Theoretical Concepts in Particle Physics L1

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The goal of these lectures is to introduce the theoretical framework of particle physics, a subject called "Quantum Field Theory." (QFT)

You have now learned that everything is built from "fundamental particles."

We will explore what this really means and why QFT is forced upon us.

We take as a starting premise that

(1) Quantum Mechanics (QM) governs the behavior of our universe at the atomic scale and (2) Special Relativity (SR) kicks in when the kinetic energy of a particle approaches its rest mass ($KE \gtrsim mc^2$).

(Or in the case of massless particles, they always travel at the "speed of light" so SR is always relevant.)

For our purposes, the key concepts we need from each subject are

- QM: probabilistic interpretation
- SR: No faster than light communication

The unique self-consistent mathematical framework that incorporates both of these principles is what we call QFT.

Outline

I. Symmetry

II. From Theory to Observables in QFT

III. Dimensional Analysis

IV. Computing Feynman Diagrams

V. Theories with Multiple Types of Particles

VI. Effective Field Theory

I. Symmetry

- Before we get into QFT, we need to take what might seem like a detour to discuss "symmetry." The idea of symmetry is very intuitive (and we will explain it more precisely in a little while). The reason it is so important in physics is due to a theoretical discovery in 1915 by mathematician Emmy Noether.

- She taught us

If a theory contains a "continuous symmetry," then it must have a "conserved charge."

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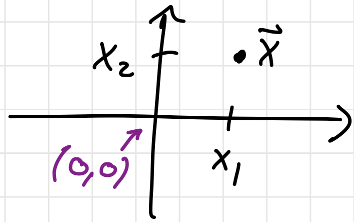
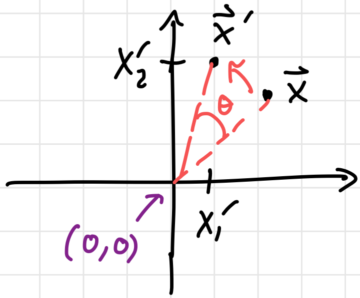
Let's unpack each of these terms:

Continuous symmetry

- Let us work in 2 dimensions for simplicity. We can specify a point on the 2D plane using a vector $\vec{x} = (x_1, x_2)$.



We want to formulate our physical laws so that they are independent of where we are in space. In other words, if we rotate the point $\vec{x}$ about the origin by an angle θ, we have



where

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta\, x_1 + \sin\theta\, x_2 \\ -\sin\theta\, x_1 + \cos\theta\, x_2 \end{pmatrix}$$

Obviously $\vec{x} \neq \vec{x}'$. However, the dot product

$$\vec{x} \cdot \vec{x} = x_1^2 + x_2^2$$

and $\vec{x}' \cdot \vec{x}' = (\cos\theta\, x_1 + \sin\theta\, x_2)^2 + (-\sin\theta x_1 + \cos\theta x_2)^2$

$$= \cos^2\theta\, x_1^2 + \sin^2\theta\, x_2^2 + 2\cos\theta\sin\theta\, x_1 x_2$$
$$+ \sin^2\theta\, x_1^2 + \cos^2\theta\, x_2^2 - 2\cos\theta\sin\theta\, x_1 x_2$$
$$= (\cos^2\theta + \sin^2\theta)(x_1^2 + x_2^2) = x_1^2 + x_2^2$$

We therefore call the combination

$\vec{x} \cdot \vec{x} = \theta$-independent an invariant.

- If we build our theory using only invariants (e.g. we only use functions that depend on $\vec{x}\cdot\vec{x}$, never using $\vec{x}$ alone), Then we are guaranteed that our theory will not have a prefered direction. In more technical terms our theory is symmetric under rotations (a continuous symmetry whose transformation depends on the continuous parameter θ).

- Emmy Noether told us that we should expect there to be an associated conserved quantity. For the case of rotations, This is angular momentum conservation! Indeed the Standard Model of Particle Physics is formulated to be rotationally invariant, and so angular momentum is always conserved.

- This simple example generalizes.

The language of symmetry and the charictorization of possible invariant combinations of generalized "vectors" is best framed in the language of so-called group theory. There are two types of groups that appear in QFT: space-time symmetry and internal symmetry.

Special relativity and Group Theory

- When we say "space-time symmetry" that is just fancy language for special relativity. The Lorentz transformations (moving clocks run slow and moving rulers shrink) can be expressed in terms of the Lorentz group. This group includes the 3D rotations and the Lorentz "boosts."

- We generalize the dot product $\vec{x}\cdot\vec{x}$ into a Lorentz invariant dot product $x_\mu x^\mu$, where x^μ is a four-vector.
  For a given point in space-time, we write $x^\mu = (ct, x_1, x_2, x_3)$, where c = speed of light. Note that ct has units of length, so this is a sensible thing to do. Then rotations mix the x_i's while boosts mix the x_i's with ct.
- If we construct our theories using these new dot products, then they will be Lorentz invariant. In fact, we should also enforce that our theories do not depend on where we perform our experiments.
  In other words, we must enforce that the formulation of our theory is invariant under a space-time translation:

$x^\mu \to x^\mu + \xi^\mu$ where ξ^μ is a four-vector that encodes the four translation parameters.

