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Hydrodynamics from an exact collision kernel

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with C. Brito and G. Soares, arXiv:2306.07423, 2311.07272

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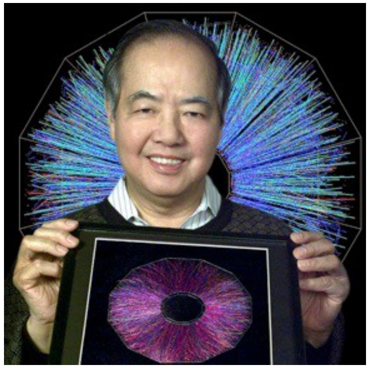
What you will see

- ✔ Motivation: heavy ion collisions and hydrodynamics
- ✔ Basics of fluid dynamics: matching conditions
- ✔ Hydrodynamic theories from kinetic theory
- ✔ Conclusions

Goal of Heavy-Ion Collisions: *Produce* and *study* QCD matter near (local) equilibrium



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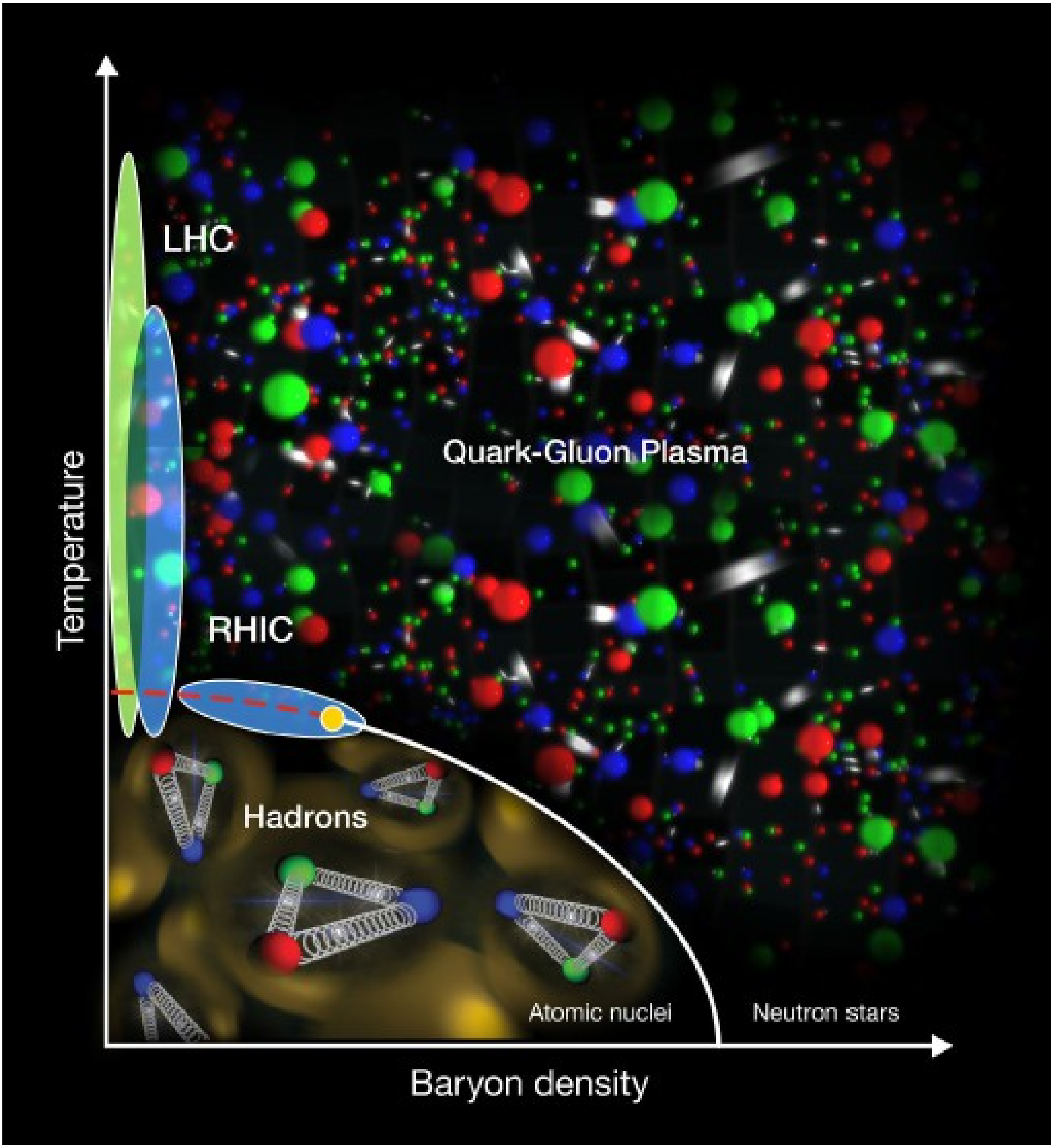
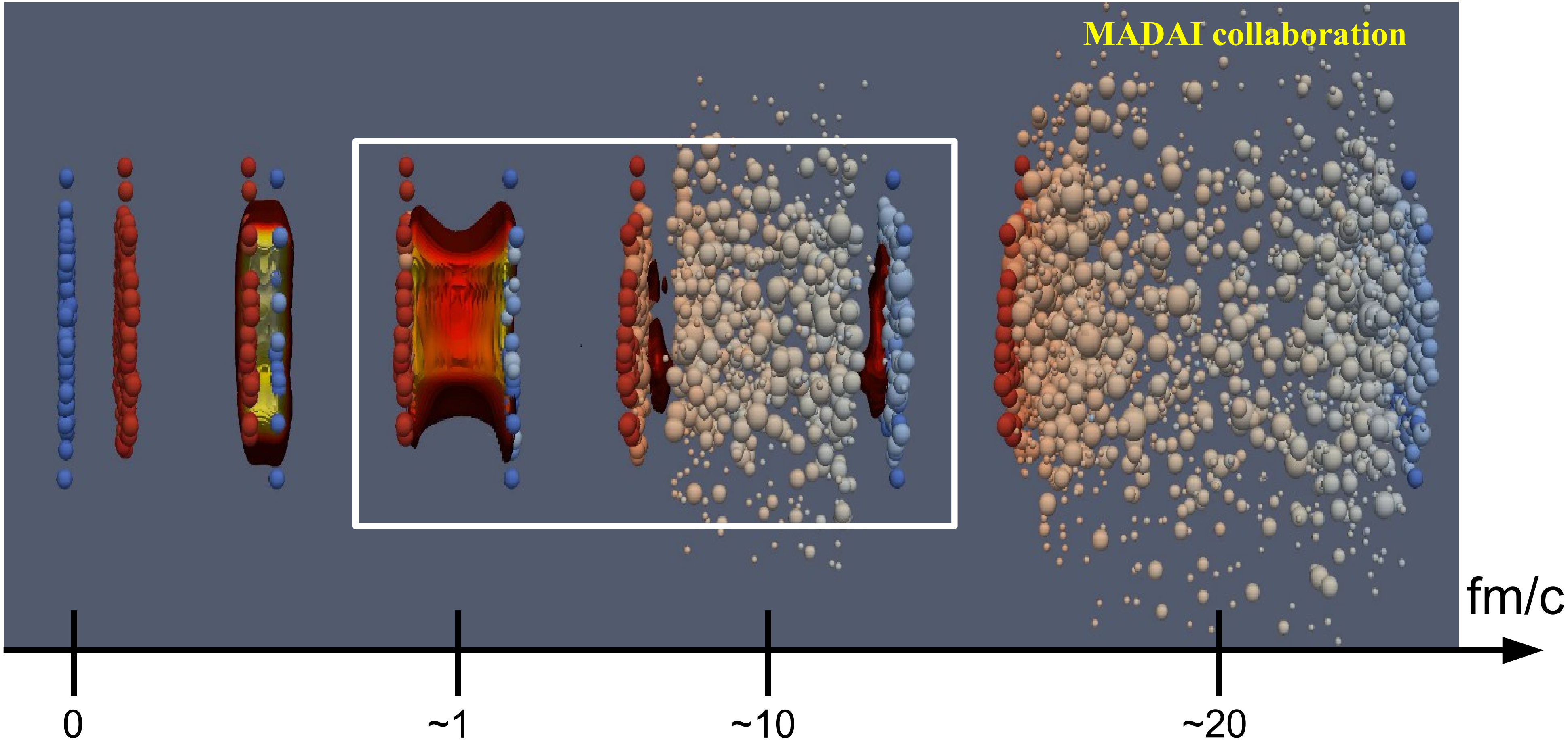


