Spectral Functions and Conformal Windows

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Asymptotic Safety meets Particle Physics & Friends
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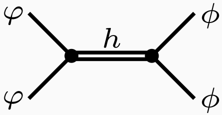
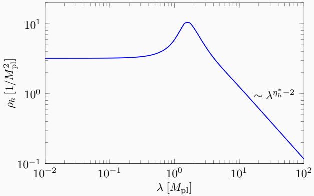
Correlation functions of the quantum effective action

- Want access to full quantum 1PI effective action Γ
- Special importance: propagator $G \sim {\Gamma^{(2)}}^{-1}$
- Important input for scattering



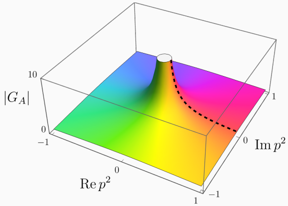
- Test properties of spectral function in conformal windows
- Understand quantum-gravity through comparison with perturbative theories

Gravity



[Fehre, Litim, Pawlowski, MR '21]

Banks-Zaks

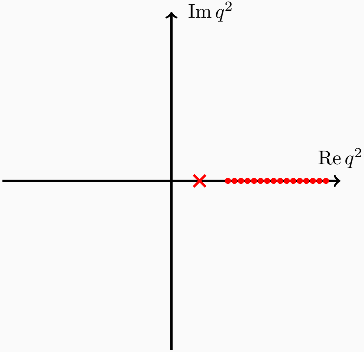


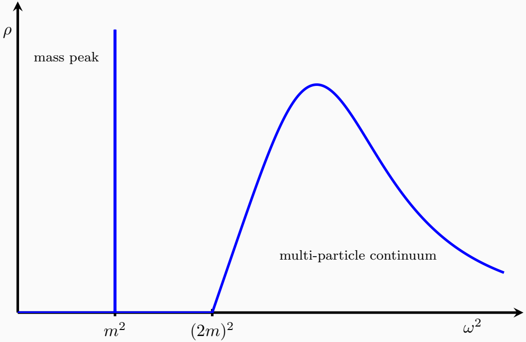
[Kluth, Litim, Reichert '22]

Källén-Lehmann spectral representation

[Källén '52; Lehmann '54]

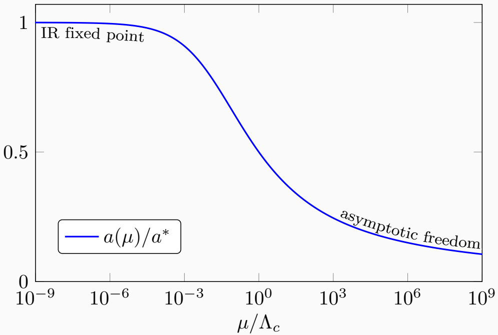
$$G(q^2) = \int_0^\infty \frac{\mathrm{d}\lambda^2}{\pi} \frac{\rho(\lambda^2)}{q^2 - \lambda^2} \qquad \text{with} \qquad \rho(\omega^2) = -\lim_{\varepsilon \to 0} \operatorname{Im} G(\omega^2 + i\varepsilon)$$

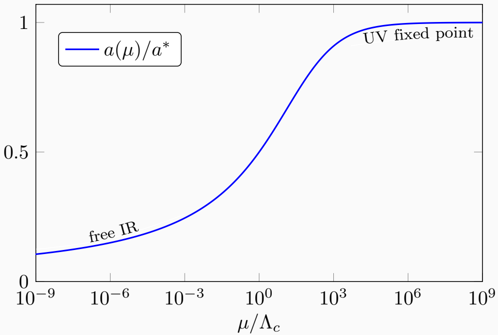




with $\rho(\omega^2) > 0$ and $\int \rho(\lambda^2)\mathrm{d}\lambda^2 = 1$

Asymptotic freedom vs safety





- Banks-Zaks with $\frac{N_f}{N_c} = \frac{11}{2} - \varepsilon$
- Wilson-Fisher in $d = 4 - \varepsilon$

- Litim-Sannino with $\frac{N_f}{N_c} = \frac{11}{2} + \varepsilon$
- Quantum Gravity in $d = 2 + \varepsilon$

Banks-Zaks fixed point

[Belavin, Migdal '74; Caswell '74; Banks, Zaks '82]