Noether's theorem applied to translations then implies that

Energy and momentum are conserved!

(always, always, always)*

Internal symmetries and charges

- Internal symmetries are more abstract.

The building blocks of QFT are quantum fields. A field is a mathematical object that encodes a value at every space-time point. The simplist example is a scalar field $\varphi(x)$. A scalar field is used to model particles without spin (e.g. the Higgs boson).

* Ignoring the expansion of the universe due to dark energy ...

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- If a scalar field is real valued, then an example of an internal symmetry is the transformation rule $\varphi(x) \to -\varphi(x)$. If we write a theory that respects this rule, then this implies we can only have even powers of $\varphi(x)$: $(\varphi(x))^2 \to (\varphi(x))^2$ is invariant. For contrast, odd powers are not invariant: $(\varphi(x))^3 \to -(\varphi(x))^3$.

  This imposes a selection rule on the theory: only processes involving even numbers of φ particles are allowed,

  e.g. $\varphi\varphi \to \varphi\varphi$ and $\varphi\varphi \to \varphi\varphi\varphi\varphi$ but not $\varphi\varphi \to \varphi\varphi\varphi$.

  (We call this group $\mathbb{Z}_2$. It is "discrete.")

- If a scalar field is complex valued, $\varphi(x) = \varphi_{real}(x) + i\,\varphi_{imag}(x)$, then we can empose that the theory is invariant under a "phase rotation":

$$\varphi(x) \to e^{i\theta}\varphi(x)$$

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, one can show that this transformation acts to rotate φ_{real} and φ_{imag} into each other.

(We call this group $U(1)$.)

To build an invariant theory, we can only use objects like

$$\varphi^*\varphi \to (e^{i\theta}\varphi)^*(e^{i\theta}\varphi) = \varphi^* e^{-i\theta} e^{i\theta}\varphi$$
$$= \varphi^*\varphi = |\varphi|^2$$

($*$ = complex conjugation)

The parameter θ is continuous, so Noether's Theorem applies.

A theory that respects this phase rotation symmetry has a conserved charge Q.

- Quantum Electrodynamics is the theory of photons and electrons. The electron has exactly such a phase rotation invariance.

In this case, the associated charge Q is the familiar electric charge!

- The phase rotation symmetry transformation is commutative: if we transform by a parameter θ_1 and then by θ_2 this is equivalent to first transforming by θ_2 and then by θ_1, since

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)} = e^{i(\theta_2+\theta_1)} = e^{i\theta_2} e^{i\theta_1}$$

We refer to groups with this property as being Abelian.

- In general, the parameter θ can be promoted to a matrix Θ. Then the phase rotation becomes a matrix too $e^{i\Theta}$.

This matrix acts on a vector $\vec{\varphi}(x)$, and we build invariants the same way: $|\vec{\varphi}(x)|^2$.

(The exponential of a matrix is defined by its Taylor expansion: $e^{A} = 1 + A + \frac{1}{2}A^2 + \dots$)

However, it is not guaranteed that the transformations commute:

$$e^{i\Theta_1} e^{i\Theta_2} = e^{i\left(\Theta_1 + \Theta_2 + \frac{1}{2}[\Theta_1, \Theta_2] + \cdots\right)}$$

while

$$e^{i\Theta_2} e^{i\Theta_1} = e^{i\left(\Theta_2 + \Theta_1 + \frac{1}{2}[\Theta_2, \Theta_1] + \cdots\right)}$$

This is the Baker-Campbell-Hausdorff formula, and $[A, B] = AB - BA$ is a commutator.

We see that $e^{i\Theta_1} e^{i\Theta_2} = e^{i\Theta_2} e^{i\Theta_1}$ if $[\Theta_1, \Theta_2] = 0$. A group with the property $[\Theta_1, \Theta_2] \neq 0$ is called non-Abelian.

- Both Abelian and non-Abelian groups appear in the Standard Model. You may see the group structure of the Standard Model expressed as

$$\underbrace{SU(3)}_{\substack{\text{Strong} \\ \text{force}}} \times \underbrace{SU(2) \times U(1)}_{\substack{\text{Electroweak} \\ \text{forces}}}$$

$U(1)$ is Abelian while $SU(2)$ and $SU(3)$ are non-Abelian

Exercise I

The 4-vector dot product is given by

$$x_1^{\mu} x_{2\mu} = c^2 t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2$$

where $x_1^{\mu} = (ct_1, x_1, y_1, z_1)$ and

$$x_2^{\mu} = (ct_2, x_2, y_2, z_2).$$

Check that $x_1^{\mu} x_{2\mu}$ is invariant under a boost in the x-direction:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

In other words, show that $x_1^{\mu} x_{2\mu} = x_1'^{\mu} x_{2\mu}'$

II. From Theories to Observables in QFT

- QFT predicts that particles can spontaneously transform into other types of particles, as long as the symmetries of the theory are respected. For the Standard Model, this means energy and momentum and charge <u>must</u> be conserved.
- There are two primary observables we compute in QFT:
  1) scattering cross sections σ
  2) decay rates Γ
- Can interpret Γ as probabilistic rate for 1 particle $\rightarrow$ other (lighter) particles

  Can interpret σ as probabilistic rate for 2 particles $\rightarrow$ same 2 particles (elastic scattering) or 2 particles $\rightarrow$ other particles (inelastic scattering).