Challenge

- Extract thermodynamic *and* transport properties of QCD matter
- Experiment is not to test QCD, but to understand it

Current Theoretical Description

Empirical: Fluid-dynamical modeling of heavy ion collisions works well at RHIC and LHC energies



Main assumption: (transient) fluid dynamics can be applied at very early times ~1 fm

Validity of fluid dynamics

- → **proximity to (local) equilibrium**
- → **“small” gradients**

Separation of scales $\longrightarrow$ **macroscopic:** L **microscopic:** ℓ

Knudsen number: $K_N \sim \dfrac{\ell}{L} \ll 1$

Why does fluid dynamics work in heavy ion collisions? Is the system produced really close to equilibrium?

What do we mean by equilibrium?

Validity of fluid dynamics

- → **proximity to (local) equilibrium**
- → **“small” gradients**

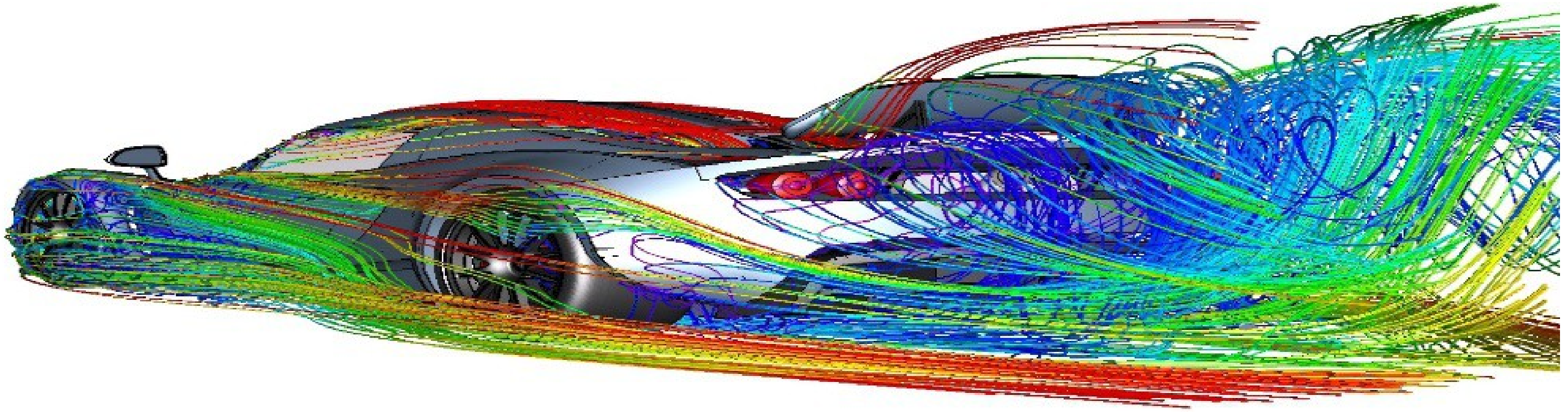
Separation of scales → **macroscopic:** L **microscopic:** ℓ

Knudsen number: $K_N \sim \frac{\ell}{L} \ll 1$

It is important to understand how fluid dynamics emerges from an underlying microscopic theory

Topic of this talk!

