LATTICE QCD AT NON-ZERO BARYON DENSITY

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(Wuppertal-Budapest collaboration)

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Outline

1) QCD in the grand canonical ensemble and the sign problem

2) The phase diagram

3) The search for Ising criticality at μ_B>0 and fluctuations

4) The equation of state of a hot-and-dense QGP and O(4) criticality

5) Summary and outlook

Why should heavy ion physicists care?

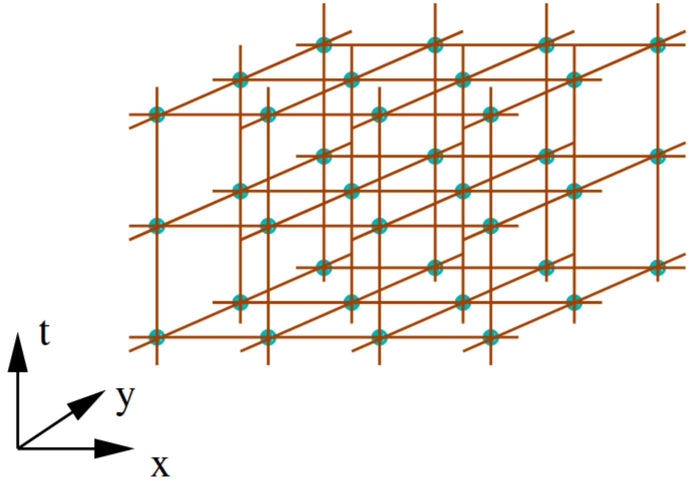
FULLY NON-PERTURBATIVE RESULTS IN FULL QCD ARE VALUEABLE

QUALITY CONTROL
IS ESSENTIAL!

The lattice formulation of QCD

Finite space-time lattice: $N_s^3 N_t$

Equilibrium physics: $T = \frac{1}{N_t a}$



1. Continuum limit:

For fixed temperature $a \to 0 \Leftrightarrow N_t \to \infty$

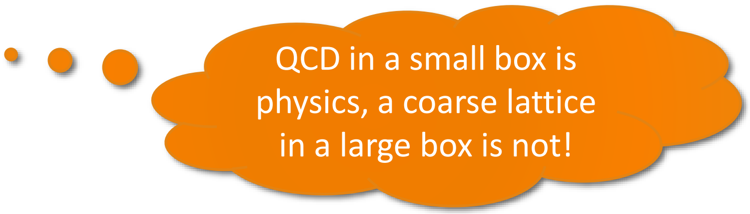
(Fixed N_t: Lower $T \Rightarrow$ Larger a (coarser))

2. Thermodynamic limit:

Size is often measured in units of 1/T

Aspect ratio: $LT = N_s/N_t$

Infinite volume limit: $LT \to \infty$



QCD in the grand canonical ensemble

$$\hat{p} \coloneqq \frac{p}{T^4} = \frac{1}{(LT)^3} \log \mathrm{Tr}\left(e^{-(H-\mu_B B-\mu_S S)/T}\right)$$ (dimensionless pressure)

$$\chi_{ij}^{BS} = \frac{\partial^{i+j}\hat{p}}{\partial\hat{\mu}_B^i \partial\hat{\mu}_S^j}$$ $$\left(\hat{\mu}_{B/S} \coloneqq \frac{\mu_{B/S}}{T}\right)$$ (generalized susceptibilities)

DERIVATIVES ⇔ FLUCTUATIONS/CORRELATIONS:

$$\chi_1^B \propto \langle B \rangle \propto n_B; \quad \chi_2^B \propto \langle B^2 \rangle - \langle B \rangle^2; \quad \chi_{11}^{BS} \propto \langle BS \rangle - \langle B \rangle \langle S \rangle$$

The QCD path integral

$$Z = \int DA_\mu D\bar{\psi} D\psi e^{-S_{YM} - \bar{\psi} M(A_\mu, m, \mu)\psi} = \int DA_\mu \mathrm{det} M(A_\mu, m, \mu)\, e^{-S_{YM}}$$

Can be simulated with Monte Carlo if $\mathrm{det}M\, e^{-S_{YM}}$ is real and positive:

- zero chemical potential $\mu = 0$
- purely imaginary chemical potential $\mathrm{R}e(\mu) = 0$
- isospin chemical potential $\mu_u = -\mu_d$

Otherwise: complex action/sign problem

$\Rightarrow$ desperate times, desperate measures

Lattice QCD at nonzero baryon density

Analytic continuation (ver. 1): Imaginary chemical potential method

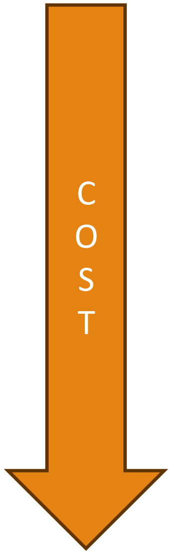
Calculate $\langle O \rangle$ at $\mathrm{Im}\mu_B$ ($\mu_B^2 < 0$), extrapolate to $\mu_B^2 > 0$

Analytic continuation (ver. 2): Taylor method

Calculate $\frac{\partial^n}{\partial \mu_B^n} \langle O \rangle$ at $\mu_B = 0$, extrapolate

Reweighting:

Simulate a different theory, correct the Boltzmann weight in observable



While cut-off and volume effects are important for every lattice result, for $\mu_B > 0$ the way we extrapolate is also an important point of quality control

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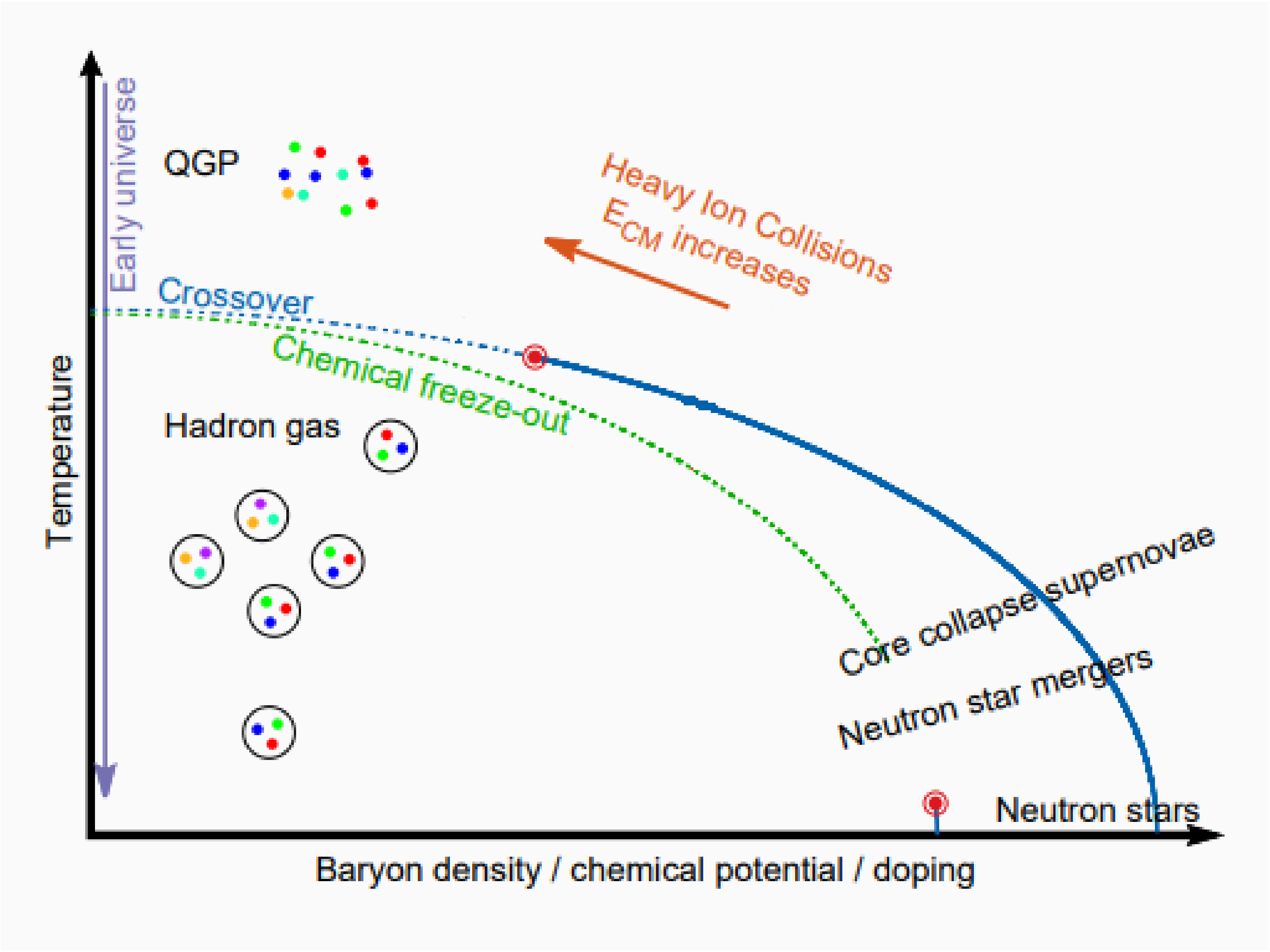
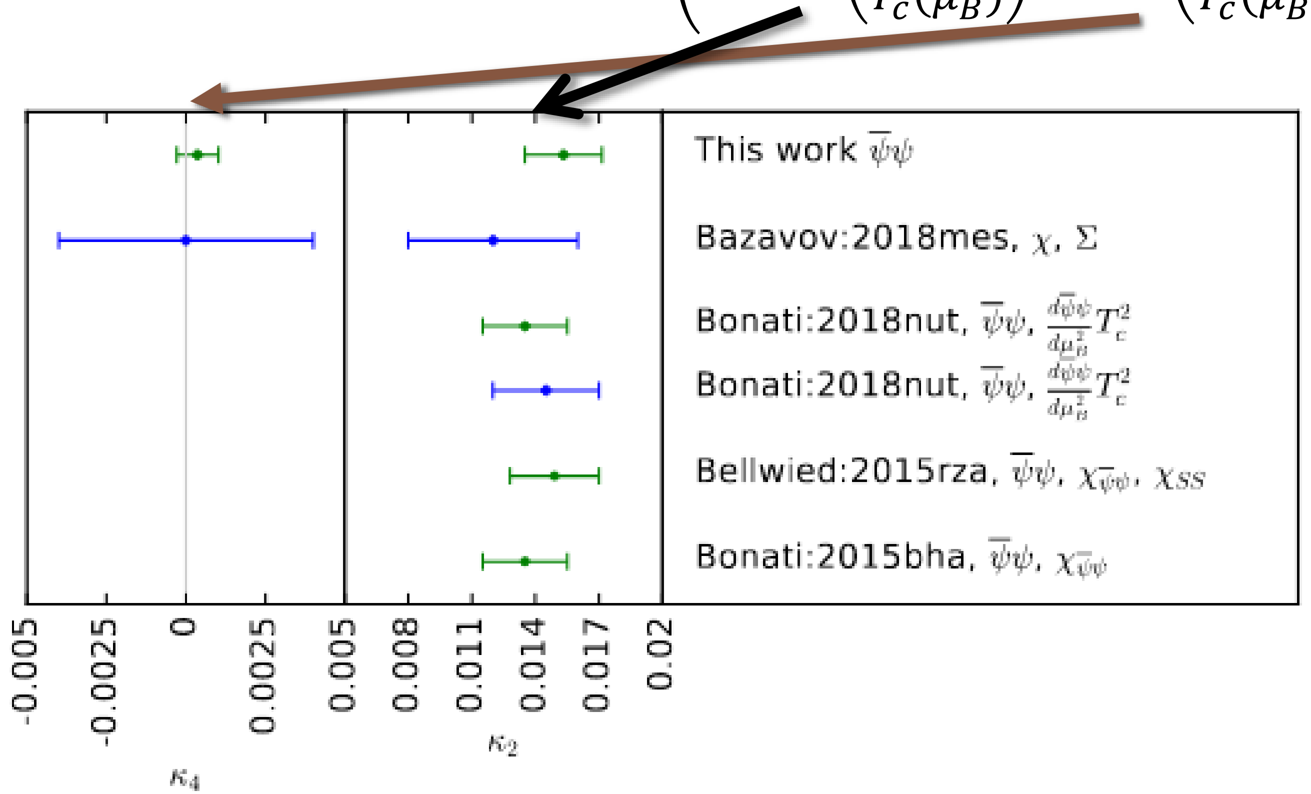
The conjectured phase diagram



