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Helical separation effect for Dirac fermions in a strong magnetic field

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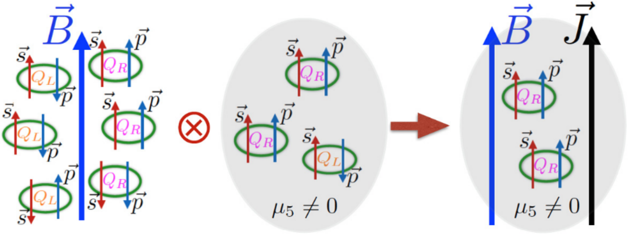
Motivation

- In the presence of a constant external magnetic field, the transverse-plane dynamics of a Dirac particle is quantized $\Rightarrow$ Landau levels.
- For strong magnetic fields, $|qB| \gg T$, the transverse dynamics is suppressed and the plasma properties are dominated by the **lowest Landau level (LLL)**.
- **Spin-orbit coupling:** LLL selects only one polarization $\Rightarrow$ **anomalous transport**.
- CME and CSE are well-known and involve the chiral (axial) current, J^μ_A.
- **Purpose of this work:** Evaluate $\langle J^\mu_H \rangle$, thereby establishing the leading-order **Helical Separation Effect (HSE)**.
- Ingredients:
  - **New conserved helicity operator** in external magnetic field;
  - **New** helicity eigenmodes;
  - Helicity imbalance via **helical chemical potential** μ_H.
- Bonus 1: thermodynamics of LLL: thermodynamic pressure P, dynamic pressure Π, shear-stress tensor π_B.
- Bonus 2: Wave-like excitations in the background $\mathbf{B}$.

- Chiral Magnetic Effect (CME): Vector current in chirally-imbalanced plasma

$$\mathbf{J}_V = \sigma_V^B \mathbf{B},$$

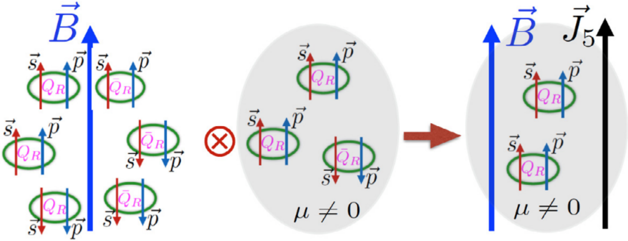
$$\sigma_V^B = \frac{q\mu_A}{2\pi^2}.$$



- Chiral Separation Effect (CSE): Axial current in charged plasma

$$\mathbf{J}_A = \sigma_A^B \mathbf{B},$$

$$\sigma_A^B = \frac{q\mu_V}{2\pi^2}.$$



New effect: Helical separation effect (HVE)

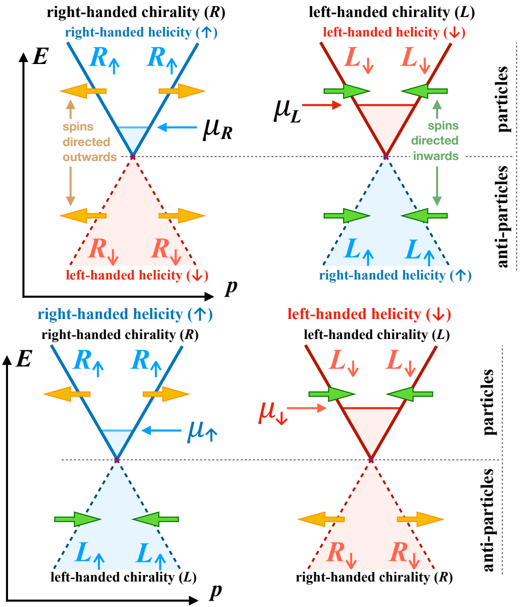
- Split particles into four groups:
  - $\mu_\uparrow^R$: particle: $R \Rightarrow \uparrow$
  - $\mu_\downarrow^L$: particle: $L \Rightarrow \downarrow$
  - $\bar{\mu}_\downarrow^R$: anti-particle: $R \Rightarrow \downarrow$
  - $\bar{\mu}_\uparrow^L$: anti-particle: $L \Rightarrow \uparrow$
- Charge densities:

$$Q_V \equiv (n_\uparrow^R + n_\downarrow^L) - (\bar{n}_\downarrow^R + \bar{n}_\uparrow^L),$$
$$Q_A \equiv (n_\uparrow^R + \bar{n}_\downarrow^R) - (n_\uparrow^L + \bar{n}_\downarrow^L),$$
$$Q_H \equiv (n_\uparrow^R + \bar{n}_\uparrow^L) - (n_\downarrow^L + \bar{n}_\downarrow^R).$$