Basics of fluid dynamics



Basics of fluid dynamics

Effective theory describing the dynamics
of a system over long-times and long-distances

Separation of scales → **macroscopic:** L **microscopic:** ℓ

Knudsen number: $K_N \sim \frac{\ell}{L} \ll 1$

Conservation laws

+

constitutive/dynamical relations

Basics of fluid dynamics

Conservation laws

energy-momentum conservation

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

Net charge conservation

$$\begin{aligned} \partial_\mu N_s^\mu &= 0 \\ \partial_\mu N_e^\mu &= 0 \\ \partial_\mu N_b^\mu &= 0 \end{aligned}$$

strangeness

electric charge

Baryon number

Tensor decomposition

$$\begin{aligned} N^\mu &= nu^\mu + \nu^\mu, \\ T^{\mu\nu} &= \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu}, \end{aligned}$$

net-charge diffusion 4-current

isotropic pressure

energy diffusion 4-current

Shear stress tensor

Projection operator: $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$

Definition of “equilibrium state”

General picture

$$n \equiv n_0(\alpha,\beta) + \delta n, \qquad \varepsilon \equiv \varepsilon_0(\alpha,\beta) + \delta\varepsilon,$$

$$P \equiv P_0(\alpha,\beta) + \Pi,$$

Define α, β, and u^μ : *matching conditions*

Landau Picture

$$\delta n \equiv 0 \quad \delta\varepsilon \equiv 0$$

$$T^{\mu}{}_{\nu} u^{\nu} \equiv \varepsilon u^{\mu}$$

$$\downarrow$$

$$h^{\mu} \doteq 0$$

Eckart Picture

$$\delta n \equiv 0 \quad \delta\varepsilon \equiv 0$$

$$N^{\mu} \equiv n u^{\mu}$$

$$\downarrow$$

$$\nu^{\mu} \doteq 0$$

Basics of fluid dynamics

Conservation laws

energy-momentum conservation

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

Net charge conservation

$$\boxed{\begin{array}{rcl} \partial_\mu N_s^\mu & = & 0 \\ \partial_\mu N_e^\mu & = & 0 \\ \partial_\mu N_b^\mu & = & 0 \end{array}} \quad \begin{array}{l} \textbf{strangeness} \\ \textbf{electric charge} \\ \textbf{Baryon number} \end{array}$$

Tensor decomposition

$$\boxed{\begin{array}{l} N^\mu = n u^\mu + \underline{\nu^\mu}^{?}; \\ T^{\mu\nu} = \varepsilon u^\mu u^\nu - \underline{P}^{?}\Delta^{\mu\nu} + \underline{h^\mu}^{?} u^\nu + \underline{h^\nu}^{?} u^\mu + \underline{\pi^{\mu\nu}}^{?}; \end{array}}$$

Challenge: closing the equations

Hydrodynamic theory

Relativistic Navier-Stokes theory

Shear Viscosity

(Resistance to deformation)

$$\boxed{\pi^{\mu\nu} \quad = \quad 2\eta\nabla^{\langle\mu}u^{\nu\rangle}}$$

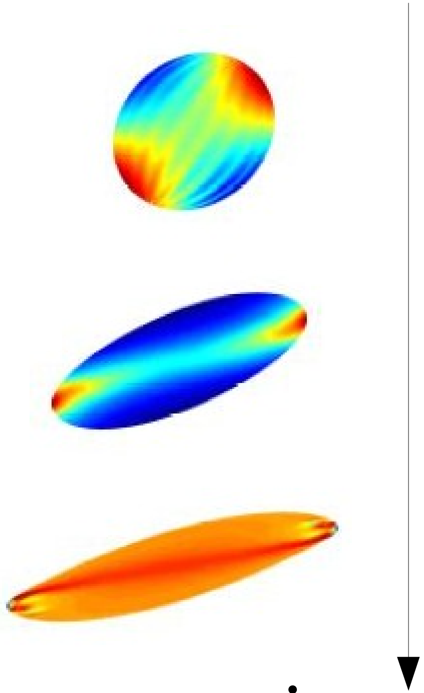
Bulk Viscosity

(Resistance to expansion)

$$\boxed{\Pi \quad = \quad -\zeta\nabla_{\mu}u^{\mu}}$$

Net-Charge Diffusion

$$\boxed{q^{\mu} \quad = \quad \kappa\nabla^{\mu}\frac{\mu_B}{T}}$$



equations violate causality and display unphysical instabilities

Israel-Stewart-like theories

Israel and Stewart, Annals Phys 118, 228 (1979) Denicol *et al* PRD 85 (2012) 114047

$$\begin{aligned}\dot{\Pi} &= -\frac{\Pi}{\tau_{\Pi}} - \beta_{\Pi}\theta - \ell_{\Pi n}\partial\cdot n - \tau_{\Pi n} n\cdot\dot{u} - \delta_{\Pi\Pi}\Pi\theta \\ &\quad -\lambda_{\Pi n} n\cdot\nabla\alpha_0 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}\ ,\end{aligned}$$