$SU(N_c)$ gauge theory with N_f quarks

$$\beta(a) = \beta_1 a^2 + \beta_2 a^3 + \ldots$$

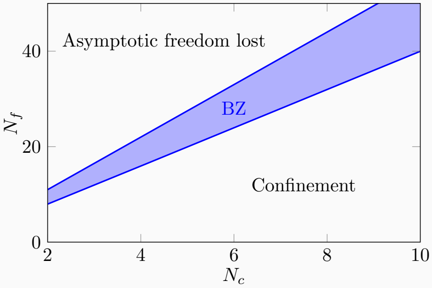
Veneziano limit gives $\beta_1 \sim -\varepsilon$ with

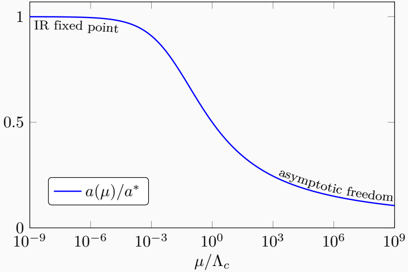
$$\varepsilon = \frac{11}{2} - \frac{N_f}{N_c} > 0$$

$$N_c \ \&\ N_f \to \infty$$

Perturbative IR fixed point

$$a_* = -\frac{\beta_1}{\beta_2} + \mathcal{O}(\beta_1^2)$$





Litim-Sannino model – asymptotically safe

[Litim, Sannino '14]

$SU(N_c)$ with N_f quarks and uncharged $N_f \times N_f$ matrix scalar

$$L \sim L_{\rm YM} + L_{\rm kin} - y\,{\rm Tr}\bar\psi_L H \psi_R - u\,{\rm Tr} H^\dagger H H^\dagger H - v\,({\rm Tr} H^\dagger H)^2$$

Veneziano limit N_c & $N_f \to \infty$ with

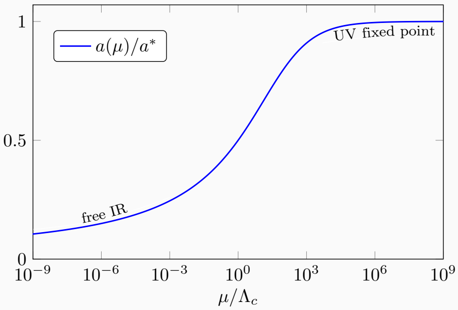
$$\varepsilon = \frac{11}{2} - \frac{N_f}{N_c} < 0$$

Perturbative UV fixed point

$$a_g^*,\ a_y^*,\ a_u^*,\ a_v^* \sim \varepsilon$$

One relevant direction

$$a_y(a_g(\mu)),\ a_u(a_g(\mu)),\ a_v(a_g(\mu))$$



Litim-Sannino model – asymptotically free

$SO(2N_c)$ with N_f quarks and uncharged $N_f \times N_f$ matrix scalar

$$L \sim L_{\mathrm{YM}} + L_{\mathrm{kin}} - y\,\mathrm{Tr}\bar{\psi}_L H \psi_R - u\,\mathrm{Tr} H^\dagger H H^\dagger H - v\,(\mathrm{Tr} H^\dagger H)^2$$

Veneziano limit N_c & $N_f \to \infty$ with

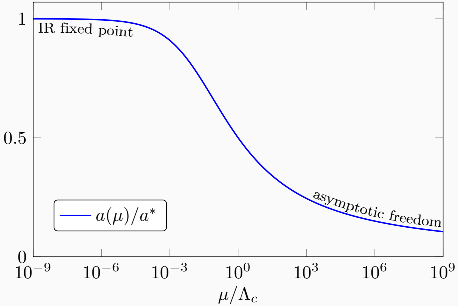
$$\varepsilon = \frac{11}{2} - \frac{N_f}{N_c} > 0$$

Perturbative UV fixed point

$$a_g^*,\ a_y^*,\ a_u^*,\ a_v^* \sim \varepsilon$$

One IR attractive direction

$$a_y(a_g(\mu)),\ a_u(a_g(\mu)),\ a_v(a_g(\mu))$$



Callan-Symanzik Resummation

[Kluth, Litim, Reichert '22]

Propagator with self-energy loop corrections

$$G_\phi(p^2, \mu^2) = \frac{1}{-p^2} \frac{1}{1 + \Pi_\phi(p^2, \mu^2)}$$

Callan-Symanzik equation to resum large logarithms $\log(p^2/\mu^2)$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}(Z_\phi G_\phi) = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(a) \frac{\partial}{\partial a} - \gamma_\phi\right) G_\phi = 0$$

Callan-Symanzik Resummation

[Kluth, Litim, Reichert '22]