- Note these observables can only be predicted probabalistically since QFT is a quantum mechanical theory.
- The primary tool for computing observables are Feynman diagrams. They compute an amplitude $\mathcal{A}$ for a given process as a perturbative expansion. Amplitudes for QFT are analogous to wave functions in quantum mechanics: probabilities for observables are proportional to $|\mathcal{A}|^2$.
- We compute Feynman diagrams using a set of Feynman rules. Each theory has its own associated Feynman rules.
- To define a theory, we must specify a function that we call the Lagrangian $\mathcal{L}$. (The idea of a Lagrangian may be familiar from a classical mechanics course.)

The Lagrangian has two parts:

1) The "kinetic" terms: these are universal terms that only require specifying the mass and spin of a given particle. Denote these by $\mathcal{L}_{kin}$.

Often "kinetic terms" only refer to those with derivatives

Ex: For our real scalar field $\varphi(x)$ (models spin-0 particles) with mass m, we have

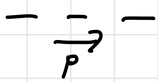
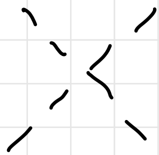
$$\mathcal{L}_{kin} = -\frac{1}{2}\varphi(x)\,\partial_\mu\partial^\mu\varphi(x) - \frac{1}{2}m^2(\varphi(x))^2$$

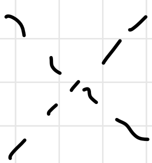
(The notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$ is short hand)

2) The "interaction" terms: These are model specific. They tell us how the particles interact with each other. They include a parameter that sets the strength of the interaction, the coupling constant. Denote by $\mathcal{L}_{int}$.

Ex: In our theory with a real scalar field, we could write $\mathcal{L}_{int} = -\frac{1}{4!}\lambda\varphi^4$.

The $\frac{1}{4!}$ is conventional (makes the Feynman rules look nicer). The λ is the coupling constant. This interaction respects the $\varphi \to -\varphi$ $\mathbb{Z}_2$ symmetry.

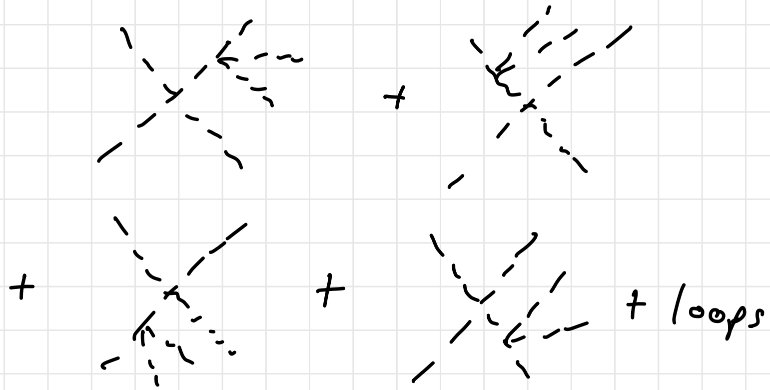
- For this theory, we have the Feynman rules:
  - propagation:  $= \Delta(p)$ (will revisit)
  - interaction:  $= -i\lambda$

- To apply these Feynman rules, our first task is to specify a process, for example $\varphi\varphi \to \varphi\varphi$, $\varphi\varphi \to \varphi\varphi\varphi$, $\varphi\varphi \to \varphi\varphi\varphi\varphi$ and so on. Then we draw all possible Feynman diagrams that can contribute:
  - $\varphi\varphi \to \varphi\varphi$  + loops

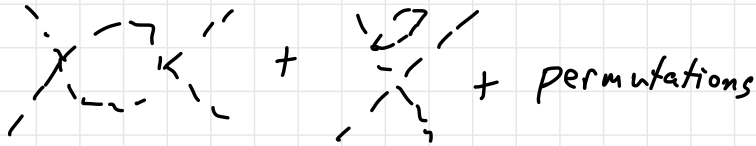
    will revisit

- $\varphi\varphi \to \varphi\varphi\varphi$ no diagrams since it violates the $\mathbb{Z}_2$ symmetry.

- $\varphi\varphi \to \varphi\varphi\varphi\varphi$



- All the diagrams we have drawn here are tree-level diagrams. We also can have diagrams with closed loops, for example $\varphi\varphi \to \varphi\varphi$ diagrams with a single loop:



- The Feynman rules tell us how to go from the diagrams to a formula for the amplitude $\mathcal{A}$.

- Critically, we must conserve energy and momentum at every interaction vertex.