- Magnetic conductivities:

$$\sigma_V^B = \frac{q\mu_A}{2\pi^2} + \frac{q\beta\mu_V\mu_H}{4\pi^2},$$
$$\sigma_A^B = \frac{q\mu_V}{2\pi^2} + \frac{q\beta\mu_A\mu_H}{4\pi^2},$$
$$\sigma_H^B = \frac{q}{\pi^2\beta}\ln 2 + \frac{q\beta\boldsymbol{\mu}^2}{8\pi^2}.$$

VEA, M. Chernodub, arXiv:2307.14987



Dirac eq. in external magnetic field

- Under minimal coupling, $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ and

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi - q\bar{\psi}A\!\!\!/\psi - M\bar{\psi}\psi. \tag{1}$$

- The Dirac equation reads

$$(i\partial\!\!\!/ - qA\!\!\!/ - M)\psi = 0, \qquad \overline{\psi}(i\overleftarrow{\partial\!\!\!/} + qA\!\!\!/ + M) = 0. \tag{2}$$

- We take A^μ in the Coulomb gauge, $A^\mu = (0, \mathbf{A})$, with $\mathbf{A} \equiv \mathbf{A}(\mathbf{x})$, such that

$$\mathbf{E} = \partial_t \mathbf{A} - \nabla A^0 = 0, \quad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{3}$$

- In this case, the Dirac Hamiltonian satisfying $H\psi = i\partial_t\psi$ reads

$$H = M\gamma_0 + \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\pi}, \tag{4}$$

  with $\boldsymbol{\pi} = -i\boldsymbol{\nabla} - q\mathbf{A}$ the generalized momentum.

- **Task:** find helicity h with $h^2 = 1/4$, s.t. $[h, H] = 0$ and

$$\lim_{\mathbf{A}\to 0} h = \frac{\mathbf{S}\cdot\mathbf{p}}{|\mathbf{p}|}, \qquad \mathbf{S} = \frac{1}{2}\gamma^5\gamma^0\boldsymbol{\gamma}. \tag{5}$$

Helicity in external magnetic field

- **Task:** find helicity h with $h^2 = 1/4$, s.t. $[h, H] = 0$ and

$$\lim_{\mathbf{A}\to 0} h = \frac{\mathbf{S}\cdot\mathbf{p}}{|\mathbf{p}|}, \qquad \mathbf{S} = \frac{1}{2}\gamma^5\gamma^0\boldsymbol{\gamma}. \tag{6}$$

- Where to find h? Look for it in H...
- Expressing $\boldsymbol{\gamma} = 2\gamma^0\gamma^5\mathbf{S}$, H can be recast as

$$H = M\gamma_0 + \gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\pi} = M\gamma_0 + 2\gamma^5\boldsymbol{S}\cdot\boldsymbol{\pi}. \tag{7}$$

- A natural choice is $h = \mathcal{N}\mathbf{S}\cdot\boldsymbol{\pi}$, with $\mathcal{N}$ s.t. $h^2 = 1/4$.
- To find $\mathcal{N}$, consider

$$H^2 = M^2 + 2M\{\gamma^0, \gamma^5\mathbf{S}\cdot\boldsymbol{\pi}\} + 4\gamma^5(\mathbf{S}\cdot\boldsymbol{\pi})^2 = M^2 + \frac{4}{\mathcal{N}^2}h^2. \tag{8}$$

- Imposing $h^2 = 1/4$, we find $\mathcal{N} = (H^2 - M^2)^{-1/2}$ and

$$h = \frac{\mathbf{S}\cdot\boldsymbol{\pi}}{\sqrt{H^2 - M^2}}. \tag{9}$$

The $V/A/H$ currents

- At the level of the Dirac theory, one may introduce the following currents:

$$J_V^\mu = \bar{\psi}\gamma^\mu\psi, \quad J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi, \quad J_H^\mu = \bar{\psi}\gamma^\mu h\psi + \overline{h\psi}\gamma^\mu\psi. \tag{10}$$