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Resummed Callan-Symanzik propagator with $\bar{a} \equiv \bar{a}(p^2)$ & $a \equiv a(\mu^2)$

$$G_\phi = \frac{1}{p^2}\frac{\mathcal{N}_\phi}{1+\sum_n \Pi_\phi^{(n)}(p^2=-\mu^2)\bar{a}(p^2)^n}\left(\frac{a(\mu^2)}{\bar{a}(p^2)}\right)^{\gamma_\phi^{(1)}/\beta_1}\prod_i\left(\frac{a(\mu^2)-a_{*,i}}{\bar{a}(p^2)-a_{*,i}}\right)^{\gamma_\phi(a_{*,i})/\theta_i}$$

Callan-Symanzik Resummation – multiple couplings

[Kluth, Litim, Reichert (in prep)]

Callan-Symanzik equation with multiple couplings

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \sum_i \beta_i \frac{\partial}{\partial a_i} - \gamma_\phi\right) G_\phi = 0$$

One monotonic coupling to re-express all others $a_i(\mu^2) = a_i(a(\mu^2))$, leads to same CS equation

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}(Z_\phi G_\phi) = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta_{\text{eff}} \frac{\partial}{\partial a} - \gamma_\phi\right) G_\phi = 0$$

Callan-Symanzik Resummation – discussion

Resummed Callan-Symanzik propagator with $\bar{a} \equiv \bar{a}(p^2)$ & $a \equiv a(\mu^2)$

$$G_\phi = \frac{1}{p^2} \frac{\mathcal{N}_\phi}{1 + \sum_n \Pi_\phi^{(n)} \bar{a}^n} \left(\frac{a}{\bar{a}}\right)^{\gamma_\phi^{(1)}/\beta_1} \prod_i \left(\frac{a - a_{*,i}}{\bar{a} - a_{*,i}}\right)^{\gamma_\phi(a_{*,i})/\theta_i}$$

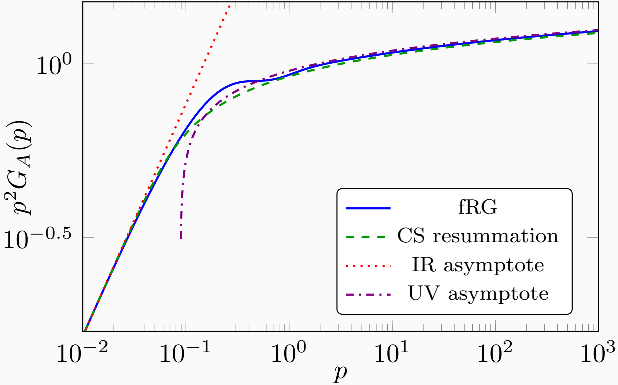
- All fixed points including complex and negative ones contribute
- Bound states can appear through the self-energies
- Self-energies and anomalous dimensions are gauge dependent for charged particles
- Banks-Zaks: 5-loop β and γ-functions & 4-loop self-energies

  [Ruijl, Ueda, Vermaseren, Vogt '17; Herzog, Ruijl, Ueda, Vermaseren, Vogt '17; Chetyrkin, Falcioni, Herzog, Vermaseren '17]

  Litim-Sannino: 433-loop β and γ-functions & no self-energies

  [Bond, Litim, Vazquez, Steudtner '18; Litim, Riyaz, Stamou, Steudtner '23]

Intermezzo – comparison to fRG compuation



[Kluth, Litim, Reichert '22]

Very good agreement between fRG and CS resummation

Existence and normalisation of spectral functions

$$G_\phi = \frac{1}{p^2} \frac{\mathcal{N}_\phi}{1 + \sum_n \Pi_\phi^{(n)} \bar{a}^n} \left(\frac{a}{\bar{a}}\right)^{\gamma_\phi^{(1)}/\beta_1} \prod_i \left(\frac{a - a_{*,i}}{\bar{a} - a_{*,i}}\right)^{\gamma_\phi(a_{*,i})/\theta_i}$$

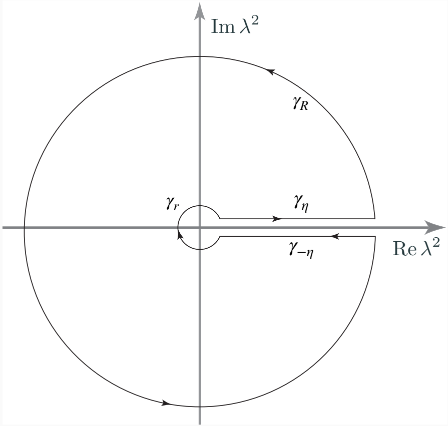
When can we use

$$G(q^2) = \int_0^\infty \frac{\mathrm{d}\lambda^2}{\pi} \frac{\rho(\lambda^2)}{q^2 - \lambda^2}$$

with

$$\rho(\omega^2) = -\lim_{\varepsilon \to 0} \operatorname{Im} G(\omega^2 + i\varepsilon)$$

