Topology and Axions from Lattice QCD

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Outline

- Motivation
- Topology in QCD with EM fields
- The topological susceptibility
- The axion-photon coupling
- Conclusions and further work
In HIC and in the early universe, strong magnetic fields of up to $eB \sim 1.0 \text{ GeV}^2$ are expected.
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Very interesting physics when combining strong interactions and electromagnetic fields!
Motivation

\[
Q_{\text{top}} = \int d^4x \, q_{\text{top}}(x), \quad q_{\text{top}} = \frac{1}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} G_{\mu\nu} G_{\rho\sigma}.
\]

Topology and axions
Topology in QCD with EM fields
In general, the expectation value of any (scalar) operator $\mathcal{O}$ under background electromagnetic fields $F_{\mu\nu}$ is of the form

$$\langle \mathcal{O} \rangle (F_{\mu\nu}) = \langle \mathcal{O} \rangle (0) + g_1 F_{\mu\nu} F^{\mu\nu} + g_2 F_{\mu\nu} \tilde{F}^{\mu\nu}$$

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plus higher order terms.
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And since $Q_{\text{top}}$ is a CP-odd operator...

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- Now, our operator is CP-even. Hence (for weak fields)
  $$\chi_{\text{top}}(F_{\mu\nu}) = \chi_{\text{top}}(0) + g'(\vec{B}^2 - \vec{E}^2).$$

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Non-trivial behaviour at finite temperature.

$\chi_{\text{top}}(0)$ has already been characterised at zero and finite $T$, both in ChPT
Cortona et al 2016 and on the lattice Borsanyi et al 2016. Small enhancement with
$\vec{B}$ from ChPT calculations Adhikari 2021.
The topological susceptibility
Lattice artifacts

- Index theorem says $\mathcal{D}$ has zero modes when $Q_{\text{top}} \neq 0$.
- Staggered operator lacks these zero modes $\rightarrow$ huge lattice artifacts, specially for high temperatures (we are talking of several orders of magnitude!).

Topology and axions
Lattice artifacts

- Index theorem says $\mathcal{D}$ has zero modes when $Q_{\text{top}} \neq 0$.
- Staggered operator lacks these zero modes $\rightarrow$ huge lattice artifacts, specially for high temperatures (we are talking of several orders of magnitude!).
- One possible solution: substitute the smallest eigenvalues of $D_{\text{stagg}}$ with their continuum values Borsanyi et al 2016.
- How? Reweighting each configuration by:

\[
\prod_f \prod_{i=1}^{2|Q_{\text{top}}|} \prod_{\sigma=\pm} \left( \frac{2m_f}{i\sigma \lambda_i + 2m_f} \right)^{n_f/4}
\]
$\chi_{\text{top}}$: preliminary results

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![Graph showing $\chi(B)/\chi(0)$ vs. $a^2\text{ fm}^2$ with markers for different categories such as regular, improved, $\Sigma_{\text{avg}}, 1206.4205$, C. limit, and ChPT (2 flavours).]
The axion-photon coupling
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- ChPT calculations show that the coupling decomposes into two terms, one model dependent and one model independent.

Current estimate from ChPT:

$$g_{a\gamma\gamma} = g_{0} + g_{QCD} = \alpha_{em}^2 \frac{\pi}{f_{a}} \left( E_{N} - 1.92^{(4)} \right)$$


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- We want to compute the QCD dependent part of the coupling $\rightarrow$ no need to include axions on the lattice!
By looking at $Z$:

\[
\frac{\delta \log Z(a)}{\delta a} \bigg|_{a=0} = \frac{\langle Q_{\text{top}} \rangle_{E,B}}{f_a} \quad \rightarrow \quad g_{a\gamma\gamma} f_a = \frac{T}{V} \frac{\partial}{\partial (E \cdot B)} \langle Q_{\text{top}} \rangle_{E,B} \bigg|_{E,B=0}
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$$\frac{\delta \log Z(a)}{\delta a} \bigg|_{a=0} = \left. \langle Q_{\text{top}} \rangle_{E,B} \frac{f_a}{f_a} \right. \quad \Rightarrow \quad g_{a\gamma\gamma} f_a = \frac{T}{V} \frac{\partial}{\partial (E \cdot B)} \langle Q_{\text{top}} \rangle_{E,B} \bigg|_{E,B=0}$$

