Pion observables calculated in Minkowski and Euclidean spaces with *Ansätze* for quark propagators

Talk presented at **ZIMÁNYI SCHOOL 2023**
Winter Workshop on Heavy Ion Physics

Budapest, Hungary, 4.-8. December 2023.

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December 4, 2023.

Overview

Introduction

Quark Propagator

Wick rotation

General properties of quark propagators

3R Quark Propagator

Electromagnetic form factor

Transition form factor

Pion distribution amplitude

Summary

Introduction

`Talk = based on` Kekez and Klabučar, Phys. Rev. D **107**, 094025 (2023).

⋆ This paper pays attention to describing the pion phenomenologically as good as possible in the chosen model framework, but this is secondary.

⋆ Its primary concern is formulating clear connections between Euclidean and Minkowski spacetime calculations of observables of $q\bar{q}$ bound states.

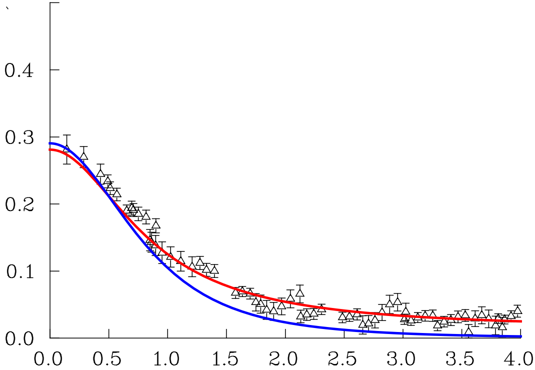
- Many QFT studies (on lattice, most of Schwinger–Dyson studies, etc.) are **not done in the physical, Minkowski spacetime, but in Euclidean.**
- ⇒ **Relating Minkowski and Euclidean spaces ("Wick rotation") must be under control. But, this is highly nontrivial in the nonperturbative case – most importantly, the nonperturbative QCD.**
- **Do nonperturbative Green's functions permit Wick rotation?**
- **For solving Bethe-Salpeter equation and calculation of processes, extrapolation to complex momenta is necessary. ⇒ Knowledge of the analytic behavior in the whole complex plane is needed.**
- **Very complicated matters ⇒ studies of Ansatz forms are instructive and can be helpful to *ab initio* studies of nonperturbative QCD Green's functions. ... and vice versa of course ...**

Lattice-inspired *Ansatz* for propagators in nonperturbative QCD

Lattice data: Parappilly *et al.*, Phys. Rev. D **73**, 054504 (2006)
compared with nonperturbatively dressed (DChSB-generated) constituent quark masses in the propagator *Ansätze* ('3R' and 'MMF') by

[1] Alkofer, Detmold, Fischer and Maris, Phys. Rev. D 70, 014014 (2004).

[2] Mello, de Melo, and Frederico, *Minkowski space pion model inspired by lattice QCD running quark mass*, Phys. Lett. B 766, 86 (2017).



$M(q)$ vs. q (both in GeV)

How Schwinger–Dyson approach generates quark propagators

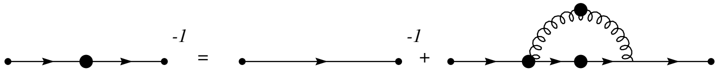
- Schwinger-Dyson (SD) approach: ranges from solving SD equations for Green's functions of non-perturbative QCD *ab initio*, to higher degrees of phenomenological modeling, *esp.* in applications including $T, \mu > 0$.

  e.g., [Alkofer, v.Smekal Phys. Rept. 353 (2001) 281], and [Roberts, Schmidt Prog.Part.Nucl.Phys. 45 (2000)S1]

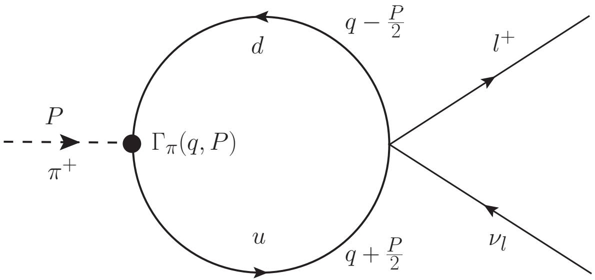
- SD approach to quark-hadron physics = nonpertubative, covariant bound state approach with strong connections with QCD.

