Semi-symmetric metric gravity

from the Friedmann-Schouten geometry with torsion to dynamical dark energy models

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1. Short review of GR
2. Geometrical preliminaries
3. Proposed theory and its cosmological applications
4. Thermodynamical interpretation
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Definition 1.1. Relativistic spacetime

A **relativistic spacetime** is a pair (M, g) consisting of

(i) a smooth 4-dimensional manifold M,

(ii) an at least piecewise C^2 (but usually smooth) Lorentzian metric g of signature $(-,+,+,+)$.

Remark: Note that the second condition puts a topological restriction on M, namely $\chi(M) = 0$.

Theorem 1.2. Existence and uniqueness of the Levi-Civita connection

On a relativistic spacetime (M, g) there exists a unique connection $\overset{\circ}{\nabla}$, which satisfies

$$\overset{\circ}{\nabla} g = 0, \quad \text{and} \quad \overset{\circ}{\nabla}_X Y - \overset{\circ}{\nabla}_Y X - [X, Y] = 0 \quad \text{for all} \quad X, Y \in \Gamma(TM).$$

Einstein's gravitational dynamics are given by

$$\overset{\circ}{R}_{\nu\sigma} - \frac{1}{2} g_{\nu\sigma} \overset{\circ}{R} + \Lambda g_{\nu\sigma} = \kappa T_{\nu\sigma}, \quad \text{where} \quad \kappa = \frac{8\pi G}{c^4} \approx 2.07 \times 10^{-43} N^{-1}.$$

One usually assumes a more rigid structure on M, namely

1. M is orientable, or equivalently allows for a globally defined volume form,
2. M has no closed timelike curves,
3. M is connected,
4. M is globally hyperbolic.

Remark: The last condition is equivalent to any of the following conditions

- the existence of a Cauchy surface,
- the existence of a $SL(2, \mathbb{C})$ spin-structure,
- $M \cong R \times \Sigma$, where Σ is a 3-dimensional oriented manifold.

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A general affine connection is fully characterized by its torsion and non-metricity

$$Q_{\mu\nu\rho} = -\nabla_\mu g_{\nu\rho}, \quad T^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho}.$$

With the help of these, the affine connection can be decomposed as

$$\Gamma^\mu{}_{\nu\rho} = \gamma^\mu{}_{\nu\rho} + \frac{1}{2}g^{\mu\lambda}\left(Q_{\nu\rho\lambda} + Q_{\rho\lambda\nu} - Q_{\lambda\nu\rho}\right) - \frac{1}{2}g^{\mu\lambda}(T_{\rho\nu\lambda} - T_{\lambda\rho\nu} + T_{\nu\rho\lambda}).$$

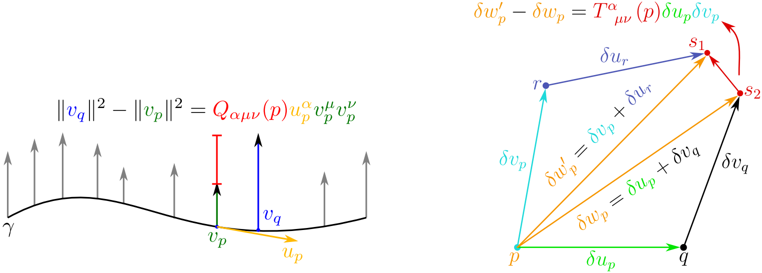
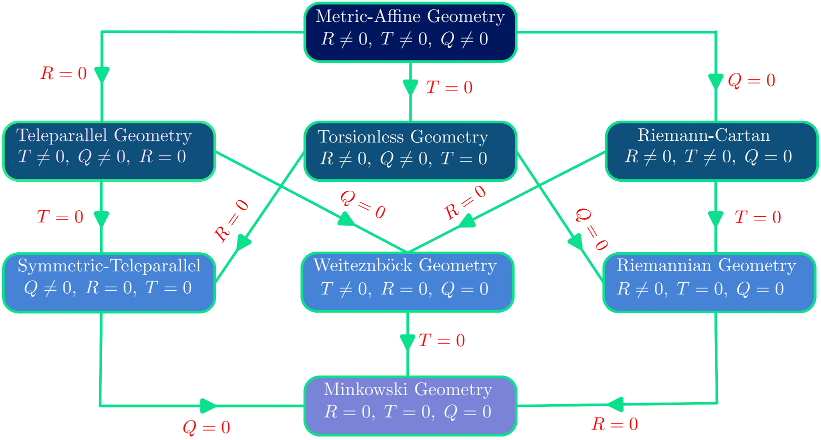


Figure 1: The effects of non-metricity (left panel) and torsion (right panel) are depicted. Non-metricity changes the lengths of vectors, while torsion measures to what extent the parallelogram law fails infinitesimally.