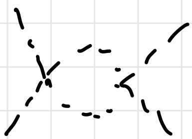
Let us do an example

The Feynman diagrams compute $i\mathcal{A}$

$\varphi\varphi \to \varphi\varphi$  $= -i\lambda = i\mathcal{A}$

$$\Rightarrow \mathcal{A} = -\lambda \Rightarrow \sigma \sim |\mathcal{A}|^2 \sim \lambda^2$$

Note 1-loop diagrams like

 $\sim (-i\lambda)^2 \times (\text{loop stuff}) \sim \lambda^2$

$$\Rightarrow \mathcal{A} = -\lambda + \mathcal{O}(\lambda^2) \Rightarrow \sigma \sim |\mathcal{A}|^2 \sim \lambda^2 + \mathcal{O}(\lambda^3)$$

$\Rightarrow$ Loops are higher order in the λ expansion.

As long as λ is "small", then the loop contribution will be a small correction to the tree-level result.

This is how we compute processes.

We call this perturbation theory.

Exercise II

Consider a theory of a real scalar field $\varphi(x)$ with the interaction Largrangian

$$\mathcal{L}_{int} = \frac{g}{3!}\varphi^3 + \frac{\lambda}{4!}\varphi^4$$

Does this theory have a $\mathbb{Z}_2$ symmetry?

What interaction vertices does this theory have?

Draw all possible Feynman diagrams for the following processes (at tree level)

1) $\varphi\varphi \to \varphi\varphi$
2) $\varphi\varphi \to \varphi\varphi\varphi$
3) $\varphi\varphi \to \varphi\varphi\varphi\varphi$

Bonus: Draw the one-loop diagrams for the same processes.

III. Dimensional Analysis

- We can make our estimate even better by appealing to the most important tool in physics: dimensional analysis.
- We use a clever trick in QFT to make dimensional analysis easy. The two fundamental constants relevant for QFT are the speed of light c (special relativity) and Planck's constant $\hbar$ (quantum mechanics). Then we do something that may cause your skin to crawl: we set $c=1$ and $\hbar=1$. This is effectively choosing a system of units: we call this system natural units.
- To understand the implications of natural units, note that $[c] = \frac{\text{length}}{\text{time}}$ in conventional units.

we use this notation for the units of whatever appears inside

Therefore, $c = 1 \Rightarrow$ length $\sim$ time

This is exactly the lesson of special relativity, and natural units bake it into our dimensional analysis.

- Following the same logic for Planck's constant, we have $[\hbar] = \text{energy} \times \text{time}$

  $\Rightarrow \hbar = 1 \Rightarrow$ energy $\sim$ 1/time

Again, this is the fundamental lesson of quantum mechanics: Energy $\Leftrightarrow$ frequency.

- When working in natural units, we choose to express energy, length, and time in terms of one of them.
- For particle physics, we typically choose to work in terms of energy, so that all dimensionful quantaties in terms of Gigaelectronvolts (GeV).
- Therefore, length $\sim$ GeV^{-1}; time $\sim$ GeV^{-1}

  Energy $\sim$ GeV

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- We can also determine some derived units:

  momentum ~ GeV ; mass ~ GeV

- A good rule of thumb is that the mass of a proton ~ 1 GeV.

- Our scalar quantum field has units too: $[\varphi] = \text{GeV}$. The Lagrangian also has units $[\mathcal{L}] = \text{GeV}^4$. This allows us to determine the units for our coupling constant

$$\mathcal{L}_{int} = -\frac{1}{4!}\lambda\varphi^4 \Rightarrow [\lambda\varphi^4] = [\lambda]\underbrace{[\varphi^4]}_{\text{GeV}^4} = \text{GeV}^4$$

$$\Rightarrow [\lambda] = \text{dimensionless}$$

- The cross section is defined as the quantum analog of a classical scattering cross section, which is determined by the cross sectional area. So σ has units of area ~ length^2 ~ Energy^{-2}

- Putting it all together, we have

$$\sigma \sim \frac{\lambda^2}{\text{Energy}^2} \quad \text{by dimensional analysis.}$$

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- What determines the energy factors in the denominator? Recall that we are colliding two φ particles at some energy E. If $E \sim m$, then $\sigma \sim \frac{\lambda^2}{m^2}$ is roughly constant. If $E >> m$, then $\sigma \sim \frac{\lambda^2}{E^2}$ and falls off quadratically. This is exactly the behavior you would find by doing a detailed QFT calculation!

Exercise III

Consider a theory with $\mathcal{L}_{int} = \frac{g}{3!} \varphi^3$

What are the units of g?

The process $\varphi\varphi \to \varphi\varphi$ comes from Feynman diagrams like  .
Using dimensional analysis, determine the energy scaling for the cross section in this theory for $\varphi\varphi \to \varphi\varphi$.

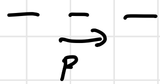
IV. Computing Feynman Diagrams

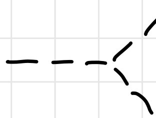
- Consider a different theory of a real scalar field φ with interaction

$$\mathcal{L}_{int} = -\frac{g}{3!}\varphi^3$$

(g is the coupling constant, with $[g] = \text{GeV}$.)