The "gap" Schwinger–Dyson equation for the quark propagator:

$$S^{-1}(p) = \not p - m - iC_F g^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S(k) \Gamma^\nu(k,p) G_{\mu\nu}(p-k) \ . \qquad (1)$$



Already just $S(p)$ enables calculating an observable - pion decay constant



... since close to the chiral limit, $\Gamma_\pi^{BS} \approx -\frac{2B(q^2)}{f_\pi}\gamma_5$ is a reasonable **approximation**,

$$\Rightarrow \quad f_\pi \approx i\frac{N_c}{2}\frac{1}{M_\pi^2}\int\frac{d^4q}{(2\pi)^4}\mathrm{tr}\left(\not{P}\gamma_5 S(q+\frac{P}{2})\left(-\frac{2B(q^2)}{f_\pi}\gamma_5\right)S(q-\frac{P}{2})\right)$$

Dynamically generated nonperturbative quark propagator

In principle, SD Eq. (1) yields the nonperturbatively dressed quark propagator

$$S(q) = \frac{A(q^2)\not{q} + B(q^2)}{A^2(q^2)q^2 - B^2(q^2)} = Z(-q^2)\frac{\not{q} + M(-q^2)}{q^2 - M^2(-q^2)} = -\sigma_V(-q^2)\not{q} - \sigma_S(-q^2)$$

[$M(x)$ = dressed quark mass function, $Z(x)$ = wave-function renormalization]

- One usually gets just a model solution for the propagator $S(q)$, since one usually simplifies SD Eq. (1) by approximations & modeling! For example:
- 1.) rainbow(-ladder) for the dressed quark–gluon vertex: $\Gamma^\mu(k,p) \to \gamma^\mu$,
- 2.) $A(k^2) = 1$, 3.) various model Ansätze for the dressed gluon propagator, $g^2 G^{\mu\nu}(k) \propto \alpha_s^{\text{eff}}(k^2)/k^2$, so that, e.g., Eq. (1) yields

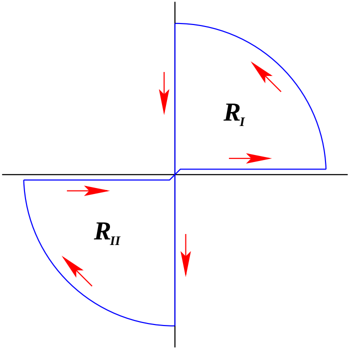
$$M(-p^2) = m + I(-p^2) = m + \int d^4k\, f(-k^2) g(-(k-p)^2)$$

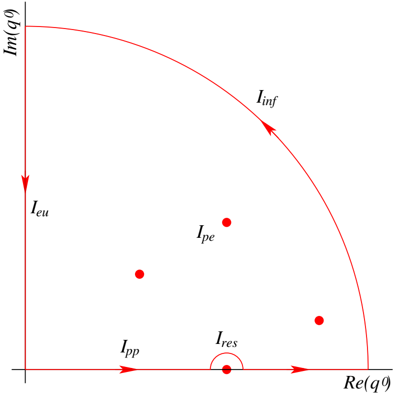
$$g(-k^2) = 4\pi C_F \frac{\alpha_s^{\text{eff}}(-k^2)}{(-k^2)} \quad f(-k^2) = \frac{3i}{(2\pi)^4} \frac{M(k^2)}{k^2 - M^2(k^2)}$$

$$I(-p^2) = \int d^3k \int dk^0\, f(-(k^0)^2 + |\mathbf{k}|^2) g(-(k^0 - p^0)^2 + |\mathbf{k} - \mathbf{p}|^2)$$

Wick rotation

- Even with such approxim. & modeling, solving SD equations like Eq. (1), and related calculations (e.g., of f_π) with Green's functions like $S(q)$, are technically very hard to do in the physical, Minkowski space-time.
- Thus, additional simplification is sought by transforming to QFT in 4-dim. Euclidean space by the Wick rotation to the imaginary time component: $q^0 \to i\, q^0$.





The issue of singularities is more problematic than in the perturbative case!

Example: simple separable approximation for low-E QCD

$$g^2 G^{\mu\nu} = g^{\mu\nu} D(p-k) \approx g^{\mu\nu} \left[D_0 f_0(-p^2) f_0(-k^2) - D_1 (p \cdot k) f_1(-p^2) f_1(-k^2) \right]$$

produces complicated singularity structure in the complex plane $z = -k^2$

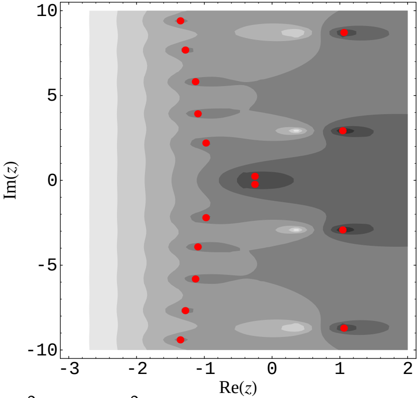
The "gap" SD equation ⇒

$$A(p^2) = 1 + a f_1(-p^2)$$
$$B(p^2) = m + b f_0(-p^2)$$

Typical phen.succes. Ansatz

$$f_0(x) = e^{-x/\Lambda_0^2}$$
$$f_1(x) = \frac{1 + e^{-x_0/\Lambda_1^2}}{1 + e^{-(x-x_0)/\Lambda_1^2}}$$

produces **poles** which are **obstacles to (any) Wick rotation!**



The contour plot of $z \mapsto \log |A(z)^2 z + B(z)^2|$ close to the origin. The red points are solutions of the equation $A(z)^2 z + B(z)^2 = 0$ and extend much further than the depicted region.