Metric-Affine Geometry
$R \neq 0,\ T \neq 0,\ Q \neq 0$
$R = 0$
$T = 0$
$Q = 0$
Teleparallel Geometry
$T \neq 0,\ Q \neq 0,\ R = 0$
Torsionless Geometry
$R \neq 0,\ Q \neq 0,\ T = 0$
Riemann-Cartan
$R \neq 0,\ T \neq 0,\ Q = 0$
$T = 0$
$R = 0$
$Q = 0$
$R = 0$
$Q = 0$
$T = 0$
Symmetric-Teleparallel
$Q \neq 0,\ R = 0,\ T = 0$
Weiteznböck Geometry
$T \neq 0,\ R = 0,\ Q = 0$
Riemannian Geometry
$R \neq 0,\ T = 0,\ Q = 0$
$T = 0$
Minkowski Geometry
$R = 0,\ T = 0,\ Q = 0$
$Q = 0$
$R = 0$

Vectorial geometries

1. **Weyl geometry**: $Q_{\mu\nu\rho} = w_\mu g_{\nu\rho}$, $T = 0$ - special case of torsionless geometry.
2. **Semi-symmetric metric geometry**: $Q = 0$, $T^\mu{}_{\nu\rho} = \pi_\rho \delta^\mu_\nu - \pi_\nu \delta^\mu_\rho$ - special case of Riemann-Cartan geometry.
3. **Schrödinger geometry**: $Q^\alpha{}_{\mu\nu} = g_{\mu\nu}\pi^\alpha - \frac{1}{2}\left(\delta^\alpha_\mu \pi_\nu + \delta^\alpha_\nu \pi_\mu\right)$, $T = 0$.

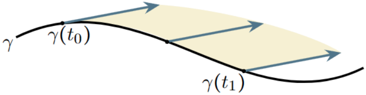
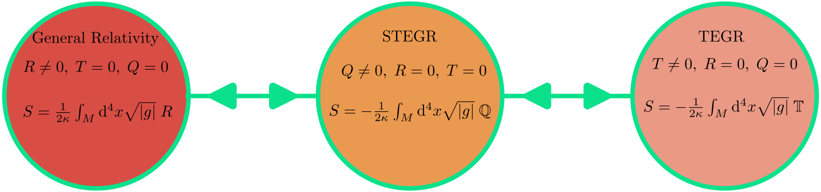


Figure 2: Illustration of the effect of non-metricity on autoparallel transport. On the left panel, one can see a Schrödinger-type non-metricity, which preserves lengths of autoparallely transported vectors, while on the right panel the effect of a general non-metricity is depicted.

Main four reasons

1. Ehlers-Pirani-Schild axiomatization of GR. arXiv:2112.14063
2. Torsion-non-metricity duality in $f(R)$ gravity. arXiv:1810.06602
3. Einstein-Cartan theory. arXiv:0711.1535
4. Non-relativistic limit. arXiv:2308.07100



Definition 2.1. Semi-symmetric metric connection

On a relativistic spacetime (M, g), a connection ∇ is called **semi-symmetric metric** if it is metric compatible and there exists $\pi \in \Gamma(T^* M)$, such that

$$\nabla_X Y - \nabla_Y X - [X, Y] = \pi(Y)X - \pi(X)Y, \forall X, Y \in \Gamma(TM).$$

In this case, torsion can be described globally and locally as

$$T(\omega, X, Y) = \pi(Y)\omega(X) - \pi(X)\omega(Y) \text{ and } T^\mu{}_{\nu\rho} = \pi_\rho \delta^\mu_\nu - \pi_\nu \delta^\mu_\rho, \text{ respectively.}$$

Theorem 2.2. Existence and uniqueness of semi-symmetric metric connection

Let (M, g, ∇) be a relativistic spacetime with a semi-symmetric metric connection and denote by $\overset{\circ}{\nabla}$ the Levi-Civita connection. Then

$$\nabla_X Y = \overset{\circ}{\nabla}_X Y + \pi(Y)X - g(X, Y)P,$$

where P is the dual vector field associated to π, i.e. $g(X, P) = \pi(X)$.

