Wilsonian renormalization group flows are generated by ordinary distributions

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Outline

- Mathematics of Euclidean Feynman functional integral.
- Mathematics of Wilsonian regularization.
- Mathematics of Wilsonian renormalization.

Recap on distribution theory

Will consider only scalar and bosonic fields for simplicity.

Will consider only flat (affine) spacetime manifold for simplicity.

- $\mathcal{E}$: space of all smooth fields over spacetime.
  They form a vector space with a topology:
  $\varphi_i \in \mathcal{E}$ $(i \in \mathbb{N}_0) \to 0$ iff all derivatives locally uniformly converge to zero.

collection of “open” sets

- $\mathcal{S}$: space of rapidly decreasing smooth fields (Schwartz fields) over spacetime.
  They form a vector space with a topology:
  $\varphi_i \in \mathcal{S}$ $(i \in \mathbb{N}_0) \to 0$ iff all derivatives uniformly converge to zero faster than polynomial.

- $\mathcal{D}$: space of compactly supported smooth fields (test fields) over spacetime.
  They form a vector space with a topology:
  $\varphi_i \in \mathcal{D}$ $(i \in \mathbb{N}_0) \to 0$ iff they stay within a compact set and $\to 0$ in $\mathcal{E}$ sense.

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}}_{\text{smooth fields}} \supset \underbrace{\mathcal{S}}_{\text{Schwartz fields}} \supset \underbrace{\mathcal{D}}_{\text{test fields}}$$

Distributions are continuous duals of $\mathcal{E}$, $\mathcal{S}$, $\mathcal{D}$.

- $\mathcal{E}'$: continuous $\mathcal{E} \to \mathbb{R}$ linear functionals.
  They are the compactly supported distributions.

- $\mathcal{S}'$: continuous $\mathcal{S} \to \mathbb{R}$ linear functionals.
  They are the tempered or Schwartz distributions.

- $\mathcal{D}'$: continuous $\mathcal{D} \to \mathbb{R}$ linear functionals.
  They are the space of all distributions.

They carry a corresponding natural topology (notion of “open” sets).

only meaningful on flat spacetime

$$\underbrace{\mathcal{E}'}_{\text{compactly supp distributions}} \subset \underbrace{\mathcal{S}'}_{\text{Schwartz distributions}} \subset \underbrace{\mathcal{D}'}_{\text{all distributions}}$$

[Of course, functions are also distributions, e.g. $\mathcal{D} \subset \mathcal{E}'$ and $\mathcal{E} \subset \mathcal{D}'$ etc.]

Recap on measure / integration / probability theory

- Let X be a set (their elements called elementary events).
- Let Σ be a collection of subsets of X such that:
  - X is in Σ,
  - for all A in Σ, their complement is in Σ.
  - for all countably infinite system $A_i \in \Sigma$ $(i \in \mathbb{N}_0)$, their union $\underset{i\in\mathbb{N}_0}{\cup} A_i$ is in Σ.

  Then, Σ is called a sigma-algebra (their elements called composite events).
  Typically, if X carries open sets (topology), the sigma-alg generated by them is taken.
- Let $\mu : \Sigma \to \mathbb{R}_0^+$ be a set-function, such that:
  - $\mu(\emptyset) = 0$,
  - for all countably infinite disjoint system $A_i \in \Sigma$ $(i \in \mathbb{N}_0)$: $\mu\big(\underset{i\in\mathbb{N}_0}{\cup} A_i\big) = \sum_{i\in\mathbb{N}_0} \mu(A_i)$,
  - $\exists$ some countably infinite system $A_i \in \Sigma$ $(i \in \mathbb{N}_0)$ with $\mu(A_i) < \infty$: $X = \underset{i\in\mathbb{N}_0}{\cup} A_i$.

  Then, μ is called a measure.

(X, Σ, μ) is called a measure space. [E.g. probability measure space if $\mu(X) =$ finite.]

- A function $f : X \to \mathbb{C}$ is called measurable iff in good terms with mesure theory.
  Theorem: f is measurable iff approximable pointwise by “histograms” with bins from Σ.

- The integral $\int\limits_{x\in X} f(x)\,\mathrm{d}\mu(x)$ is defined via the histogram “area” approximations.
  Theorem: this is well-defined.

- Let $(X.\Sigma,\mu)$ be a measure space and (Y,Δ) an other one, with unspecified measure.
  Let $C : X \to Y$ be a measurable mapping.
  Then, one can define the pushforward (or marginal) measure $C_*\mu$ on Y.
  [For all $B \in \Delta$ one defines $(C_*\mu)(B) := \mu\big(\overset{-1}{C}(B)\big)$.]

- Pushforward (marginal) measure means simply transformation of integration variable.
  If forgetful transformation, the “forgotten” d.o.f. are “integrated out”.

- If μ is a probability measure, $M(y) := \int\limits_{x\in X} \mathrm{e}^{\mathrm{i}(y|x)}\,\mathrm{d}\mu(x)$ is its Fourier transform.

Mathematics of Euclidean Feynman functional integral

- Take an Euclidean action $S = T + V$, with kinetic + potential term splitting.

- Then, T has a propagator $K(\cdot,\cdot)$ which is positive definite: with notation $T(\varphi) = -\int \varphi\Box\varphi$, then K satisfies
  - $\Box_x K(x,y) = \delta_y(x)$,
  - for all $j \in \mathcal{S}$ rapidly decreasing sources: $(K|j\otimes j) \geq 0$.

- Due to above, the function $\mathcal{S} \to \mathbb{C},\ j \mapsto \mathrm{e}^{-(K|j\otimes j)}$ has "quite nice" properties.