General properties a quark propagator should have:

- $S(q) \to S_{\rm free}(q)$ because of asymptotic freedom
  $\Rightarrow \sigma_{V,S}(-q^2) \to 0$ for $q^2 \in \mathbb{C}$ and $q^2 \to \infty$
- $\sigma_{V,S}(-q^2) \to 0$ cannot be analytic over the whole complex plane
- positivity violating spectral density $\leftrightarrow$ confinement

Try Ansätze of the form (meromorphic parametrizations, like Alkofer&al. [1]):

$$S(p) = \frac{1}{Z_2}\sum_{j=1}^{n_p} r_j \left(\frac{\not p + a_j + ib_j}{p^2 - (a_j + ib_j)^2} + \frac{\not p + a_j - ib_j}{p^2 - (a_j - ib_j)^2} \right)$$

Dressing f′nctns are thus
$$\sigma_V(x) = \frac{1}{Z_2}\sum_{j=1}^{n_p} \frac{2r_j(x + a_j^2 - b_j^2)}{(x + a_j^2 - b_j^2)^2 + 4a_j^2 b_j^2}$$

(e.g., see Ref. [1]) and
$$\sigma_S(x) = \frac{1}{Z_2}\sum_{j=1}^{n_p} \frac{2r_j a_j(x + a_j^2 + b_j^2)}{(x + a_j^2 - b_j^2)^2 + 4a_j^2 b_j^2}$$

Constraints:

$$\sum_{j=1}^{n_p} r_j = \frac{1}{2} \qquad \sum_{j=1}^{n_p} r_j a_j = 0$$

3R quark propagator - 3 poles on the real axis

Parameters (yielding 3R propagator of Alkofer & al., and larger f_π): $n_p = 3$, $a_1 = 0.341$, $a_2 = -1.31$, $a_3 = -1.35919$, $b_1 = 0$, $b_2 = 0$, $b_3 = 0$, $r_1 = 0.365$, $r_2 = 1.2$, $r_3 = -1.065$, $Z_2 = 0.982731$.

$\Rightarrow$ **No obstacles to Wick rotation** $\Rightarrow \pi$ **decay constant calculated equivalently**

* **in Minkowski space:**

$$f_\pi^2 = -i \frac{N_c}{4\pi^3 M_\pi^2} \int_0^\infty \xi^2 \, d\xi \int_{-\infty}^{+\infty} dq^0 \, B(q^2) \, \mathrm{tr} \left(\not{P} \gamma_5 S(q + \frac{P}{2}) \gamma_5 S(q - \frac{P}{2}) \right)$$

where $q^2 = (q^0)^2 - \xi^2$, $\xi = |\mathbf{q}|$, **and** $q \cdot P = M_\pi q^0$, **OR**

* **in Euclidean space:**

$$f_\pi^2 = \frac{3}{8\pi^3 M_\pi^2} \int_0^\infty dx \, x \int_0^\pi d\beta \, \sin^2 \beta \, B(q^2) \mathrm{tr} \left(\not{P} \gamma_5 S(q + \frac{P}{2}) \gamma_5 S(q - \frac{P}{2}) \right)$$

where $q^2 = -x$ **and** $q \cdot P = -i M_\pi \sqrt{x} \cos \beta$.

3R quark propagator Ansatz

Numerically, for $\xi = 0.5$

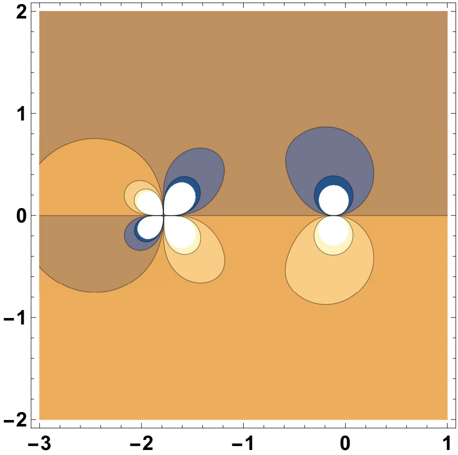
$$|I_{pp} + I_{res} + I_{inf} + I_{eu} - I_{pe}| \sim 10^{-8}$$

Particular integrals for $\xi = 0.5$

$$\begin{aligned} I_{pp} &\approx 4 \cdot 10^{-12} \;, \\ I_{res} &= -0.0221314i \\ I_{inf} &\approx 4 \cdot 10^{-12} \;, \\ I_{eu} &= 0.0221314i \\ I_{pe} &= 0 \end{aligned}$$

