XII Bolyai–Gauss–Lobachevsky Conference (BGL-2024): Non-Euclidean Geometry in Modern Physics and Mathematics, Budapest

Looking for Carroll particles in two time spacetime

Alexander Kamenshchik

University of Bologna and INFN, Bologna

May 1-3, 2024

Based on

Alexander Kamenshchik and Federica Muscolino,
Looking for Carroll particles in two time spacetime,
Phys. Rev. D 109 (2024) 2, 025005

Content

1. Introduction
2. Two time physics
3. Carroll symmetry
4. Carroll particle in two time spacetime: classical theory
5. Carrol particle in two time spacetime: quantum theory
6. Concluding remarks

Introduction

- Theories with extra spatial dimensions have become quite traditional since the times when they were put forward in the famous works by Kaluza and Klein.
- Theories with more than one time dimensions look much less intuitive and plausible.
- An impressive series of papers devoted to the so called two time or 2T physics was produced by I. Bars and his co-authors beginning from 1996.
- Classical and quantum physics of simple systems such as non-relativistic particle, massive and massless relativistic particles, harmonic oscillator, hydrogen-like atoms can be described in the framework of 2T-physics from a unifying point of view.

- In the book
  I. Bars and J. Terning, Extra Dimensions in Space and Time, Springer, New York, 2010
  different one time physical systems arise as some "shadows" in the Plato's cave, which is nothing but the world with one additional temporal dimension and one additional spatial dimension.
- The language of the two time physics is quite adapted also for the description of field theories and of the gravity.
- A new approach to cosmology, inspired by two time physics has open the way to an interesting treatment of the problem of passing through the cosmological singularities.
- Relations between the two time physics and the Carroll symmetry were not explored before.

Two time physics

- From the point of view of the 2T Physics, usual physical systems living in a one time world represent projections from the spacetime with one additional temporal dimension and one additional spatial dimension.
- These additional dimensions are introduced to construct a new gauge theory, based on the localization of the phase-space symmetry described by the symplectic group $Sp(2, \mathbb{R})$.
- The usual physics with 1T is obtained by means of a gauge fixing.

The phase-space coordinates for the two time world

$$X^M = \left(X^{0'}, X^{1'}, X^{\mu}\right) \qquad P^M = \left(P^{0'}, P^{1'}, P^{\mu}\right).$$

The indices $0'$ and $1'$ label an extra time and an extra space dimensions.

The extra space dimension is necessary to get the right number of degrees of freedom in the 1T theory.

The index $\mu = 0, \ldots, d-1$ labels usual coordinates in one time world.

$$X_i^M = \left(X^M, P^M\right),$$

where $i = 1, 2$ labels mean the position and momentum respectively.

The two types of phase variables can be mixed through $Sp(2, \mathbb{R})$ transformations.

The worldline action for a free particle in a flat two time spacetime

$$S = \frac{1}{2}\int d\tau\ \epsilon^{ij}\eta_{MN}\ \partial_\tau X_i^M X_j^N,$$

where $\eta_{MN} = \text{Diag(-1,1,-1,1,\ldots,1)}$ is the flat metric, with signature $(2, d)$, and ϵ^{ij} is the antisymmetric tensor with $\epsilon^{12} = 1$. τ is a proper time parameter.

The action is invariant under the global $Sp(2,\mathbb{R})$ transformations

$$\delta_\omega X_i^M = \epsilon_{ij}\omega^{jk} X_k^M .$$

The transformation parameters ω^{jk} are symmetric in j,k.
When $\omega^{ij} \to \omega^{ij}(\tau)$, we need to introduce a connection that takes into account the new gauge symmetry.
The covariant derivative is

$$\partial_\tau X_i^M \to D_\tau X_i^M = \partial_\tau X_i^M - \epsilon_{ij} A^{jk}(\tau) X_k^M ,$$

where $A^{jk}(\tau)$ is symmetric in the indices i,j and belongs to the adjoint representation of the Lie algebra of $Sp(2,\mathbb{R})$ (that we call $\mathfrak{sp}(2,\mathbb{R})$).
It transforms as a gauge field under the $Sp(2,\mathbb{R})$ group

$$\delta_\omega A^{ij}(\tau) = \partial_\tau \omega^{ij} + \omega^{ik}\epsilon_{kl} A^{lj} + \omega^{jk}\epsilon_{kl} A^{li} .$$

The worldline action invariant under these gauge transformations is

$$S = \frac{1}{2}\int d\tau\ \epsilon^{ij}\eta_{MN} D_\tau X_i^M X_j^N$$
$$= \int d\tau\ \left[\eta^{MN}\partial_\tau X_M P_N - A^{ij}(\tau) Q_{ij}\right],$$

where

$$Q_{11} = \frac{1}{2}X\cdot X, \quad Q_{22} = \frac{1}{2}P\cdot P,$$
$$Q_{12} = Q_{21} = \frac{1}{2}X\cdot P$$

are the $\mathfrak{sp}(2,\mathbb{R})$ conserved currents o constraints.