bulk

$$\begin{aligned}\dot{n}^{\langle\mu\rangle} &= -\frac{n^{\mu}}{\tau_n} + \beta_n\nabla^{\mu}\alpha_0 - n_{\nu}\omega^{\nu\mu} - \delta_{nn}n^{\mu}\theta - \ell_{n\Pi}\nabla^{\mu}\Pi \\ &\quad +\ell_{n\pi}\Delta^{\mu\nu}\partial_{\lambda}\pi_{\nu}^{\lambda} + \tau_{n\Pi}\Pi\dot{u}^{\mu} - \tau_{n\pi}\pi_{\nu}^{\mu}\dot{u}^{\nu} \\ &\quad -\lambda_{nn}n^{\nu}\sigma_{\nu}^{\mu} + \lambda_{n\Pi}\Pi\nabla^{\mu}\alpha_0 - \lambda_{n\pi}\pi^{\mu\nu}\nabla_{\nu}\alpha_0\end{aligned}$$

diffusion

$$\begin{aligned}\dot{\pi}^{\langle\mu\nu\rangle} &= -\frac{\pi^{\mu\nu}}{\tau_{\pi}} + 2\beta_{\pi}\sigma^{\mu\nu} + 2\pi_{\alpha}^{\langle\mu}\omega^{\nu\rangle\alpha} - \tau_{\pi n}n^{\langle\mu}\dot{u}^{\nu\rangle} \\ &\quad +\ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi_{\alpha}^{\langle\mu}\sigma^{\nu\rangle\alpha} \\ &\quad +\lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}\ .\end{aligned}$$

shear

Transient equations; second order in a gradient expansion

can be causal and stable!

BDNK theory

Benfica, Disconzi, Noronha Phys.Rev.D 98 (2018) 10, 104064

Constitutive equations that also include time-like derivatives

$$\delta n = \xi^{(\alpha)} D\alpha - \xi^{(\beta)} \frac{D\beta}{\beta} - \xi^{(\theta)}\theta, \quad \delta\varepsilon = \chi^{(\alpha)} D\alpha - \chi^{(\beta)} \frac{D\beta}{\beta} - \chi^{(\theta)}\theta,$$

$$\Pi = \zeta^{(\alpha)} D\alpha - \zeta^{(\beta)} \frac{D\beta}{\beta} - \zeta^{(\theta)}\theta,$$

scalars

$$\nu^\mu = \kappa^{(\alpha)} \nabla^\mu \alpha - \kappa^{(\beta)} \left(\frac{1}{\beta} \nabla^\mu \beta + Du^\mu \right),$$

$$h^\mu = \lambda^{(\alpha)} \nabla^\mu \alpha - \lambda^{(\beta)} \left(\frac{1}{\beta} \nabla^\mu \beta + Du^\mu \right),$$

4-vectors

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},$$

Shear-stress tensor

Can be causal and stable; Landau/Eckart matching conditions excluded

We can study this problem in kinetic theory

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}}=C\left[f\right]$$

Boltzmann eq.

???

Chapman-Enskog series

Method of moments

...

$$\tau_{\Pi}\dot{\Pi}+\Pi=-\zeta\theta+\ \ldots$$

$$\tau_{\pi}\dot{\pi}^{\langle\mu\nu\rangle}+\pi^{\mu\nu}=2\eta\sigma^{\mu\nu}+\ \ldots$$

“Hydrodynamics”

Today: exactly solvable microscopic interaction

Boltzmann

Relativistic Boltzmann equation

$$k^\mu \partial_\mu f_{\mathbf{k}} = C\left[f\right]$$

momentum distribution

collision term

Collision term – elastic 2-to-2 collisions

$$C\left[f\right] = \frac{1}{\nu} \int dK' dP dP' W_{\mathbf{kk'}\to\mathbf{pp'}} \left(f_{\mathbf{p}} f_{\mathbf{p'}} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k'}} - f_{\mathbf{k}} f_{\mathbf{k'}} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p'}} \right)$$

$$\boxed{\tilde{f}_{\mathbf{k}} \equiv 1 - a f_{\mathbf{k}}}$$

Transition rate

$$W_{\mathbf{kk'}\to\mathbf{pp'}} = \underbrace{s\sigma(s,\Theta)}_{\text{cross section – microscopic information}} (2\pi)^6 \, \delta^{(4)} \left(k^\mu + k'^\mu - p^\mu - p'^\mu\right)$$

Close to local equilibrium

$$f_{\mathbf{k}} = \underset{\downarrow}{f_{0\mathbf{k}}}\left(1 + \underset{\downarrow}{\tilde{f}_{0\mathbf{k}}\phi_{\mathbf{k}}}\right)$$

equilibrium *distribution* **non-equilibrium *correction***

Collision term near equilibrium:

$$C\left[f\right] = \underbrace{\frac{1}{\nu}\int dK'dPdP'W_{\mathbf{kk'}\to\mathbf{pp'}}f_{0\mathbf{k}}f_{0\mathbf{k'}}\tilde{f}_{0\mathbf{p}}\tilde{f}_{0\mathbf{p'}}\left(\phi_{\mathbf{p}} + \phi_{\mathbf{p'}} - \phi_{\mathbf{k}} - \phi_{\mathbf{k'}}\right)}_{\textit{Linear}\ \text{collision operator: }\ \hat{L}\phi_{\mathbf{k}}} + \mathcal{O}\left(\phi^2\right)$$

