Conformal bootstrap meets (quantum) phase transition

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at "Asymptotic safety, particle physics and friends"

[JR, N. Su, arxiv: 2311.00933]

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Why CFT?

One of the modern views of quantum field theory, which is especially popular in the field of statistical physics and condensed matter physics, is that quantum field theory can be interpreted as the renormalization group flows connecting different conformal field theories (CFTs).

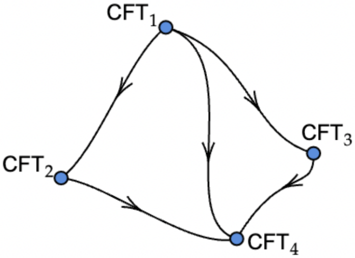


Figure: CFTs as special points of quantum field theories

Why CFT?

Physical systems with completely different microscopic origin can have the same critical behaviour.

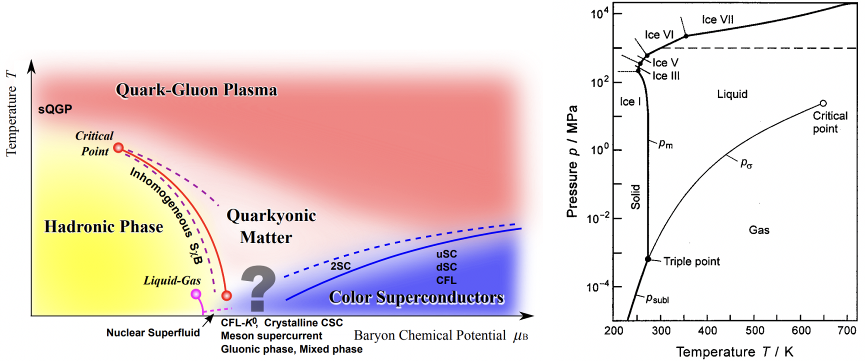


Figure: Left: the QCD phase diagram [Rept. Prog. Phys. 74 (2011) 014001]. Right: phase diagram of water. [Journal of physical and chemical reference data, vol 31 (2002) 387.]

Why CFT?

The conformal symmetry fix fully the spatial dependence of certain correlations function. The two point function becomes

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{|x-y|^{2\Delta_\phi}}.$$

The three point function becomes

$$\begin{aligned}&\langle O_1(x_1)O_2(x_2)O(x_3)\rangle \\ &= \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}}.\end{aligned}$$

Why Bootstrap?

$$\mathcal{L} = \frac{1}{2}\vec{\nabla}\phi^i \cdot \vec{\nabla}\phi^i + m^2 \sum_i (\phi^i)^2 + \lambda(\sum_i (\phi^i)^2)^2 + \dots$$

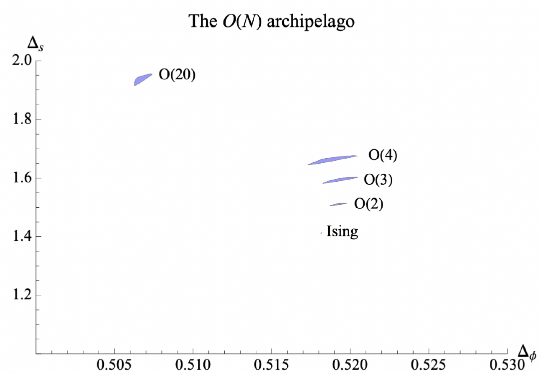


Figure: The O(N) CFTs, [Kos, F., Poland, D., Simmons-Duffin, D., & Vichi, A. (2015),JHEP, 2015(11), 1-2]

Why Bootstrap?

The Gross-Neveu-Yukawa CFTs,

$$\mathcal{L} = \bar{\psi}^i \gamma \cdot \partial \psi^i + \frac{1}{2}\partial_\mu \phi \partial_\mu \phi + \phi \bar{\psi}^i \psi^i$$

Why Bootstrap?