Locally, we have $\Gamma^\mu{}_{\nu\rho} = \gamma^\mu{}_{\nu\rho} - \pi^\mu g_{\rho\nu} + \pi_\nu \delta^\mu_\rho$.

Theorem 2.3. Yano

Let (M, g, ∇) be a relativistic spacetime with a semi-symmetric connection that is metric-compatible and denote with $\overset{\circ}{\nabla}$ the Levi-Civita connection. Moreover, denote by *Riem* the curvature tensor of the semi-symmetric metric connection ∇ and by $\overset{\circ}{Riem}$ the curvature tensor of the Levi-Civita connection $\overset{\circ}{\nabla}$. Then, the following equation

$$\begin{aligned} Riem(\omega, Z, X, Y) = \overset{\circ}{Riem}(\omega, Z, X, Y) &- \omega(S(Y, Z)X) \\ &+ \omega(S(X, Z)Y) - \omega(g(Y, Z)A(X)) \\ &+ \omega(g(X, Z)A(Y)), \end{aligned}$$

is satisfied for all one-forms ω, and vector fields X, Y, Z, where

$$S(X, Y) = \left(\overset{\circ}{\nabla}_X \pi\right)(Y) - \pi(X)\pi(Y) + \frac{1}{2}\pi(P)g(X, Y)$$

and A is a $(1, 1)-$tensor field defined by

$$g(A(X), Y) = S(X, Y).$$

In a local coordinate system, the Riemann tensor of a semi-symmetric metric connection takes the form

$$Riem^{\mu}{}_{\nu\rho\sigma} = \overset{\circ}{Riem}{}^{\mu}{}_{\nu\rho\sigma} - S_{\sigma\nu}\delta^{\mu}_{\rho} + S_{\rho\nu}\delta^{\mu}_{\sigma} - g_{\sigma\nu}S_{\rho\lambda}g^{\lambda\mu} + g_{\rho\nu}S_{\sigma\lambda}g^{\lambda\mu},$$

where the tensor $S_{\sigma\nu}$ is defined as

$$S_{\sigma\nu} = \overset{\circ}{\nabla}_{\sigma}\pi_{\nu} - \pi_{\sigma}\pi_{\nu} + \frac{1}{2}g_{\sigma\nu}\pi_{\lambda}\pi^{\lambda}.$$

We thus immediately obtain that the Ricci tensor and scalar of a semi-symmetric metric connection are given by

(*i*) Ricci tensor: $R_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} - 2\overset{\circ}{\nabla}_{\nu}\pi_{\mu} + 2\pi_{\nu}\pi_{\mu} - 2g_{\mu\nu}\pi_{\lambda}\pi^{\lambda} - g_{\mu\nu}\overset{\circ}{\nabla}_{\lambda}\pi^{\lambda}$.

(*ii*) Ricci scalar: $R = \overset{\circ}{R} - 6\overset{\circ}{\nabla}_{\alpha}\pi^{\alpha} - 6\pi_{\alpha}\pi^{\alpha}$.

Upon symmetrization, there is a formal analogy with the Weyl geometry, where we have

(*i*) Ricci tensor: $R_{\mu\nu} = \overset{\circ}{R}_{\mu\nu} + g_{\mu\nu}\overset{\circ}{\nabla}_{\alpha}w^{\alpha} + 3\overset{\circ}{\nabla}_{\nu}w_{\mu} - \overset{\circ}{\nabla}_{\mu}w_{\nu} - 2g_{\mu\nu}w_{\rho}w^{\rho} + 2w_{\mu}w_{\nu}$.

(*ii*) Ricci scalar: $R = \overset{\circ}{R} + 6\overset{\circ}{\nabla}_{\alpha}w^{\alpha} - 6w_{\alpha}w^{\alpha}$.

