Teleparallel Geometries and Degrees of Freedom

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Gravity is very successfully described by the General Relativity theory of Albert Einstein. It is one of the best and most beautiful theories we have. Still, we are stubbornly trying to modify it.

There are mysteries in cosmology. What are the Dark Sectors? Was there inflation, and if yes then how? And if the problems such as H_0 tension are real, what are we making out of that?

On top of that, there are singularities, inherent and unavoidable. They are mostly hidden whenever one can imagine. But don't we want to have a better understanding of what is going on?

And let alone the puzzle of quantum gravity, together with our pathological belief in the mathematically horrendous quantum field theory approach.

And the amazing news we get is that it is extremely difficult to meaningfully modify the theory of General Relativity.

Simple models such as $f(R)$ are almost nothing new, and can be reformulated as an extra universal force mediated by a scalar field on top of the usual gravity. Deeper attempts at modifying it require exquisite care to not encounter with ghosts, or other bad instabilities, or total lack of well-posedness, or no reasonable cosmology available, or.... you name it!

And having the miserable lack of an undoubtful success, it makes all the good sense to try whatever crazy new geometry one can think of. And let it lead us to a better understanding.

On top of the usual curvature, one can consider two other geometric quantities related to the spacetime connection:

$$\text{torsion} \qquad T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$$

and

$$\text{nonmetricity} \qquad Q_{\alpha\mu\nu} = \bigtriangledown_{\alpha} g_{\mu\nu}.$$

Then it is easy to see that

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_{\mu} g_{\nu\beta} + \partial_{\nu} g_{\mu\beta} - \partial_{\beta} g_{\mu\nu} \right) \\ + \frac{1}{2} \left(T^{\alpha}{}_{\mu\nu} + T_{\nu}{}^{\alpha}{}_{\mu} + T_{\mu}{}^{\alpha}{}_{\nu} \right) - \frac{1}{2} \left(Q_{\mu\nu}{}^{\alpha} + Q_{\nu\mu}{}^{\alpha} - Q^{\alpha}{}_{\mu\nu} \right).$$

One possible alternative approach is to describe gravity in terms of different geometry.

Metric teleparallel (torsion)

In the orthonormal-tetrad-based description of gravity, one can naturally have torsionful connections without curvature or non-metricity by

$$\Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{A}\partial_{\mu}e^{A}_{\nu}.$$

Note the zero spin connection! (pure tetrad approach)
At least locally, every connection of this sort can be written like this, for some particular tetrad.

If we go beyond TEGR, or just reproducing GR, this framework is about more than just a metric. In general, different tetrads for the same metric are physically different objects.

I do not agree with the common opinion that it is necessary to have a locally Lorentz covariant description of teleparallel gravity, nor with another frequent opinion that such a description is severely problematic.

The zero-spin-connection tetrad has a clear geometric meaning: it is a covariantly constant basis of vector fields.

I would like to stress that, due to its very geometric meaning,

the teleparallel connection should not be invariant under local transformations of its defining tetrad.

However, there is no problem of rewriting the whole story in terms of some another tetrad. Moreover, it can sometimes be very convenient to do so.

To fix the notations, recall that the quest for TEGR action can start from observing that a metric-compatible connection $\Gamma^{\alpha}_{\mu\nu}$ with torsion differs from the Levi-Civita one $\overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}$ by a contortion tensor:

$$\Gamma^{\alpha}_{\mu\nu} = \overset{(0)}{\Gamma}{}^{\alpha}_{\mu\nu}(g) + K^{\alpha}{}_{\mu\nu}$$

which is defined in terms of the torsion tensor $T^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ as

$$K_{\alpha\mu\nu} = \frac{1}{2}\left(T_{\alpha\mu\nu} + T_{\nu\alpha\mu} + T_{\mu\alpha\nu}\right).$$

It is antisymmetric in the lateral indices because I ascribe the left lower index of a connection coefficient to the derivative, e.g. $\bigtriangledown_{\mu} T^{\nu} \equiv \partial_{\mu} T^{\nu} + \Gamma^{\nu}_{\mu\alpha} T^{\alpha}$.