Assume no complex structures for the moment



Existence and normalisation of spectral functions

$$G_\phi = \frac{1}{p^2}\frac{\mathcal{N}_\phi}{1+\sum_n \Pi_\phi^{(n)}\bar{a}^n}\left(\frac{a}{\bar{a}}\right)^{\gamma_\phi^{(1)}/\beta_1}\prod_i\left(\frac{a-a_{*,i}}{\bar{a}-a_{*,i}}\right)^{\gamma_\phi(a_{*,i})/\theta_i}$$

Existence in the IR

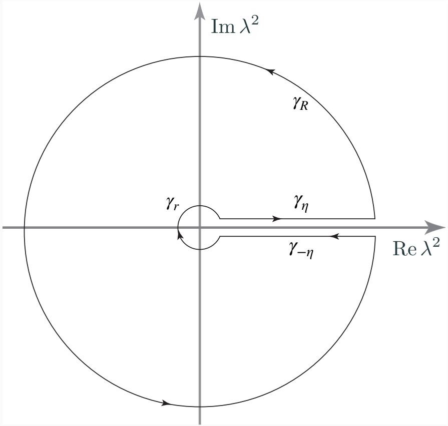
- At most singular with $G_\phi \sim 1/p^2$ for $p \to 0$
- Exception: $G_\phi \sim 1/p^{2n}$

Existence in the UV

- Decay for $p \to \infty$: $G_\phi <$ const

Normalisation determined by $p \to \infty$ behaviour

$$\int \rho_\phi \mathrm{d}\lambda^2 = \begin{cases} 0 & G_\phi < 1/p^2 \\ 1 \quad \text{if} & G_\phi = 1/p^2 \\ \infty & G_\phi > 1/p^2 \end{cases}$$



Existence and normalisation of spectral functions

[Kluth, Litim, Reichert (in prep)]

IR safe & UV free

Existence IR:

$$\gamma_\phi^* \geq 0$$

Existence UV:

trivial

Normalisable:

$$\gamma_\phi^{(1)} = 0$$

UV safe & IR free

Existence IR

$$\gamma_\phi^{(1)}/\beta_1 \geq 0$$

Existence UV

$$\gamma_\phi^* < 1$$

Normalisable

$$\gamma_\phi^* = 0$$

Existence and normalisation of spectral functions

[Kluth, Litim, Reichert (in prep)]

IR safe & UV free	UV safe & IR free
Existence IR:	Existence IR
$\gamma_\phi^* \geq 0$	$\gamma_\phi^{(1)}/\beta_1 \geq 0$
Existence UV:	Existence UV
trivial	$\gamma_\phi^* < 1$
Normalisable:	Normalisable
$\gamma_\phi^{(1)} = 0$	$\gamma_\phi^* = 0$

$0 \leq \gamma_\phi < 1$ necessary for existence (gauge dependent if ϕ carries gauge charge)

Normalisability

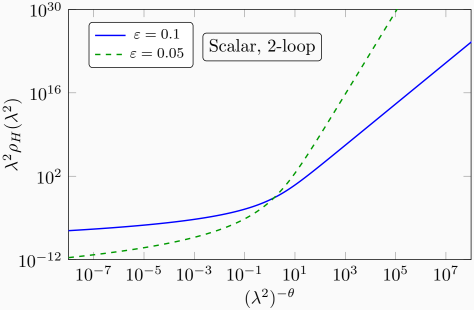
There is no perturbative fundamental scalar field with a normalisable spectral function

Normalisability

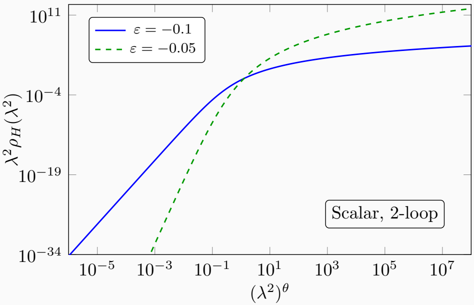
There is no perturbative fundamental scalar field with a normalisable spectral function

Example: Litim-Sannino model at 211 loop order

IR free & UV safe



IR safe & UV free



[Kluth, Litim, Reichert (in prep)]