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First of all, we *postulate* that

$$R_{(\nu\sigma)} - \frac{1}{2} g_{\nu\sigma} R = 8\pi T_{\nu\sigma}.$$

Post-Riemannian expansion leads to

$$\mathring{R}_{\nu\sigma} - \frac{1}{2} g_{\nu\sigma} \mathring{R} - \mathring{\nabla}_{\sigma}\pi_{\nu} - \mathring{\nabla}_{\nu}\pi_{\sigma} + 2\pi_{\sigma}\pi_{\nu} + 2g_{\sigma\nu}\mathring{\nabla}_{\lambda}\pi^{\lambda} + g_{\nu\sigma}\pi^{\rho}\pi_{\rho} = 8\pi T_{\nu\sigma}.$$

Observations

1. In the limit $\pi \to 0$, we recover GR as expected.
2. The torsion vector is fully determined by a vectorial part, and it has contributions to the usual EFE, which could be thought of as a geometric type dark energy.
3. There are no dynamics for torsion! This has to be imposed separately.

Assume homogeneous, isotropic, *spatially flat* FLRW metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j.$$

The matter content is given by a perfect fluid with energy-momentum tensor

$$T_{\nu\sigma} = \rho u_\nu u_\sigma + p(u_\nu u_\sigma + g_{\nu\sigma}).$$

The problem is taken into account in a comoving frame

$$u_\nu = (-1, 0, 0, 0) \iff u^\nu = (1, 0, 0, 0).$$

The cosmological principle implies

$$\pi_\nu = (-\omega(t), 0, 0, 0) \iff \pi^\nu = (\omega(t), 0, 0, 0).$$

Theorem 3.1. Friedmann equations in semi-symmetric metric gravity

$$3H^2 = 8\pi\rho - 3\omega^2 + 6H\omega,$$

$$2\dot{H} + 3H^2 = -8\pi p + 4H\omega - \omega^2 + 2\dot{\omega}.$$

The previously mentioned interpretation can now be made explicit as

$$3H^2 = 8\pi\rho - 3\omega^2 + 6H\omega = 8\pi\left(\rho + \rho_{eff}\right) = 8\pi\rho_{tot},$$

$$2\dot{H} + 3H^2 = -8\pi p + 4H\omega - \omega^2 + 2\dot{\omega} = -8\pi\left(p + p_{eff}\right) = -8\pi p_{tot},$$

where we have denoted

$$\rho_{eff} = \frac{1}{8\pi}\left(6H\omega - 3\omega^2\right), \quad p_{eff} = -\frac{1}{8\pi}\left(4H\omega - \omega^2 + 2\dot{\omega}\right),$$

while $\rho_{tot} = \rho + \rho_{eff}$, and $p_{tot} = p + p_{eff}$.

We impose the conditions

$$\frac{1}{8\pi}\left(6H\omega - 3\omega^2\right) = \lambda, \quad -\frac{1}{8\pi}\left(4H\omega - \omega^2 + 2\dot{\omega}\right) = \frac{2}{3}K\lambda.$$

To get rid of signs and factors, we redefine $\Lambda = 8\pi\lambda, k = -K$ to obtain

$$3\omega\left(2H - \omega\right) = \Lambda, \quad 4H\omega - \omega^2 + 2\dot{\omega} = \frac{2}{3}k\Lambda,$$

respectively, where k and $\Lambda \geq 0$ are constants. Eliminating H yields the equation

$$2\dot{\omega} + \omega^2 + 2(1-k)\frac{\Lambda}{3} = 0,$$

which admits an analytical solution

$$\omega(t) = \sqrt{\frac{2(k-1)\Lambda}{3}}\tanh\left[\frac{\sqrt{(k-1)\Lambda}}{\sqrt{6}}\left(t - t_0\right)\right].$$

The Hubble function is given by

$$H(t) = \frac{\sqrt{\Lambda}}{2\sqrt{6(k-1)}} \tanh\left[\frac{\sqrt{(k-1)\Lambda}(t-t_0)}{\sqrt{6}}\right] \times \left\{\coth^2\left[\frac{\sqrt{(k-1)\Lambda}(t-t_0)}{\sqrt{6}}\right] + 2(k-1)\right\}.$$

The matter density $8\pi\rho = 3H^2 - \Lambda$ reads

$$8\pi\rho(t) = \frac{\Lambda}{8(k-1)}\left\{\coth\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right] - 2(k-1)\tanh\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right]\right\}^2.$$

Pressure can also be obtained analytically

$$8\pi p(t) = \frac{\Lambda}{24(k-1)}\left\{(4k-7)\operatorname{csch}^2\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right] + 4(k-1)^2\operatorname{sech}^2\left[\sqrt{\frac{(k-1)\Lambda}{6}}(t-t_0)\right] + 4k^2 - 4k - 3\right\}.$$