- Bochner-Khinchin theorem: because of
  - "quite nice" properties of $j \mapsto \mathrm{e}^{-(K|j\otimes j)}$,
  - "quite nice" properties of the space $\mathcal{S}$,

  $\exists$ a unique measure γ_T on $\mathcal{S}'$, whose Fourier transform is $j \mapsto \mathrm{e}^{-(K|j\otimes j)}$.
  It is the Feynman measure for free theory: $\int\limits_{\phi\in\mathcal{S}'} (\ldots)\,\mathrm{d}\gamma_T(\phi) = \int\limits_{\phi\in\mathcal{S}'} (\ldots)\,\mathrm{e}^{-T(\phi)}\ \text{“}\mathrm{d}\phi\text{”}$.

- One is tempted to define Feynman measure of the interacting theory via

$$\int\limits_{\phi\in\mathcal{S}'} (\ldots)\,\mathrm{e}^{-V(\phi)}\,\mathrm{d}\gamma_T(\phi) \qquad \left[= \int\limits_{\phi\in\mathcal{S}'} (\ldots)\,\mathrm{e}^{-(T(\phi)+V(\phi))}\ \text{“}\mathrm{d}\phi\text{”}\right]$$

Mathematics of Wilsonian regularization

- Problem, the interacting Feynman measure $\mathrm{e}^{-V}\,\gamma_T$ is undefined:

$$\int_{\phi\in\mathcal{S}'} (\ldots) \qquad \underbrace{\mathrm{e}^{-V(\phi)}}_{\substack{\text{lives on function}\\ \text{sense fields}}} \qquad \underbrace{\mathrm{d}\gamma_T(\phi)}_{\substack{\text{lives on distribution}\\ \text{sense fields}}}$$

  Because V is spacetime integral of pointwise product of fields, e.g. $V(\varphi)=\int\varphi^4$.
  How to bring e^{-V} and γ_T to common grounds?

- Physicist solution: brute force, i.e. Wilsonian regularization.
  Take a continuous linear mapping C: (distributional fields) $\to$ (function sense fields).
  Take the pushforward Gaussian measure $C_*\gamma_T$, which lives on the image space of C.
  Integrate e^{-V} against that:

$$\int_{\varphi\in\mathrm{Ran}(C)} (\ldots)\; \mathrm{e}^{-V(\varphi)}\; \mathrm{d}\big(C_*\gamma_T\big)(\varphi) \qquad \Big[= \int_{\varphi\in\mathrm{Ran}(C)} (\ldots)\; \mathrm{e}^{-(T_C(\varphi)+V(\varphi))}\; \text{“}\mathrm{d}\varphi\text{”} \Big]$$

a space of UV regularized fields

[Schwartz kernel theorem: C is convolution by a test function, if translationally invariant.
I.e., it is a momentum space damping, or coarse-graining of fields.]

Mathematics of Wilsonian renormalization

- What do we do with the C-dependence? What is the physics / mathematics behind?

- Take a family V_C $(C \in \{$coarse-grainings$\})$ of interaction terms.
  We say that it is a Wilsonian renormalization group (RG) flow iff:
  $\exists$ some continuous functional z : $\{$coarse-grainings$\} \to \mathbb{R}$, such that
  $\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
  $z(C'')_*\,(\mathrm{e}^{-V_{C''}}\,C''_*\gamma_T)$ is pushforward of $z(C)_*\,(\mathrm{e}^{-V_C}\,C_*\gamma_T)$ by C'.
  [z is called the running wave function renormalization factor.]

- If $\mathcal{G}_C = (\mathcal{G}_C^{(0)}, \mathcal{G}_C^{(1)}, \mathcal{G}_C^{(2)}, \ldots)$ are the moments of $\mathrm{e}^{-V_C}\,C_*\gamma_T$, then
  $\exists$ some continuous functional z : $\{$coarse-grainings$\} \to \mathbb{R}$, such that
  $\forall$ coarse-grainings C, C', C'' with $C'' = C'\,C$:
  $z(C'')^n\,\mathcal{G}_{C''}^{(n)} \;=\; z(C)^n \otimes^n C'\,\mathcal{G}_C^{(n)}$ for all $n = 0, 1, 2, \ldots$
  [Valid also in Lorentz signature and on manifolds, for formal moments (correlators).]

- Theorem [A.L., Z.Tarcsay `arXiv:2303.03740`]:
  In case of any bosonic fields over flat spacetime,
  $\exists$ C-independent distributional correlator $G = (G^{(0)}, G^{(1)}, G^{(2)}, \ldots)$, such that
  $\mathcal{G}_C^{(n)} = z(C)^{-n} \otimes^n C\,G^{(n)}$ holds. [I.e., Wilsonian RG flow ↔ distribution.]

Summary

- Wilsonian RG flow of correlators can be defined in any signature and on manifolds.

- Under mild conditions, they originate from a distributional correlator (UV limit).

  [$\sim$ existence theorem for multiplicative renormalization.]

- Likely to be generically true (on manifolds, in any signature).

Backup slides

Followed guidelines

Do not use (unless emphasized):

- Structures specific to an affine spacetime manifold.
- Known fixed spacetime metric / causal structure.
- Known splitting of Lagrangian to free + interaction term.

Consequences:

- Cannot go to momentum space, have to stay in spacetime description.
- Cannot refer to any affine property of Minkowski spacetime, e.g. asymptotics.
  (No Schwartz functions.)
- Cannot use Wick rotation to Euclidean signature metric.
- Even if Wick rotated, no free + interaction splitting, so no Gaussian Feynman measure.
- Can only use generic, differential geometrically natural objects.