The gauge fields A^{ij} are not dynamical and play the role of Lagrange multipliers.
When a gauge is chosen, the following constraints must be satisfied

$$X \cdot X = 0,$$
$$X \cdot P = 0,$$
$$P \cdot P = 0.$$

These constraints lead to a non-trivial parameterization of the 1T spacetime only when the starting theory has more than one timelike dimension.

When the gauge is fixed and the constraints are satisfied, one gets the right number of 1T variables
$X_i^M(\tau) = X_i^M(\vec{x}(\tau), \vec{p}(\tau))$.
The action is

$$S = \int d\tau \left(\dot{\vec{x}} \cdot \vec{p} - H \right),$$

where H is the Hamiltonian of the 1T theory.
Different gauge fixings correspond to different choices of the Hamiltonian (and different choices of the time).
The different systems in the 1T physics are described by a unique two time model.
These systems are dual to each other under local $Sp(2, \mathbb{R})$ transformations.

Carroll symmetry

It is well known that the Poincaré group possesses the contraction, obtained by sending the speed of light to infinity $c \to \infty$. This limit leads to the Galilean group that describes non-relativistic models.

What does happen when we consider the opposite limit: $c \to 0$?

Let us define the new variables

$$t = \frac{1}{c} x_0 \quad \vec{\hat{v}} = \frac{1}{c} \vec{\beta}, \quad b = \frac{1}{c} a_0,$$

requiring that they remain constant after c is sent to zero.

We get the following transformations

$$\begin{cases} t' = t + \vec{\hat{v}} \cdot (R\vec{x}) + b, \\ \vec{x}' = R\vec{x} + \vec{a}. \end{cases}$$

These transformations form the Carroll group.

J.-M. Lévy-Leblond, Une nouvelle limite non-relativiste du groupe de Poincaré, Ann. Inst. Henri Poincaré **3**, 1 (1965).

N. D. Sen Gupta, On an Analogue of the Galilei Group, Nuovo Cimento **44**, 512 (1966).

Carroll Lie algebra

$$
\begin{aligned}
&[L^{ij}, L^{kl}] = \delta^{ik} L^{jl} + \delta^{jl} L^{ik} - \delta^{il} L^{jk} - \delta^{jk} L^{il}, \\
&[L^{ij}, P^k] = \delta^{ik} P^j - \delta^{jk} P^i, \\
&[L^{ij}, B^k] = \delta^{ik} B^j - \delta^{jk} B^i, \\
&[L^{ij}, H] = 0, \\
&[P^i, P^j] = 0, \\
&[P^i, B^j] = \delta^{ij} H, \\
&[P^i, H] = 0, \\
&[B^i, B^j] = 0, \\
&[B^i, H] = 0.
\end{aligned}
$$

Carroll particle in two time spacetime: classical theory

The Carrol particle with non-zero energy should be always in rest.
The Carroll particle with zero energy is always moving.
These two cases are not connected.
The Carroll boosts do not change the value of the energy in contrast to the Lorentzian and Galilean boosts.

The Lie algebra generators corresponding to the Lorentz boosts have the form:

$$t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t},$$

the Galilean boosts can be represented by the generators

$$t\frac{\partial}{\partial x},$$

while the Carroll boosts are

$$x\frac{\partial}{\partial t}.$$

The Hamiltonian is always proportional to the operator

$$\frac{\partial}{\partial t}.$$

The Carroll boost commutes with the Hamiltonian in contrast to the Lorentz boost and Galilei boost.
One cannot change the value of the energy making a boost in the Carroll world and should treat the cases of the vanishing and non-vanishing energy separately.
the conservation of the energy-momentum tensor implies the disappearance of the flux of energy, if the energy is different from zero.
The action for the Carroll particle can be represented as

$$S = -\int d\tau \left\{ \dot{t}E - \dot{x} \cdot p - \lambda \left(E - E_0 \right) \right\},$$

where τ is the proper time, t is the physical time, E represents the classical Hamiltonian and x^i and p^i are the space coordinates and the momenta, for $i = 1, \ldots, d-1$.