Intermediate summary

- Fully analytic computation of propagator

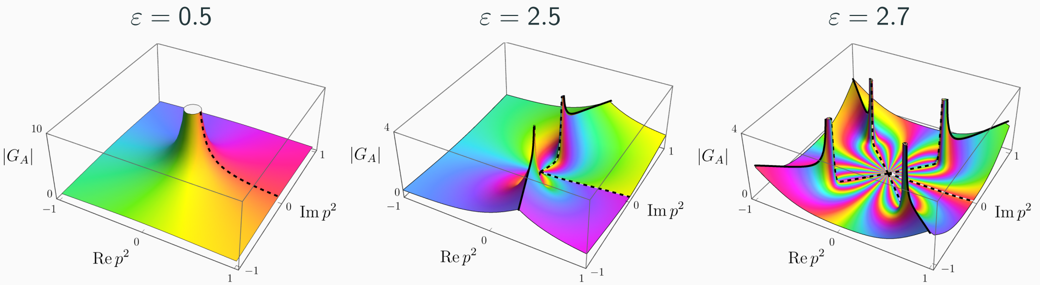
$$G_\phi = \frac{1}{p^2} \frac{\mathcal{N}_\phi}{1 + \sum_n \Pi_\phi^{(n)} \bar{a}^n} \left(\frac{a}{\bar{a}}\right)^{\gamma_\phi^{(1)}/\beta_1} \prod_i \left(\frac{a - a_{*,i}}{\bar{a} - a_{*,i}}\right)^{\gamma_\phi(a_{*,i})/\theta_i}$$

- Momentum-dependent coupling $\bar{a}(p)$ carries all momentum dependences
- $0 \leq \gamma_\phi < 1$ necessary for existence of spectral function
- No normalisable fundamental spectral functions

Can we constrain the size of conformal windows via

- non-analyticities in the complex momentum plane?
- positivity of the spectral function?

Propagator in the complex plane (Banks-Zaks at 2-loop)



[Kluth, Litim, MR '22]

- Branch cuts and violation of KL spectral representation for large ε
- Branch cuts are inherited from the running coupling

Existence of branch cuts in the coupling

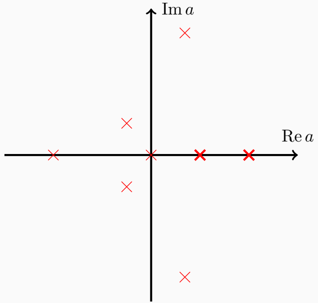
[Kluth, Litim, MR '22]

Perturbative n-loop beta function

$$\beta(a) = d_a a + \beta_1 a^2 + \beta_2 a^3 + \cdots + \beta_n a^{n+1}$$

Absence of branch cuts guaranteed if (a_0 initial value)

$$\left| \sum_i^{a_{i,*} > a_0} \frac{1}{\theta_i} \right| > 1$$



- Gauge independent statement due to universal critical exponents $\theta_i = \partial_a \beta(a)|_{a_{i,*}}$
- Only real and positive fixed points contribute
- Absence of branch cuts guaranteed for $\varepsilon \to 0$ since $\theta_{BZ/LS} \to 0$

Size of conformal Banks-Zaks window

$N_c = 3$	$\varepsilon_{\text{branch-cut}}$	ε_{max}
2-loop	2.2723	2.8158
3-loop	2.6798	3.5520
4-loop	2.6817	3.0538
5-loop	–	1.2019
5-loop Padé [1,3]	–	2.2183
5-loop Padé [2,2]	–	1.6993
5-loop Padé [3,1]	–	0.7304

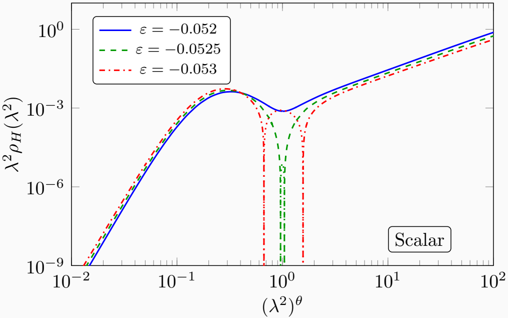
[Kluth, Litim, MR '22]

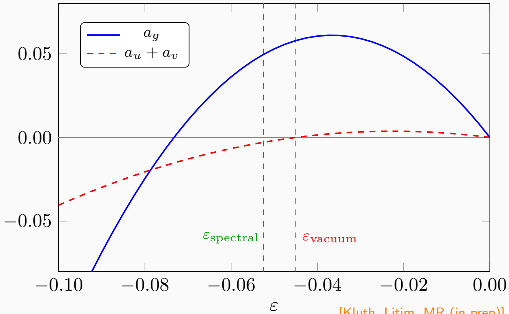
Existence of branch cuts does not post a tighter constrain on BZ window at 5-loop