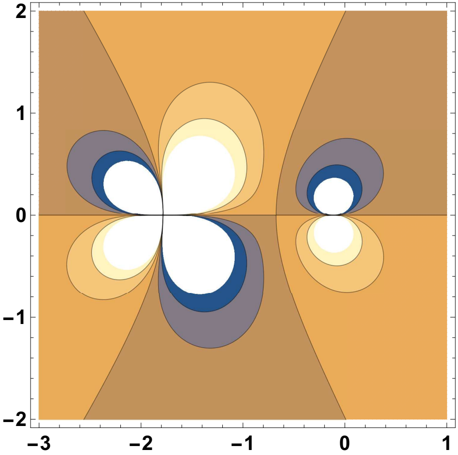
$\Rightarrow f_\pi = 0.072\ \mathrm{GeV}$ in both Euclidean and Minkowski space.

3R Quark Propagator



Contour plot of $\text{Im}(\sigma_V(z))$ in the complex z-plane:
the first two poles on the real axis are very close - "glued together" .

3R Quark Propagator



A similar contour plot of $\mathrm{Im}(\sigma_S(z))$ in the complex z-plane.

3R Quark Propagator

Spectral representation of the 3R quark propagator is defined by the spectral density

$$\rho(\sigma^2) = \sum_{j=1}^{3} A_j^{-1} \delta(\sigma^2 - M_j^2)$$

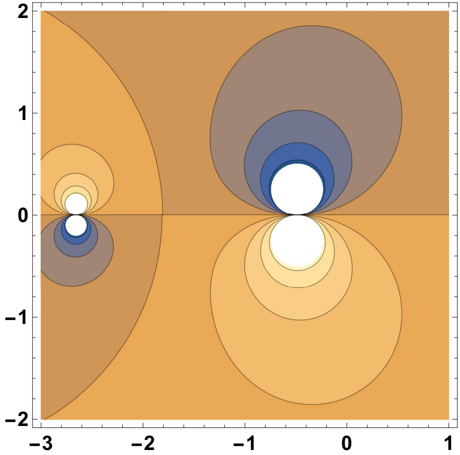
where $A_1 = 1.35$, $A_2 = 0.41$, $A_3 = -0.46$, $M_j = B_j/A_j$, $j = 1, 2, 3$.

Although the propagator functions A and B from $S^{-1}(q) = A(-q^2)\not{q} - B(-q^2)$ exhibit different structure of the poles,

$$\begin{aligned} A(x) &= \frac{\sigma_V(x)}{\sigma_S^2(x) + x\,\sigma_V^2(x)} \\ B(x) &= \frac{\sigma_S(x)}{\sigma_S^2(x) + x\,\sigma_V^2(x)} \end{aligned}$$

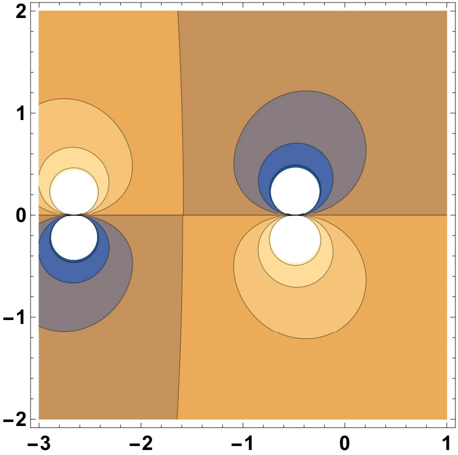
the poles are still on the real axis:

3R Quark Propagator



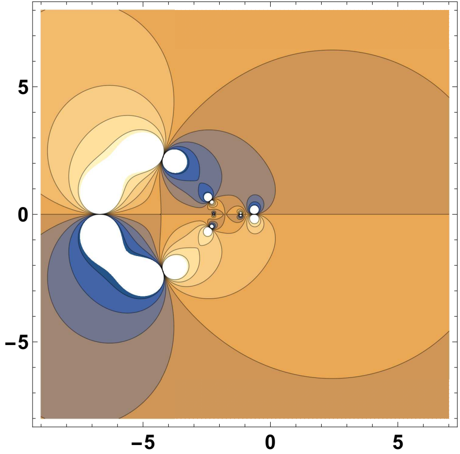
Contour plot of Im($A(z)$) in the complex z-plane: all poles are still on the real axis.

Similarly with B in the 3R Quark Propagator:



Contour plot of $\mathrm{Im}(B(z))$ in the complex z-plane: all poles are still on the real axis.

But a modest change of parameters spoils 3R Quark Propagator:



Contour plot of $\mathrm{Im}(B(z))$ in the complex z-plane. Parameters changed to $a_1 = -1.31 \;\; \rightarrow \;\; a_1 = -2 \Rightarrow$ complicated analytic structure, some poles of $B(z)$ are now moved off the real axis. Wick rotation does not go any more!

