Universes without Time and Consequences

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Gödel's universe best description

- It exists a first Gödel's metric on a set M which satisfies EFE:

$$ds^2 = (dx^0)^2 - (dx^1)^2 + \frac{e^{2x^1}}{2}(dx^2)^2 - (dx^3)^2 + 2e^{x^1}dx^0dx^2;$$

- Gödel's transformation of coordinates $\bar{M} \to M$

$$\begin{cases} x^0 = 2t - \phi\sqrt{2} + 2\sqrt{2}\arctan\left(\tan\left(\frac{\phi}{2}\right)e^{-2r}\right), \phi \neq \pi;\ x^0 = 2t \ if \ \phi = \pi \\ x^1 = \ln\left[\cosh(2r) + \cos\phi\sinh(2r)\right] \\ x^2 = \dfrac{\sqrt{2}\sin\phi\sinh(2r)}{\cosh(2r) + \cos\phi\sinh(2r)} \\ x^3 = 2y. \end{cases}$$

- Gödel's second metric on $\bar{M}$,

$$ds^2 = 4\left[dt^2 - dr^2 - dy^2 + (\sinh^4 r - \sinh^2 r)d\phi^2 + 2\sqrt{2}\sinh^2 r d\phi dt\right].$$

The new metric on $\bar{M}$ allows the orientation in time for vectors, highlighting time-like future oriented loops and closed future oriented time-like chain of curves on M.

Our Universe without Time in $f(R) = R$ and $f(R) = R^2$ Gravity

- First metric

$$ds^2 = e^{x^3}(dx^0)^2 + (dx^1)^2 + (dx^2)^2 - (dx^3)^2$$

which satisfies EFE on $M = \mathbf{R}^4$.

- The change of coordinates $(t, r, \phi, y) \to (x^0, x^1, x^2, x^3)$:

$$F : \begin{cases} x^0 = t, \ \ t \in \mathbf{R}, \\ x^1 = r \sin\phi, \ \ r > 0, \ \ \phi \in \mathbf{R}, \\ x^2 = r\cos\phi, \\ x^3 = y, \ \ y \in \mathbf{R}. \end{cases}$$

- Second metric

$$d\bar{s}^2 = e^y dt^2 + dr^2 + r^2 d\phi^2 - dy^2$$

on the set $\bar{M}$.

Some details about the first metric in f(R)=R gravity

- $M = \mathbf{R}^4$ and

$$ds^2 = e^{x^3}(dx^0)^2 + (dx^1)^2 + (dx^2)^2 - (dx^3)^2$$

therefore the only nonzero Ricci tensor components are

$$R_{00} = \frac{1}{4}e^{x^3};\ \ R_{33} = -\frac{1}{4},$$

that is

$$R = \frac{1}{2}.$$

The exotic matter in f(R)=R gravity

Replacing, the Einstein's field equations are

$$R_{ij} - \frac{1}{2} R \ g_{ij} = 8\pi G \ T_{ij}$$

where

$$T_{ij} = \frac{1}{8\pi G} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Some details about the first metric in $f(R) = R^2$ gravity

In modified gravity, when $f(R) = R^2$, the Einstein field equations should have the form

$$f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij} + \Lambda g_{ij} = 8\pi G T_{ij}.$$

After computations, we obtain for the cosmological constant $\Lambda = \dfrac{1}{4}$ the equality

$$2RR_{ij} - \frac{1}{2}R^2 g_{ij} + \frac{1}{4}g_{ij} = 8\pi G T_{ij},$$

where T_{ij} has the unexpected same form as in the case of the usual Einstein field equations, that is

$$T_{ij} = \frac{1}{8\pi G}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

I. Organizing our Universe without Time - establishing the time and space coordinates

• For the coordinates x^1 and x^2 when $r > 0;\ \phi \in \mathbf{R}$ it can be seen a 2π periodicity of x^1 and x^2 when r is fixed.
x^1 and x^2 become the cylindrical coordinates for the initial set $M(x^0, x^1, x^2, x^3)$.

• The points (t, r, ϕ, y) and $(t, r, \phi + 2k\pi, y),\ k \in \mathbf{Z}$ in $\bar{M}$ describe a same point (x^0, x^1, x^2, x^3) in M.