Outline

Will attempt to set up eom for the key ingredient for the quantum probability space of QFT.

I. On Wilsonian regularized Feynman functional integral formulation:

- Can be substituted by regularized master Dyson-Schwinger equation for correlators.
- For conformally invariant or flat spacetime Lagrangians, showed an existence condition for regularized MDS solutions, provides convergent iterative solver method.

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II. On Wilsonian renormalization group flows of correlators:

- They form a topological vector space which is
  Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
- On flat spacetime for bosonic fields: in bijection with distributional correlators.

[`arXiv:2303.03740` *with Zsigmond Tarcsay*]

Part I:

On Wilsonian regularized Feynman functional integral formulation

The classical field theory scene

$\mathcal{M}$ a smooth orientable oriented manifold (wannabe spacetime, but no metric, yet).

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Field configurations:

$$\underbrace{(v,\nabla)}_{=:\ \psi} \;\in\; \underbrace{\Gamma\Big(\,V(\mathcal{M})\times_{\mathcal{M}}\mathrm{CovDeriv}\big(V(\mathcal{M})\big)\,\Big)}_{=:\ \boldsymbol{\mathcal{E}}}$$

Real topological affine space with the $\mathcal{E}$ smooth function topology.

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$$\underbrace{(\delta v, \delta C)}_{=:\ \delta\psi} \ \in\ \underbrace{\Gamma\Big(V(\mathcal{M}) \times_{\mathcal{M}} T^{*}(\mathcal{M}){\otimes}V(\mathcal{M}){\otimes}V^{*}(\mathcal{M})\Big)}_{=:\ \mathcal{E}}$$

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Test field variations: $\delta\psi_{T} \in \mathcal{D}$, compactly supported ones from $\mathcal{E}$ with $\mathcal{D}$ test funct. top.

Informal Feynman functional integral in Lorentz signature

Fix a reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$ for bringing the problem from $\boldsymbol{\mathcal{E}}$ to $\mathcal{E}$, and take $J_1, ..., J_n \in \mathcal{E}'$. Then, $\psi \mapsto (J_1|\psi{-}\psi_0) \cdot ... \cdot (J_n|\psi{-}\psi_0)$ defines a $\boldsymbol{\mathcal{E}} \to \mathbb{R}$ polynomial observable.

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Feynman type quantum vacuum expectation value of this is postulated as:

$$\int\limits_{\psi\in\boldsymbol{\mathcal{E}}} (J_1|\psi{-}\psi_0) \cdot ... \cdot (J_n|\psi{-}\psi_0) \;\; \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)}\, \mathrm{d}\lambda(\psi) \;\; \Big/ \;\; \int\limits_{\psi\in\boldsymbol{\mathcal{E}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)}\, \mathrm{d}\lambda(\psi)$$

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Partition function often invoked to book-keep these (formal Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$):

$$Z_{\psi_0}: \quad \mathcal{E}' \longrightarrow \mathbb{C}, \quad J \longmapsto Z_{\psi_0}(J) := \int\limits_{\psi\in\boldsymbol{\mathcal{E}}} \mathrm{e}^{\mathrm{i}\,(J|\psi-\psi_0)} \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)} \, \mathrm{d}\lambda(\psi),$$

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and from this one can define

$$G^{(n)}_{\psi_0} \quad := \quad \left((-\mathrm{i})^n \, \frac{1}{Z_{\psi_0}(J)} \, \partial^{(n)}_J Z_{\psi_0}(J) \right)\Bigg|_{J=0}$$

n-field correlator, and their collection $G_{\psi_0} := (G^{(0)}_{\psi_0}, G^{(1)}_{\psi_0}, \ldots, G^{(n)}_{\psi_0}, \ldots) \in \bigoplus\limits_{n\in\mathbb{N}_0} \overset{n}{\otimes} \mathcal{E}$.

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Above quantum expectation value expressable via distribution pairing: $\big(J_1 {\otimes} \dots {\otimes} J_n \,\big|\, G^{(n)}_{\psi_0}\big)$.

Well known problems:

- No “Lebesgue” measure λ in infinite dimensions.
- Neither $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ is meaningful. (Can be repaired to some extent in Euclidean signature.)
- Neither the Fourier transform of this undefined measure is meaningful.

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Rules in informal QFT:

- as if λ existed as *translation invariant* (Lebesgue) measure,
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Textbook “theorem”: because of above rules, one has
$Z:\,\mathcal{E}'\to\mathbb{C}$ is Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ “$\Longleftrightarrow$” it satisfies master-Dyson-Schwinger eq

$$\Big(\,\mathbf{E}\big((-\mathrm{i})\partial_J+\psi_0\big)\;+\;\hbar\,J\,\Big)Z(J)\quad=\quad 0\quad(\forall J\in\mathcal{E}')$$

where $E(\psi):=DS(\psi)$ is the Euler-Lagrange functional at $\psi\in\boldsymbol{\mathcal{E}}$.

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Does this informal PDE have a meaning? [Yes, on the correlators $G=\big(G^{(0)},G^{(1)},\ldots\big)$.]

Rigorous definition of Euler-Lagrange functional

- Let a Lagrange form be given, which is

$$\mathrm{L}:\; V(\mathcal{M}) \;\oplus\; T^*(\mathcal{M})\otimes V(\mathcal{M}) \;\oplus\; T^*(\mathcal{M})\wedge T^*(\mathcal{M})\otimes V(\mathcal{M})\otimes V^*(\mathcal{M}) \;\longrightarrow\; \overset{\dim(\mathcal{M})}{\wedge}\, T^*(\mathcal{M})$$

pointwise bundle homomorphism.

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- Lagrangian expression:

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where $F(\nabla)$ is the curvature tensor.