The actual phase diagram at small μ_B

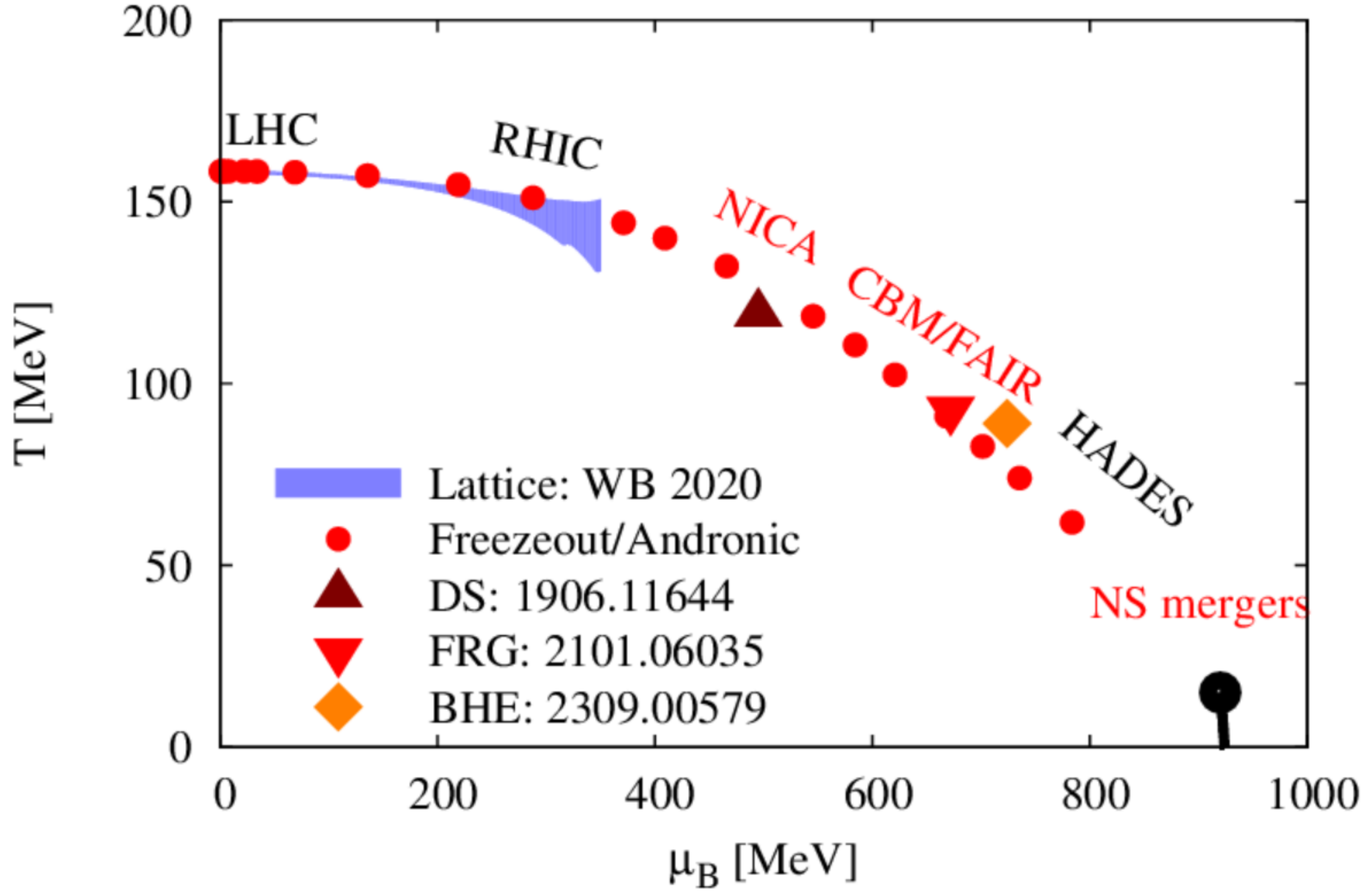
The crossover line: $T_c(\mu_B) = T_c(0)\left(1 - \kappa_2\left(\frac{\mu_B}{T_c(\mu_B)}\right)^2 - \kappa_4\left(\frac{\mu_B}{T_c(\mu_B)}\right)^4 - \cdots\right)$



[Wuppertal-Budapest, PRL125 (2020)]: continuum, $n_S = 0$, $LT = 4$

$\mu_B > 0$ quantities with very good quality control!

The crossover line vs chemical freeze-out (CFO)



Small μ_B: T(crossover) ≈ T(freeze-out)
[P. Braun-Munzinger et al, PLB (2004)]

Large μ_B: T(crossover) > T(freeze-out)
[Floerchinger, Wetterich, NPA (2012)]

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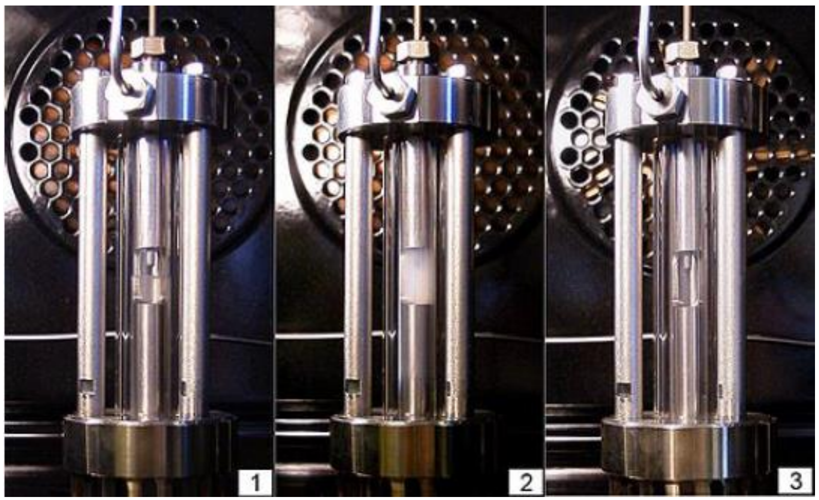
One way: fluctuations

Experiment: tune to criticality



$T < T_c$ $T \approx T_c$ $T > T_c$

HEAT THE SYSTEM

Picture from Wikipedia

Lattice/Taylor: try to see it from far away

$$\chi_n^B = \left(\frac{\partial^n \hat{p}}{\partial \hat{\mu}_B^n}\right)_{\mu_B=0}$$