The curvature tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\nu\beta} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}$$

for the two different connections obviously has a quadratic in K expression in the difference. Then making necessary contractions, such as $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$, we can come to

$$0 = R = \overset{(0)}{R} + \mathbb{T} + 2 \overset{(0)}{\bigtriangledown}_\mu T^\mu .$$

Here $T_\mu \equiv T^{\alpha}{}_{\mu\alpha}$ is the torsion vector while the torsion scalar

$$\mathbb{T} \equiv \frac{1}{2} S_{\alpha\mu\nu} T^{\alpha\mu\nu}$$

is given in terms of the superpotential

$$S_{\alpha\mu\nu} \equiv K_{\mu\alpha\nu} + g_{\alpha\mu} T_\nu - g_{\alpha\nu} T_\mu .$$

Due to the basic relation above, the Einstein-Hilbert action $-\int d^4x\sqrt{-g}\overset{(0)}{R}$ is equivalent to the TEGR one, $\int d^4x\|e\|\mathbb{T}$. They are the same, up to the surface term $\mathbb{B} \equiv 2\overset{(0)}{\bigtriangledown}_\mu T^\mu$.

Of course, this equivalence disappears when we go to modified gravity, for example the $f(\mathbb{T})$ gravity:

$$S = \int f(\mathbb{T}) \cdot \|e\| d^4x.$$

Actually, the work of varying this action can be simplified a lot by using this observation.

But many problems await us!
The pesky strong coupling issues...

After some little exercise, the equation of motion can be written as

$$f' \overset{(0)}{G}_{\mu\nu} + \frac{1}{2}\left(f - f'\mathbb{T}\right) g_{\mu\nu} + f'' S_{\mu\nu\alpha}\partial^{\alpha}\mathbb{T} = \kappa \mathcal{T}_{\mu\nu}$$

with $\mathcal{T}_{\mu\nu}$ being the energy-momentum tensor of the matter.
This is a very convenient form of equations!

If $f'' \neq 0$, then the antisymmetric part of the equations takes the form of

$$(S_{\mu\nu\alpha} - S_{\nu\mu\alpha})\partial^{\alpha}\mathbb{T} = 0.$$

It can be thought of as related to Lorentzian degrees of freedom.

And we see that solutions with constant $\mathbb{T}$ are very special and do not go beyond the usual GR, unless we are to study perturbations around them.

The number of degrees of freedom is not very well known. And the main reason is a variable rank of the algebra of Poisson brackets of constraints.
But, what is for sure, is that there must be at least one extra mode.

Still, the trivial Minkowski $e^A_\mu = \delta^A_\mu$ is obviously in a strong coupling regime for a model with $f(0) = 0$ in vacuum. Indeed, then $\mathbb{T} \propto (\partial\delta e)^2$, and for the quadratic action we just take $f(\mathbb{T}) = f_0 + f_1\mathbb{T} + \mathcal{O}(\mathbb{T}^2)$ which means accidental restoration of the full Lorentz symmetry, and linearised GR.
Therefore, all the standard properties of gravitational waves are there. This absence of contradiction to experiments is highly problematic.

Moreover, the strong coupling issue is there also for the standard cosmology.

One can look for generalisations. For example, a model of $f(\mathbb{T}, \mathbb{B})$ type. Those go beyond one of the main initial motivations for $f(\mathbb{T})$ gravity, for they produce 4-th order equations of motion.

It is unclear whether they can avoid the Ostrogradski-type ghosts, unless in the case of $f(\overset{(0)}{R})$. However, what is clear is that they inherit all the troubles of $f(\mathbb{T})$ gravity. Indeed, they obviously can be rewritten as $f(\mathbb{T}, \overset{(0)}{R})$, with all the issues of rather chaotic remnant symmetries.

All in all, these $f(\mathbb{T})$-type models are very problematic because they break the local Lorentz symmetry not strongly enough.

Symmetric teleparallel (nonmetricity)

Then yet another option is non-metricity.
But can it do much better?