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}} \approx \hat{L}\phi_{\mathbf{k}}$$

This form of the Boltzmann equation is usually employed in the derivation of fluid dynamics

cross section

$$\boxed{\sigma_T(s) = \frac{\lambda^2}{32\pi s} \equiv \frac{g}{s}}$$

Self-interacting $\lambda\varphi^4$ scalar field theory

Eigenfunctions and Eigenvalues can be calculated exactly

(*massless*, *classical* limits)

$$\hat{L}\left[L_{n\mathbf{p}}^{(2\ell+1)} p^{\langle\mu_1}\cdots p^{\mu_\ell\rangle}\right] = \chi_{n\ell} L_{n\mathbf{p}}^{(2\ell+1)} p^{\langle\mu_1}\cdots p^{\mu_\ell\rangle}$$

Laguerre polynomials

Irreducible tensors

$$\chi_{n\ell} = -\frac{g\mathcal{M}}{2}\left(\frac{n+\ell-1}{n+\ell+1} + \delta_{\ell 0}\delta_{n0}\right)$$

GSD, J. Noronha, 2209.10370

All hydrodynamic theories *and* their transport coefficients can be derived analytically

Definition of “equilibrium state”

in the Boltzmann equation

Landau picture

$$
\begin{aligned}
\int dK\ E_{\mathbf{k}} f_{0\mathbf{k}} \phi_{\mathbf{k}} &= 0, \\
\int dK\ k^{\mu} E_{\mathbf{k}} f_{0\mathbf{k}} \phi_{\mathbf{k}} &= 0,
\end{aligned}
$$

Eckart picture

$$
\begin{aligned}
\int dK\ E_{\mathbf{k}}^{2} f_{0\mathbf{k}} \phi_{\mathbf{k}} &= 0, \\
\int dK\ k^{\mu} f_{0\mathbf{k}} \phi_{\mathbf{k}} &= 0,
\end{aligned}
$$

Definition of “equilibrium state”

in the Boltzmann equation

- Ensemble of matching conditions:

$$\left\{\begin{aligned} &\int dP\, E_{\mathbf{p}}^{q} \delta f_{\mathbf{p}} \equiv 0, \\ &\int dP\, E_{\mathbf{p}}^{s} \delta f_{\mathbf{p}} \equiv 0, \\ &\int dP\, E_{\mathbf{p}}^{z} p^{\langle\mu\rangle} \delta f_{\mathbf{p}} \equiv 0, \end{aligned}\right.$$

- ***q,s,z*** **are free parameters**
- **Landau and Eckart are special cases**
- **not the most general case**

- additional non-conserved fields required!
- how to do this outside of kinetic theory?

Relativistic Navier-Stokes theory

Perturbative solution

$$\epsilon E_{\mathbf{p}} D f_{\mathbf{p}} + \epsilon p^{\mu} \nabla_{\mu} f_{\mathbf{p}} = C[f_{\mathbf{p}}] \quad \left\{ \begin{aligned} f_{\mathbf{p}} &= \sum_{i=0}^{\infty} \epsilon^i f_{\mathbf{p}}^{(i)}, \\ D f_{\mathbf{p}} &= \sum_{i=0}^{\infty} \epsilon^i D^{(i)} f_{\mathbf{p}}, \end{aligned} \right.$$

$\epsilon \downarrow$ Knudsen number

Solve it order by order

0-th order: local equilibrium, $f_{\mathbf{p}}^{(0)} = f_{0\mathbf{p}}$

1-st order: $\frac{1}{4} L_{1\mathbf{p}}^{(3)} p_{\langle\mu\rangle} \nabla^{\mu} \alpha - \beta p_{\langle\mu} p_{\nu\rangle} \sigma^{\mu\nu} = \hat{L} \phi_{\mathbf{p}}$

Relativistic Navier-Stokes theory

1-st order: $\frac{1}{4}L^{(3)}_{1\mathbf{p}}p_{\langle\mu\rangle}\nabla^{\mu}\alpha - \beta p_{\langle\mu}p_{\nu\rangle}\sigma^{\mu\nu} = \hat{L}\phi_{\mathbf{p}}$

$$\phi_{\mathbf{k}} = \underbrace{\phi^{\mathrm{hom}}_{\mathbf{k}}}_{a+b_{\mu}k^{\mu}} + \hat{L}^{-1}\left(\frac{1}{4}L^{(3)}_{1\mathbf{k}}k_{\langle\mu\rangle}\nabla^{\mu}\alpha - \beta k_{\langle\mu}k_{\mu\rangle}\sigma^{\mu\nu}\right),$$

$$\Downarrow$$

$$\phi_{\mathbf{k}} = a + b_{\mu}k^{\mu} + \frac{1}{4\chi_{11}}L^{(3)}_{1\mathbf{k}}k_{\langle\mu\rangle}\nabla^{\mu}\alpha - \frac{\beta}{\chi_{02}}k_{\langle\mu}k_{\mu\rangle}\sigma^{\mu\nu},$$

↙

$a = 0$ and $b^{\mu} = z\nabla^{\mu}\alpha/(4\chi_{11})$

matching conditions

Relativistic Navier-Stokes theory

1-st order: $\frac{1}{4} L^{(3)}_{1\mathbf{p}} p_{\langle\mu\rangle} \nabla^\mu \alpha - \beta p_{\langle\mu} p_{\nu\rangle} \sigma^{\mu\nu} = \hat{L} \phi_{\mathbf{p}}$