Quark propagator model of Mello, de Melo, and Frederico - MMF

[2] Mello, de Melo, and Frederico, Phys. Lett. **B766**, 66 (2017)

$$M(x) = (m_0 - i\varepsilon) + m^3 \left[x + \lambda^2 - i\varepsilon\right]^{-1}$$

$$Z(x) = 1$$

Model parameters: $m_0 = 0.014$ GeV, $m = 0.574$ GeV, and $\lambda = 0.846$ GeV
Asymptotic expansions about ∞ and 0:

$$M(x) = m_0 + \frac{m^3}{x} - \frac{\lambda^2 m^3}{x^2} + \mathcal{O}((\frac{1}{x})^3) \ , \qquad \text{for } x \to \infty \ ,$$

$$M(x) = \left(m_0 + \frac{m^3}{\lambda^2}\right) - \frac{m^3 x}{\lambda^4} + \frac{m^3 x^2}{\lambda^6} + \mathcal{O}(x^3) \ , \qquad \text{for } x \to 0 \ ,$$

Quark propagator model of Mello, de Melo, and Frederico - MMF

The quark dressing functions σ_V and σ_S:

$$\sigma_V(x) = \sum_{j=1}^{3} \frac{b_{Vj}}{x + a_j}$$

$$\sigma_S(x) = \sum_{j=1}^{3} \frac{b_{Sj}}{x + a_j}$$

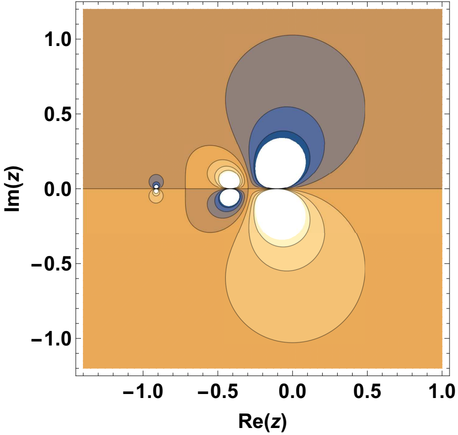
The mass parameters:

$$a_1 = 0.1046 \ \mathrm{GeV}^2$$

$$a_2 = 0.4160 \ \mathrm{GeV}^2$$

$$a_3 = 0.9110 \ \mathrm{GeV}^2$$

Quark propagator model of Mello, Melo, and Frederico - MMF



Contour plot of $\text{Im}(\sigma_V(z))$ in the complex z-plane.

Pion Decay Constant in the MMF model for the quark propagator

$$f_\pi = i\frac{N_c}{2}\frac{1}{M_\pi^2}\int\frac{d^4q}{(2\pi)^4}\mathrm{tr}\left(\not{P}\gamma_5 S(q+\frac{P}{2})\left(-\frac{2B(-q^2)}{f_\pi}\gamma_5\right)S(q-\frac{P}{2})\right)$$

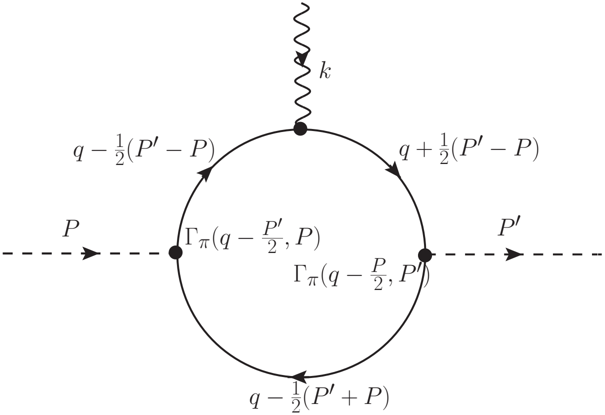
(1) Calculation using FeynCalc and Package-X (or LoopTools)

$$f_\pi = 87.5599\ \mathrm{MeV}$$

(2) Euclidean integration

- "naïve" Wick rotation ($q^0 \to -iq^0$) is correct here
- two nontrivial integration (red variables)
  $q = \left(q^0, \xi\sin(\vartheta)\cos(\varphi), \xi\sin(\vartheta)\sin(\varphi), \xi\cos(\vartheta)\right)$
- numerical integration using Mathematica
- the same result for f_π

Electromagnetic form factor



Generalized impulse approximation to the charged pion electromagnetic form factor $F_\pi(Q^2)$.