So for homogeneous, static and weak EM fields

$$\frac{T}{V} \langle Q_{\text{top}} \rangle_{E,B} \approx \frac{g_{a\gamma\gamma} f_a}{e^2} e^2 E \cdot B \quad \text{and} \quad g_{a\gamma\gamma} < 0.$$
$g_{a\gamma\gamma}^{QCD}$: preliminary results

$40^3 \times 48, T = 0$

$\langle a \rangle$

$e^2 \vec{E} \cdot \vec{B} \text{ GeV}^4 \times 10^{-3}$

regular

improved
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However, some observables like $\chi_{top}$ can be very sensitive to isospin asymmetries. In particular for the coupling (LO ChPT):

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However, some observables like $\chi_{top}$ can be very sensitive to isospin asymmetries. In particular for the coupling (LO ChPT):

$$\frac{2}{5} \frac{4m_d + m_u}{m_u + m_d} \approx 1.21.$$ 

Thus, we can take into account isospin violation by rescaling our result by that factor:

$$g^{QCD}_{a\gamma\gamma} = \begin{cases} 
-0.023(2) & \text{this work,} \\
-0.0243(5) & \text{ChPT.}
\end{cases}$$
Conclusions and further work
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▶ We have shown:

- that the reweighting is necessary for a controlled continuum limit of \( \chi_{\text{top}}(B)/\chi_{\text{top}}(0) \).
- First signals of an enhancement of \( \chi_{\text{top}} \) with the magnetic field at low temperatures.
- A preliminary result for the continuum limit of \( g_{a\gamma\gamma}^{QCD} \).

▶ Further work:

- Generate more statistics for both observables and perform the continuum limit for \( \chi_{\text{top}} \) for the rest of temperatures.
- Implement the reweighting technique for \( g_{a\gamma\gamma}^{QCD} \).
Thank you for your attention!
Backup slides
EM and QCD Topology

- EM fields can induce topologies in the gluon sector. But how? \(\Rightarrow\) Index theorem.
- The index theorem says (for QCD):

\[
\text{Index}(\mathcal{D}) \equiv n_- - n_+ = Q_{top}
\]

Since in QCD \(\langle Q_{top} \rangle = 0\), we don’t see imbalances in chirality.
- But after including electromagnetic fields the situation is different:

\[
\text{Index}(\mathcal{D}) \equiv n_- - n_+ = Q_{top} + Q_{U(1)}.
\]

We have two different topological contributions to the zero modes.
- Path integral favours as little zero modes as possible: \(\det M \uparrow \uparrow\).
- Hence, it selects gluon field configurations such that:

\[
Q_{U(1)} \uparrow \iff Q_{top} \downarrow.
\]
Simulation setup for $\chi_{top}$

- Improved staggered quarks with 2+1 flavours and physical quark masses.
- $N_s \times N_t = 24^3 \times 6, 24^3 \times 8, 28^3 \times 10, 36^3 \times 12$.
- $T = 110-300$ MeV, eB = 0, 0.5, 0.8 GeV$^2$.
- Gradient Flow used to reduce the UV fluctuations and control the topology
  Lüscher 2010.
Simulation setup for $g_{a\gamma\gamma}^{QCD}$

- Improved staggered quarks with 2+1 flavours and physical quark masses.
- $N_S \times N_t = 24^3 \times 32, 32^3 \times 48, 40^3 \times 48.$
- $T = 0.$
- We keep $\mathbf{E} \cdot \mathbf{B}$ in the linear response region.
- Imaginary electric fields (sign problem).
- Gradient Flow used to reduce the UV fluctuations and control the topology.