density parameters as a function of the redshift z in the Barthel-Kropina and ΛCDM cosmological models.

Results

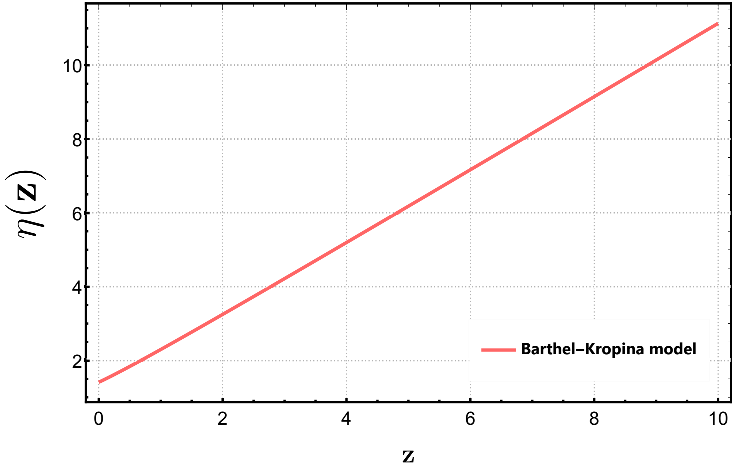

Figure: The evolution of the coefficient $\eta(z) = (1+z)(1+\psi(z))$ of the one form β of the Kropina metric.

Results

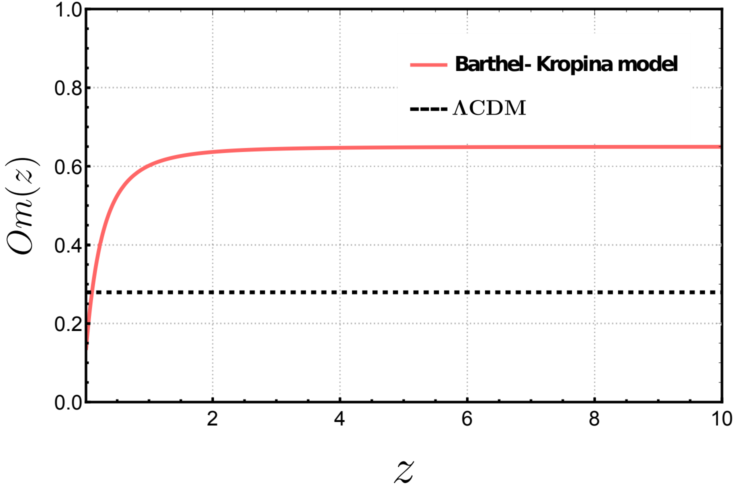

Figure: The evolution of the $Om(z)$ function in the Barthel-Kropina and ΛCDM models.

Conclusions

1. Finsler geometry represents an interesting extension of Riemann geometry, and it can open some new perspectives on gravitational theories
2. It allows a natural embedding of new degrees of freedom that significantly enlarge the physical space of the physical variables
3. Finsler geometry in its Barthel-Kropina version provides a natural explanation for the accelerated expansion of the Universe, and it gives an excellent description of the observational data

Conclusions

1. However, more tests of the Barthel-Kropina geometric gravity models are necessary to confirm/infirm the basic theory
2. A further interesting field of investigation is related to the black hole solutions in Barthel-Kropina geometry
3. There are a large number of astrophysical effects that allow a detailed testing of the validity of the black hole solutions
4. The tests are local (Solar System) level, or involve high energy astrophysical processes

Conclusions

1. From the comparison with the observational data one can obtain some strong constraints on the model parameters
2. The astrophysical tests impose some important restrictions on the free parameters of the Finsler type geometric theories of gravity
3. These restrictions must be combined with the cosmological predictions to give a more detailed picture of the full potential of the theory