The scale factor of this cosmological model is given by

$$a(t) = a_0 \sinh^{2(k-1)}\left[\sqrt{\frac{(k-1)\Lambda}{6}}\,(t-t_0)\right] \times \cosh^{\sqrt{6}}\left[\sqrt{\frac{(k-1)\Lambda}{6}}\,(t-t_0)\right].$$

Large time limits yield

$$\lim_{t\to\infty} \rho(t) = \frac{(3-2k)^2}{8(k-1)}\Lambda, \quad \lim_{t\to\infty} p(t) = \frac{4k^2-4k-3}{24(k-1)}\Lambda.$$

Comments

1. For $k = \frac{3}{2}$ in the large time limit we have $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \rho(t) = 0$,indicating that the Universe ends in a vacuum state. For other values of k the cosmological evolution ends in constant density and pressure thermodynamic phase.
2. The deceleration parameter can also be computed analytically. In particular, it can be shown that $\lim_{t\to\infty} q(t) = -1$: the Universe ends in a de Sitter type phase.

We assume a linear equation of state

$$P_{eff}(z) = -\sigma(z) r_{eff}(z) - \lambda,$$

and consider CPL parametrization

$$\sigma(z) = \sigma_0 + \sigma_a \frac{z}{1+z},$$

where σ_0 and σ_a are constants.

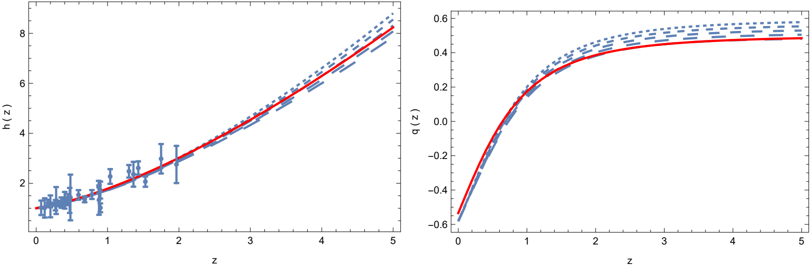

Figure 3: Variations as a function of the redshift z of the dimensionless Hubble function (left panel), and of the deceleration parameter $q(z)$ (right panel) for Model II, for $\lambda = 0.79$, $r(0) = 0.311$, $\sigma_0 = -0.10$, and different values of σ_a: $\sigma_a = 0.04$ (dotted curve), $\sigma_a = 0.06$ (short dashed curve), $\sigma_a = 0.08$ (dashed curve), $\sigma_a = 0.10$ (long dashed curve), and $\sigma_a = 0.12$ (ultra-long dashed curve), respectively. The predictions of the ΛCDM model are represented by the red curve.

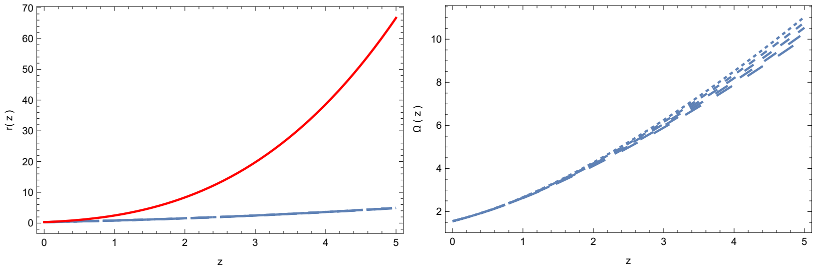

Figure 4: Variations as a function of the redshift z of the dimensionless matter density $r(z)$ (left panel), and of the torsion vector component $\Omega(z)$ (right panel) for Model II, for $\lambda = 0.67$, $r(0) = 0.311$, $\sigma_0 = -0.10$, and different values of σ_a:$\sigma_a = 0.04$ (dotted curve), $\sigma_a = 0.06$ (short dashed curve), $\sigma_a = 0.08$ (dashed curve), $\sigma_a = 0.10$ (long dashed curve), and $\sigma_a = 0.12$ (ultra-long dashed curve), respectively. The predictions of the ΛCDM model are represented by the red curve.