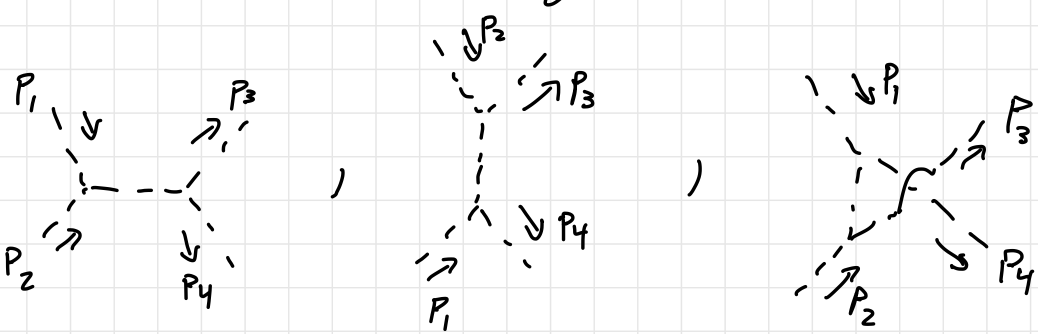
For this theory, the Feynman rules are

- propagation :  $= \Delta(p)$

- interaction :  $= -ig$

- Again, let us consider $\varphi\varphi \to \varphi\varphi$:

There are three diagrams :



Note that we are adding a new feature : the momentum of the particles.

The amplitude we compute for each diagram takes the form $i\mathcal{A} = (-ig)^2 \Delta(p)$.

So we have to understand what this $\Delta(p)$ does.

Energy and Momentum Conservation

- Recall that the Feynman rules require that energy and momentum are conserved at every vertex. Let us take a moment to discuss how to capture this requirement in the language of four-vectors.
- Special relativity tells us that space and time mix under boosts $\Rightarrow$ we introduce $x^\mu = (t, x_1, x_2, x_3)$

(Note $c=1$ here)

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- Under boosts, energy and momentum transform into each other. We therefore combine them into a four-vector

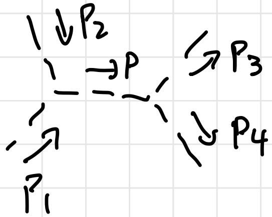
$$p^\mu = (E, p_1, p_2, p_3) \qquad (\text{again with } c=1)$$

Then it is easy to state energy and momentum conservation for our process

$$\varphi\varphi \to \varphi\varphi: \quad p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu .$$

(This should be read as four equations, one for each value of the μ index.)

- Now we have what we need to understand our Feynman diagrams in more detail. Let us start with



Momentum conservation at the left vertex $\Rightarrow p = p_1 + p_2$

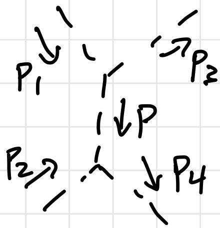
and the right vertex $\Rightarrow p = p_3 + p_4$

which is consistent with overall four-momentum conservation $p_1 + p_2 = p_3 + p_4$

Then the amplitude for this diagram is $\boxed{\text{IV}\,4}$

$$iA = (-ig)^2 \Delta(p_1 + p_2)$$

Similarly

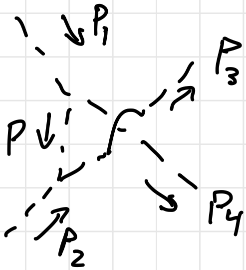


$$\Rightarrow \quad p = p_1 - p_3 \quad \text{or} \quad p = p_4 - p_2$$

so that $p_1 - p_3 = p_4 - p_2 \Leftrightarrow p_1 + p_2 = p_3 + p_4$

$$\Rightarrow \quad iA = (-ig)^2 \Delta(p_1 - p_3)$$

and finally



$$\Rightarrow \quad p = p_1 - p_4 = p_3 - p_2$$

$$\Rightarrow \quad iA = (-ig)^2 \Delta(p_1 - p_4)$$

- All that is left to go from the diagrams to the complete mathematical expression for the amplitudes is we need to know what $\Delta(p)$ is. This is a very important object known as the Feynman propagator.

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The propagator is determined by the kinetic terms in $\mathcal{L}_{kin}$.

For the scalar field $\mathcal{L}_{kin} = -\frac{1}{2}\varphi\,\partial^\mu\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2$

To derive the propagator, we "invert" the operator $\mathcal{O}_{kin} = \partial^\mu\partial_\mu + m^2$

But how do we make sence of $\dfrac{1}{\partial^\mu\partial_\mu + m^2}$?

We work in Fourier space where $\partial^\mu \to ip^\mu$

$$\Rightarrow \frac{1}{\partial^\mu\partial_\mu + m^2} \to \frac{1}{-p^2+m^2} \qquad (p^2 = p^\mu p_\mu)$$

A proper derivation fixes the normalization

$$\Rightarrow \Delta(p) = \frac{i}{p^2-m^2}$$

(Actually, the propagator is really $\Delta(p) = \dfrac{i}{p^2-m^2+i\varepsilon}$ but you'll have to wait until your QFT course to learn about the "$i\varepsilon$")

So we see that $\Delta(p)$ is really just a function of p^2.