Constitutive relations

$$\delta n = 0, \quad \nu^\mu = z \frac{3}{g\beta^2} \nabla^\mu \alpha,$$

$$\delta \varepsilon = 0, \quad h^\mu = (z-1) \frac{12}{g\beta^3} \nabla^\mu \alpha,$$

$$\pi^{\mu\nu} = \frac{96}{g\beta^3} \sigma^{\mu\nu}.$$

matching condition

$$\int dP \, E^z_{\mathbf{p}} p^{\langle\mu\rangle} \delta f_{\mathbf{p}} \equiv 0,$$

- transport coefficients depend on matching conditions

BDNK theory from kinetic theory

$$\Pi = \frac{\chi}{3}\left(\frac{D\beta}{\beta} - \frac{\theta}{3}\right), \quad \delta n = \xi\left(\frac{D\beta}{\beta} - \frac{\theta}{3}\right), \quad \delta\varepsilon = \chi\left(\frac{D\beta}{\beta} - \frac{\theta}{3}\right),$$

$$\nu^\mu = \kappa\left(\frac{\nabla^\mu\beta}{\beta} + Du^\mu\right), \quad h^\mu = \lambda\left(\frac{\nabla^\mu\beta}{\beta} + Du^\mu\right),$$

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}.$$

no chemical potential terms ...

Transport coefficients

$$\xi = \frac{12}{g\beta^2}(q-1)(s-1), \quad \chi = \frac{36}{g\beta^3}(q-2)(s-2),$$

$$\kappa = \frac{12}{g\beta^2}z, \quad \lambda = \frac{48}{g\beta^3}(z-1),$$

$$\eta = \frac{48}{g\beta^3}.$$

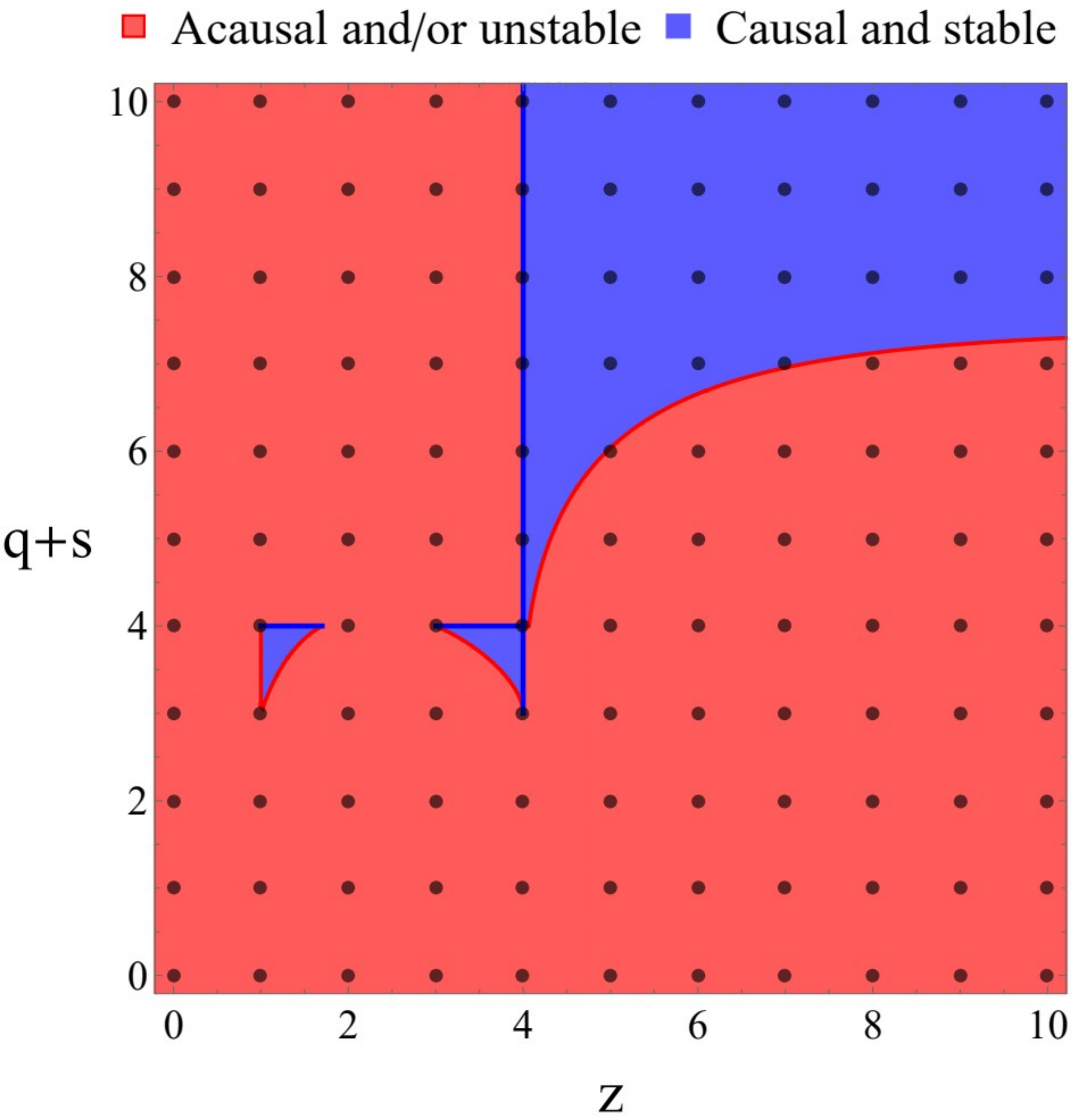
matching conditions

$$\int dP\, E_{\mathbf{p}}^q \delta f_{\mathbf{p}} = 0 \quad \int dP\, E_{\mathbf{p}}^s \delta f_{\mathbf{p}} = 0,$$

$$\int dP\, E_{\mathbf{p}}^z p^{\langle\mu\rangle} \delta f_{\mathbf{p}} = 0,$$

Linear Stability and Causality

stability of perturbations around global equilibrium (fluid at rest)