To as large n as possible…

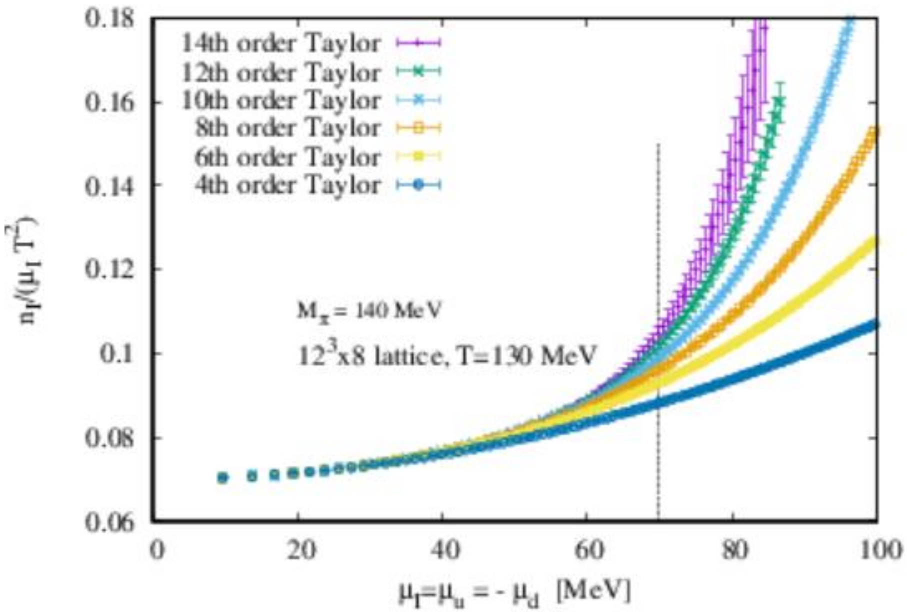
To hopefully see a divergence…

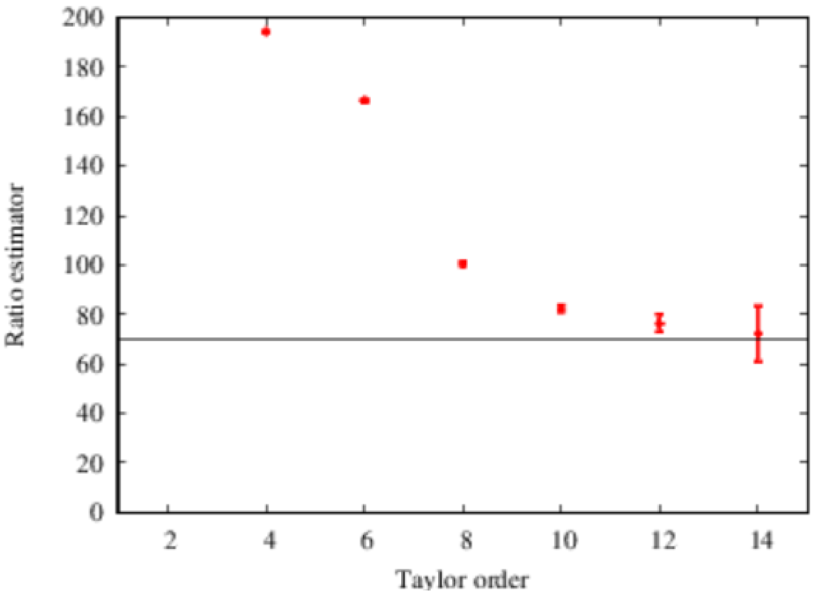
2^{nd} order known since 2012
4^{th} order known since 2015
6^{th} order: now

Is this even possible?

A case study: pion condensation [Wuppertal-Budapest, 2308.06105]

- Instead of μ_B, introduce μ_I (prefers π^+ over π^-)
- Second order transition at low T and $\mu_I \approx m_\pi/2 \approx 70\text{MeV}$ [Son, Stephanov, PRL (2001)] [Brandt, Endrődi, PRD (2





Eventually finds the correct value. 6th order gives $170\text{MeV} \gg 70\text{MeV}$
No high orders in μ_B: analysis of the radius of convergence from Taylor data is premature
Warning: the ratio estimator is not always applicable [Giordano & Pásztor, PRD99(2019)]

The HRG as a non-critical baseline

Hadron resonance gas (HRG) model

$$p_{QCD} \approx \sum_H p_H^{free}$$

- sum over stable hadrons and resonances
- heavy ion phenomenology uses the HRG as a non-critical baseline (non-trivial: see, e.g., [Braun-Munzinger et al, NPA1008(2021)])
- in lattice QCD: can use grand canonical ensemble
- minimum goal: establish deviations from HRG (with good quality control!)

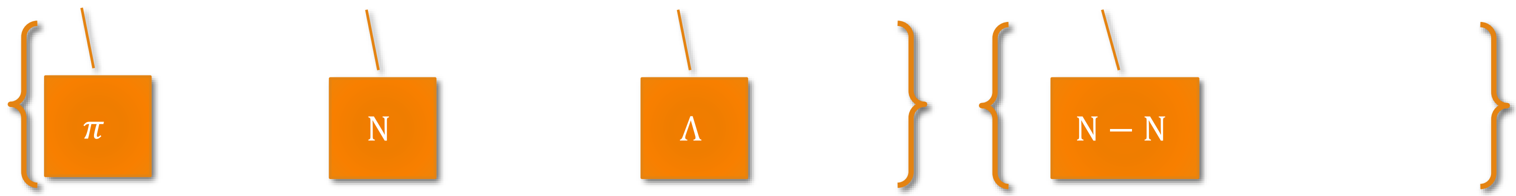
WHAT ARE THE CORRECTIONS TO THE HRG? ARE THEY LARGE?

Corrections to the HRG

$$\hat{p}(T,\mu_B,\mu_S) = P^{BS}_{00}(T) + \cdots + P^{BS}_{10}(T)\cosh(\hat{\mu}_B) + P^{BS}_{11}(T)\cosh(\hat{\mu}_B - \hat{\mu}_S) + \cdots + P^{BS}_{20}(T)\cosh(2\hat{\mu}_B) + \cdots$$

$$\chi^B_1(T,\mu_B,\mu_S) = P^{BS}_{10}(T)\sinh(\hat{\mu}_B) + P^{BS}_{11}(T)\sinh(\hat{\mu}_B - \hat{\mu}_S) + \cdots + 2P^{BS}_{20}(T)\sinh(2\hat{\mu}_B) + \cdots$$

$$\chi^B_2(T,\mu_B,\mu_S) = P^{BS}_{10}(T)\cosh(\hat{\mu}_B) + P^{BS}_{11}(T)cosh(\hat{\mu}_B - \hat{\mu}_S) + \cdots + 4P^{BS}_{20}(T)cosh(2\hat{\mu}_B) + \cdots$$



Higher order ⇒ larger beyond HRG corr.

S-matrix formalism

[Dashen et al, PR187 (1969)]

Repulsive interactions ⇒ negative sector

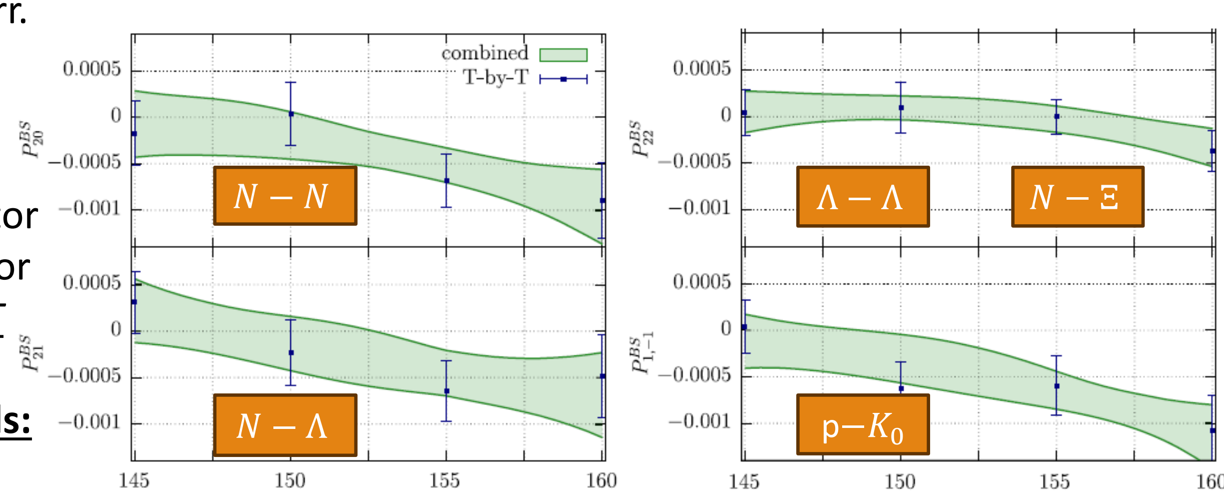
Attractive interactions ⇒ positive sector

B=2,3 etc. are *exp. suppressed at low T*

Lattice data vs HRG + repulsion models:

[Huovinen, Petreczky PLB777 (2017)]

[Vovchenko, Pásztor et al, PLB 775 (2017)]