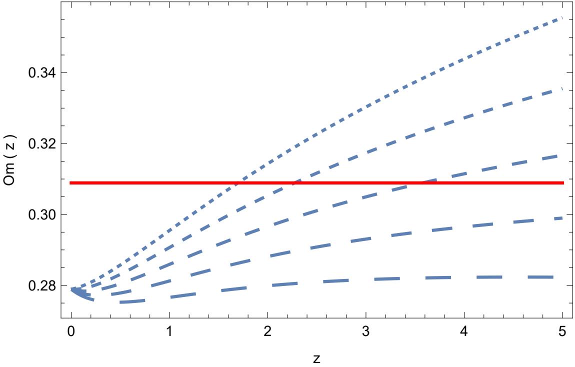

Figure 5: Behavior of the function $Om(z)$ for Model II, for $\lambda = 0.79$, $r(0) = 0.311$, $\sigma_0 = -0.10$, and different values of σ_a: $\sigma_a = 0.04$ (dotted curve), $\sigma_a = 0.06$ (short dashed curve), $\sigma_a = 0.08$ (dashed curve), $\sigma_a = 0.10$ (long dashed curve), and $\sigma_a = 0.12$ (ultra-long dashed curve), respectively. The predictions of the ΛCDM model are represented by the red curve.

We impose a polytropic equation of state $P_{eff} = K r_{eff}^2$.

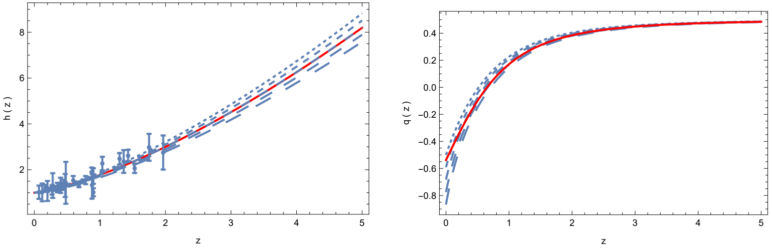

Figure 6: Variations of the dimensionless Hubble function $h(z)$(left panel), and of the deceleration parameter $q(z)$ (right panel) for Model III with $K = -2$ and initial conditions $\Omega(0) = 0.35$ (dotted curve), $\Omega(0) = 0.37$ (short dashed curve), $\Omega(0) = 0.39$ (dashed curve) , $\Omega(0) = 0.41$ (long dashed curve), $\Omega(0) = 0.43$ (ultra-long dashed curve),respectively. The observational data for the Hubble function are represented with their error bars, while the red curve depicts the predictions of the ΛCDM model.

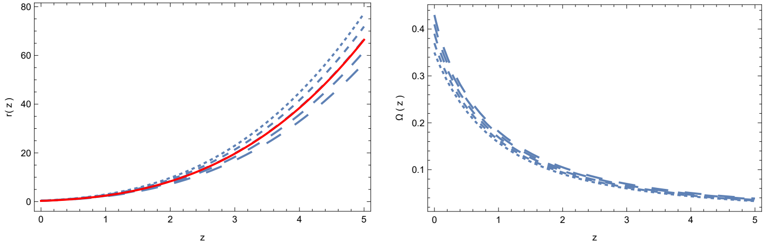

Figure 7: Variations of the dimensionless matter energy density $r(z)$ (left panel), and of the torsion vector $\Omega(z)$ (right panel) for Model III with $K = -2$ and initial conditions $\Omega(0) = 0.35$ (dotted curve), $\Omega(0) = 0.37$ (short dashed curve), $\Omega(0) = 0.39$ (dashed curve) , $\Omega(0) = 0.41$ (long dashed curve), $\Omega(0) = 0.43$ (ultra-long dashed curve),respectively. The red curve represents the predictions of the ΛCDM model.

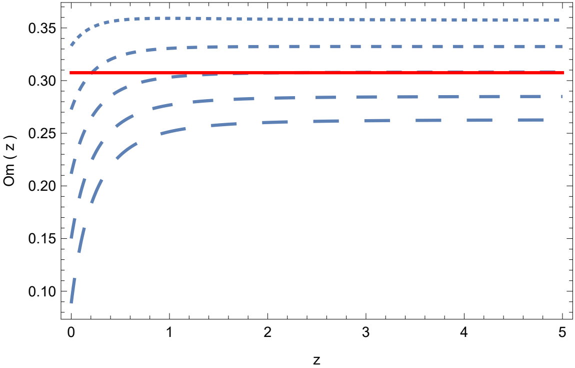

Figure 8: Behavior of the function *Omz* for Model III with $K = -2$ and initial conditions $\Omega(0) = 0.35$ (dotted curve), $\Omega(0) = 0.37$ (short dashed curve), $\Omega(0) = 0.39$ (dashed curve), $\Omega(0) = 0.41$ (long dashed curve), $\Omega(0) = 0.43$ (ultra-long dashed curve), respectively. The red curve represents the predictions of the ΛCDM model.