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- Action functional:

$$S:\quad \underbrace{\Gamma\big(V(\mathcal{M}) \times_{\mathcal{M}} \mathrm{CovDeriv}(V(\mathcal{M}))\big)}_{=:\; \boldsymbol{\mathcal{E}}} \;\longrightarrow\; \mathrm{Meas}(\mathcal{M},\mathbb{R}), \quad \underbrace{(v,\nabla)}_{=:\; \psi} \;\longmapsto\; \Big(\mathcal{K} \mapsto S_{\mathcal{K}}(v,\nabla)\Big)$$

where $S_{\mathcal{K}}(v,\nabla) := \int\limits_{\mathcal{K}} \mathrm{L}(v,\nabla v, F(\nabla))$ for all $\mathcal{K} \subset \mathcal{M}$ compact.

Action functional $S : \boldsymbol{\mathcal{E}} \to \mathrm{Meas}(\mathcal{M}, \mathbb{R})$ Fréchet differentiable, its Fréchet derivative

$$DS : \quad \boldsymbol{\mathcal{E}} \times \mathcal{E} \longrightarrow \mathrm{Meas}(\mathcal{M}, \mathbb{R}), \quad (\psi, \delta\psi) \longmapsto \Big(\mathcal{K} \mapsto \big(DS_{\mathcal{K}}(\psi) \,\big|\, \delta\psi\big)\Big)$$

is the usual Euler-Lagrange integral on $\mathcal{K}$ $+$ usual boundary integral on $\partial\mathcal{K}$.

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Euler-Lagrange functional:
We restrict DS from $\boldsymbol{\mathcal{E}} \times \mathcal{E}$ to $\boldsymbol{\mathcal{E}} \times \mathcal{D}$, to make the EL integral over full $\mathcal{M}$ finite.

$$E : \quad \boldsymbol{\mathcal{E}} \times \mathcal{D} \longrightarrow \mathbb{R}, \quad \big(\psi,\, \delta\psi_T\big) \longmapsto \big(E(\psi) \,\big|\, \delta\psi_T\big) := \big(DS_{\mathcal{M}}(\psi) \,\big|\, \delta\psi_T\big)$$

Bulk Euler-Lagrange integral remains, no boundary term. Meaningful on full $\mathcal{M}$, real valued.

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Observables are the $O : \boldsymbol{\mathcal{E}} \to \mathbb{R}$ continuous maps.

Rigorous definition of master Dyson-Schwinger equation

- Want to rephrase informal MDS operator on Z to n-field correlators $G = \big(G^{(0)}, G^{(1)}, ...\big)$.
  These sit in the tensor algebra $\mathcal{T}(\mathcal{E}) := \bigoplus\limits_{n\in\mathbb{N}_0} \hat{\otimes}_\pi^n \, \mathcal{E}$ of field variations.
  More precisely, they sit in a graded-symmetrized subspace, e.g. $\bigvee(\mathcal{E})$ or $\bigwedge(\mathcal{E})$ of $\mathcal{T}(\mathcal{E})$.
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  Unity $\mathbb{1} := (1, 0, 0, 0, ...)$.
  Left-multiplication L_x by a fix element x meaningful and continuous linear.
  Left-insertion ι_p (tracing out) by $p \in \big(\mathcal{T}(\mathcal{E})\big)' \equiv \mathcal{T}_a(\mathcal{E}')$ also meaningful, continuous linear.
  Usual graded-commutation: $\big(\iota_p \, L_{\delta\psi} \pm L_{\delta\psi} \, \iota_p \big) \, G \;=\; (p|\delta\psi) \, G \quad (\forall p \in \mathcal{E}', \; \delta\psi \in \mathcal{E}, \; G\,)$.

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$$\forall\, \psi \in \boldsymbol{\mathcal{E}} : \quad \underbrace{O_{\psi_0}(\psi{-}\psi_0)}_{=\,O(\psi)} \;=\; \Big(\mathbf{O}_{\psi_0} \,\Big|\, (1,\, \overset{1}{\otimes}(\psi{-}\psi_0),\, \overset{2}{\otimes}(\psi{-}\psi_0),\, \ldots) \Big).$$

Similarly $E: \boldsymbol{\mathcal{E}} \to \mathcal{D}'$, $\psi \mapsto E(\psi)$, let $E_{\psi_0} := E \circ (\mathrm{I}_{\mathcal{E}} + \psi_0)$ the same re-expressed on $\mathcal{E}$.

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For fixed $\delta\psi_T\in\mathcal{D}$ one has $(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)\in\mathcal{T}_a(\mathcal{E}')$, i.e. one can left-insert with it: $\iota_{(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)}$ meaningfully acts on $\mathcal{T}(\mathcal{E})$.

The master Dyson-Schwinger (MDS) equation is:

we search for (ψ_0, G_{ψ_0}) such that:
$$\underbrace{G^{(0)}_{\psi_0}}_{=:\, b\, G_{\psi_0}} = 1,$$

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[Feynman type quantum vacuum expectation value of O is then $\big(\mathbf{O}_{\psi_0} \,\big|\, G_{\psi_0}\big)$.]

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$$\Downarrow$$

MDS operator at $\psi_0 = 0$ reads

$$\big(\mathbf{M}_{\psi_0,\delta\psi_T}\,G\big)^{(n)}(x_1,\dots,x_n) =$$

$$\int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\,\Box_y G^{(n+1)}(y,x_1,\dots,x_n)\,\mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\,G^{(n+3)}(y,y,y,x_1,\dots,x_n)\,\mathrm{v}(y)$$

$$-\mathrm{i}\,\hbar\;\underbrace{n\,\tfrac{1}{n!}\textstyle\sum_{\pi\in\Pi_n}\delta\psi_T(x_{\pi(1)})\,G^{(n-1)}(x_{\pi(2)},\dots,x_{\pi(n)})}_{=\,(L_{\delta\psi_T}G)^{(n)}(x_1,\dots,x_n)}$$

Pretty much well-defined, and clear recipe, if field correlators were *functions*.