Disappearance of fixed point via merger is essential

Positivity of spectral function $SO(2N_c)$ Litim-Sannino

ε-expansion at 322 loop order



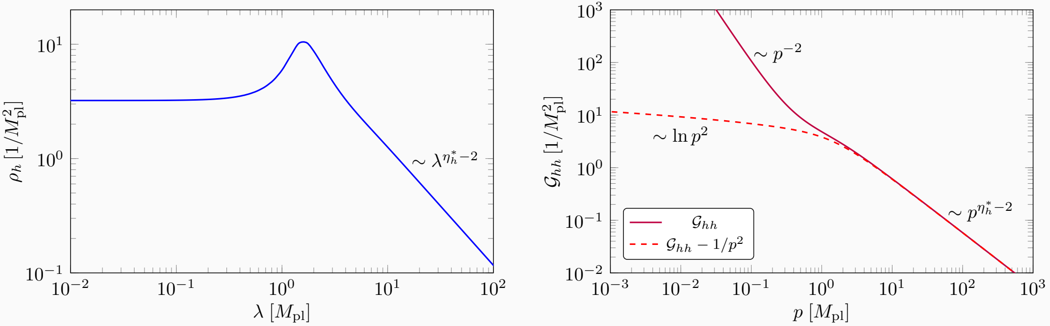


[Kluth, Litim, MR (in prep)]

- Loss of vacuum stability at $\varepsilon = -0.0450$
- Loss of positivity of spectral function at $\varepsilon = -0.0525$

Comparison to Quantum Gravity

Quantum Gravity results from spectral RG



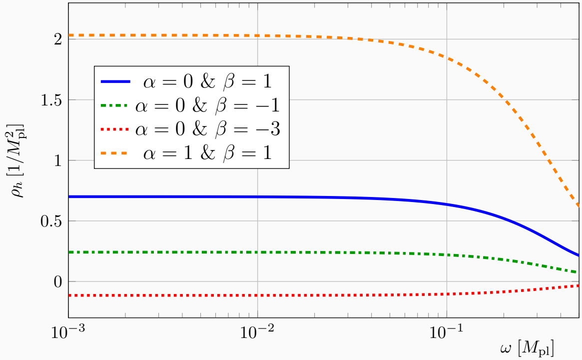
[Fehre, Litim, Pawlowski, MR '21]

- Massless graviton delta-peak with positive multi-graviton continuum
- Non-normalisable spectral function
- No ghosts and no tachyons $\longrightarrow$ no indications for unitarity violation
- Good agreement with reconstruction result and EFT

[Bonanno, Denz, Pawlowski, MR '21]

Gauge dependence of graviton spectral function

Gauge-fixing $S_{\text{gf}} = \frac{1}{\alpha}\int_x F_\mu^2$ with $F_\mu = \bar\nabla^\nu h_{\mu\nu} - \frac{1+\beta}{4}\bar\nabla_\mu h^\nu{}_\nu$

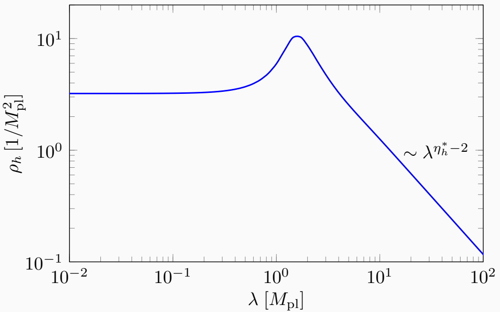


[Pawlowski, MR '23]

Propagator is gauge-dependent but pole structure is typically not

Spectral functions of massless particles

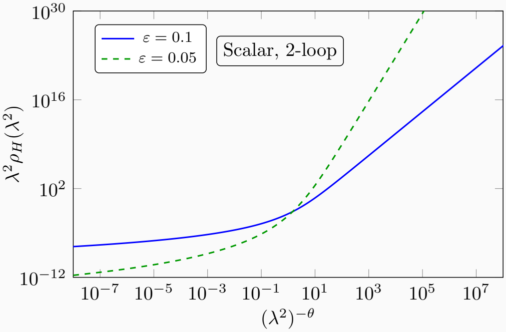
Gravity



[Fehre, Litim, Pawlowski, MR '21]

On-shell δ-peak at $\lambda = 0$

Litim-Sannino



[Kluth, Litim, MR (in prep)]