$E_0 \neq 0$ represents the rest energy of the Carroll particle and λ plays the role of a Lagrange multiplier.
This action is invariant under the transformations generated by

$$L^{ij} = x^i p^j - x^j p^i, \quad B^i = E x^i, \quad p^i \text{ and } E.$$

Their Poisson brackets satisfy the Carroll algebra.
The equations of motion:

$$\dot{t} = \lambda, \qquad \dot{E} = 0,$$
$$\dot{x}^i = 0, \quad \dot{p}^i = 0.$$

We would like to obtain this action from the 2T action.

Let us introduce the light cone coordinates

$$X^{+} = \frac{1}{2}\left(X^{1'} + X^{0'}\right), \quad X^{-} = \frac{1}{2}\left(X^{1'} - X^{0'}\right).$$

We fix the gauge fields as

$$A_{11} = A_{12} = 0, \; A_{22} = \lambda = const.$$

The two time coordinate and momenta are

$$\begin{aligned} X^{+} &= E_0 t, \\ X^{-} &= \frac{x_i p^i}{E_0} + \frac{t}{E_0}\left(E - E_0 + \frac{p_i p^i}{2}\right), \\ X^{0} &= \sqrt{x_i x^i}, \\ X^{i} &= x^i + t p^i, \end{aligned}$$

$$
\begin{aligned}
P^{+} &= E_0, \\
P^{-} &= \frac{1}{E_0}\left(E - E_0 + \frac{p_i p^i}{2}\right), \\
P^{0} &= 0, \\
P^{i} &= p^i.
\end{aligned}
$$

In terms of this parametrization (gauge-fixing) the constraints are

$$
\begin{aligned}
X \cdot X &= -2t^2(E - E_0), \\
X \cdot P &= -2t(E - E_0), \\
P \cdot P &= -2(E - E_0)
\end{aligned}
$$

and are satisfied if and only if $E = E_0$.

Substituting the above parametrization into the 2 time action we obtain one-time action for a Carroll particle.
The system possesses also the symmetry with respect to $SO(2, d)$ two time Lorentz group.
The generators of the group $SO(2, d)$ are

$$L^{MN} = X^M P^N - X^N P^M,$$

and are invariant under $Sp(2, \mathbb{R})$ transformations.

Written in terms of our reparametrization the are

$$
\begin{aligned}
L^{ij} &= x^i p^j - x^j p^i, \\
L^{0i} &= \sqrt{x^j x_j}\; p^i, \\
L^{+i} &= -E_0 x^i \\
L^{-i} &= -\frac{E - E_0}{E_0} x^i - \frac{p_j p^j}{2E_0} x^i + \frac{p^j x_j}{E_0} p^i, \\
L^{+-} &= -p^i x_i, \\
L^{-0} &= -\sqrt{x_i x^i} \left(\frac{E - E_0}{E_0} + \frac{p_i p^i}{2E_0} \right), \\
L^{+0} &= -E_0 \sqrt{x_i x^i}.
\end{aligned}
$$

The Poisson brackets of these generators do not form an $\mathfrak{so}(2, d)$ algebra, unless the constraint $E - E_0 = 0$ is satisfied.

A direct computation shows that

$$\{L^{-i}, L^{-j}\} = -2\frac{E - E_0}{E_0} L^{ij},$$

which is a new element of the algebra.
When $E - E_0 = 0$ these Poisson brackets vanish and all the generators form the $\mathfrak{so}(2, d)$ algebra, described by

$$\begin{aligned}\{L^{MN}, L^{RS}\} \\ = \eta^{MR} L^{NS} + \eta^{NS} L^{MR} \\ - \eta^{MS} L^{NR} - \eta^{NR} L^{MS}.\end{aligned}$$

Carrol particle in two time spacetime: quantum theory

The commutation relation for the position and momentum operators in the standard $d-1$-dimensional space are

$$[x^i, p^j] = i\ \delta^{ij}.$$

When we quantize some classical functions of these operators, the problem of the choice of the ordering arises.
All the operators should be Hermitian, but this requirement is not sufficient.