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Workaround in QFT: Wilsonian regularization using coarse-graining (UV damping).

Wilsonian regularized master Dyson-Schwinger equation

- When $\mathcal{E}$ (resp $\mathcal{D}$) are smooth sections of some vector bundle,
  denote by $\mathcal{E}^{\times}$ (resp $\mathcal{D}^{\times}$) the smooth sections of its densitized dual vector bundle.
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Coarse-graining ops are natural generalization of convolution by test functions to manifolds.

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$$\forall\, \delta\psi_T \in \mathcal{D} : \qquad \underbrace{\Big(l_{\gamma(\kappa)\,(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T\,)} \;-\; \mathrm{i}\,\hbar\, L_{C_\kappa \delta\psi_T} \Big)}_{=:\, \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}} \,\mathcal{G}_{\psi_0,\kappa} \quad = \quad 0.$$

Originally: Feynman integral “$\Longleftrightarrow$” MDS equation.

Wilsonian regularized Feynman integral:
integrate only on the image space $C_\kappa[\mathcal{D}^{\times\prime}] \subset \mathcal{E}$ of some coarse-graining operator C_κ.

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Brings back problem from distributions to smooth functions, but depends on regulator κ.

Smooth function solution to free KG regularized MDS eq: $\mathcal{G}_{\psi_0,\kappa} = \exp(\mathcal{K}_{\psi_0,\kappa})$ where

$$\begin{aligned}
\mathcal{K}^{(0)}_{\psi_0,\kappa} &= 0, \\
\mathcal{K}^{(1)}_{\psi_0,\kappa} &= 0, \\
\mathcal{K}^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\,\mathtt{K}^{(2)}_{\psi_0,\kappa} && \longleftarrow \text{(smoothed symmetric propagator)} \\
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But what we do with κ dependence? (Rigorous Wilsonian renormalization?)

Part II:

On Wilsonian RG flows of correlators

Informal Wilsonian RG flows of Feynman measures

Fix a reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$ to bring the problem from $\boldsymbol{\mathcal{E}}$ to $\mathcal{E}$.

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A family of actions S_{ψ_0, C_κ} $(C_\kappa \in$ coarse-grainings$)$ is Wilsonian RG flow iff:
$\forall$ coarse-grainings C_κ, C_μ, C_ν with $C_\nu = C_\mu C_\kappa$ one has that
$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\psi_0, C_\nu}}\lambda_{C_\nu}$ is the pushforward of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\psi_0, C_\kappa}}\lambda_{C_\kappa}$ by C_μ. $\longleftarrow$ RGE

Rigorous definition will be this, but expressed on the formal moments (n-field correlators).

Rigorous Wilsonian RG flows

Definition:

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Space of Wilsonian RG flows is nonempty:

For any distributional correlator G, the family

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Theorem[A.L., Z.Tarcsay]:

1. On manifolds it is Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
2. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form of $(*)$.

Sketch of proof for 1.:

- Coarse-grainings have a natural partial ordering of being less UV than an other:
  $C_\nu \preceq C_\kappa$ iff $C_\nu = C_\kappa$ or $\exists\, C_\mu :\ C_\nu = C_\mu C_\kappa$.
- With this, the space of Wilsonian RG flows is seen to be projective limit of copies of $\mathcal{T}(\mathcal{E})$.
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- On flat spacetime, convolution ops by test functions $C_f := f \star (\cdot)$ exist and commute.
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  That is, one has corresponding $\mathcal{G}^{(n)}_{f_1,\dots,f_n}$ symmetric n-linear map in $f_1, ..., f_n$.
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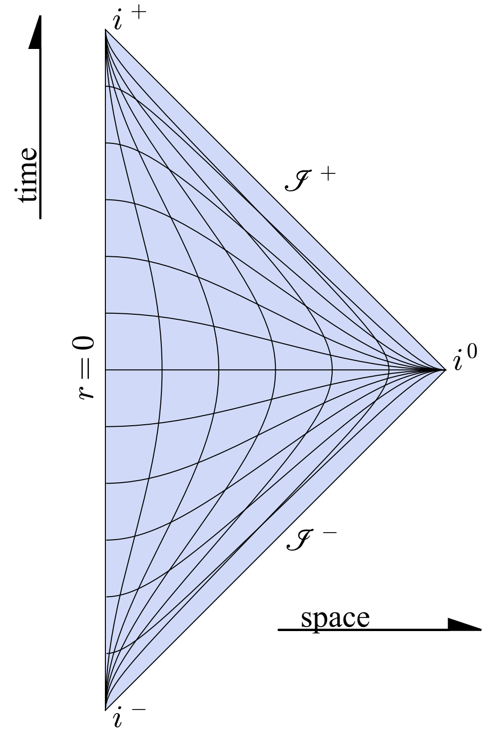
An improved Banach-Steinhaus theorem (the key lemma – A.L., Z.Tarcsay):
If a sequence of n-variate distributions pointwise converge on $\otimes^n\mathcal{D}$, it does also on full $\mathcal{D}_n$.
Therefore, by ordinary Banach-Steinhaus thm, the limit is an n-variate distribution.

Existence condition for regularized MDS solutions

If Euler-Lagrange functional $E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ conformally invariant:
re-expressable on Penrose conformal compactification.