• The coordinates (r, ϕ, y) in $\bar{M}$ determine completely the coordinates (x^1, x^2, x^3) in M, therefore the coordinates x^0 and t play the same role, i.e. $t-$lines of matter in $\bar{M}$ are x^0- lines of matter in M.

$$ds^2 = e^{x^3}(dx^0)^2 + ((dx^1)^2 + (dx^2)^2 - (dx^3)^2)$$

$$d\bar{s}^2 = e^y dt^2 + (dr^2 + r^2 d\phi^2 - dy^2)$$

II. Organizing our Universe without Time: establishing the orientation in time

- Denote the second metric coefficients by $\bar{g}_{ij} = \bar{g}_{ij}(t, r, \phi, y)$.
- $v = (v^0, v^1, v^2, v^3)$ is a time-like vector if $\bar{g}_{ij} v^i v^j > 0$.
- The vector $e = (e^0, e^1, e^2, e^3) = (1, 1, 1, 0)$ has the property

$$\bar{g}_{ij} e^i e^j = e^y + 1 + r^2 > 0,$$

that is e is a time-like vector.

- For a time-like vector v we say that v is future pointing if $\bar{g}_{ij} e^i v^j > 0$.
- If $\bar{g}_{ij} e^i v^j < 0$ the vector v is called past pointing.

Let us observe that if v is time-like and future pointing, then $-v$ is still time-like but past pointing.
It is easy to see that $-e$ is past pointing because

$$\bar{g}_{ij} e^i (-e^j) = -e^y - 1 - r^2 < 0.$$

This way $\bar{M}$ becomes time orientable.

Theorem 1: *M allows time-like future oriented loops.*
Proof. Consider the curve $\alpha(s) := (0, R, s, 0)$ of $\bar{M}$.
Its velocity vector is $\dot{\alpha}(s) = (v^0, v^1, v^2, v^3) = (0, 0, 1, 0)$. We have

$$d\bar{s}^2(\dot{\alpha}(s), \dot{\alpha}(s)) = \bar{g}_{ij} v^i v^j = \bar{g}_{22}(0, R, s, 0)(v^2)^2 = R^2 \cdot 1 > 0,$$

i.e. this vector is a time-like one. More,

$$d\bar{s}^2(\dot{\alpha}(s), e) = \bar{g}_{ij} v^i e^j = \bar{g}_{22}(0, R, s, 0) v^2 e^2 = R^2 \cdot 1 \cdot 1 > 0,$$

that is the vector $\dot{\alpha}(s)$ is future pointing.
• Subsequently, the image in M of the α-curve from $\bar{M}$ *will be considered a future oriented time-like curve of M.*
In M the values at 0 and 2π shows a same point, therefore this curve starting from a point of M is returning at the same point of M.
We have obtained a time-like future oriented loop of M.

I. The Existence of Closed Future Oriented Time-Like Chain of Curves in M

Obseve that the points $\bar{E}_1(t_1, R, 0, 0)$ and $\bar{E}_2(t_2, R, 0, 0)$ from $\bar{M}$ induce in M the points $E_1(t_1, R, 0, 0)$ and $E_2(t_2, R, 0, 0)$.

Theorem 2: i) If $R^2 > \frac{1}{2\pi}|t_2 - t_1|$, the two points E_1 and E_2 of M can be joined by a time-like future oriented curve in M.
ii) M allows time-like future oriented closed chain of curves.

Proof: i) The curve

$$\gamma(s) = \left(t_1 + \frac{t_2 - t_1}{2\pi}s, R, s, 0\right) \subset \bar{M}$$

is time-like because the vector
$\dot{\gamma}(s) = (w^0, w^1, w^2, w^3) = \left(\frac{t_2 - t_1}{2\pi}, 0, 1, 0\right)$ has the property

$$d\bar{s}^2(\dot{\gamma}(s), \dot{\gamma}(s)) = \bar{g}_{ij} w^i w^j = \bar{g}_{00}(w^0)^2 + \bar{g}_{22}(w^2)^2 = e^0 \cdot \left(\frac{t_2 - t_1}{2\pi}\right)^2 + R^2 \cdot 1 > 0$$

II. The Existence of Closed Future Oriented Time-Like Chain of Curves in M

More,

$$d\bar{s}^2(\dot{\gamma}(s), e) = \bar{g}_{ij} w^i e^j = \bar{g}_{00} w^0 e^0 + \bar{g}_{22} w^2 e^2 = e^0 \cdot \frac{t_2 - t_1}{2\pi} \cdot 1 + R^2 \cdot 1 \cdot 1$$

which is positive according to the statement, that is $\dot{\gamma}(s)$ is pointing the future.