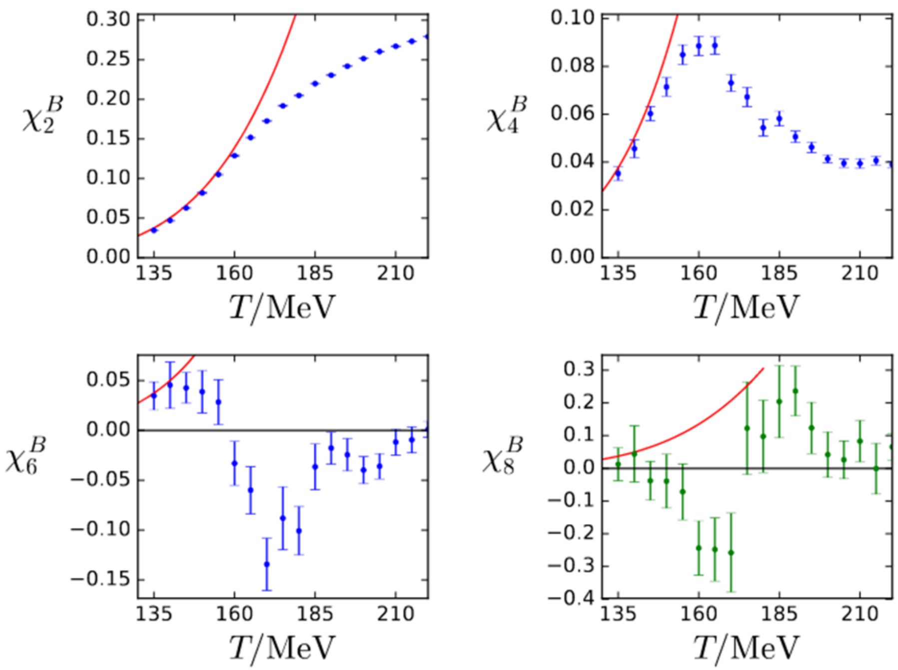


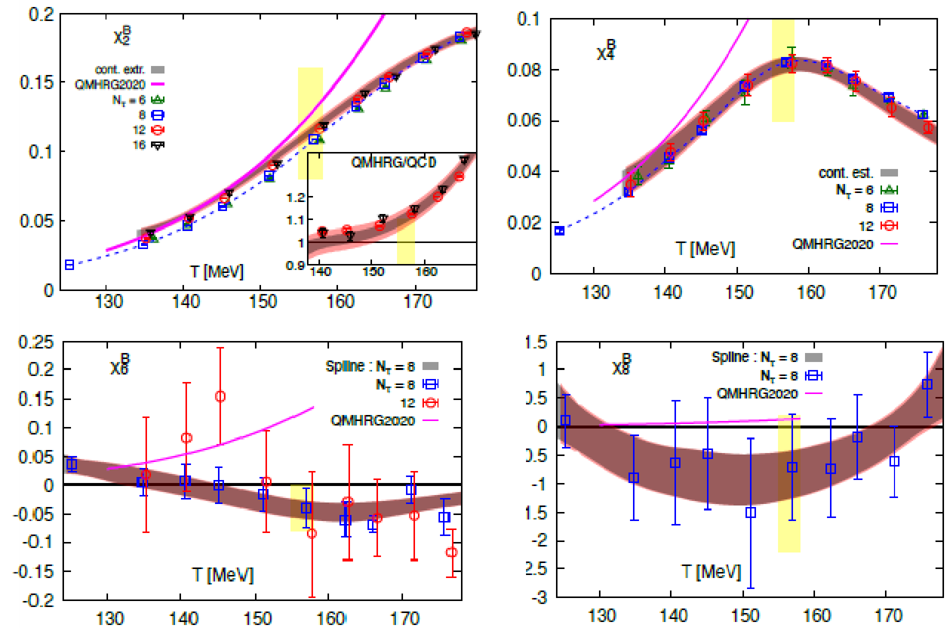
[Wuppertal-Budapest, PRD104(2021)]

Taylor coefficients of the pressure

From imaginary chemical potential

From zero chemical potential

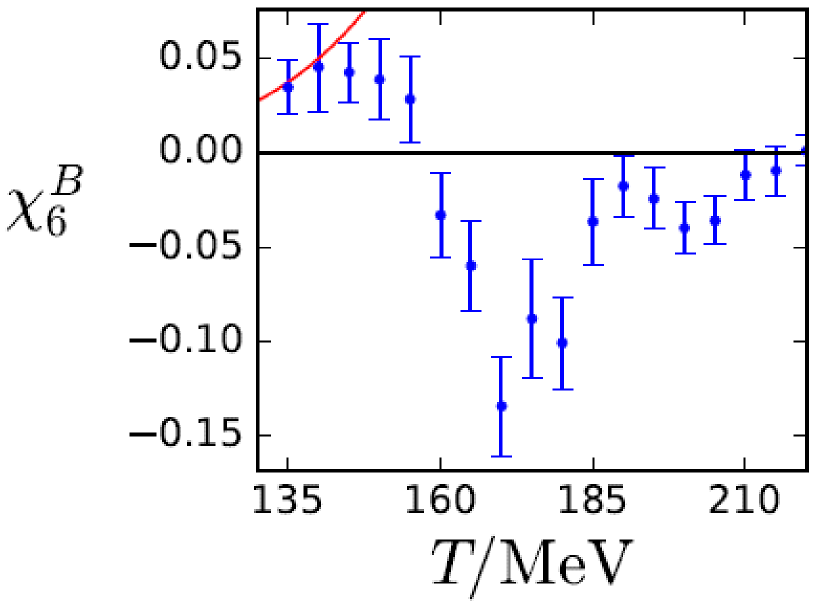




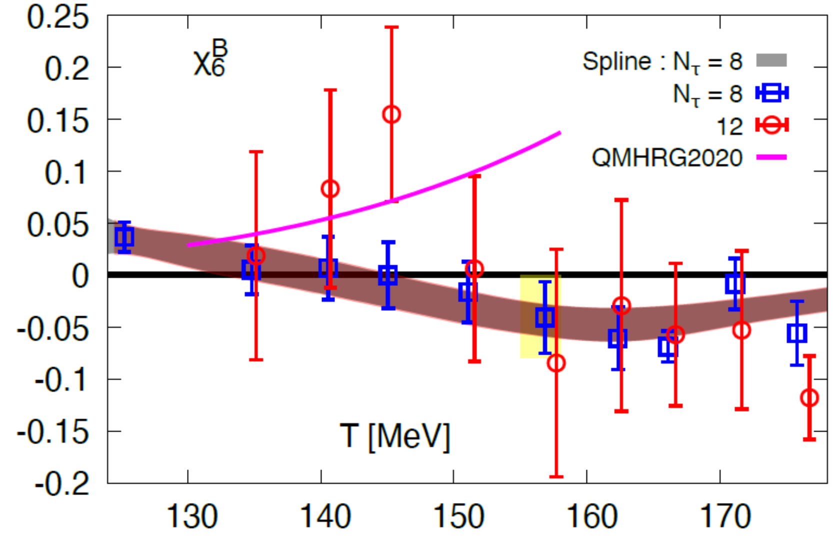
[Wuppertal-Budapest, JHEP (2018)] (LT=4, N_t=12) [HotQCD, PRD105 (2022)] (LT=4, N_t=6,8,(12))

6^{th} order: zoom in to see discrepancies

From imaginary chemical potential



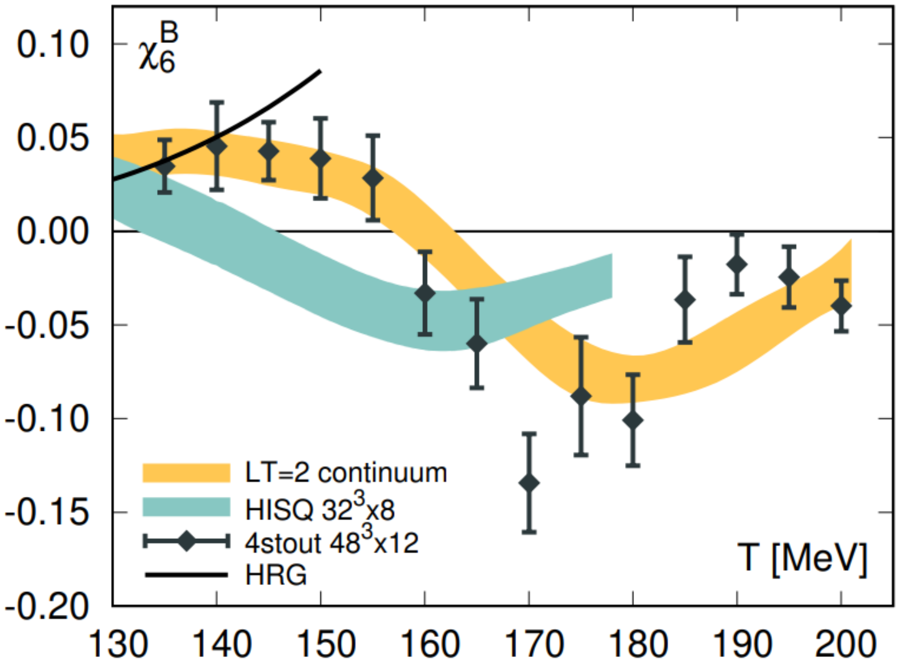
From zero chemical potential (Taylor)



[Wuppertal-Budapest, JHEP (2018)] (LT=4, N_t=12) [HotQCD, PRD105 (2022)] (LT=4, N_t=8,(12))

- N_t=12 (left, WB) agrees with the HRG (value, slope) better than N_t=8 (right, HotQCD) at low T
- At T=145-155MeV: N_t=12>0 and N_t=8<0

6^{th} order order: new dataset



New dataset:
Taylor, LT=2, continuum (new discretization)

Lower T: cut-off effects dominate
Smaller T means larger a for fixed N_t
5 points at least 1σ below: $\left(\frac{1-0.68}{2}\right)^5 \approx 10^{-4}$