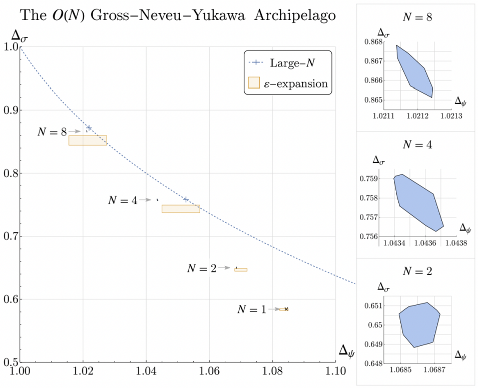


Figure: Gross-Neveu-Yukawa CFTs, [Erramilli, R. S., Iliesiu, L. V., Kravchuk, P., Liu, A., Poland, D., & Simmons-Duffin, D, JHEP, 2023(2), 1-49]

The N=1 was worked out using superconformal bootstrap in [JR, N. Su, JHEP 06 (2021) 154].

Why Bootstrap?

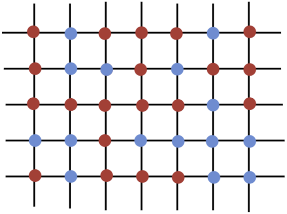
The critical exponents were determined to high precision:

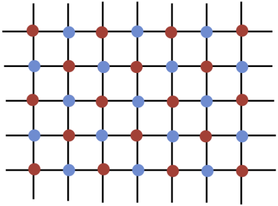
$$
\begin{aligned}
\eta_\sigma &= 0.168887(17)\,,\\
\eta_\psi &= 0.168887(17)\,,\\
1/\nu &= 1.415557(8)\,,\\
\omega &= 0.8869(25)\,.
\end{aligned}
$$

For comprehensive review of conformal bootstrap, see [D Poland, S. Rychkov, A. Vichi, arXiv:1805.04405], and [S. Rychkov, N. Su, arxiv: 2311.15844].

How does CFT emerge in nature?

We consider the structural phase transition as an example,


a)

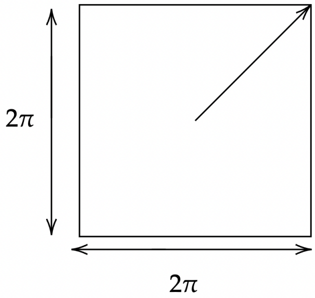

b)

At low temperature, the ordered phase, the density of the red atoms at point (x, y) is given by

$$\rho(x, y) = \frac{1 + (-1)^{x+y}}{2} = \frac{1}{2} + \frac{1}{2} e^{i\pi x + i\pi y}$$

How does CFT emerge in nature?

The second term is the order parameter, located at edge of the Brillouin zone,



Treat the coefficient of the second term as the order parameter of the phase transition

$$\phi e^{i\pi x+i\pi y}$$

, under spatial translation $x \to x+1$, the "scalar field" ϕ change sign.

How does CFT emerge in nature?

One can write the corresponding Landau theory preserving the Z_2 symmetry

$$\mathcal{L} = \frac{1}{2}\vec{\nabla}\phi \cdot \vec{\nabla}\phi + m^2\phi^2 + \lambda_4\phi^4 + \lambda_6\phi^6 + \dots$$

The order parameter forms an unfaithful representation of the space group, which can be classified mathematically, see ["Isotropy Subgroups of the 230 Crystallographic Space Groups" by Stokes and Hatch.] There are only 132 Landau theories, one can analysis the corresponding perturbative RG in $4-\epsilon$ expansion, one get

How does CFT emerge in nature?