Subsequently, we have in M a time-like future oriented curve connecting $(t_1, R, 0, 0)$ by $(t_2, R, 0, 0)$.

Therefore E_1 and E_2 in M, are connected by a time-like future oriented curve which tell us precisely that the event E_2 occurs after the event E_1.

III. The Existence of Future Oriented Time-Like Closed Chain of Curves in M

ii) Using the previous idea we can create same type time-like future oriented curves between E_2 and E_3 and between E_3 and E_1.
The concatenation of the three time-like future oriented curves is a the time-like future oriented closed "chain of curves".

Taking into consideration how events occur in an order related to the future pointing time-like tangent vectors of the curves, we can conclude that neither t or x^0 can be proper time coordinates, because if it is so, moving forward in time we return in our past.

Therefore no global time-coordinate exists in the universe we presented.

The Existence of a Wormhole Solution related to our Universe without Time

Consider the metric

$$d\bar{s}^2 = e^y dt^2 + dr^2 + (r^2 + a^2)d\phi^2 - dy^2$$

suggested by the previous second metric of our universe without time.
The metric above describes a wormhole solution - Ellis type.
When the parameter a approaches 0, i.e. $a \to 0$, this metric becomes exactly the second metric studied before, i.e.

$$d\bar{s}^2 = e^y dt^2 + dr^2 + r^2 d\phi^2 - dy^2.$$

The wormhole solution creates our universe without time when the wormhole disappears.

I. Some details about the wormhole solution in f(R)=R gravity

$$\Gamma^r_{\phi\phi} = -r;\ \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{r}{r^2+a^2};\ \Gamma^t_{yt} = \Gamma^t_{ty} = \frac{1}{2};\ \Gamma^y_{tt} = \frac{1}{2}e^y.$$

It results

$$R_{tt} = \frac{1}{4}e^y;\ R_{rr} = \frac{-a^2}{(r^2+a^2)^2},\ R_{\phi\phi} = \frac{-a^2}{r^2+a^2},\ R_{yy} = \frac{-1}{4}$$

and

$$R = \frac{1}{2} - \frac{2a^2}{(r^2+a^2)^2}.$$

Therefore, the Einstein field equations

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = 8\pi G T_{ij}$$

are satisfied for the cosmological constant $\Lambda = \frac{1}{4}$ and "the matter" induced by the matrix

II. Some details about the wormhole solution in f(R)=R gravity

$$A(r) = \frac{1}{8\pi G} \begin{pmatrix} \Psi(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Psi(r) \end{pmatrix} \text{ where } \Psi(r) = \frac{1}{4} + \frac{a^2}{(r^2 + a^2)^2},$$

in the form $T_{ij} = (a_{ij}) \cdot A(r)$.
Let us observe the geometric description of the Ψ function,

$$\Psi(r) = \frac{1}{2}(1 - R).$$

This shows that the stress-energy tensor has a geometric matter depending by the Ricci scalar of curvature R.

III. Some details about the wormhole solution in $f(R) = R^2$ gravity

In this case the computations lead to $\Lambda = 0$ and

$$2RR_{ij} - \frac{1}{2}R^2 g_{ij} = 8\pi G T_{ij},$$

where "the matter" is induced by the matrix

$$B(r) = \frac{R(1-R)}{16\pi G} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

in the form

$$T_{ij} = (a_{ij}) \cdot B(r).$$

Indicating the traversable wormhole

The wormhole metric is obtained for $t = 0$ and $y = 0$, i.e.

$$ds^2 = dr^2 + (r^2 + a^2)d\phi^2$$

which is in fact the metric of a catenoid. Explanations below:

$$\begin{cases} x = a\cosh\dfrac{x^1}{a}\cos x^2 \\ y = a\cosh\dfrac{x^1}{a}\sin x^2 \\ z = x^1 \end{cases}$$

and its corresponding metric is

$$ds^2 = \cosh^2\frac{x^1}{a}(dx^1)^2 + a^2\cosh^2\frac{x^1}{a}(dx^2)^2.$$

$$\begin{cases} r = a\sinh\dfrac{x^1}{a} \\ \phi = x^2 \end{cases}$$

The Existence of an Expanding Universe without Time

The first metric

$$ds^2 = e^{x^3}(dx^0)^2 + e^{x^0}[(dx^1)^2 + (dx^2)^2 - (dx^3)^2]$$

satisfies Einstein's field equations on $M = \mathbf{R}^4$.
The change of coordinates $(t, r, \phi, y) \to (x^0, x^1, x^2, x^3)$:

$$F : \begin{cases} x^0 = t, \ \ t \in \mathbf{R}, \\ x^1 = r \sin\phi, \ \ r > 0, \ \ \phi \in \mathbf{R}, \\ x^2 = r\cos\phi, \\ x^3 = y, \ \ y \in \mathbf{R} \end{cases}$$

transforms the initial metric into the metric

$$d\bar{s}^2 = e^y dt^2 + e^t[dr^2 + r^2 d\phi^2 - dy^2] := \bar{g}_{ij} d\bar{u}^i \bar{u}^j$$

on the set $\bar{M}$ described by the new coordinates (t, r, ϕ, y).
We can obtain similar results related to time-like future pointing loops and closed curves.

Some details related to the first metric

$$R_{00} = \frac{1}{4}e^{x^3 - x^0} - \frac{3}{4};\ R_{11} = -\frac{3}{4}e^{x^0 - x^3} = R_{22};\ R_{33} = -\frac{1}{4} + \frac{3}{4}e^{x^0 - x^3}.$$

$$R = \frac{1}{2}e^{-x^0} - 3e^{-x^3}.$$

Replacing, the Einstein's field equations

$$R_{ij} - \frac{1}{2}R\ g_{ij} = 8\pi G\ T_{ij}$$

are satisfied for the exotic matter represented by the tensor

$$T_{ij} = \frac{1}{32\pi G}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3e^{x^0 - x^3} - 1 & 0 & 0 \\ 0 & 0 & 3e^{x^0 - x^3} - 1 & 0 \\ 0 & 0 & 0 & -3e^{x^0 - x^3} \end{pmatrix}.$$

Properties of our Expanding Universe without Time

• The orientation in time is done in the same manner as in our previous example.

• The existence of the time-like future oriented loops in M is done same way, using the time-like future oriented curve $\alpha(s) = (0, R, s, 0) \subset \bar{M}$

• The existence of closed time-like future oriented chain of curves in M starts from the same points $\bar{E}_1$ and $\bar{E}_2$ in $\bar{M}$ having the coordinates $(t_1, R, 0, 0)$ and $(t_2, R, 0, 0)$ respectively and the time-like future oriented curve

$$\gamma(s) = \left(t_1 + \frac{t_2 - t_1}{2\pi} s, R, s, 0 \right) \subset \bar{M}.$$

• ..... same type computation

• ... same conclusions...

The Mathematical Structure of the Massless Scalar Field of our Expanding Universe without Time

In the general case the massless scalar field (MSF) is described by a function $u(t,x,y,z)$ satisfying the massless Klein-Gordon PDE

$$e^z u_{tt} + e^t (u_{xx} + u_{yy} - u_{zz}) = 0.$$

To solve it, we propose the solution in the form

$$u(t,x,y,z) = a(t)b(x)c(y)d(z).$$

When we separate the variables we obtain constant left and right members. In fact there are three constants k, k_1 and k_2 which appear.

How we separate the variables? Obtaining k and a(t)

We can arrange the equation in the form

$$\frac{a''(t)}{e^t a(t)} = \frac{-b''(x)c(y)d(z) - b(x)c''(y)d(z) + b(x)c(y)d''(z)}{e^z b(x)c(y)d(z)}.$$

The first member depends on t and the second member depends on x, y, z, therefore both member are constant. The constant can be choose k. Therefore we obtain two equations; the first one

$$a''(t) - ke^t a(t) = 0$$

can be seen as a Bessel type ODE with the solution

$$a(t) = \phi_0 \sum_0^\infty \frac{k^n}{(n!)^2} e^{nt}, \ k \neq 0.$$

If $k = 0$ then $a(t) = A_1 t + A_2$.

How we separate the variables? Obtaining k_1

The second one is deduced from

$$ke^z b(x)c(y)d(z) + b(x)c''(y)d(z) - b(x)c(y)d''(z) = -b''(x)c(y)d(z)$$

in the form

$$-\frac{b''(x)}{b(x)} = \frac{ke^z c(y)d(z) + c''(y)d(z) - c(y)d''(z)}{c(y)d(z)} = k_1.$$

Who is b_{k_1}?