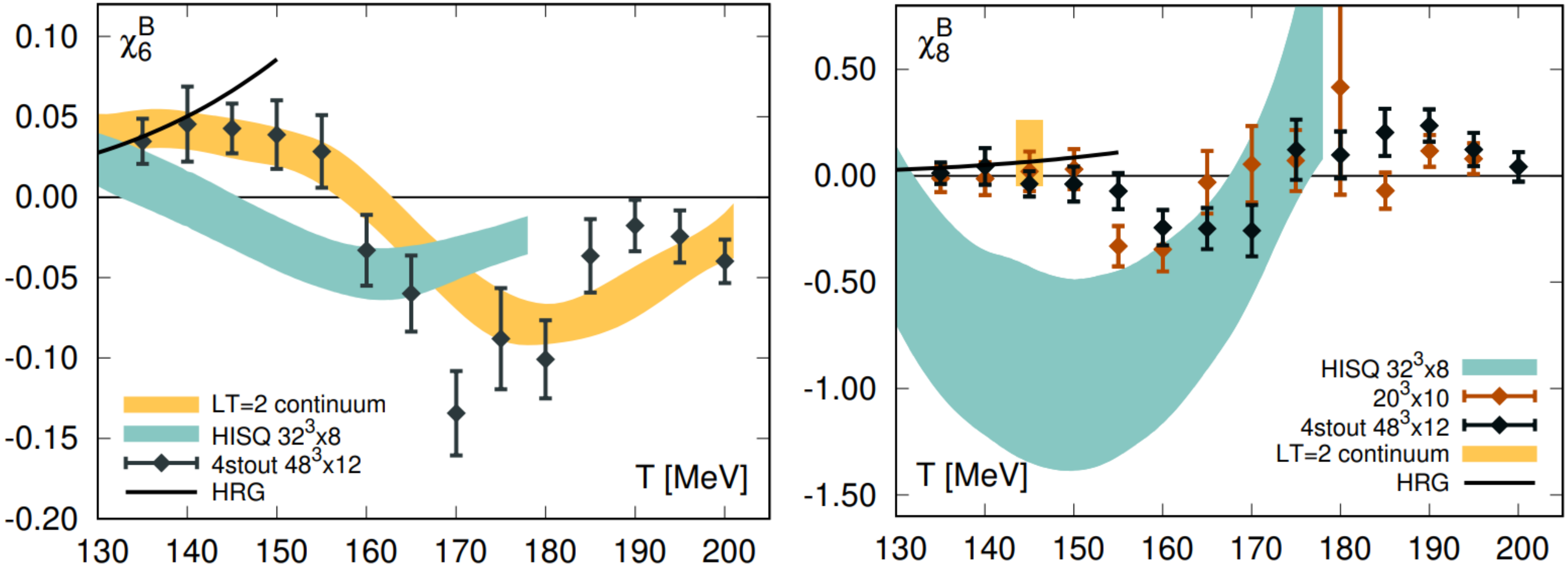
Higher T: finite volume effects dominate
T_c depends on L

T>145MeV: HRG ≠ LATTICE
T<145MeV: HRG = LATTICE
(within errors)

No sign of a CEP in the Taylor coefficients up to 6^{th} order

[D. Pesznyák, Tuesday]

6th and 8th order order: new dataset



[D. Pesznyák, Tuesday]

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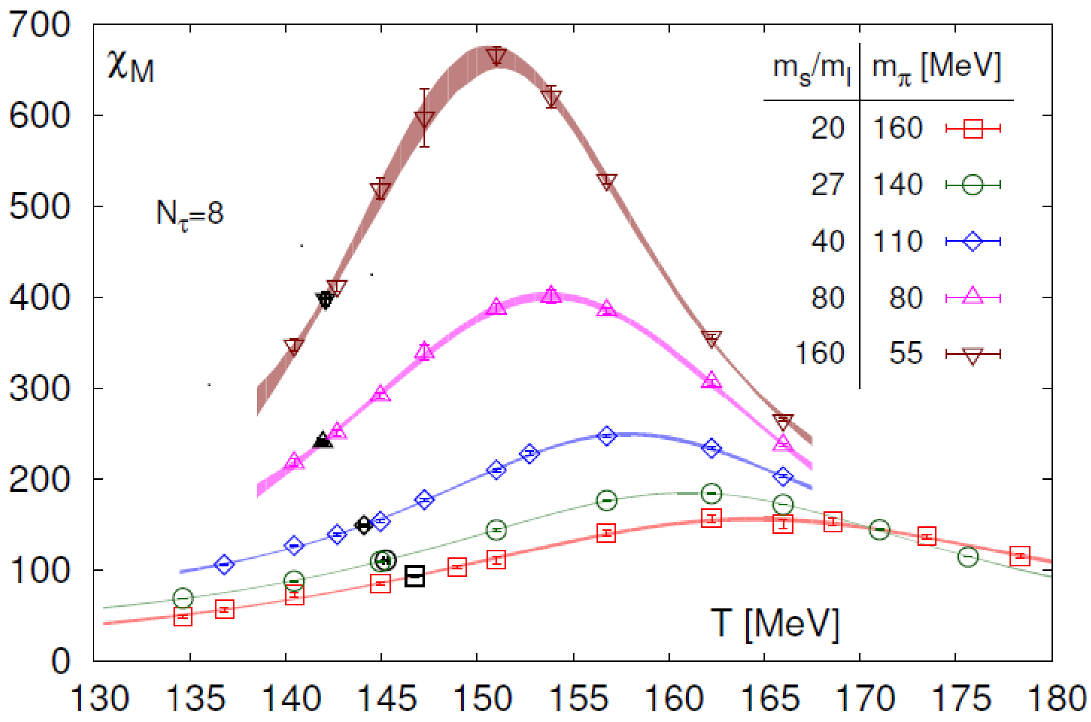
Chiral criticality and the equation of state

Smaller-than-physical quark mass @ $\mu_B = 0$



Plot from [HotQCD, PRL123 (2019)]

See also [Kotov, Lombardo, Trunin, PLB823 (2021)]:

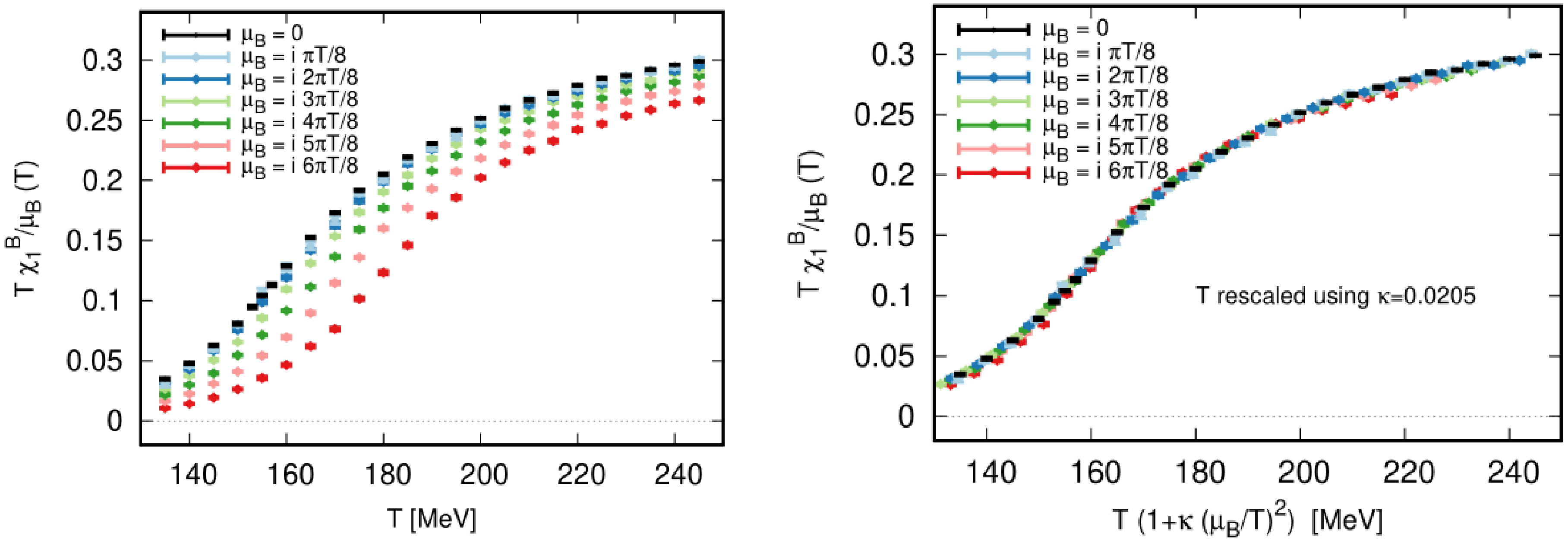
scaling for heavier-than-physical quark masses

Chiral criticality and the equation of state

<u>T and μ_B dependence with physical masses</u>

- Empirically: approximate scaling variable $T(1+\kappa_2\hat{\mu}_B^2)$

  $\Rightarrow$ transition not sharpening for small $\hat{\mu}_B^2$



[Wuppertal-Budapest, PRL126 (2021)]

I strongly suspect that the mechanism behind this collapse is chiral scaling.