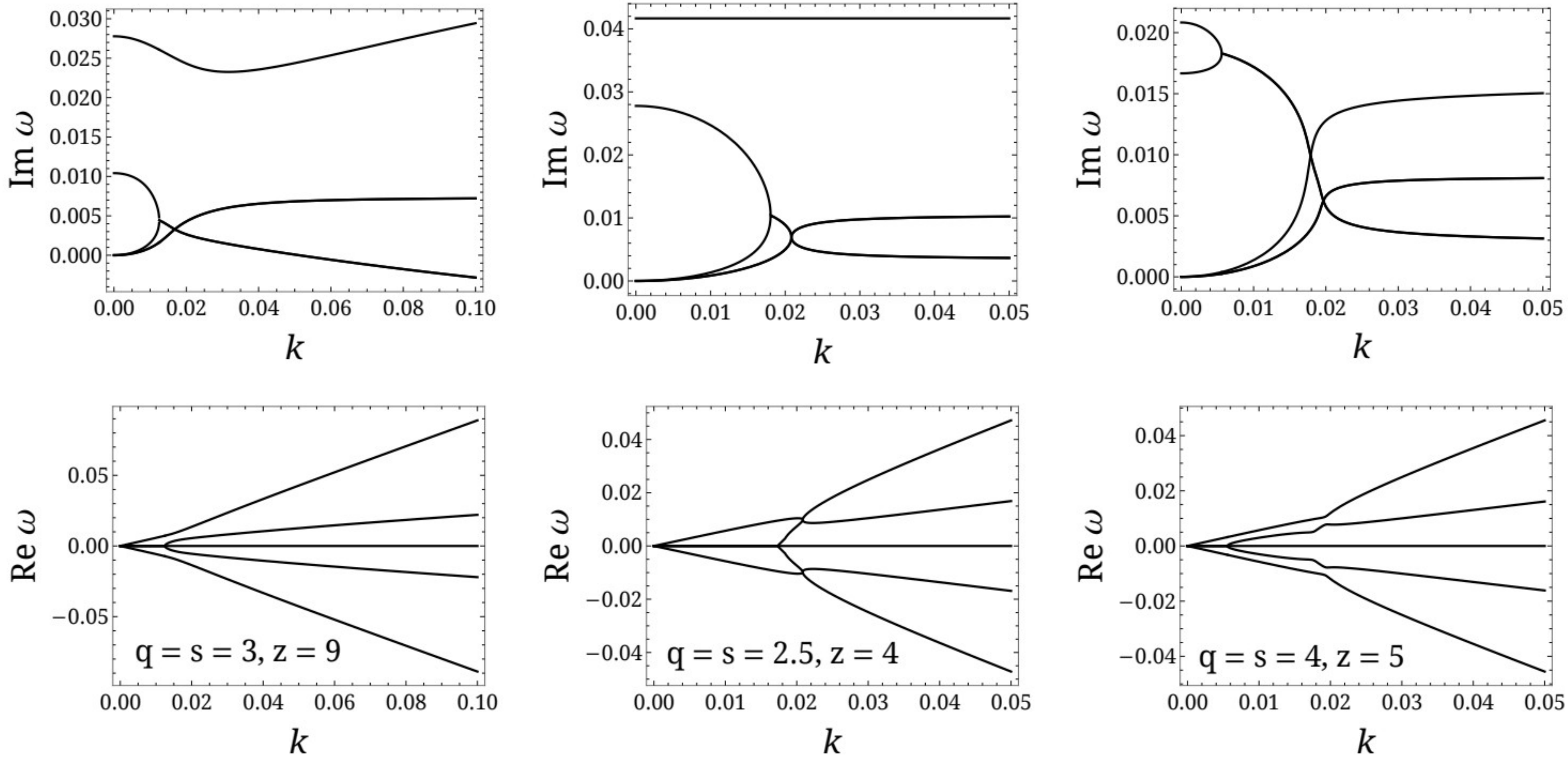
$$\int dP\, E^q_{\mathbf{p}} \delta f_{\mathbf{p}} \equiv 0,$$

$$\int dP\, E^s_{\mathbf{p}} \delta f_{\mathbf{p}} \equiv 0,$$

$$\int dP\, E^z_{\mathbf{p}} p^{\langle\mu\rangle} \delta f_{\mathbf{p}} \equiv 0,$$

Dispersion Relations: ω(k)