No.	Name	Images
1	Ising	A2a
2	XY	B4a, B6b, B8a, B12a, B12b, B24a
3	N=3 Cubic	C24a, C24c, C48a
4	XY^2	D32e, D64a, D64b, D64d, D72b, D128a, D144a
5	N=4 Cubic	D192a, D192c, D384a
6	XY^3	E96k, E192j, E768b, E768c, E1536a

Many of these crystal universalities have been studied case by case earlier, see for example [A. Aharony, Phys. Rev. Lett. 31 (1973) 1494.]

the Cubic CFT

The Lagrangian of the Cubic CFT can be written as

$$\mathcal{L} = \frac{1}{2}\vec{\nabla}\phi^i \cdot \vec{\nabla}\phi^i + m^2 \sum_i (\phi^i)^2 + u(\sum_i (\phi^i)^2)^2 + v \sum_i (\phi^i)^4 + \ldots$$

The symmetry group is the Signed symmetric group $(Z_2)^N \rtimes S_N$.
This theory has been studied using perturbative QFT in $4-\epsilon$ expansion up to six loops [L.Ts. Adzhemyan, E.V. Ivanova, M.V. Kompaniets, A. Kudlis, A.I. Sokolov, Nucl. Phys. B 940 (2019) 332-350]

the Cubic CFT

What do we expect from perturbative calculation, [Pelissetto, Vicari, Phys.Rept.368:549-727,2002][[A. Aharony, Phys. Rev. Lett. 31 (1973) 1494.]]. N_c is approximately 2.86.

the Cubic CFT

What do we expect from perturbative calculation, [Pelissetto, Vicari, Phys.Rept.368:549-727,2002][[A. Aharony, Phys. Rev. Lett. 31 (1973) 1494.]]. N_c is approximately 2.86. This is a prefect playground for conformal perturbation theory.

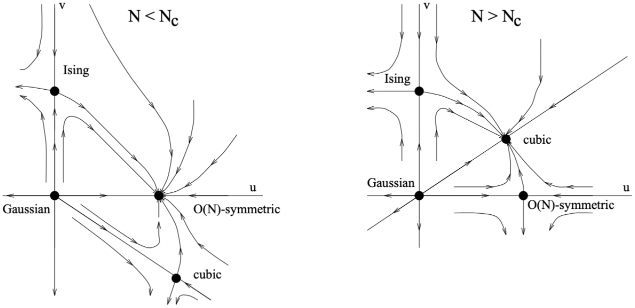


Figure 14: RG flow in the coupling plane (u, v) for $N < N_c$ and $N > N_c$ for magnetic systems with cubic anisotropy.

the Cubic CFT

The operators of the O(3) CFT are given by their O(3) irreps

$$\begin{aligned} j &= 1 \quad \phi^i \\ j &= 0 \quad \phi^i\phi^i \\ j &= 2 \quad t_2^{ij} = \phi^i\phi^j - \text{trace} \\ j &= 4 \quad t_4^{ijkl} = \phi^i\phi^j\phi^k\phi^l - \text{trace} \end{aligned} \tag{1}$$

The Cubic can be understood as the theory one get by turning on a slight relevant perturbation form the O(3) CFT.

$$S_{Cubic} = S_{O(3)} + \int dx^3 t_4^{1111}$$

. Here

$$t_4^{ijkl} = \phi^i \phi^j \phi^k \phi^l - \text{trace}$$

is in the rank-4 symmetric traceless representation of O(3).
When $\Delta_{t_4} > 3$, the O(3) CFT is stable. When $\Delta_{t_4} < 3$, the Cubic CFT is stable. See the phase diagram.

Conformal perturbation: 1st order

Here is a small review about conformal perturbation theory (see for example [Zohar Komargodski, David Simmons-Duffin, J.Phys.A 50 (2017) 15, 154001]). Imagine deform the CFT partition function by (with $O(x)$ a marginal operator)

$$e^{g \int_V dx^D O(x)}$$

Consider the observable

$$\begin{aligned}\langle O|0\rangle_{V,g} = & \langle O(\infty) e^{g \int_V dx^D O(x)} \rangle \\ & = \langle O(\infty)\rangle + g \int_V dx^D \langle O(x) O(\infty)\rangle \\ & + \frac{1}{2} g^2 \int_V dx^D \int_V dx_1^D \langle O(x) O(x_1) O(\infty)\rangle + \cdots\end{aligned}$$

Here $O(\infty) = limit_{x\to\infty} x^{2\Delta_O} O(x)$.