O(4) scaling and collapse plots at μ_B>0

Observation: $\chi_1^B/(\hat{\mu}_B)$ collapses as a function of $T\left(1+\kappa\hat{\mu}_B{}^2\right)$ but χ_2^B does not

Why? One possibility: scaling near the chiral limit (Kadanoff scaling ansatz)

$$p_{QCD}(T,\mu_B,m) - p_{QCD}(0,0,m) \sim f_{sing}(h,t) \sim t^{2-\alpha} F\left(\frac{h}{t^{\beta\delta}}\right)$$

where $h \sim m$ and t $\sim T - T_{ch}(1-\kappa\,\hat{\mu}_B^2)$

$$\Rightarrow \frac{1}{\hat{\mu}_B}\frac{\partial}{\partial\hat{\mu}_B} f_{sing} = (2-\alpha)t^{1-\alpha}F\left(\frac{h}{t^{\beta\delta}}\right)(2\kappa) + t^{1-\alpha-\beta\delta}F'\left(\frac{h}{t^{\beta\delta}}\right)(-\beta\delta)(2\kappa)$$

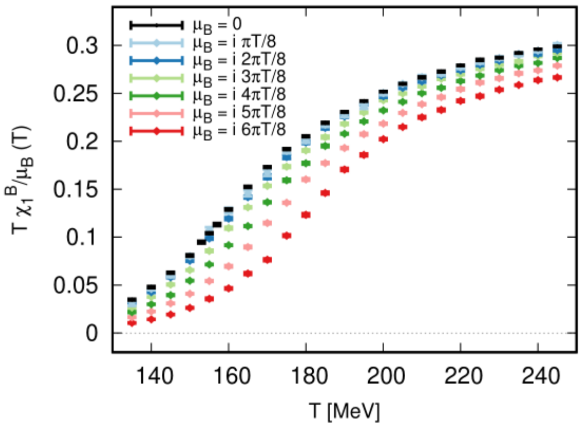
$\Rightarrow$ a function of the scaling variables h and t only

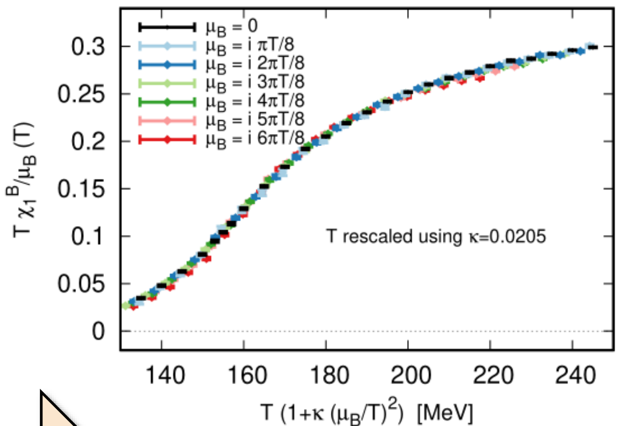
$$\frac{\partial^2}{\partial\hat{\mu}_B{}^2} f_{sing} = (2\kappa)G(h,t) + (2\,\kappa\,\hat{\mu}_B)^2\frac{\partial G}{\partial t}$$

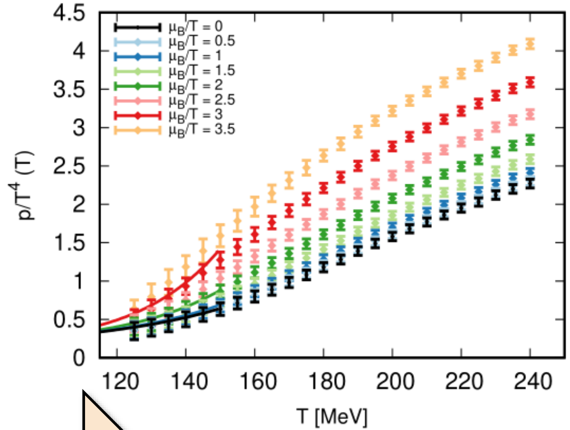
$\Rightarrow$ not a function of h and t only

Similar for the chiral condensate: here Σ/f_π^4 collapses but Σ/T^4 doesn't

Alternative expansion scheme







Approximate collapse in the Im(μ_B) data

Account for small deviations systematically, extrapolate to real μ_B

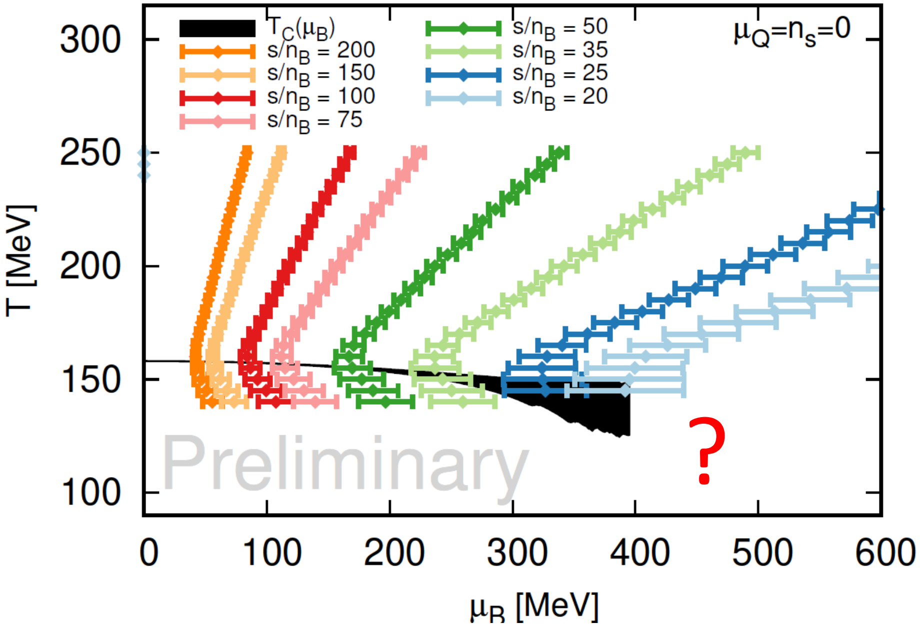
continuum, $LT = 4$, $\mu_S = 0$: [Wuppertal-Budapest, PRL126 (2021)]

continuum, $LT = 4$, $n_S = 0$: [Wuppertal-Budapest, PRD105 (2022)]

Also, small nonzero n_S

Precise EoS from extrapolations

Isentropes (resummation)



RHIC freeze-out [STAR, PRC96 (2017)]

$$\sqrt{s} = 19.6\text{GeV} \leftrightarrow \mu_B \approx 200\text{MeV}$$

$$\sqrt{s} = 11.5\text{GeV} \leftrightarrow \mu_B \approx 300\text{MeV}$$

$$\sqrt{s} = 7.7\text{GeV} \leftrightarrow \mu_B \approx 400\text{MeV}$$

No sign of critical lensing within errors

New preliminary dataset.
Improvement compared to last year comes from more accurate EoS at $\mu_B = 0$

More direct methods

Freely tune T and μ_B on the lattice?

Desirable:

No ill-posed analytic continuation

Data closer to conjectured CEP

Common lore:

Impossible

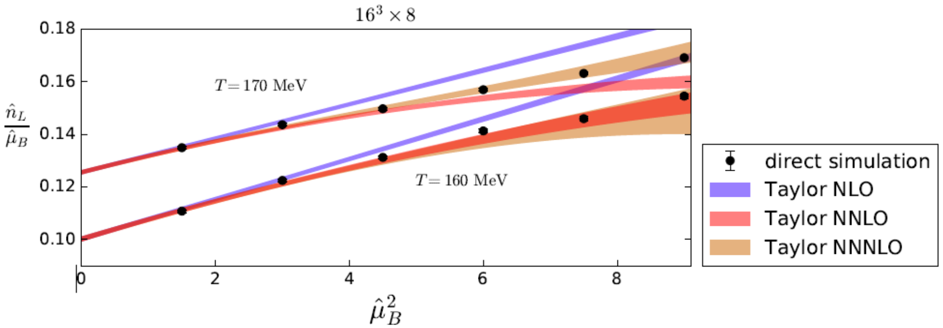
Truth:

Possible (with reweighting), but expensive

Increasingly more feasible

Many technical developments:
[JHEP05 (2020)] [PRD105 (2022)] [PRD107 (2023)] [2308.06105]

One application: cross-check QGP EoS



[Wuppertal-Budapest, PRD 107 (2023)]

For $T \geq 145\text{MeV}$:

4^{th} order Taylor accurate up to $\mu_B = 2T$

Alternative expansion at least up to $\mu_B = 3T$

Future: scan low T and larger μ_B in small volume

Summary and outlook

The phase diagram

- Curvature at μ_B=0 very well established and small
- Fourth order at μ_B=0 is also very small.
- Where does the crossover line deviate from the freeze-out curve?

QGP equation of state

- $\mu_B/T<2$ from 4th order Taylor expansion (continuum)
- $\mu_B/T<3$-3.5 from alternative expansion scheme (continuum)
- Direct simulations agree with expansions, provided the order is high enough
- No sign of critical lensing in the QGP EoS (within errors)
- Where do the more direct methods say at lower T?

Search for the CEP

- Continuum 6th order and 8th order fluctuations for the 1st time
- Some previous calculations had large cut-off effects
- No deviations from the HRG for T<145MeV in cumulants up to 8th order
- Can this (or absence of lensing) be converted to an exclusion region for the CEP?
- Need better algorithms to go to 10th and 12th order

What does this mean?