Electromagnetic form factor

$$\langle \pi^+(P')|J^\mu(0)|\pi^+(P)\rangle = (P^\mu + P'^\mu)F_\pi(Q^2) = i(\mathcal{Q}_u - \mathcal{Q}_d)\frac{N_c}{2}\int \frac{d^4q}{(2\pi)^4}\times$$

$$\times\mathrm{tr}\Big\{\bar\Gamma(q-\tfrac{P}{2},P')S(q+\tfrac{1}{2}(P'-P))\Gamma^\mu(q+\tfrac{1}{2}(P'-P),q-\tfrac{1}{2}(P'-P))$$
$$\times S(q-\tfrac{1}{2}(P'-P))\Gamma(q-\tfrac{1}{2}P',P)S(q-\tfrac{1}{2}(P+P'))\Big\}$$

- $\Gamma^\mu(p',p)$ – dressed quark γ vertex, modeled by Ball-Chiu vertex
- The proper perturbative QCD asymptotics:

$$F_\pi(Q^2) = 16\pi\frac{\alpha_s(Q^2)}{Q^2}f_\pi^2 \propto \frac{1}{Q^2\ln(Q^2)} \quad \text{for} \;\; Q^2\to\infty$$

cannot be expected with the *present* Ansätze, but presently available experimental data are anyway well above the pQCD predictions.

* In the case of the MMF quark propagator, we used three different methods of calculation. All three yielded the same results.

Various calculation methods employed by Kekez and Klabučar, Phys. Rev. D **107**, 094025 (2023)

(1) Calculation using FeynCalc [**?**, **?**] and Package-X [**?**, **?**]

(2) "Euclidean" integration

- 3 nontrivial integrations (over q^0, ξ, ϑ)

(3) Minkowski space integration utilizing light-cone momenta

Dressed quark–quark–photon vertex $\Gamma^\mu(p', p)$

Quark–quark–photon vertex: Ball–Chiu vertex

$$\Gamma^\mu(p', p) = \frac{1}{2}\left[A(-p'^2) + A(-p^2)\right]\gamma^\mu + \frac{(p' + p)^\mu}{(p'^2 - p^2)}\left\{\left[A(-p'^2) - A(-p^2)\right]\frac{(\not{p}' + \not{p})}{2} - \left[B(-p'^2) - B(-p^2)\right]\right\}$$

Ward–Takahashi identity: $(p' - p)_\mu \Gamma^\mu(p', p) = S^{-1}(p') - S^{-1}(p)$

For the model of Mello, Melo & Frederico, $\Gamma^\mu(p', p) = \gamma^\mu - \dfrac{m^3(p'^\mu + p^\mu)}{(p'^2 - \lambda^2)(p^2 - \lambda^2)}$

Electromagnetic form factor

Poles of the integrand:

$$(q_0)_{1,2} = \mp\sqrt{M_q^2 + \xi^2 - \xi\sqrt{Q^2}\cos\vartheta + Q^2/4}$$

$$(q_0)_{3,4} = \mp\sqrt{M_q^2 + \xi^2 + \xi\sqrt{Q^2}\cos\vartheta + Q^2/4}$$

$$(q_0)_{5,6} = \frac{1}{2}\left(\sqrt{4M_\pi^2 + Q^2} \mp 2\sqrt{M_q^2 + \xi^2}\right)$$

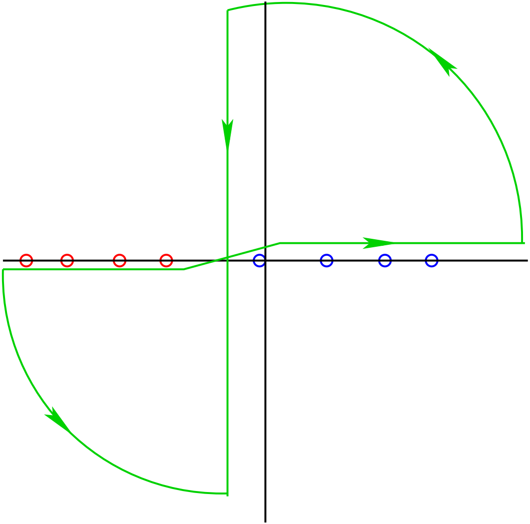
$$(q_0)_{7,8} = \frac{1}{4}\left(\sqrt{4M_\pi^2 + Q^2} \mp \sqrt{16M_q^2 + 16\xi^2 + 8\xi\sqrt{Q^2}\cos\vartheta + Q^2}\right)$$

$$(q_0)_{9,10} = \frac{1}{4}\left(\sqrt{4M_\pi^2 + Q^2} \mp \sqrt{16M_q^2 + 16\xi^2 - 8\xi\sqrt{Q^2}\cos\vartheta + Q^2}\right)$$

$$M_q^2 \in \{a_1, a_2, a_3, \lambda^2\}$$

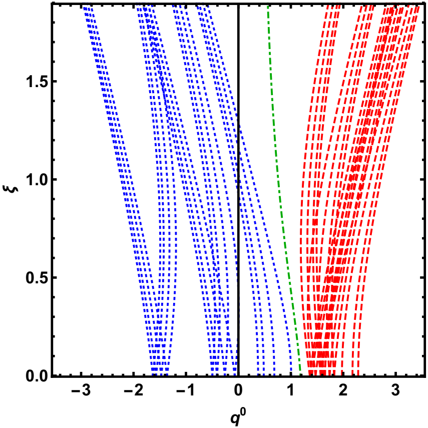
Electromagnetic form factor

Loop integration: q^0 complex plane Wick rotation:
$q^0 = (q^0)_c - iq_4, \quad -\infty < q_4 < +\infty$