- Fundamentally, the propagator is the object that allows us to connect two points in spacetime in such a way that is consistent with causality (no faster than light communication).
- It also forces upon us the idea of "virtual particles". To understand what we mean by this, note that special relativity tells us $p^\mu = (E, \vec{p})$ where $E = \sqrt{\vec{p}^{\,2} + m^2}$

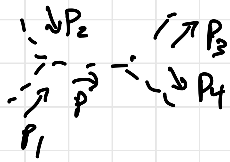
Also note that $p_\mu = (E, -\vec{p})$

This minus sign is critical in SR. Go read about it!

Then $p^2 = p^\mu p_\mu = E^2 - \vec{p}^{\,2} = \vec{p}^{\,2} + m^2 - \vec{p}^{\,2} = m^2$.

$(c=1)$

As we stated before, dot products like $p^\mu p_\mu$ are invariant quantaties. In the case of momentum, $p^2 = m^2$ in any Lorentz frame. This is the definition of the invariant mass m.

- To understand what a virtual particle is, let us return to our diagram  .

In this case

$$p = p_1 + p_2 = (E_1 + E_2, \vec{p}_1 + \vec{p}_2)$$

We can choose a useful Lorentz frame, the center-of-mass frame defined so that $\vec{p}_1 = -\vec{p}_2$. In this frame

$$p = (E_1 + E_2, 0)$$

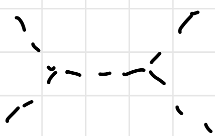
$$\Rightarrow p^2 = (p_1 + p_2)^2 = (E_1 + E_2)^2 = \left(\sqrt{\vec{p}_1^{\,2} + m^2} + \sqrt{\vec{p}_2^{\,2} + m^2}\right)^2$$

$$= 4(\vec{p}_1^{\,2} + m^2) = \underbrace{4E^2}_{\text{in cm frame}} > m^2$$

This tells us that the φ particle in the propagator has $p^2 \neq m^2$, even though the mass of this particle is m^2.

We call a particle with $p^2 \neq m^2$ a virtual particle.

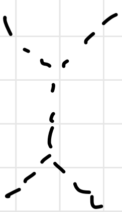
- Virtual particles only occur on the inside of Feynman diagrams. But there is no contridiction because the particles we observe are associated with the external lines in the diagram, for which we always have $p^2 = m^2$.
- The $2 \to 2$ scattering process is so important in particle physics, that we give each type of diagram a special name: (explicit evaluations are in center of mass frame)

 : s-channel

$(\vec{p}^{\,2} = \vec{p}_1^{\,2} = \vec{p}_2^{\,2})$

$$s = (p_1 + p_2)^2 = 4(\vec{p}^{\,2} + m^2) > 0$$

(Note that $s = E^2_{collision} \Rightarrow$ LHC collision energy often stated as $\sqrt{s} = 13.6$ TeV)

 : t-channel

$$t = (p_1 - p_3)^2 = -2\,\vec{p}^{\,2}(1 - \cos\theta) < 0$$

θ: angle between incoming and outgoing particles

 : u-channel

$$u = (p_1 - p_4)^2 = -2\vec{p}^{\,2}(1+\cos\theta) < 0$$

(Derive these yourself!)

Then $s+t+u = 4m^2 \Rightarrow$ only two of them are independent.

In other words, the two kinematic parameters are the energy of the collision and the angle of the final state particle direction (for $2\to2$ scattering).

The variables s, t, u are Lorentz invariants and are often called the Mandelstam variables.

- For the scalar theory we have been studying, the amplitudes are

$$A_s = -g^2\frac{1}{s-m^2}, \quad A_t = -g^2\frac{1}{t-m^2}, \quad A_u = -g^2\frac{1}{u-m^2}$$

So the total amplitude is $A = -g^2\left(\frac{1}{s-m^2} + \frac{1}{t-m^2} + \frac{1}{u-m^2}\right)$.

Exercise IV

Consider a process $\varphi(p_1)\,\varphi(p_2) \to \varphi(p_3)\,\varphi(p_4)$

Express the Mandelstam variables

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2$$

in the center of mass frame where $\vec{p}_1 = -\vec{p}_2$ and $\vec{p}_3 = -\vec{p}_4$

Hint: write $p_1 = (\sqrt{p^2+m^2}, 0, 0, p)$

$p_2 =$ you fill in

$p_3 = (\sqrt{p^2+m^2}, 0, p\sin\theta, p\cos\theta)$

$p_4 =$ you fill in

ensuring that total energy and momentum is conserved: $p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu$

Note $s \geq 4m^2$

$$-(s - 4m^2) \leq t < 0$$

$$-(s - 4m^2) \leq u < 0$$

V. Theories with Multiple Types of Particles

- Let us study a new theory, with two types of real scalar fields. Let φ correspond to a particle with mass m and Φ correspond to a particle with mass M. Each field has its own kinetic terms, and so there is a propagator for each.