Conformal perturbation: 1st order

The integrated three point functions have UV divergence

$$\int_V dx^D \int_V dx_1^D \langle O(x)O(x_1)O(\infty)\rangle = V\int_V dx^D \langle O(0)O(x)O(\infty)\rangle$$
$$= V\int_V dx^D \frac{C_{OOO}}{|x|^D} = A\cdot log(u),$$

with $A = S_{d-1}C_{OOO}$.

Take the relation between the bare coupling and the renormalized coupling to be

$$g_0 = g(u) + \frac{1}{2}g(u)^2 A\cdot log(u) + \dots,$$

to cancel the divergent piece, we get from $\frac{dg_0}{dlog(u)} = 0$ that

$$\frac{dg}{dlog(u)} = \beta(g) = -\frac{1}{2}S_{D-1}C_{OOO}g^2 + \dots.$$

Conformal perturbation: 1st order

To perturb by a slightly relevant operator with scaling dimension $\Delta_O = D - \delta$.

$$g_0 = u^{\delta}\left(g(u) + g(u)^2\frac{1}{\delta} + \dots\right),$$

$$\frac{dg}{dlog(u)} = \beta(g) = -\delta g + \beta_1 g^2 \dots,$$

with

$$\beta_1 = -\frac{1}{2}\int_V dx^D \int_V dx_1^D \langle O(x)O(x_1)O(\infty)\rangle_{\delta^{-1}} = -\frac{1}{2}S_{d-1}C_{OOO}$$

Higher order terms can be calculated using integrated four point functions.

Conformal perturbation: 1st order

What about anomalous dimension? To calculate the anomalous dimension, just like in any perturbation theory, we deform the

$$e^{\lambda \int_V dx^D \Phi(x) + g \int_V dx^D O(x)},$$

and calculate the beta function for λ. Consider the observable, (pick the leading order in λ)

$$\langle \Phi | 0 \rangle_{V,g} = \; V\lambda \langle \Phi(\infty)\Phi(0)\rangle + V\lambda g \int dx^D \langle \Phi(\infty) O(x) \Phi(0)\rangle + \cdots ,$$

to make it finite, λ needs renormalization and we get

$$\beta_\lambda = -\lambda S_{D-1} C_{\Phi\Phi O} \cdot g$$

The scaling dimension of the Φ field is

$$\Delta_\Phi(g) = \Delta_\Phi - S_{D-1} C_{\Phi\Phi O} \cdot g + \ldots = \Delta_\Phi + 2\frac{C_{\Phi\Phi O}}{C_{OOO}}\delta + \ldots$$

Conformal perturbation: 1st order

The sub-leading correction in the conformal perturbation theory depend on the renormalisation scheme, for example, the sub-leading term in the beta function is from

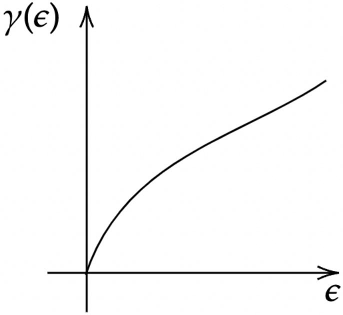
$$\int_V dx_1{}^D dx_2{}^D \langle O(0)O(x_1)O(x_2)O(\infty)\rangle \sim B \cdot log^2(u) + C \cdot log(u) \quad (2)$$

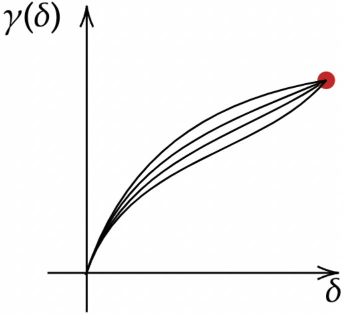
The C coefficient appears in the 2nd order beta function ($C = \frac{1}{6}\beta_2$). The OPE contains an unfixed contact term

$$O(x)O(y) \sim \alpha \cdot \delta^D(x-y)O(y) \quad (3)$$

Conformal perturbation: 1st order

What's the difference between the current case and the $4-\epsilon$ expansion. In the case of $4-\epsilon$ expansion, we have a one parameter family of CFTs parametrised by ϵ. The whole line is physical. In the case of conformal perturbation theory, we only have the end points, the UV CFT and the IR CFT.