Sound modes for several matching conditions



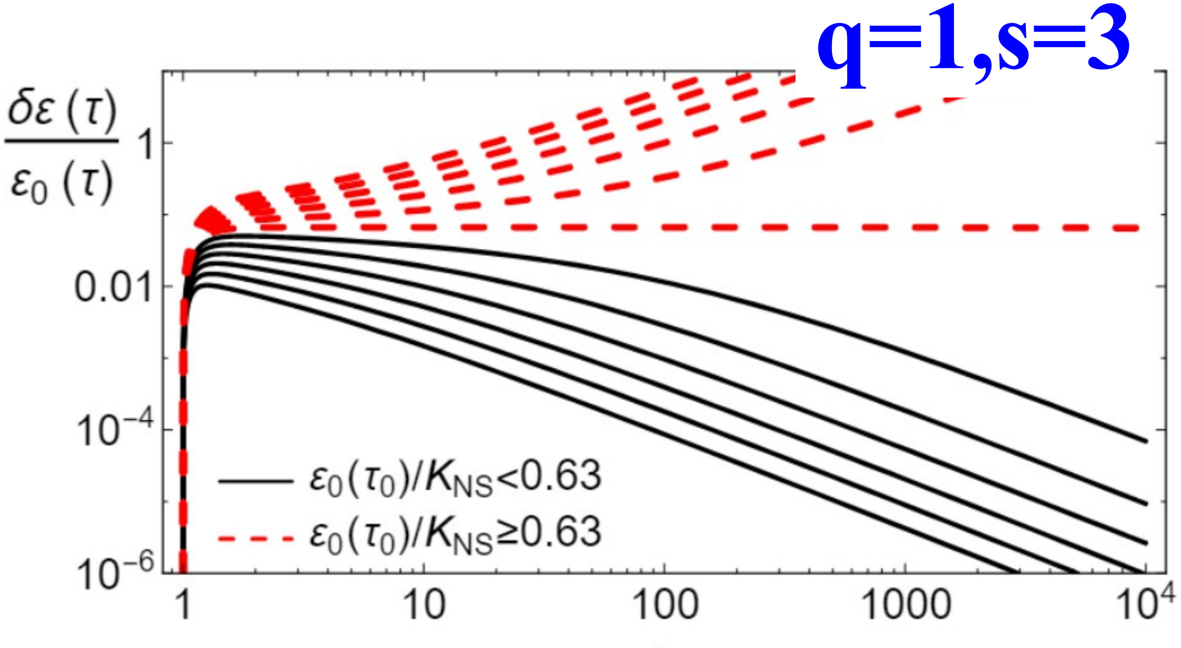
Dispersion relations are significantly effected by matching conditions

Solutions in Bjorken flow

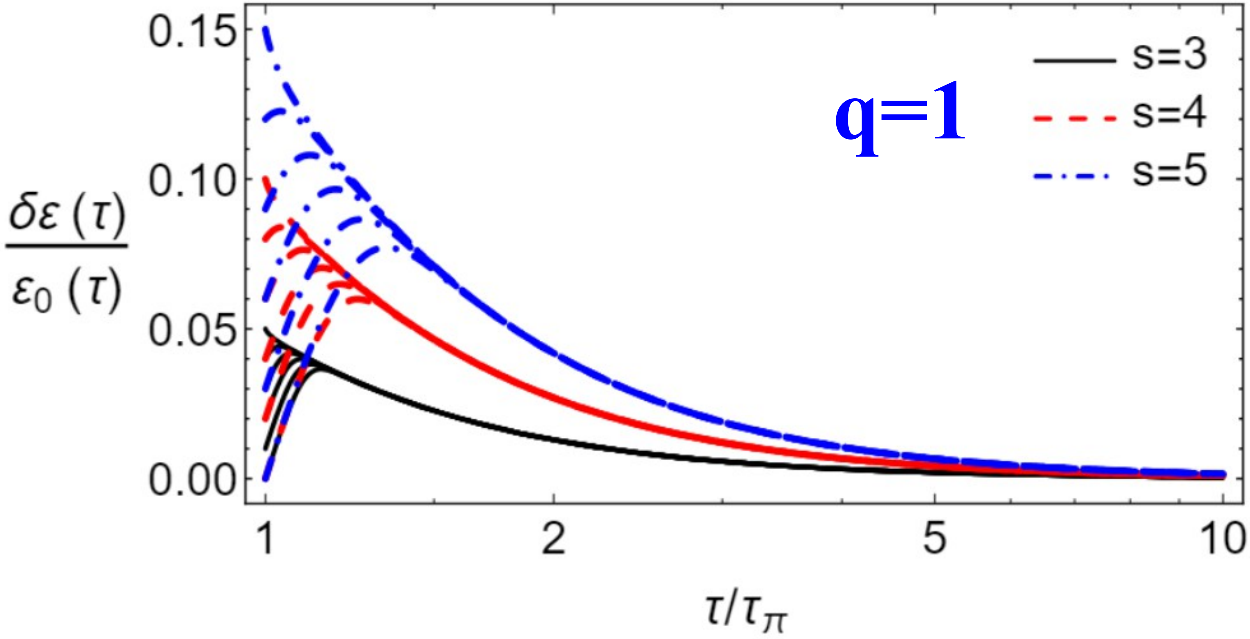
$$\dot{n}_0 + \frac{n_0}{\tau} = 0,$$

$$\dot{\varepsilon}_0 + \dot{\delta\varepsilon} + \frac{4}{3\tau}(\varepsilon_0 + \delta\varepsilon) - \frac{64}{g\beta^3\tau^2} = 0,$$

$$\delta\varepsilon = -\frac{36}{g\beta^3}(s-2)\left(\frac{\dot{\beta}}{\beta} - \frac{1}{3\tau}\right).$$



$$K_{\rm NS}^2 \equiv \frac{g}{64}\varepsilon_0(\tau_0)^3\beta(\tau_0)^3\tau_0,$$



Open questions:

- Is there an optimal matching condition?
How do we select it?

- How do we define general matching conditions
outside of kinetic theory?

these issues will have to be solved for implementations of BDNK theory

Conclusions

- Hydrodynamics can be systematically derived from the Boltzmann equation.

- For ultra-relativistic scalar particles self-interacting via a quartic potential, theories can be exactly derived.

- Matching conditions can significantly affect the magnitude of some transport coefficients.