No on-shell δ-peak

Both theories are UV asymptotically safe and free in the IR

Spectral functions of massless particles – QG and QED

One-loop effective action of QG

$$\Gamma_{\text{1-loop,QG}} \sim G_{\mathrm{N}} R \log(\frac{\Box}{\mu^2}) R + \dots$$

Graviton propagator

$$\mathcal{G}_{hh}(p^2) \sim \frac{1}{p^2 + \# G_{\mathrm{N}} \ln(p^2/\mu^2) p^4}$$

Graviton spectral function

$$\rho_h(\lambda^2) \sim \delta(\lambda^2) + \# G_{\mathrm{N}} 2\pi + \dots$$

One-loop effective action of QED

$$\Gamma_{\text{1-loop,QED}} \sim F_{\mu\nu} \log(\frac{m_e^2 + \Box}{m_e^2}) F_{\mu\nu} + \dots$$

Photon spectral function

$$\rho_\gamma(\lambda^2) \sim \delta(\lambda^2) + \#\theta(\lambda^2 - m_e^2) + \dots$$

Mass dimension of G_{N}, electron mass, lack of photon-self interactions preserve on-shell δ peak

Summary

Banks-Zaks & Litim-Sannino

- Analytic and perturbatively controlled approach to correlation and spectral functions
- $0 \leq \gamma_\phi < 1$ necessary for existence of spectral function
- Normalisability of fundamental spectral function impossible in perturbation theory
- Positivity and non-analyticities offer independent constraints on conformal windows

Asymptotically safe quantum gravity

- Direct computation of graviton spectral function with spectral fRG
- Well-behaved spectral function without cuts in the complex plane
- Key step towards scattering processes and unitarity

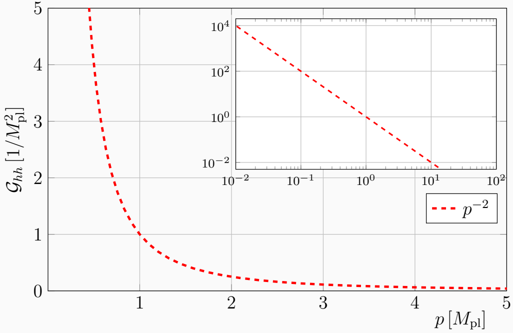
Thank you for your attention!

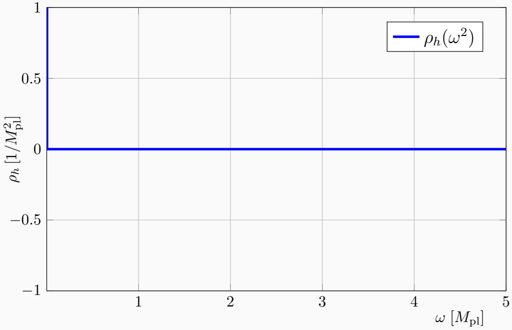
Back-up slides

Classical graviton spectral function

Einstein-Hilbert action: $S_{\text{EH}} = \frac{1}{16\pi G_{\text{N}}} \int_x \sqrt{g}\,(2\Lambda - R)$

Flat Minkowski background: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$



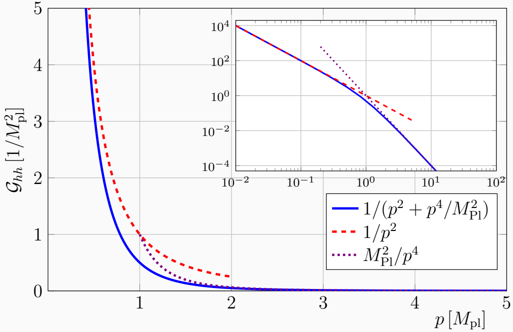


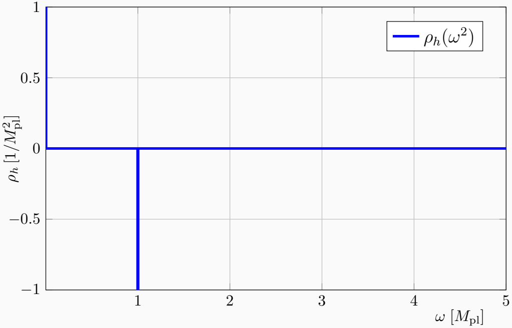
$$\mathcal{G}_{hh}(p^2) \sim \frac{1}{p^2}$$

$$\rho_h(\omega^2) \sim \delta(\omega^2)$$

Classical graviton spectral function

Higher-derivative action: $S_{\text{HD}} = S_{\text{EH}} + \int_x \sqrt{g}\,(aR^2 + bC^2_{\mu\nu\rho\sigma})$