If we have only non-metricity and no torsion, then

$$\Gamma^{\alpha}_{\mu\nu} = \mathring{\Gamma}^{\alpha}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}$$

with the disformation being

$$L_{\alpha\mu\nu} = \frac{1}{2}\left(Q_{\alpha\mu\nu} - Q_{\mu\alpha\nu} - Q_{\nu\alpha\mu}\right)$$

with the non-metricity $Q_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu}$.
Note the zero affine connection! (coincident gauge)

One easily relates the two curvature tensors and finds that

$$0 = \mathbb{R} = \mathring{\mathbb{R}} + \mathbb{Q} + \mathbb{B}$$

with

$$\mathbb{Q} = \frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} - \frac{1}{2} Q_{\alpha\mu\nu} Q_{\mu\alpha\nu} - \frac{1}{4} Q_{\mu} Q^{\mu} + \frac{1}{2} Q_{\mu} \tilde{Q}^{\mu},$$

$$\mathbb{B} = g^{\mu\nu} \mathring{\bigtriangledown}_{\alpha} L^{\alpha}{}_{\mu\nu} - \mathring{\bigtriangledown}^{\beta} L^{\alpha}{}_{\alpha\beta} = \mathring{\bigtriangledown}_{\alpha} \left(Q^{\alpha} - \tilde{Q}^{\alpha} \right)$$

where $Q_{\alpha} \equiv Q_{\alpha}{}^{\mu}{}_{\mu}$ and $\tilde{Q}_{\alpha} \equiv Q^{\mu}{}_{\mu\alpha}$.

Then all the game starts to resemble the metric teleparallel.

Note also that the coincident-gauge STEGR is nothing but the non-covariant $\Gamma\Gamma$ action of Einstein.

The diffeomorphisms become broken in $f(\mathbb{Q})$. Though it can hardly be in a stably broken manner either. And the trivial Minkowski, i.e. the Minkowski metric in Cartesian coordinates, is an obvious strong coupling regime again.

There is still much to do.

There is a similar issue of fixing a gauge, pure tetrad in metric teleparallel and coincident gauge in symmetric teleparallel. From $\Gamma = 0$ by $x \longrightarrow \xi(x)$ one can get

$$\Gamma^{\alpha}_{\mu\nu} = \left[(\partial\xi)^{-1}\right]^{\alpha}_{\beta} \partial_\mu \partial_\nu \xi^{\beta}.$$

Note the second derivatives here!

Actually, the symmetric teleparallel geometry can be described in terms of a set of 1-forms that form a covariantly-constant basis

$$e^n_\mu \equiv \frac{\partial \xi^n}{\partial x^\mu},$$

or a (co-)tetrad with zero spin-connection, and the affine connection coefficients are

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_n \partial_\mu e^n_\nu.$$

Basically, the ζ^n are a set of coordinates in which the spacetime connection is zero.

General teleparallel geometries

A teleparallel geometry, i.e. zero curvature, means that there exists a basis of covariantly conserved vectors. It means

$$\nabla_\mu e^n_\nu = 0$$

in the sense of four independent 1-forms, or equivalently a soldering form which corresponds to zero spin connection.

Or equivalently, $\Gamma^\alpha_{\mu\nu} = e^\alpha_n \partial_\mu e^n_\nu$. The geometry is invariant under global transformations of the defining tetrad. It allows some people to talk about conservation laws in teleparalell gravity.
Of course, in GR-equivalent models, this is an abstract and unobservable feature of the chosen geometry. Therefore, I am not for that. I rather prefer to admit that things like conserved energy simply do not exist.

In metric teleparallel, the usual approach is that this tetrad as a dynamical variable is absolutely free (for sure, apart from non-degeneracy), while the metric is defined as

$$g_{\mu\nu} = \eta_{mn} e^m_\mu e^n_\nu,$$

so that an arbitrary tetrad is orthonormal by definition.

In symmetric teleparallel, the tetrad is holonomic, i.e. it is a basis of coordinate vectors

$$e^n_\mu = \frac{\partial \xi^n}{\partial x^\mu},$$

while the metric is an independent variable.

On metric teleparallel again

Coming back to torsion...