Corrections to the HRG are exp. suppressed at low T, thus

χ_6^B and χ_8^B agree with HRG at say T=140MeV

$\Rightarrow$ they will also agree everywhere below

The HRG cannot be exact at any T, since it misses effects that we know exist in full QCD, like N-N scattering. Correct way to proceed:

1) Demonstrate discrepancy between QCD and HRG at some order (χ_{10}^B? χ_{12}^B?)
2) Only then go to lower T

If at T=130MeV the fluctuations χ_6^B and χ_8^B agree with the HRG, that does NOT imply that there is no CEP at this temperature or above, either:

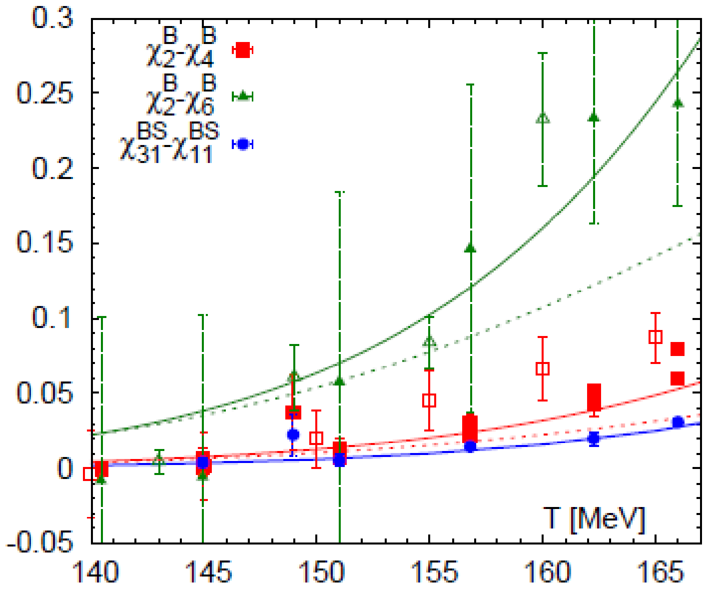
1) There is no CEP at this T or above, OR:
2) There is a CEP, but its effect on χ_6^B and χ_8^B is smaller than the error bars. If this is the case, the signal for the CEP will be stronger in say χ_{10}^B and χ_{12}^B

Repulsive hadronic models vs lattice data

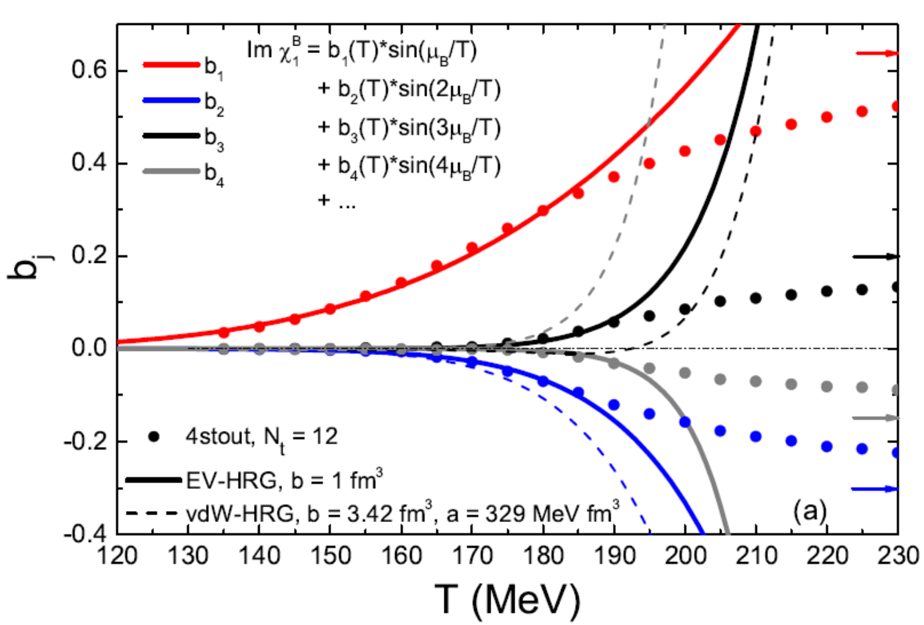
Repulsive core of NN interactions is very well established,
and the HRG model does not take it into account at all!

[Huovinen, Petreczky PLB777 (2017)]

[Vovchenko, Pásztor et al, PLB 775 (2017)]



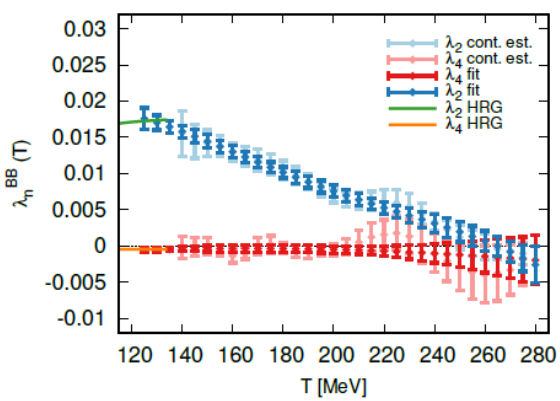
LT=4, N_t=8, Taylor VS repulsive mean field

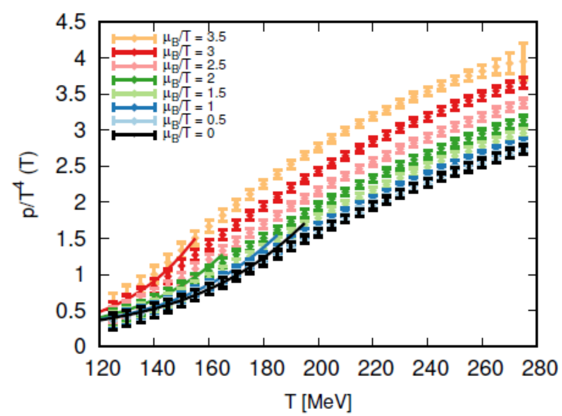


LT=4, N_t=12, Imμ_B VS excluded volume or VdW HRG

Resummed EoS: some details

- Systematically improvable ansatz: $F(T,\mu_B) = F(T',0) \qquad T' = T(1 - \lambda_2(T)\hat{\mu}_B^2 - \lambda_4(T)\hat{\mu}_B^4 - \cdots)$
- This ansatz together with a choice of the observable F defines an extrapolation scheme (resummation)
- A good choice for $\langle S \rangle = 0$ is $F = \frac{c_1^B(T,\hat{\mu}_B)}{c_1^B(T\to\infty,\hat{\mu}_B)}$ where $c_1^B \coloneqq \left(\frac{d\hat{p}}{d\,\hat{\mu}_B}\right)_{\langle S \rangle = 0}$
- The normalization makes sure the infinite temperature behavior is correct
- The ansatz itself exploits the existence of the approximate scaling variable
- Already the leading order, with λ_2 only generates terms to all orders in the Taylor expansion of $\hat{p}$
- Analysis is like the extrapolation of $T_c(\hat{\mu}_B)$
- Result: λ_4 is very small, while λ_2 has a very simple temperature dependence

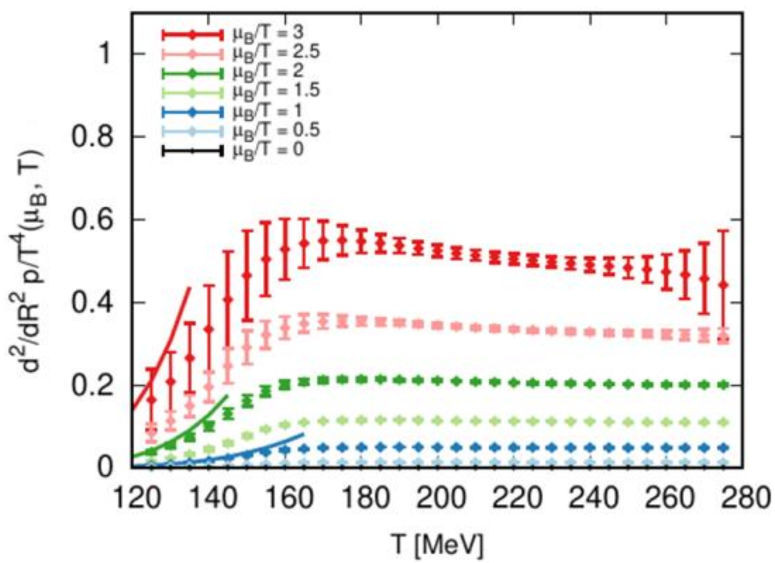


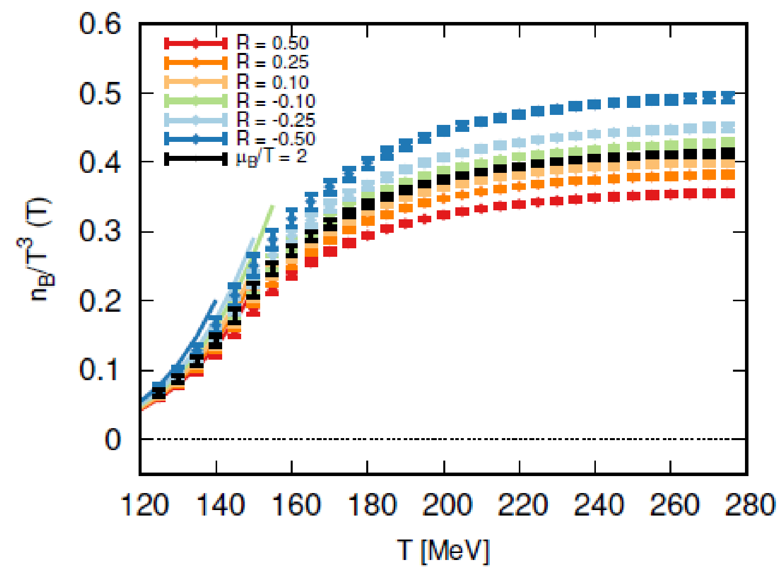


Beyond strangeness neutrality

Makes it possible to take small local fluctuations of strangeness into account in hydrodynamics:

$$\hat{p}(T, \mu_B, R) \approx \hat{p}(T, \mu_B, 0) + \frac{1}{2}\frac{d^2\hat{p}}{dR^2} R^2 \quad \text{where} \quad R = \frac{n_S}{n_B}$$





[Borsányi et al, PRD105 (2022)]