That is always a compact manifold, with cone condition boundary.

$E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ reformulable over this base manifold.

In such situation, $\mathcal{E} = \mathcal{D}$ and have nice properties:
countably Hilbertian nuclear Fréchet (CHNF) space.

$$F_0 \ \supset\ F_1 \ \supset\ \ldots\ \supset\ F_m \ \supset\ \ldots\ \supset\ \mathcal{E}$$

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without penalty, one can equip $\mathcal{T}(\mathcal{E})$ with a better topology, inheriting CHNF topology.

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Regularized MDS operator is then a Hilbert-Schmidt linear map

$$\mathbf{M}_{\psi_0,\kappa} : \quad H_m \otimes F_m \longrightarrow H_0, \quad \mathcal{G} \otimes \delta\psi_T \longmapsto \mathbf{M}_{\psi_0,\kappa,\delta\psi_T} \mathcal{G}$$

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Theorem: one can legitimately trace out $\delta\psi_T$ variable to form

$$\hat{\mathbf{M}}^2_{\psi_0,\kappa} : \quad H_m \longrightarrow H_m, \quad \mathcal{G} \longmapsto \sum_{i \in \mathbb{N}_0} \mathbf{M}^{\dagger}_{\psi_0,\kappa,\delta\psi_{T\,i}} \mathbf{M}_{\psi_0,\kappa,\delta\psi_{T\,i}} \mathcal{G}$$

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(i) the iteration

$$\mathcal{G}_0 \;:=\; \mathbb{1} \text{ and } \mathcal{G}_{l+1} \;:=\; \mathcal{G}_l - \tfrac{1}{T}\,\hat{\mathbf{M}}^2_{\psi_0,\kappa}\mathcal{G}_l \qquad (l = 0, 1, 2, ...)$$

is always convergent if $T >$ trace norm of $\hat{\mathbf{M}}^2_{\psi_0,\kappa}$.

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Use for lattice-like numerical method in Lorentz signature?
(Treatment can be adapted to flat spacetime also, because Schwartz functions are CHNF.)

Structure of model building in fundamental physics

Relativistic or non-relativistic point mechanics:

- Take Newton equation over a fixed spacetime and fixed potentials.
- Solution space to the equation turns out to be a symplectic manifold.
- One can play classical probability theory on the solution space:
  - Elements of solution space X are elementary events.
  - Collection of Borel sets Σ of X are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is classical probability space.

Relativistic or non-relativistic quantum mechanics:

- Take Dirac etc. equation over a fixed spacetime and fixed potentials.
- Finite charge weak solution space to the equation turns out to be a Hilbert space.
- One can play quantum probability theory on the solution space:
  - One dimensional subspaces of the solution space $\mathcal{H}$ are elementary events, X.
  - Collection of all closed subspaces Σ of $\mathcal{H}$ are composite events.
  - A state is a probability measure W on Σ, i.e. (X, Σ, W) is quantum probability space.

Fréchet derivative in top.vector spaces

Let F and G real top.affine space, Hausdorff.
Subordinate vector spaces: $\mathbb{F}$ and $\mathbb{G}$.

A map $S:\ F \to G$ is Fréchet-Hadamard differentiable at $\psi \in F$ iff:
there exists $DS(\psi):\ \mathbb{F} \to \mathbb{G}$ continuous linear, such that for all sequence $n \mapsto h_n$ in $\mathbb{F}$, and nonzero sequence $n \mapsto t_n$ in $\mathbb{R}$ which converges to zero,

$$(\mathbb{G})\lim_{n\to\infty}\left(\frac{S(\psi + t_n\, h_n) - S(\psi)}{t_n} - DS(\psi)\, h_n\right) \quad = \quad 0$$

holds.

Fréchet derivative of action functional

Fréchet derivative of $S:\ \boldsymbol{\mathcal{E}} \longrightarrow \mathrm{Meas}(\mathcal{M},\mathbb{R})$ is

$$DS:\ \boldsymbol{\mathcal{E}} \times \mathcal{E} \longrightarrow \mathrm{Meas}(\mathcal{M},\mathbb{R}),\ (\psi,\delta\psi) \longmapsto \Big(\mathcal{K} \mapsto \big(DS_{\mathcal{K}}(\psi)\big|\delta\psi\big)\Big)$$

For $\underbrace{(v,\nabla)}_{=:\psi} \in \boldsymbol{\mathcal{E}}$ given,

$$\underbrace{(\delta v,\delta C)}_{=:\delta\psi} \ \mapsto\ \big(DS_{\mathcal{K}}(v,\nabla)\,\big|\,(\delta v,\delta C)\big) =$$

$$\int\limits_{\mathcal{K}}\Big(D_1\mathrm{L}(v,\nabla v,P(\nabla))\,\delta v + D_2^a\mathrm{L}(v,\nabla v,P(\nabla))\,(\nabla_a\delta v + \delta C_a v) + 2\,D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))\,\tilde{\nabla}_{[a}\delta C_{b]}\Big)$$

$$= \int\limits_{\mathcal{K}}\Big(D_1\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\,\delta v - \big(\tilde{\nabla}_a D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\big)\,\delta v\Big) +$$

$$\Big(D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\,\delta C_a v - 2\,\big(\tilde{\nabla}_a D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\big)\,\delta C_b\Big)$$

$$+\ m\int\limits_{\partial\mathcal{K}}\Big(D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[ac_1\ldots c_{m-1}]}\,\delta v + 2\,D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))_{[ac_1\ldots c_{m-1}]}\,\delta C_b\Big)$$

$$(m := \dim(\mathcal{M}))$$

[usual Euler-Lagrange bulk integral + boundary integral]

Distributions on manifolds

$W(\mathcal{M})$ vector bundle, $W^{\times}(\mathcal{M}) := W^{*}(\mathcal{M}) \otimes \overset{\dim(\mathcal{M})}{\wedge} T^{*}(\mathcal{M})$ its densitized dual.
$W^{\times\times}(\mathcal{M}) \equiv W(\mathcal{M})$.