$f(\mathbb{T})$ gravity is also very popular for cosmology with a simple solution of the form

$$ds^2 = a^2(\tau)\left(d\tau^2 - dx^i dx^i\right)$$

in terms of the following tetrad Ansatz:

$$e^A_\mu = a(\tau) \cdot \delta^A_\mu .$$

It is **not enough** to choose just some possible tetrad for the most general perturbed metric like

$$
\begin{aligned}
e_0^{\emptyset} &= a(\tau)\cdot(1+\phi)\\
e_i^{\emptyset} &= 0\\
e_0^{a} &= a(\tau)\cdot(\partial_a\zeta + v_a)\\
e_j^{a} &= a(\tau)\cdot\left((1-\psi)\delta_j^a + \partial^2_{aj}\sigma + \partial_j c_a + \frac{1}{2}h_{aj}\right).
\end{aligned}
$$

Instead, one must use the most general Ansatz for the tetrad perturbation

$$
\begin{aligned}
e_0^{\emptyset} &= a(\tau)\cdot(1+\phi)\\
e_i^{\emptyset} &= a(\tau)\cdot(\partial_i\beta + u_i)\\
e_0^{a} &= a(\tau)\cdot(\partial_a\zeta + v_a)\\
e_j^{a} &= a(\tau)\cdot\left((1-\psi)\delta_j^a + \partial^2_{aj}\sigma + \epsilon_{ajk}\partial_k s + \partial_j c_a + \epsilon_{ajk}w_k + \frac{1}{2}h_{aj}\right).
\end{aligned}
$$

Under infinitesimal diffeomorphisms $x^\mu \to x^\mu + \xi^\mu(x)$ with ξ^0 and $\xi^i \equiv \partial_i \xi + \tilde{\xi}_i$, one can simply derive the following transformation laws:

$$
\begin{array}{rcl}
\phi & \longrightarrow & \phi - \xi^{0\prime} - H\xi^0 \\
\psi & \longrightarrow & \psi + H\xi^0 \\
\sigma & \longrightarrow & \sigma - \xi \\
\beta & \longrightarrow & \beta - \xi^0 \\
\zeta & \longrightarrow & \zeta - \xi' \\
c_i & \longrightarrow & c_i - \tilde{\xi}_i \\
v_i & \longrightarrow & v_i - \tilde{\xi}'_i .
\end{array}
$$

Gauge invariant combinations and possible gauge choices are obvious.

After careful calculations, there are no new dynamical modes in linear perturbations!
Out of 6 new variables: 5 constrained variables and 1 dropping off, from every equation ("remnant symmetry"?).

Therefore, no new degrees of freedom at the linear level. Hence, the strong coupling problem. Predictions are not reliable.

A very interesting unreliable prediction is non-zero gravitational slip:

$$\phi - \psi = -\frac{12 f_{TT} H(H' - H^2)}{f_T} \zeta$$

where

$$\triangle \zeta = -3 \left(\psi' + H\phi - \frac{H' - H^2}{H} \psi \right).$$

In order to see the extra dynamical modes, one might go for other backgrounds.

Due to the "remnant symmetry", we can take another solution for Minkowski metric

$$e^A_\mu = \begin{pmatrix} \cosh(\lambda) & \sinh(\lambda) & 0 & 0 \\ \sinh(\lambda) & \cosh(\lambda) & 0 & 0 \\ 0 & 0 & \cos(\psi) & -\sin(\psi) \\ 0 & 0 & \sin(\psi) & \cos(\psi) \end{pmatrix}$$

with arbitrary functions $\lambda(t, x, y, z)$ and $\psi(t, x, y, z)$.

It has $\mathbb{T} = 0$ and is a solution as long as $f(0) = 0$.

For linear Lorentzian perturbations one gets equations for the perturbations of $\mathbb{T}$

$$
\begin{aligned}
-\psi_z \mathbb{T}_t - \lambda_y \mathbb{T}_x + \lambda_x \mathbb{T}_y + \psi_t \mathbb{T}_z &= 0,\\
\psi_y \mathbb{T}_t - \lambda_z \mathbb{T}_x - \psi_t \mathbb{T}_y + \lambda_x \mathbb{T}_z &= 0,\\
-\lambda_y \mathbb{T}_t - \psi_z \mathbb{T}_x + \lambda_t \mathbb{T}_y + \psi_x \mathbb{T}_z &= 0,\\
-\lambda_z \mathbb{T}_t + \psi_y \mathbb{T}_x - \psi_x \mathbb{T}_y + \lambda_t \mathbb{T}_z &= 0.
\end{aligned}
$$

In generic enough a situation we get $\mathbb{T} =$const. However, in case of only a boost or only a rotation, perturbations of non-constant $\mathbb{T}$ are possible.