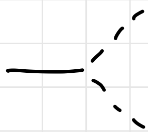
- We choose the interaction Lagrangian to be

$$\mathcal{L}_{int} = -\frac{a}{2}\Phi\varphi^2 \qquad (a \text{ is the coupling constant})$$

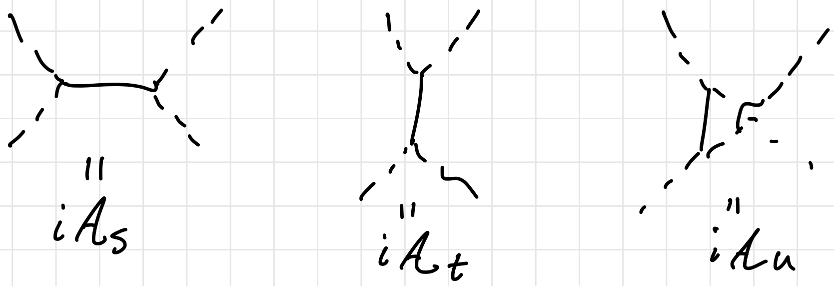
The Feynman rules are

φ propagator: $= \Delta_\varphi$

Φ propagator:  $= \Delta_\Phi$

Interaction:  $= -ia$

- Let us study the $2 \to 2$ process $\varphi\varphi \to \varphi\varphi$ in this theory:



Following the same logic as before, we can evaluate these amplitudes:

$$A_s = -a^2 \frac{1}{s - M^2}$$

$$A_t = -a^2 \frac{1}{t - M^2}$$

$$A_u = -a^2 \frac{1}{u - M^2}$$

- Recall that $t < 0$ and $u < 0$, so these amplitudes are well defined in the entire physical region.

- However $s \geq 4m^2 > 0$. So if we are working with a theory such that $M^2 > 4m^2$, then there exists a physically allowed value of s such that $s_{res} = M^2 \Rightarrow \mathcal{A}_s(s_{res}) = -a^2 \frac{1}{M^2 - M^2} \to \infty$.

  $\uparrow$ for resonance

  This is a pole in the propagator.

  The presence of a pole corresponds to a particle "going on-shell".

  In other words, the position of the pole (in the complex plane) tells us the mass of the propagating particle.

- Have we lost all our predictive power?

  The resolution is that the Φ particle can decay. When a particle can decay, its propagator is modified: we must use the Breit-Wigner propagator instead

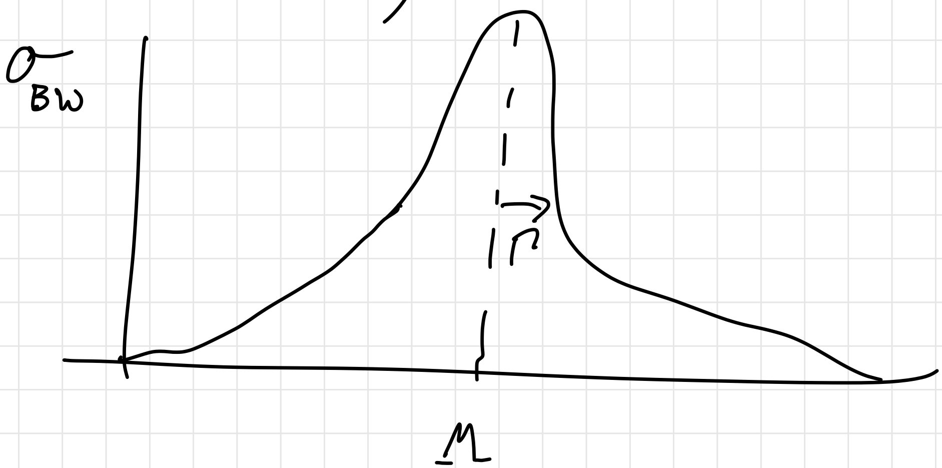
$$\Delta_{BW} = \frac{i}{p^2 - M^2 + iM\Gamma}$$

V 4

Then the cross section takes the form $\sigma_{BW} \sim |\Delta_{BW}|^2 \sim \frac{1}{(p^2-M^2)^2+M^2\Gamma^2}$ (work it out!)

Here Γ is the decay width of the particle, and can be computed from Feynman diagrams like  .

This "resolves" the pole, and now the s-channel diagram is finite:

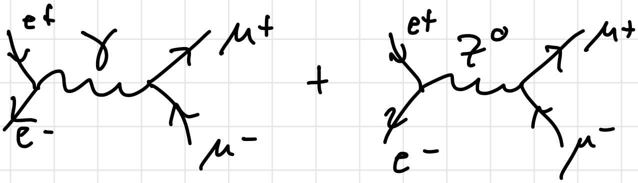


- Look up Z-boson line shape on google images to find one of the most beautiful plots in particle physics.

Searching for resonant features like the one above is one way we search for new particles.