Correspondingly: $\mathcal{E}^{\times}$ and $\mathcal{D}^{\times}$ are densitized duals of $\mathcal{E}$ and $\mathcal{D}$.

$\mathcal{E} \times \mathcal{D}^{\times} \to \mathbb{R}$, $(\delta\psi, p_{T}) \mapsto \int\limits_{\mathcal{M}} \delta\psi\, p_{T}$ and $\mathcal{D} \times \mathcal{E}^{\times} \to \mathbb{R}$, $(\delta\psi_{T}, p) \mapsto \int\limits_{\mathcal{M}} \delta\psi_{T}\, p$ jointly sequentially continuous.

Therefore, continuous dense linear injections $\mathcal{E} \to \mathcal{E}^{\times\prime}$ and $\mathcal{D} \to \mathcal{D}^{\times\prime}$.
(hance the name, distributional sections)

Let $A:\ \mathcal{E} \to \mathcal{E}$ continuous linear.
It has formal transpose iff there exists $A^{t}:\ \mathcal{D}^{\times} \to \mathcal{D}^{\times}$ continuous linear, such that $\forall \delta\psi \in \mathcal{E}$ and $p_{T} \in \mathcal{D}^{\times}$: $\int\limits_{\mathcal{M}} (A\,\delta\psi)\, p_{T} = \int\limits_{\mathcal{M}} \delta\psi\, (A^{t}\, p_{T})$.
Topological transpose of formal transpose $(A^{t})' :\ (\mathcal{D}^{\times})' \to (\mathcal{D}^{\times})'$ is the distributional extension of A. Not always exists.

Fundamental solution on manifolds

Let $E : \boldsymbol{\mathcal{E}} \times \mathcal{D} \to \mathbb{R}$ be Euler-Lagrange functional, and $J \in \mathcal{D}'$.

$\mathtt{K}_{(J)} \in \boldsymbol{\mathcal{E}}$ is solution with source J, iff $\forall \delta\psi_T \in \mathcal{D} :\ (E(\mathtt{K}_{(J)})\,|\,\delta\psi_T) = (J|\delta\psi_T)$.

Specially: one can restrict to $J \in \mathcal{D}^\times \subset \mathcal{E}^\times \subset \mathcal{D}'$.

A continuous map $\mathtt{K} : \mathcal{D}^\times \to \boldsymbol{\mathcal{E}}$ is fundamental solution, iff for all $J \in \mathcal{D}^\times$ the field $\mathtt{K}(J) \in \boldsymbol{\mathcal{E}}$ is solution with source J.

May not exists, and if does, may not be unique.

If $\mathtt{K}_{\psi_0} : \mathcal{D}^\times \to \mathcal{E}$ vectorized fundamental solution is linear (e.g. for linear $E_{\psi_0} : \mathcal{E} \to \mathcal{D}'$): $\mathtt{K}_{\psi_0} \in \mathcal{L}in(\mathcal{D}^\times, \mathcal{E}) \subset (\mathcal{D}^\times)' \otimes (\mathcal{D}^\times)'$ is distribution.

Particular solutions to the free MDS equation

Distributional solutions to free MDS equation: $G_{\psi_0} = \exp(K_{\psi_0})$ where

$$\begin{aligned} K^{(0)}_{\psi_0} &= 0, \\ K^{(1)}_{\psi_0} &= 0, \\ K^{(2)}_{\psi_0} &= \mathrm{i}\,\hbar\,\mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

Smooth function solutions to free regularized MDS equation: $G_{\psi_0} = \exp(K_{\psi_0,\kappa})$ where

$$\begin{aligned} K^{(0)}_{\psi_0,\kappa} &= 0, \\ K^{(1)}_{\psi_0,\kappa} &= 0, \\ K^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\,(C_\kappa \otimes C_\kappa)\mathtt{K}^{(2)}_{\psi_0} \\ K^{(n)}_{\psi_0,\kappa} &= 0 \qquad\qquad (n \geq 2) \end{aligned}$$

[Here $C_\kappa(\cdot) := \eta \star (\cdot)$ is convolution by a test function η.]

Renormalization from functional analysis p.o.v.

Let $\mathbb{F}$ and $\mathbb{G}$ real or complex top.vector space, Hausdorff loc.conv complete.

Let $M : \mathbb{F} \rightarrowtail \mathbb{G}$ densely defined linear map (e.g. MDS operator).

Closed: the graph of the map is closed.

Closable: there exists linear extension, such that its graph closed (unique if exists).

Closable $\Leftrightarrow$ where extendable with limits, it is unique.

Multivalued set:
$\mathrm{Mul}(M) := \big\{ y \in \mathbb{G} \,\big|\, \exists\, (x_n)_{n\in\mathbb{N}}$ in $\mathrm{Dom}(M)$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} M x_n = y \big\}$.

$\mathrm{Mul}(M)$ always closed subspace.

Closable $\Leftrightarrow$ $\mathrm{Mul}(M) = \{0\}$.

Maximally non-closable $\Leftrightarrow$ $\mathrm{Mul}(M) = \overline{\mathrm{Ran}(M)}$. Pathological, not even closable part.

Polynomial interaction term of MDS operator maximally non-closable!