For example,

$$p^2 r \to p_i r p^i,$$

which is clearly Hermitian. This ordering is not unique ordering providing the Hermiticity. We can choose another form of the operator

$$p^2 r \to r p_i r^{-1} p^i r = p_i r p^i - \frac{d-3}{2r}.$$

In a more general case

$$p^2 r \to r^\lambda p_i r^{1-2\lambda} p^i r^\lambda = p_i r p^i + \frac{\lambda(\lambda - d + 2)}{2r}.$$

We have to resort to the covariant quantization in the 2T spacetime.

The generators L^{MN} which become operators should constitute the Lie algebra with respect to the commutators.

This requirement also does not define the ordering in the quantum generators in a unique way and one should use also the properties of the Casimir operators of the unitary representations of both the groups $SO(2, d)$ and $Sp(2, \mathbb{R})$. The constraints play the role of the generators of the symmetry with respect to the $Sp(2, \mathbb{R})$ group. They should be applied to the acceptable quantum states of the system according to the prescription of the Dirac quantization of systems with first-class constraints:

$$Q|\Psi\rangle = 0.$$

The same should be valid also for the Casimir operators.

If we choose the basis of the Hermitian quantum generators of the $Sp(2, \mathbb{R})$ group as follows

$$J_0 = \frac{1}{4}(P^2 + X^2), \;\; J_1 = \frac{1}{4}(P^2 - X^2),$$
$$J_2 = \frac{1}{4}(X \cdot P + P \cdot X),$$

Then

$$[J_0, J_1] = iJ_2, \;\; [J_0, J_2] = -iJ_1,$$
$$[J_1, J_2] = -iJ_0.$$

The quadratic Casimir operator is defined as

$$C_2(Sp(2, \mathbb{R})) = J_0^2 - J_1^2 - J_2^2.$$

Using the commutation rules

$$[X^M, P^N] = i\eta^{MN},$$

we can show that

$$C_2(Sp(2,\mathbb{R}) = \frac{1}{4}\left(X^M P^2 X_M - (X \cdot P)(P \cdot X) + \frac{d^2}{4} - 1\right).$$

On the other hand one defines the quadratic Casimir operator for the $SO(2, d)$ group as

$$C_2(SO(2,d)) = \frac{1}{2} L_{MN} L^{MN}$$

and the direct calculation shows that

$$C_2(SO(2,d)) = 4C_2(Sp(2,\mathbb{R}) + 1 - \frac{d^2}{4}.$$

If the generators of the $Sp(2,\mathbb{R})$ select quantum states and their quadratic Casimir operator should be equal to zero, the quadratic Casimir operator on the same quantum states treated as belonging to a representation of the $SO(2,d)$ group should be equal to $1-\frac{d^2}{4}$.
It is this requirement that fixes the ordering in the generators of the $SO(2,d)$ group.
In paper
I. Bars, Conformal symmetry and duality between free particle, H - atom and harmonic oscillator, Phys. Rev. D **58**, 066006 (1998)
this technique was implemented to reproduce the quantization scheme and the spectrum for the hydrogen-like atom.

If we manage to fix the ordering in the generators of $SO(2, d)$ group at $\tau = 0$, then the same ordering will be conserved.
Our parametrization of the variables X^M, P^M at $\tau = 0$ coincides with that used for the description of the hydrogen atom provided we have already put $E = E_0$.
It is amazing because these physical systems are quite different and their actions are also different.
We can use this fact to quantize our Carroll particle.
It does not mean that we shall obtain the discrete spectrum.
The combination of the squared momentum and the inverse radius is not connected with the Hamiltonian.
The momentum is not connected with the velocity (which is equal to zero).

What is the role of the momentum?
It enters into the commutation relations and, hence, the Heisenberg inequality of uncertainties

$$\Delta x^i \cdot \Delta p^j \geq \frac{1}{4}\delta^{ij}$$

is valid.
In contrast to the standard non-relativistic quantum mechanics, we can choose the quantum states with a dispersion of the coordinate Δx as small as we wish, because the growth of the dispersion of the momentum Δp is not important.
Thus, a particle can be localized with an arbitrary high precision.

Concluding remarks

- We have found such a parametrization of the phase space variables in two time spacetime, which permits to describe a Carroll particle in rest in the one time spacetime.
- In quantum theory we have seen an amusing correspondence between our parametrization and that used for the description and quantization of the hydrogen atom.
- The case of the always moving particle (Carroll tachyon) is more complicated. It is under study.
- Another direction of research: field-theoretical systems with Carroll symmetry in the two time world.