In particular, for $\lambda(z)$ and no rotation, we get a new mode with strange Cauchy data of $C_1(y,z)$ and $C_2(x,y,z)$.

The Hamiltonian analysis of $f(\mathbb{T})$ is tricky. There are two contradictory claims (in 4D):

1. It has 5 d.o.f., i.e. three extra propagating degrees of freedom. (Li, Miao, Miao 2011; Blagojevic, Nester 2020)

2. It has 3 d.o.f., i.e. one extra propagating degree of freedom. (Ferraro, Guzman 2018)

The last version of the first claim is probably the most accurate one, even though not without its shortcomings. In particular, no attention is payed to singular surfaces in the phase space, jumps in the ranks of Poisson brackets algebra, and so on.

At the same time, to the best of my knowledge, no one has ever seen the full set of three new modes in practical calculations.

The main mistake in the second claim was in neglecting the spatial derivatives of $\mathbb{T}$ in the Poisson brackets.
And indeed, our "almost one" new mode was seen around the non-trivial Minkowski background with $\mathbb{T} = 0$.

In cosmological tasks, the $\mathbb{T}$ scalar does naturally have a time-like gradient, and therefore can be taken for a time variable.
Does this mean a possible existence of preferred foliation in this case?

As a final topic, let me briefly mention another way of modifying the TEGR, that is the New GR. The idea is to take the TEGR action

$$S = \int d^4x \sqrt{-g} \cdot \mathfrak{T}$$

with a modified torsion scalar $\mathfrak{T} = \frac{1}{2} T_{\alpha\mu\nu} \mathfrak{S}^{\alpha\mu\nu}$, no longer equivalent to the Ricci scalar, with a new "superpotential"

$$\mathfrak{S}_{\alpha\mu\nu} = \frac{a}{2} T_{\alpha\mu\nu} + \frac{b}{2} \left(T_{\mu\alpha\nu} - T_{\nu\alpha\mu} \right) + c \left(g_{\alpha\mu} T_{\nu} - g_{\alpha\nu} T_{\mu} \right).$$

The case of $a = b = c = 1$ is TEGR.

The case of $a + b = 2c$ is the so-called one-parameter New GR which used to be a preferred one due to several reasons.

1. Unless we go for unnaturally complicated tetrads, the static spherically-symetric solutions are just the same as in GR. Moreover, the spatially-flat cosmology has got absolutely the same linear perturbations in the metric sector. I would say, all in all it is kind of boring.

2. There were (quite murky) arguments by van Nieuwenhuizen that in all the other cases we would have ghosts. However, he was also against the benign story of "tachyons", and even more importantly, he excluded lots of models for "nonlocality" which was brought about by himself due to using the spin projectors in the action.

Another problem of special New GR cases is that they are also prone to strong coupling issues. Even though, precisely the one-parameter New GR model has a smooth Minkowski space limit from the spatially-flat cosmology.

The general New GR theory, i.e. when

$$a \neq \pm b, \quad a + b \neq 2c, \quad a + b \neq 6c,$$

exhibits robust dynamical properties, at least to the number of degrees of freedom. Out of 16 tetrad components, 4 are pure gauge due to diffeomorphism invariance, 4 more are constrained by gauge symmetries "hitting twice", and finally there are 8 dynamical modes; precisely the numbers we would expect from symmetry considerations. The dynamical stability is an open and very interesting question.

Conclusions

It is very interesting to study the teleparallel modifications of gravity, however problematic they might be.

The main ingredient is a new connection which defines a notion of vectors parallel at a distance, and is quite esoteric for GR-equivalent models.

Modifications violate the local Lorentz symmetry or diffeomorphism symmetry, or both, by introducing a covariantly constant tetrad, either orthonormal (in metric teleparalell) or a coordinate one (in symmetric telperallel), or something more general.

The simplest modifications of this sort are unreliable because the symmetry violation is not robust enough.

Thank you!