Exercise V

The process $e^+e^- \to \mu^+\mu^-$ has two contributions in the Standard Model



γ = photon

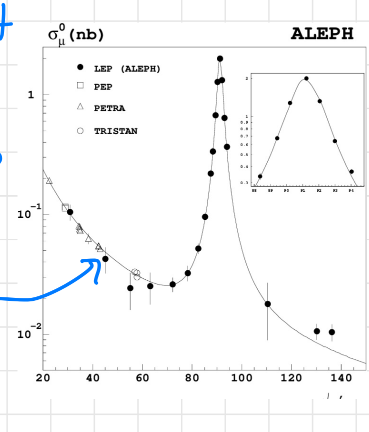
Z = Z-boson

The photon propagator $\sim 1/p^2$ (photon is massless)

Using your favorite software package, plot the Breit-Wigner line shape, also including a contribution for the photon assuming the couplings are the same for simplicity (they do take different values in the Standard Model)

$$\sigma \sim |\Delta_\gamma - \Delta_Z|^2$$

Does it look like this?



with $\Delta_\gamma \sim 1/p^2$

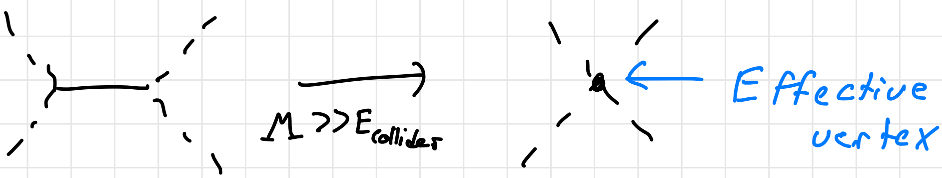
$$\Delta_Z \sim \frac{1}{p^2 - m_Z^2 + i\Gamma_Z m_Z}$$

$m_Z = 91$ GeV, $\Gamma_Z = 2.5$ GeV

VI. Effective Field Theory

- We can use our theory with $\mathcal{L}_{int} = -\frac{a}{2}\varphi^2\Phi$ to get some insight into another very important QFT concept called Effective Field Theory (EFT).
- Imagine that we have a $\varphi\varphi$-collider that operates at an energy $E_{collider} \ll M$. This implies that a Taylor expansion of the propagator should give us a good approximation for our process: $\frac{1}{p^2 - M^2} = \frac{-1}{M^2} + \ldots$

  This has a natural interpretation in terms of Feynman diagrams:

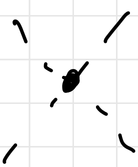


- To understand the implications of this expansion, let us do some dimensional analysis. Using the same arguments as before, we know that $[a] \sim \text{GeV}$.
- The "natural" expectation is that all dimensionful quantaties should be proportional to the largest mass scale in the theory (since this corresponds to the shortest distance, ie, it is the most fundamental)

So we expect $a = (\text{numerical prefactor}) \times \Lambda$.

* Note that m defies this expectation: this is the famous Hierarchy problem!

- We can now estimate the size of the EFT correction to $\varphi\varphi \to \varphi\varphi$

 $\sim \dfrac{a^2}{\Lambda^2} \sim$ order one

This is independent of Λ.

VI 3

- Let us assume our theory also includes a Φ self-interaction $\mathcal{L}_{int} = -\frac{a}{2}\varphi^2\Phi - \frac{b}{2}\Phi^3$

What about a process with more φ's?

$\varphi\varphi \to \varphi\varphi\varphi\varphi$:

Again $[b] \sim GeV \Rightarrow$ expect $b = (\text{number}) \times M$

Then the amplitude goes like

$$A \sim \frac{a^3 b}{(M^2)^3} \sim \frac{M^4}{M^6} \sim \frac{1}{M^2}$$

So as $M \to \infty$, the contribution to $\varphi\varphi \to \varphi\varphi\varphi\varphi$ drops off.

This is a general phenomenon known as heavy particle decoupling.

- This is a direct concequence of reductionism. It tells us that we do not need to know about the existance of particles with masses far beyond our experimental reach in order to make predictions at accessable energies.
- All QFTs are really EFTs in this sence. They are simply systematic approximations of a more fundamental description that allow us to make predictions at experiments like the LHC.
- EFT also allows us a way to introduce deviations to our lower energy descriptions by systematically including effects supressed by $1/M^2$ (and higher powers). Many searches being done at the LHC rely on exactly this approach.

Exercise VI

Compute $\varphi\varphi \to \varphi\varphi$ amplitude in the theory with φ and Φ. Expand the resulting at energy $E \ll M$, keeping only the leading term. Call this $\mathcal{A}_{Fund}$

Next, compute the amplitude $\varphi\varphi \to \varphi\varphi$ in an EFT with only φ, assuming
$\mathcal{L}^{EFT}_{int} = \frac{g}{3!}\varphi^3 + \frac{\lambda}{4!}\varphi^4$. Call this $\mathcal{A}_{EFT}$.

Then determine g and λ in the EFT by equating $\mathcal{A}_{Fund} = \mathcal{A}_{EFT}$. (This is called "matching.") This relates the EFT parameters we measure at experiments to the more fundamental description with the heavy particle Φ. (Hint: you might find that you do not need g and λ both non-zero. In fact you could anticipate this from symmetry properties of the fundamental theory.)