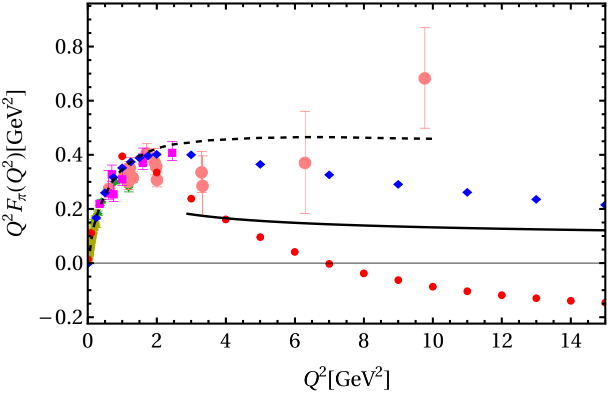
Electromagnetic form factor

$\vartheta = \frac{\pi}{3}$



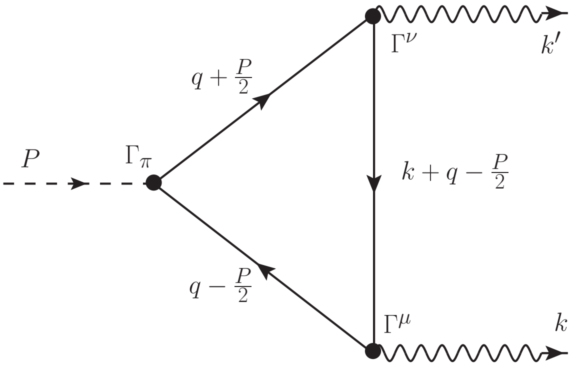
All in units of GeV. Dot–dashed green line represents $(q^0)_c$, blue dotted lines represent odd-indexed poles (set $\mathcal{A}$), and red dashed lines represents even–indexed poles (set $\mathcal{B}$).

Electromagnetic form factor - data and calculations compared



Charged π EM form factor $\times Q^2$. Experimental points are shown by olive triangles [102], green diamonds [103], pink circles [104-106], and magenta squares [107-110]. Solid **red circles** and **blue diamonds** are our results from 3R and MMF *Ansätze*, respectively. The **black dashed line** is the result of Mello et al. [19]. The **black solid line** corresponds to the perturbative QCD result with asymptotic PDA.

Transition form factor for flavorless pseudoscalar mesons



- Diagram for $\pi^0 \to \gamma\gamma$ decay, and for the $\gamma^\star \pi^0 \to \gamma$ process if $k'^2 \neq 0$
- ... also for η and η', but even just π^0 is challenging enough for now

Transition form factor

$$S_{fi} = (2\pi)^4 \delta^{(4)}(P - k - k') e^2 \varepsilon^{\alpha\beta\mu\nu} \varepsilon_\mu^\star(k,\lambda) \varepsilon_\nu^\star(k',\lambda') T_{\alpha\beta}(k^2, k'^2)$$

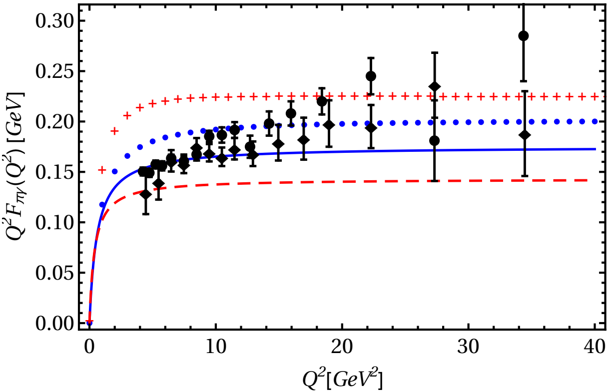
$$\begin{aligned} T^{\mu\nu}(k,k') &= \varepsilon^{\alpha\beta\mu\nu} k_\alpha k'_\beta \, T(k^2, k'^2) \\ &= -N_c \frac{\mathcal{Q}_u^2 - \mathcal{Q}_d^2}{2} \int \frac{d^4q}{(2\pi)^4} \mathrm{tr}\{\Gamma^\mu(q - \frac{P}{2}, k + q - \frac{P}{2}) S(k + q - \frac{P}{2}) \\ &\times \Gamma^\nu(k + q - \frac{P}{2}, q + \frac{P}{2}) S(q + \frac{P}{2}) \left(-\frac{2B(q^2)}{f_\pi}\gamma_5\right) S(q - \frac{P}{2})\} \\ &+ (k \leftrightarrow k', \mu \leftrightarrow \nu) \ . \end{aligned}$$

The π^0 transition form factor: $F_{\pi\gamma}(Q^2) = |T(-Q^2, 0)|$

* UV limit: asymptotically, $F_{\pi\gamma}(Q^2) \to 2f_\pi / Q^2$ for $Q^2 \to \infty$
(Caveat: persistent nonperturb. effects by Eichmann & *al*, PLB 774 (2017) 425)
* The current data (up to 35 GeV^2) do not show agreement with this limit yet.