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It can be easily shown that if we impose conservation of matter, the theory reduces to Einstein relativity. Thus, we interpret the non-conservation to be related to particle creation. Hence, we introduce the **thermodynamical quantities**

1. Particle flux $N^\mu \equiv nu^\mu$, where n is the particle number density.
2. Entropy flux vector $S^\mu \equiv su^\mu = n\sigma u^\mu$, where s is the entropy density, and σ is the entropy per particle.

The particle flux satisfies

$$\nabla_\mu N^\mu = \dot{n} + 3Hn = n\Psi,$$

where Ψ is the particle generation rate. From the second law of thermodynamics we have

$$\nabla_\mu S^\mu = n\dot{\sigma} + n\sigma\Psi \geq 0.$$

The total thermodynamic energy balance equation, $u_\mu \nabla_\nu T^{\mu\nu} = 0$, gives the generalized energy conservation equation in the presence of particle creation

$$\dot{\rho} + 3H(\rho + p + p_c) = 0.$$

The Gibbs law in presence of matter creation is given by

$$nTd\left(\frac{s}{n}\right) = nTd\sigma = d\rho - \frac{\rho + p}{n} dn.$$

From the Friedmann equations, upon some manipulations we can read off

$$p_c = \frac{\omega}{8\pi}\left[2\frac{\dot{H}}{H} - \frac{2\dot{\omega}}{H} + 2H - 2\omega\right].$$

Some easy algebra gives the particle creation rate

$$\Psi = -3H\frac{p_c}{\rho + p} = -\frac{3H\omega}{8\pi(\rho + p)}\left[2\frac{\dot{H}}{H} - \frac{2\dot{\omega}}{H} + 2H - 2\omega\right].$$

The entropy production rate is given by

$$\nabla_\mu S^\mu = -3n\sigma H \frac{p_c}{(\rho+p)} = \frac{3n\sigma H\omega}{8\pi(\rho+p)}\left[2qH + \frac{2\dot{\omega}}{H} + 2\omega\right].$$

We assume the equations of state

$$\rho = \rho(n, T), \quad p = p(n, T).$$

In this case, the temperature evolution takes the form

$$\frac{\dot{T}}{T} = \left(\frac{\partial p}{\partial \rho}\right)_n \frac{\dot{n}}{n} = c_s^2 \frac{\dot{n}}{n} = c_s^2 \left(\Psi - 3H\right) = -3c_s^2 H\left(1 + \frac{p_c}{\rho+p}\right), \qquad (1)$$

where $c_s^2 = \left(\partial p/\partial \rho\right)_n$ is the speed of sound. Hence, in the semi-symmetric metric gravity theory, the time variation of the temperature of the newly created particles is given by

$$\frac{\dot{T}}{T} = 3c_s^2 H\left\{1 + \frac{\omega}{8\pi(\rho+p)}\left[2qH + \frac{2\dot{\omega}}{H} + 2\omega\right]\right\}.$$

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Definition 5.1. Einstein metric

A semi-Riemannian metric g on a smooth manifold M is called an **Einstein metric** if there exists a smooth function $\lambda : M \to \mathbb{R}$, such that

$$\overset{\circ}{R}_{\mu\nu} = \lambda g_{\mu\nu}. \tag{2}$$

Remark: Note that for a general affine connection ∇ equation (2) does not make sense, as $R_{\mu\nu}$ is not symmetric in general. Hence, we will symmetrize in our following definition accordingly.

Definition 5.2. Generalized Einstein manifold

A relativistic spacetime (M, g) equipped with an affine connection ∇ is called a **generalized Einstein manifold** if there exists a smooth function $\lambda : M \to \mathbb{R}$ such that

$$R_{(\mu\nu)} = \lambda g_{\mu\nu}.$$

Proposition 5.3. Characterization of semi-symmetric Einstein manifolds

Let (M, g) be a relativistic spacetime equipped with a semi-symmetric metric connection. Then the following are equivalent:

(i) (M, g) is a generalized Einstein-manifold.