MDS operator:

$$\mathbf{M}: \quad \mathcal{D}\otimes\mathcal{T}(\mathcal{E})\to\mathcal{T}(\mathcal{E}), \quad G\mapsto\mathbf{M}\,G$$

linear, everywhere defined continuous. So,

$$\mathbf{M}: \quad \mathcal{T}(\mathcal{D}^{\times\prime})\rightarrowtail\mathcal{D}'\otimes\mathcal{T}(\mathcal{D}^{\times\prime}), \quad G\mapsto\mathbf{M}\,G$$

linear, densely defined.

Similarly: $\mathbf{M}_\kappa$ regularized MDS operator (κ: a fix regularizator).

Not good equation:

$G \in \mathcal{T}(\mathcal{D}^{\times\prime})$? $\qquad G^{(0)} = 1$ and $\exists\, \mathcal{G}_\kappa \to G$ approximator sequence, such that :

$$\lim_{\kappa\to\delta} \mathbf{M}\, \mathcal{G}_\kappa = 0.$$

All G would be selected, because $\mathrm{Mul}()$ set of interaction term is full space.

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Can be good:

$$G \in \mathcal{T}(\mathcal{D}^{\times\,\prime})\ ? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that } : \quad \forall\, \kappa :\ \mathbf{M}_\kappa\, \mathcal{G}_\kappa = 0.$$

That is, as implicit function of κ, not as operator closure kernel.

Running coupling:
If in $\mathbf{M}_\kappa$ EL terms are combined with κ-dependent weights $\gamma(\kappa)$.
(Not just with real factors.)
E.g.:

$$(\gamma, G) \in \mathcal{T}(\mathcal{D}^{\times\,\prime})\ ? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that } : \quad \forall\, \kappa :\ \mathbf{M}_{\gamma(\kappa),\kappa}\, \mathcal{G}_\kappa = 0.$$

Feynman integral “$\Longleftrightarrow$” MDS equation.

Wilsonian regularized Feynman integral:
integrate not on $\mathcal{E}$, only on the image space $C_\kappa[\mathcal{E}]$ of a smoothing operator $C_\kappa : \mathcal{E} \to \mathcal{E}$.

[Smoothing operator: $\sim$ convolution, can be generalized to manifolds. Does UV damping.]
Automatically knows RGE relations.

Wilsonian regularized Feynman integral “$\Longleftrightarrow$” regularized MDS equation + RGE:

$$\left(\psi_0, \kappa \mapsto \gamma(\kappa), \kappa \mapsto \mathcal{G}_{\psi_0,\kappa}\right) = ? \text{ such that : } \qquad \underbrace{\mathcal{G}^{(0)}_{\psi_0,\kappa}}_{=:\, b\,\mathcal{G}_{\psi_0,\kappa}} = 1,$$

$$\forall\kappa : \forall\,\delta\psi_T \in \mathcal{D} : \qquad \underbrace{\Big(\iota_{\gamma(\kappa)\,(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)} - \mathrm{i}\,\hbar\,L_{C_\kappa\delta\psi_T} \Big)}_{=:\,\mathbf{M}_{\psi_0,\kappa,\delta\psi_T}}\,\mathcal{G}_{\psi_0,\kappa} = 0,$$

$$\text{RGE} \longrightarrow \qquad \forall\mu,\kappa : \quad \mathcal{G}^{(n)}_{\psi_0,(C_\mu\kappa)} = \left(\otimes^n C_\mu\right)\mathcal{G}^{(n)}_{\psi_0,\kappa}.$$

Running coupling is meaningful. Conjecture: RG flow of $\mathcal{G}_{\psi_0,\kappa} \leftrightarrow$ distributional G_{ψ_0}.
(Conjecture proved for flat spacetime for bosonic fields.)

Some complications on topological vector spaces

Careful with tensor algebra! Schwartz kernel theorems:

$$\begin{array}{ccccccc}
\hat{\otimes}^n_\pi \mathcal{E} & \equiv & \mathcal{E}_n & \equiv & (\hat{\otimes}^n_\pi \mathcal{E}')' & \equiv & \mathcal{L}in(\mathcal{E}', \hat{\otimes}^{n-1}_\pi \mathcal{E}) \\
 & & & & & & \\
(\hat{\otimes}^n_\pi \mathcal{E})' & \equiv & \mathcal{E}'_n & \equiv & \hat{\otimes}^n_\pi \mathcal{E}' & \equiv & \mathcal{L}in(\mathcal{E}, \hat{\otimes}^{n-1}_\pi \mathcal{E}') \\
 & & & & & & \\
\hat{\otimes}^n_\pi \mathcal{D} & \leftarrow & \mathcal{D}_n & \equiv & (\hat{\otimes}^n_\pi \mathcal{D}')' & & \\
 & \text{cont.bij.} & & & & & \\
(\hat{\otimes}^n_\pi \mathcal{D})' & \rightarrow & \mathcal{D}'_n & \equiv & \hat{\otimes}^n_\pi \mathcal{D}' & \equiv & \mathcal{L}in(\mathcal{D}, \hat{\otimes}^{n-1}_\pi \mathcal{D}')
\end{array}$$

$\mathcal{E} \times \mathcal{E} \to F$ separately continuous maps are jointly continuous.
$\mathcal{E}' \times \mathcal{E}' \to F$ separately continuous bilinear maps are jointly continuous.
For mixed, no guarantee.
For $\mathcal{D}$ or $\mathcal{D}'$ spaces, joint continuity from separate continuity of bilinear forms not automatic.
For mixed, even less guarantee.

But as convergence vector spaces, everything is nice with mixed $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{D}$, $\mathcal{D}'$ multilinears (separate sequential continuity $\Leftrightarrow$ joint sequential continuity).