It results
$$-\frac{b''(x)}{b(x)} = k_1,\ \ k_1 \in \mathbf{R}.$$
If $k_1 = 0$ the solution is $b_0 = B_1 x + B_2$.
If $k_1 \neq 0$ the solution of this ODE depends on the the sign of k_1.
If $k_1 = -l^2$ the equation to solve is $b''(x) - l^2 b(x) = 0$, while if $k_1 = l^2$ the equation is $b''(x) + l^2 b(x) = 0$.
In the first case the solution is $b_-(x) = Ae^{lx} + Be^{-lx}$, while in the second case the solution is $b_+(x) = A \sin lx + B \cos lx$.

How we separate the variables? Obtaining k_2 and $c_{k_2}(y)$

If we continue when $k \neq 0$ we have

$$k_1 c(y)d(z) = ke^z c(y)d(z) + c''(y)d(z) - c(y)d''(z)$$

which leads to

$$k_1 - \frac{c''(y)}{c(y)} = ke^z - \frac{d''(z)}{d(z)}.$$

Both members are constant, therefore we have to study the following two equations

$$\frac{c''(y)}{c(y)} = k_2$$

and

$$ke^z - \frac{d''(z)}{d(z)} = k_1 - k_2.$$

The first differential equation is studied exactly as we did it for the function $b(x)$. According to the constant k_2 which can be 0, negative or positive, we have the solutions denoted by $c_0(y)$, $c_-(y)$ and $c_+(y)$.

How we separate the variables? Obtaining α and $d_\alpha(z)$

We continue when $k \neq 0$. The equation

$$\frac{d''(z)}{d(z)} = \alpha + ke^z, \ \alpha := k_2 - k_1$$

is transformed into the Bessel type ODE

$$v^2\phi''(v) + v\phi'(v) - 4(kv^2 + \alpha)\phi(v) = 0.$$

It is shown that this equation allows non-zero solutions if and only if $\alpha \in \left\{0, \frac{1}{4}\right\}$. If $\alpha = 0$ then

$$d_{\alpha=0}(z) = D\sum_{0}^{\infty} \frac{k^n}{(n!)^2} e^{nz}.$$

If $\alpha = \frac{1}{4}$ then

$$d_{\alpha=1/4}(z) = E\sum_{0}^{\infty} \frac{k^n}{(n+1)(n!)^2} e^{(n+1/2)z}.$$

The Massless Scalar Field solution

If $k = 0$, $d''(z) = \alpha d(z)$ has a solution of type d_α as above. For $\alpha = 0$ the $d_0(z) = D_1 z + D_2$.
We can conclude: If $k = 0$ the solution is

$$u(t,x,y,z) = (A_1 t + A_2) b_{k_1}(x) c_{k_2}(y) d_\alpha(z).$$

If $k \neq 0$ the solution according to α is

$$u_{\alpha=0}(t,x,y,z) = \phi_0 D \sum_0^\infty \frac{k^n}{(n!)^2} e^{nt} b_{k_1}(x) c_{k_2}(y) \sum_0^\infty \frac{k^m}{(m!)^2} e^{mz}$$

$$u_{\alpha=1/4}(t,x,y,z) = \phi_0 E \sum_0^\infty \frac{k^n}{(n!)^2} e^{nt} b_{k_1}(x) c_{k_2}(y) \sum_0^\infty \frac{k^m}{(m+1)(m!)^2} e^{(m+1/2)z}$$

The Massless Scalar Field exists and it is nonzero if and only if $\alpha = 0$

i) Now consider $x = y = 0$. If $k = 0$, the vacuum $u_0(t, z)$ corresponding to the PDE

$$e^z u_{tt} - e^t u_{zz} = 0$$

is described by the product of two first degree polynomials, i.e.

$$u_0(t, z) = (C_1 t + C_2)(C_3 z + C_4), \ \ C_m \in \mathbf{R}.$$

ii) If $k \neq 0$ the vacuum satisfying the previous PDE, now denoted by $u_k(t, z)$, has the form

$$u_k(t, z) = C \sum_{n,m=0}^{\infty} \frac{k^{n+m}}{(n!)^2 (m!)^2} e^{nt+mz}.$$

$u_0(x, z)$ and $u_k(x, z)$ have to be the particular solutions of the general solution $u(t, x, y, z)$ obtained for $x = y = 0$.

It results that the vacuum exists if and only if $\alpha = 0$.