Equation of state (summary)

1. Realize the existence of the approximate scaling variable
2. Turn it into a systematically improvable extrapolation ansatz [Borsányi et al, PRL126 (2021)]
3. Validate the scheme by comparison with direct simulation results at non-zero density on finite (but reasonable) lattices [Borsányi et al, PRD107 (2023)]
4. Calculate the coefficients of the validated extrapolation scheme in the continuum in conditions relevant for heavy ion phenomenology. [Borsányi et al, PRD105(2022)]
5. Realize that the finite μ_B part is so precise that the errors are dominated by μ_B=0, so make the μ_B=0 equation of state more precise. **[P. Parotto, Tue 16:30 , QCD at finite T and μ]**

$\Rightarrow$ A PRECISE EQUATION OF STATE FOR THE RHIC BES RANGE

Reweighting

Fields: ϕ Target theory: $Z_t = \int D\phi\, w_t(\phi)$ Simulated theory: $Z_s = \int D\phi\, w_s(\phi)$

$$\langle O \rangle_t = \frac{\int D\phi\, w_t(\phi) O(\phi)}{\int D\phi\, w_t(\phi)} = \frac{\int D\phi \frac{w_t(\phi)}{w_s(\phi)} w_s(\phi) O(\phi)}{\int D\phi \frac{w_t(\phi)}{w_s(\phi)} w_s(\phi)} = \frac{\left\langle \frac{w_t}{w_s} O \right\rangle_s}{\left\langle \frac{w_t}{w_s} \right\rangle_s} \quad \text{and} \quad \frac{Z_t}{Z_s} = \left\langle \frac{w_t}{w_s} \right\rangle_s$$

Two problems (usually exponentially hard in the volume) can arise:

- sign problem: $\frac{w_t}{w_s} \in \Rightarrow$ large signal to noise ratios

- overlap problem: tails of $\mathrm{P}\left(\frac{w_t}{w_s}\right)$ do not decay fast enough $\Rightarrow$ potentially incorrect results

Two choice of w_s that eliminate this overlap problem:

- phase reweighting: $w_s = e^{-S_{YM}} |\det M| \quad \Rightarrow \quad \frac{Z_t}{Z_s} = \left\langle e^{i\,\theta} \right\rangle_s$

- sign reweighting: $w_s = e^{-S_{YM}} |\operatorname{Re}\det M| \quad \Rightarrow \quad \frac{Z_t}{Z_s} = \langle \pm \rangle_s$

Staggered rooting and low T difficulties

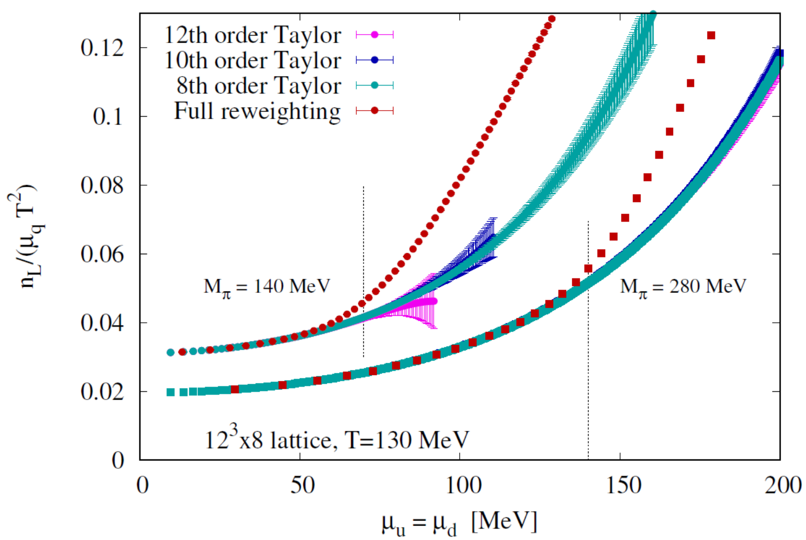
Say I want N_f=2+1 with staggered: $Z = \int DU (\det M_{ud}(U,\mu))^{\frac{1}{2}} (\det M_s(U))^{\frac{1}{4}} e^{-S_{YM}(U)}$

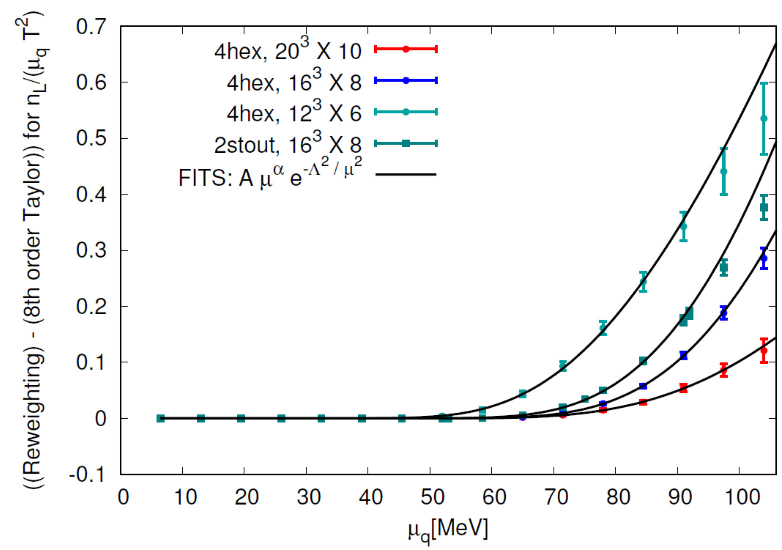
Determinant complex, so sqrt ambiguous. Standard choice: continuously connect to the positive root at μ=0

We empirically observe that this leads to non-analytic behavior (essential singularity) at μ=0

The non-analytic part is suppressed for $\mu<m_\pi$

The amplitude of the non-analytic part decreases with the lattice spacing

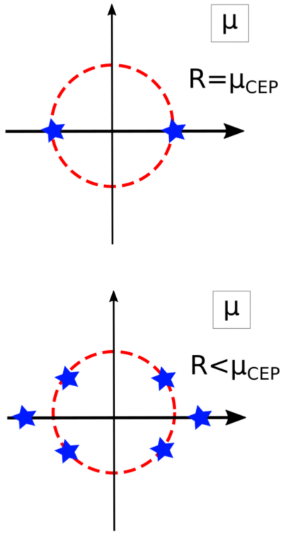




Radius of convergence

$$\hat{p} = \hat{p}(T, \mu_B = 0) + \frac{1}{2}\chi_2^B \hat{\mu}_B^2 + \frac{1}{4!}\chi_4^B \hat{\mu}_B^4 + \cdots \quad \text{converges for } |\hat{\mu}_B| < R = ?$$

Motivation: Inside the radius of convergence of the Taylor expansion there can be no singularities in the complex μ_B plane, and thus also no CEP on the real μ_B line



- For a long time (≈15years) ratio estimators were used
- For complex singularities (expected, e.g., for $T \approx T_{crossover}$) doesn't converge

[Vovchenko et al, PRD97 (2018)] [Giordano & Pásztor, PRD99(2019)]

- There are also possible issues with lattice artefacts

[Giordano et al, PRD101 (2020)] [Borsányi et al, 2308.06105]

- For reliable estimation, needs many more orders
- Higher orders not available in the continuum
- Can be phenomenologically estimated from O(4) scaling + other assumptions

[Mukherjee & Skokov, PRD103 (2021)]

⇒ All current lattice estimates of R should be considered preliminary/exploratory estimates, with inadequate quality control (⇒ MORE WORK)

O(4) scaling and collapse plots at μ_B>0

Empirical observations from imaginary μ_B data:

- Σ/f_π^4 collapses as a function of $T\left(1+\kappa\left(\frac{\mu_B}{T}\right)^2\right)$ but Σ/T^4 does not
- $\chi_1^B/(\mu_B/T)$ collapses as a function of $T\left(1+\kappa\left(\frac{\mu_B}{T}\right)^2\right)$ but χ_2^B does not

BUT WHY?

One possible explanation is scaling near the chiral limit:

$$p_{QCD}(T,\mu_B,m)-p_{QCD}(0,0,m)\sim f_{sing}(h,t)\sim t^{2-\alpha}F\left(\frac{h}{t^{\beta\delta}}\right)$$ where $h\sim m$ and t $\sim T-T_{ch}(1-\kappa(\mu_B/T_{ch})^2)$

$$\Rightarrow \Sigma_{sing}=m\frac{\partial}{\partial m}f_{sing}=t^{2-\alpha}\frac{h}{t^{\beta\delta}}F'\left(\frac{h}{t^{\beta\delta}}\right)$$

$\Rightarrow$ near T_{ch} near the chiral limit, Σ/f_π^4 is a function of the scaling variables h and t only, while Σ/T^4 is not

$$\Rightarrow \frac{1}{(\mu_B/T_{ch})}\frac{\partial}{\partial(\mu_B/T_{ch})}f_{sing}=(2-\alpha)t^{1-\alpha}F\left(\frac{h}{t^{\beta\delta}}\right)(2\kappa)+t^{1-\alpha-\beta\delta}F'\left(\frac{h}{t^{\beta\delta}}\right)(-\beta\delta)(2\kappa)\coloneqq(2\kappa)G(h,t)$$

$\Rightarrow$ again, a function of h and t only, while

$$\frac{\partial^2}{\partial(\mu_B/T_{ch})^2}f_{sing}=(2\kappa)G(h,t)+\left(\frac{(2\kappa)\mu_B}{T_{ch}}\right)^2\frac{\partial G}{\partial t}$$

$\Rightarrow$ not a function of h and t only