Conformal perturbation: 1st order

Notice the beta function is

$$\beta(g) = -\delta g + \beta_1 g^2.$$

This tells us

$$\omega = \frac{d\beta(g)}{dg}|_{g=g_*} = \delta.$$

This tells us $Y_4^{O(3)} - \omega^{Cubic(3)} \approx 0$. In a recent work by Martin Hasenbusch, he showed [arxiv:2307.05165]

$$Y_4 - \omega = 0.00081(7)$$

.

The Lagrangian

$$\mathcal{L} = \frac{1}{2}\vec{\nabla}\phi^i \cdot \vec{\nabla}\phi^i + m^2 \sum_i (\phi^i)^2 + u(\sum_i (\phi^i)^2)^2 + v \sum_i (\phi^i)^4 + \ldots .$$

The symmetry breaking coupling is

$$t_4(x) = \sum_i (\phi^i)^4 - \text{traces}. \tag{4}$$

The RG flow from the O(3) CFT to the Cubic(3) CFT is triggered by the this operator. (One of the 9 components of the $j = 4$ irrep.)

Conformal perturbation: 1st order

The ϕ^i operator (order parameter critical exponents) and the $s = \sum_i (\phi^i)^2$ operator receives no correction in the leading order of δ. Since the OPE $C_{\phi\phi t_4}$ and C_{sst_4} vanish because of the O(3) symmetry.
The scalar bilinear operators in the t_2 irrep, however, receives correction at leading order. The off diagonal components of a symmetry traceless matrix, form a 3 dimensional irrep of the Cubic(3) group.

$$w \in \{\phi_1\phi_2, \phi_2\phi_3, \phi_1\phi_3\}.$$

The trace part form a two dimensional irrep of the Cubic(3) group,

$$t' \in \{\phi_1^2 - \phi_2^2, \phi_2^2 - \phi_3^2\}.$$

The two irreps receive different correction.

Conformal perturbation: 1st order

In the formula

$$\gamma_\Phi(g) = -S_{D-1} C_{\Phi\Phi O} \cdot g + \ldots = 2\frac{C_{\Phi\Phi O}}{C_{OOO}}\delta + \ldots,$$

we have assumed the normalisation $\langle \Phi | \Phi \rangle = \langle O | O \rangle = 1$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \tilde{w}^a, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \tilde{t'}^a,$$

with normalisation $(\tilde{w}^a)^\dagger \tilde{w}^a = (\tilde{t'}^a)^\dagger \tilde{t'}^a = 1$. Similarly for t_4. We get [JR, N. Su, arxiv: 2311.00933]

$$\frac{\gamma_w}{\gamma_{t'}} = \frac{\tilde{w}^a \tilde{w}^b \tilde{t_4}^c \left\{ \frac{t_2, t_2}{a, b} \middle| \frac{t_4}{c} \right\}}{\tilde{t'}^a \tilde{t'}^b \tilde{t_4}^c \left\{ \frac{t_2, t_2}{a, b} \middle| \frac{t_4}{c} \right\}} = -\frac{2}{3}.$$

Conformal perturbation: 1st order

In the formula

$$\gamma_\Phi(g) = -S_{D-1} C_{\Phi\Phi O} \cdot g + \ldots = 2 \frac{C_{\Phi\Phi O}}{C_{OOO}} \delta + \ldots ,$$

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This is confirmed in Lattice simulation [Martin Hasenbusch, private communication]