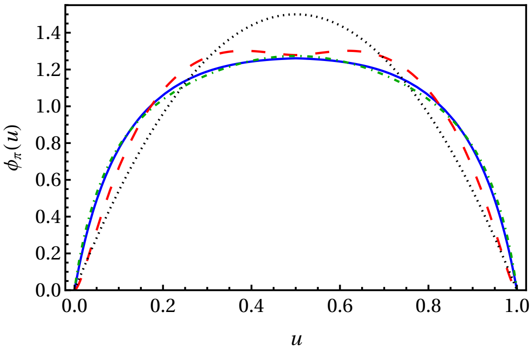
In the chiral limit, the π^0 decay amplitude to two real photons: $T(0,0) = \frac{1}{4\pi f_\pi}$

Transition form factor



Blue dots represent π^0 transition form factor from MMF quark-propagator Ansatz. Red pluses are calculated from 3R Ansatz. The blue solid line and red dashed line represent the Brodsky-Lepage interpolation formula with f_π from MMF and 3R quark-propagator models, respectively. Solid circles and diamonds (with error bars) are the data points of BABAR [116] and Belle [117] Collaborations, respectively.

Pion distribution amplitude



Blue and red lines = obtained from MMF and 3R *Ansätze*, respectively. Black dotted line represents the asymptotic form, $\phi_\pi^{\rm as}(u) = 6u(1-u)$. Dash-dotted green line (very close to the solid blue one) is the PDA from the state-of-the-art SDE pion bound state, Eq. (22) in *Roberts, Symmetry 12 (2020) 9, 1468.*

$$\phi_\pi(u) = i\frac{N_c}{8\pi f_\pi}\mathrm{tr}\left(\gamma_+\gamma_5\int\frac{dq_-}{2\pi}\int\frac{d^2q_\perp}{(2\pi)^2}S(q+\frac{P}{2})\Gamma_\pi(q,P)S(q-\frac{P}{2})\right)$$

Pion distribution amplitude

- For high values of Q^2:

$$F_{\pi\gamma}(Q^2) = \frac{2f_\pi}{3Q^2} \int_0^1 \frac{du\,\phi_\pi(u)}{1-u}$$

- The asymptotic form of the pion distribution amplitude gives

$$\int_0^1 \frac{du\,\phi_\pi^{\rm as}(u)}{1-u} = 3$$

- This actual ϕ_π gives

$$\int_0^1 \frac{du\,\phi_\pi(u)}{1-u} = 3.07$$

- Asymptotic form of the transition form factor:

$$F_{\pi\gamma}(Q^2) \sim \frac{2f_\pi}{Q^2}$$

Summary

- The analytic structure of quark propagators and some observables has been investigated in the nonperturbative regime of QCD for two Ansätze/propagator models which permitted us formulating a clear connections between Euclidean and Minkowski spacetime calculations for all observables mentioned below.
- The propagator model of Mello, de Melo, and Frederico [2] was found relatively successful phenomenologically. Already they calculated pion decay constant and electromagnetic form factor $F_\pi(Q^2)$ of the charged pion. However, we found their result increasingly inaccurate for $Q^2 > 2$ GeV^2, since their $Q^2\, F_\pi(Q^2)$ was not falling noticeably, but remained almost a constant. We checked the correctness of our $Q^2 F_\pi(Q^2)$ by redoing the calculations in three independent ways.
- Our present work (PRD **107** (2023) 094025) also found empirically reasonable values, for both Ansätze MMF and 3R, of the pion charge radius r_π and the slope parameter a of the transition form factor for small timelike momenta.
- We also calculated, for both considered quark propagator models, the $\pi^0\gamma$ transition form factor $F_{\pi\gamma}(Q^2)$ up to 40 GeV^2.
- Our pion distribution amplitudes, especially the one from MMF Ansatz, are close to PDA obtained from state-of-the-art SDE pion bound state.

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[2] C.S. Mello, J.P.B.C. de Melo, and T. Frederico, "Minkowski space pion model inspired by lattice QCD running quark mass", Phys. Lett. B 766, 86 (2017).

[3] G. P. Lepage and S. J. Brodsky, "Exclusive processes in perturbative quantum chromodynamics," *Phys. Rev.* **D22** (1980) 2157.

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[6] R. J. Holt and C. D. Roberts, "Distribution Functions of the Nucleon and Pion in the Valence Region," *Rev.Mod.Phys.* **82** (2010) 2991–3044, arXiv:1002.4666 [nucl-th].