(ii) $\mathring{R}_{\mu\nu} - \mathring{\nabla}_\mu \pi_\nu - \mathring{\nabla}_\nu \pi_\mu + 2\pi_\nu \pi_\mu + \frac{1}{2} g_{\mu\nu} \mathring{\nabla}_\lambda \pi^\lambda - \frac{1}{2} g_{\mu\nu} \pi_\lambda \pi^\lambda = \frac{1}{4} g_{\mu\nu} \mathring{R}$.

Trivially, a relativistic spacetime (M, g) equipped with a semi-symmetric connection and an Einstein metric is a generalized Einstein manifold iff

$$\mathring{\nabla}_\mu \pi_\nu - \mathring{\nabla}_\nu \pi_\mu + 2\pi_\nu \pi_\mu + \frac{1}{2} g_{\mu\nu} \mathring{\nabla}_\lambda \pi^\lambda - \frac{1}{2} g_{\mu\nu} \pi_\lambda \pi^\lambda = 0.$$

Theorem 5.4. Existence of Einstein manifolds with torsion

There exists a generalized Einstein manifold (M, g), where g is neither an Einstein metric, nor static.

Fix $M = \mathbb{R}^4$ and equip with with the metric

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j,$$

where we assume that $\frac{\dot{a}}{a} = H_0$ is a non-zero constant. Moreover, we equip $\left(\mathbb{R}^4, ds^2\right)$ with a semi-symmetric connection. The constructed tuple is a generalized Einstein manifold iff

$$\mathring{R}_{\mu\nu} - \mathring{\nabla}_\mu \pi_\nu - \mathring{\nabla}_\nu \pi_\mu + 2\pi_\nu \pi_\mu + \frac{1}{2} g_{\mu\nu} \mathring{\nabla}_\lambda \pi^\lambda - \frac{1}{2} g_{\mu\nu} \pi_\lambda \pi^\lambda = \frac{1}{4} g_{\mu\nu} \mathring{R}$$

is satisfied. As the metric possesses high symmetry, we choose

$$\pi_\mu = (\psi(t), 0, 0, 0) \iff \pi^\mu = (-\psi(t), 0, 0, 0).$$

Hence, we have to satisfy the following system of differential equations for $\psi(t)$

$$-3\frac{\ddot{a}}{a} - \dot{\psi} - \dot{\psi} + 2\psi^2 - \frac{1}{2}\left(-\dot{\psi} - 3\frac{\dot{a}}{a}\psi\right) - \frac{1}{2}\psi^2 = -\frac{6}{4}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right),$$

$$a\ddot{a} + 2\dot{a}^2 + 2a\dot{a}\psi + \frac{1}{2}a^2\left(-\dot{\psi} - 3\frac{\dot{a}}{a}\psi\right) + \frac{1}{2}a^2\psi^2 = \frac{6}{4}a^2\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right).$$

By introducing the Hubble parameter $H_0 = \frac{\dot{a}}{a}$, we obtain

$$-\frac{3}{2}\dot{H}_0 - \frac{3}{2}H_0^2 + \frac{3}{2}H_0^2 - \frac{3}{2}\dot{\psi} + \frac{3}{2}\psi^2 + \frac{3}{2}H_0\psi = 0,$$

$$-\frac{1}{2}\dot{H}_0 - \frac{1}{2}H_0^2 + \frac{1}{2}H_0^2 + \frac{1}{2}H_0\psi - \frac{1}{2}\dot{\psi} + \frac{1}{2}\psi^2 = 0.$$

Our assumption that H_0 is constant implies

$$-\frac{3}{2}\dot{\psi} + \frac{3}{2}\psi^2 + \frac{3}{2}H_0\psi = 0,$$

$$\frac{1}{2}H_0\psi - \frac{1}{2}\dot{\psi} + \frac{1}{2}\psi^2 = 0.$$

We can see that the two equations are identical. Hence, we can solve for example the first one, i.e.

$$\dot{\psi} - \psi^2 - H_0\psi = 0,$$

which is a Bernoulli type differential equation. The solution is given by

$$\psi(t) = -\frac{H_0 e^{H_0(t_0+t)}}{e^{H_0(t_0+t)} - 1}.$$