Conformal bootstrap

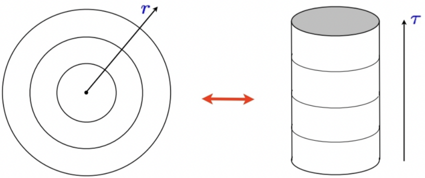
We need to OPE ratio

$$\frac{C_{t_2 t_2 t_4}}{C_{t_4 t_4 t_4}}$$

in the O(3) fixed point. The denominator appears in the OPE of $t_4 \times t_4$, so that we need to include t_4 in our bootstrap system.
With this OPE ratio, we conclude [JR, N. Su, arxiv: 2311.00933]

$$\Delta_w \approx 1.1988(24), \quad \text{and} \quad \Delta_{t'} \approx 1.2256(36). \tag{5}$$

Large charge expansion

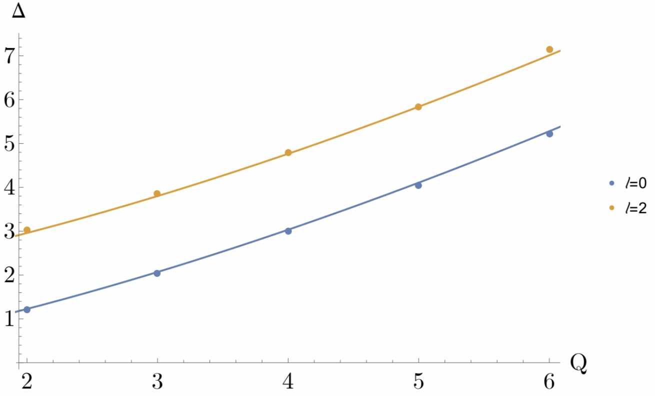


One can introduce charge density on the cylinder, the ground state energy on the cylinder gives the scaling dimension of charge Q operators. [D.T. Son, M. Wingate, Annals Phys. 321 (2006) 197-224], [S. Hellerman, D. Orlando, S. Reffert, and M. Watanabe, JHEP 12 (2015), p. 071] [A. Monin, Phys.Rev. D94.8 (2016), p. 085013] [A. Monin, D. Pirtskhalava, R. Rattazzi, and F. K. Seibold., JHEP 06 (2017), p. 011]. Using the effective theory, one get the following formula

$$\Delta_Q = c_{3/2} Q^{3/2} + c_{1/2} Q^{1/2} - 0.094 + \sqrt{\frac{l(l+1)}{2}}$$

Large charge expansion

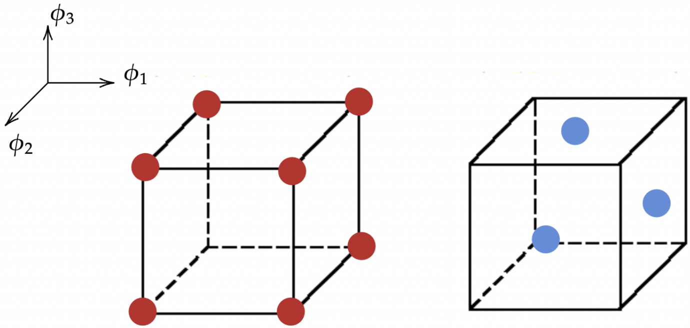
The $t_4 \times t_4$ OPE allows us to access operators in the j=5,6,8 channels, this allows us the check some predictions of large Q effective theory.



(The Q=8 is not stable. Warning! Some channels have small lower operator with smaller OPE, probably due to sharing effect.)

Conformal perturbation: 0th order

The mean field theory tells us when $v > 0$, we have the so called corner Cubic phase, when $v < 0$, we have the so called face-cubic phase