$$\mathcal{G}_{hh}(p^2) \sim \frac{1}{p^2} - \frac{1}{M_{\text{Pl}}^2 + p^2}$$

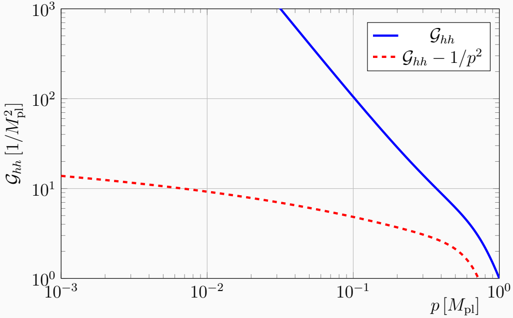
$$\rho_h(\omega^2) \sim \delta(\omega^2) - \delta(\omega^2 - M_{\text{Pl}}^2)$$

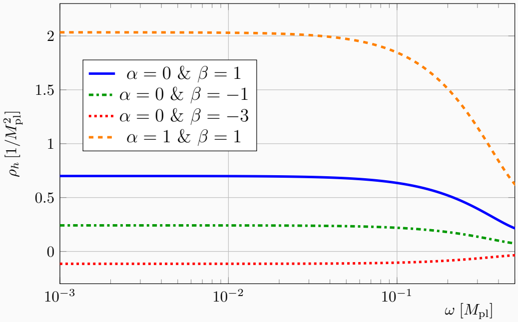
EFT graviton spectral function

One-loop effective action:

$$\Gamma_{\text{1-loop}} = S_{\text{EH}} + \int_x \sqrt{g}\,\big(c_1 R \ln(\Box) R + c_2 C_{\mu\nu\rho\sigma} \ln(\Box) C^{\mu\nu\rho\sigma}\big) + \dots$$

Gauge-fixing $\quad S_{\text{gf}} = \frac{1}{\alpha}\int_x F_\mu^2 \quad$ with $\quad F_\mu = \bar{\nabla}^\nu h_{\mu\nu} - \frac{1+\beta}{4}\bar{\nabla}_\mu h^\nu{}_\nu$





$$\mathcal{G}_{hh}(p^2) \sim \frac{1}{p^2 + \ln(p^2)p^4} \qquad \rho_h(\omega^2) \sim \delta(\omega^2) + 2\pi + 4\pi w^2 \ln(w^2) + \dots$$

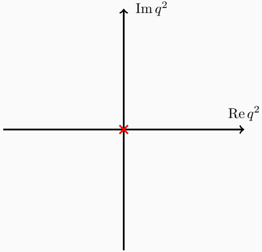
New Lorentzian formulation

[Fehre, Litim, Pawlowski, MR '21]

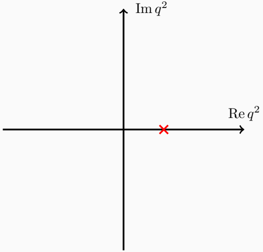
- Callan-Symanzik cutoff $R_k \sim k^2$ allows use of spectral representation
- Dimensional regularisation of UV divergences in $d = 4 - \varepsilon$

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr}\, \mathcal{G}_k \, \partial_t R_k - \partial_t S_{\mathrm{ct},k}$$

[Braun, Chen, Fu, Geißel, Horak, Huang, Ihssen, Pawlowski, MR, Rennecke, Tan, Töpfel, Wessely, Wink '22]



p^2



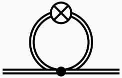
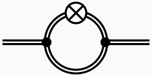
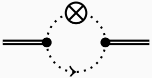
$p^2 + k^2$

Lorentzian setup

- Einstein-Hilbert action expanded about flat Minkowski background
- Directly compute flow of spectral function

$$\partial_t \rho_h = -2\,\mathrm{Im}\,\mathcal{G}_{hh}^2 \left(\partial_t \Gamma_{\mathrm{TT}}^{(hh)} + \partial_t R_k\right)$$

- Flow of transverse-traceless graviton two point function

$\partial_t \Gamma_k^{(hh)} = -\frac{1}{2}$  $+$  $-\,2$  $-\,\partial_t S_{\mathrm{ct},k}^{(hh)}$

- Schematically

 $= \prod_{i=1}^{3} \int_0^\infty \frac{\mathrm{d}\lambda_i}{\pi} \lambda_i\, \rho_h(\lambda_i) \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{V_{\text{3-point}}(p,q)}{(q^2+\lambda_1^2)(q^2+\lambda_2^2)\left((p+q)^2+\lambda_3^2\right)}$