Conformal perturbation: 0th order

Remember that

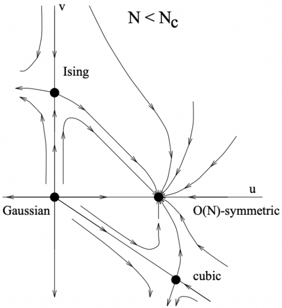
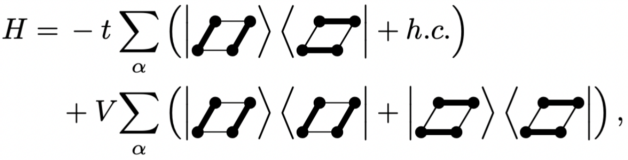
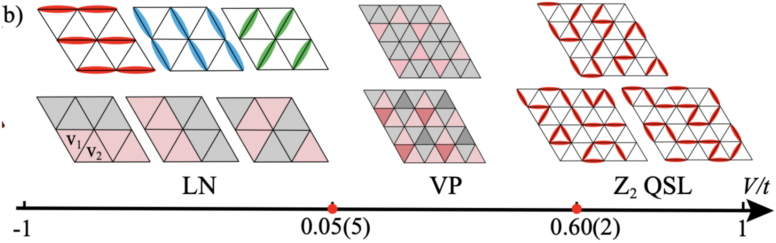


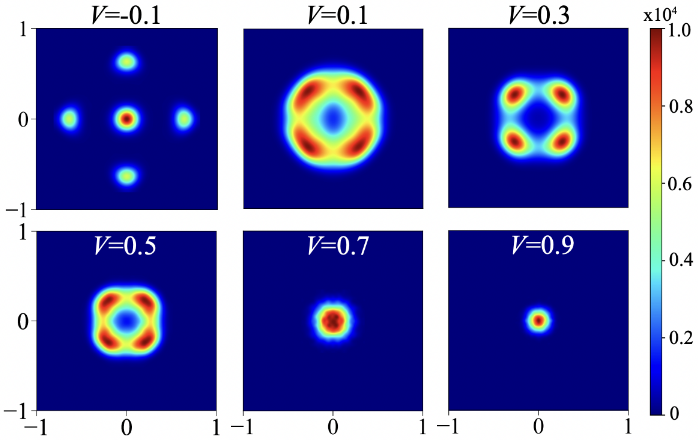
Figure 14: RG flow in the coupling plane (u, v) for $N < N_c$ and $N > N_c$ for magnetic systems with cubic anisotropy.

The Cubic CFT appears in certain quantum phase transitions





Conformal perturbation: 0th order



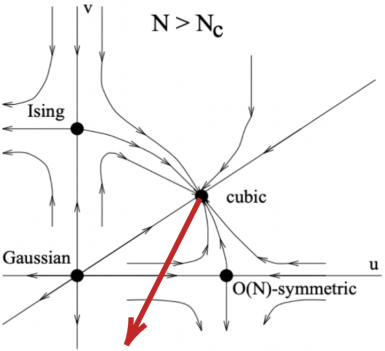
[Z. Yan, X. Ran, Y Wang, R Samajdar, JR, S. Sachdev, Y. Qi, Z Meng, arxiv: 2205.04472]

Conformal perturbation: 0th order

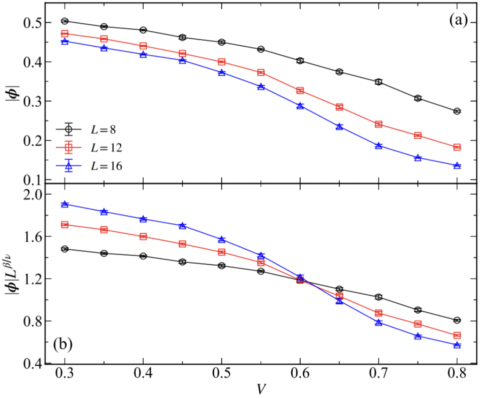
Lattice operator can have non-trivial mix with the field theory operator

$$O_{lattice} \sim a \sum_i (\phi^i)^2 + b \sum_i (\phi^i)^4,$$

when moving changing the coupling on the lattice. The $\sum_i (\phi^i)^4$ mixing cause a first order phase transition



Data collapsing with the O(3) exponents,



Conformal perturbation: 0th order

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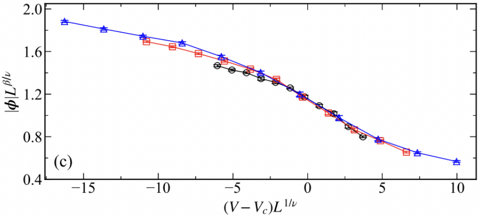

Thank you!