- The vector current J_V^μ is conserved by virtue of the Dirac eq., $\partial\!\!\!/\psi = -i(qA\!\!\!/ + M)\psi$:

$$\partial_\mu J_V^\mu = \bar{\psi}\partial\!\!\!/\psi + \overline{\partial\!\!\!/\psi}\psi = -i\bar{\psi}(qA\!\!\!/ + M)\psi - \overline{i(qA\!\!\!/ + M)\psi}\psi = 0. \tag{11}$$

- The conservation of the axial current is broken by the mass term:

$$\partial_\mu J_A^\mu = i\bar{\psi}\{\gamma^5, qA\!\!\!/ + M\}\psi = 2iM\bar{\psi}\gamma^5\psi. \tag{12}$$

- The divergence of J_H^μ reads:

$$\partial_\mu J_H^\mu = \bar{\psi}\partial\!\!\!/ h\psi + \overline{\partial\!\!\!/\psi}h\psi + \text{h.c.} \tag{13}$$

- Because $[H, h] = 0$, if $H\psi = i\partial_t\psi$ then $Hh\psi = i\partial_t(h\psi)$, s.t.

$$\partial\!\!\!/ h\psi = -i(qA\!\!\!/ + M)h\psi \quad \Rightarrow \quad \bar{\psi}\partial\!\!\!/ h\psi = -\overline{\partial\!\!\!/\psi}h\psi \quad \Rightarrow \quad \partial_\mu J_H^\mu = 0. \tag{14}$$

- **J_H^μ is conserved in a background magnetic field, even at finite M!**

Fermions in a constant background magnetic field, $\mathbf{B} = B\mathbf{k}$

- We take $\mathbf{B} = B\mathbf{k}$ with $B = \text{const}$ and $A^\mu = Bx\delta^\mu_y$, leading to

$$H = M\gamma^0 + 2\gamma^5 \mathbf{S}\cdot\mathbf{P} - 2qBx\gamma^5 S^y, \qquad \mathbf{P} = -i\boldsymbol{\nabla}. \tag{15}$$

- Consider now particle solutions of the Dirac equation satisfying:

$$HU_j = E_j U_j, \quad P^y U_j = p^y_j U_j, \quad P^z U_j = p^z_j U_j, \quad hU_j = \lambda_j U_j. \tag{16}$$

- From $h = \mathcal{N}^{-1}\mathbf{S}\cdot\boldsymbol{\pi}$ and $H = M\gamma^0 + 2\gamma^5\mathbf{S}\cdot\boldsymbol{\pi}$, one can write

$$h = \frac{\gamma^5}{2}\frac{H - M\gamma^0}{\sqrt{H^2 - M^2}} \qquad \Rightarrow \qquad hU_j = \frac{1}{2}\begin{pmatrix} 0 & \mathcal{K}^+_j/\mathcal{K}^-_j \\ \mathcal{K}^-_j/\mathcal{K}^+_j & 0 \end{pmatrix} U_j, \tag{17}$$

where $\mathcal{K}^\pm_j = \sqrt{1 \pm M/E_j}$.

- The mode solutions we are looking for are

$$U_j = \frac{e^{-iE_j t}}{\sqrt{2}}\begin{pmatrix} \mathcal{K}^+_j \\ 2\lambda_j \mathcal{K}^-_j \end{pmatrix} \otimes \phi_j(\mathbf{x}), \quad \phi_j(\mathbf{x}) = \frac{e^{ip^y_j y + ip^z_j z}}{2\pi}\begin{pmatrix} f^+_j \\ f^-_j \end{pmatrix}, \tag{18}$$

where $f^\pm_j \equiv f^\pm_j(x)$ must be found by imposing $\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\phi_j = 2\lambda_j\sqrt{E_j^2 - M^2}\phi_j$.

- The equation $\boldsymbol{\sigma} \cdot \boldsymbol{\pi} \phi_j = 2\lambda_j \sqrt{E_j^2 - M^2} \phi_j$ leads to

$$\begin{pmatrix} 2\lambda_j \sqrt{E_j^2 - M^2} - p_j^z & i\sqrt{2|qB|}(\partial_{\xi_j} - \frac{\sigma}{2}\xi_j) \\ i\sqrt{2|qB|}(\partial_{\xi_j} + \frac{\sigma}{2}\xi_j) & 2\lambda_j \sqrt{E_j^2 - M^2} + p_j^z \end{pmatrix} \begin{pmatrix} f_j^+ \\ f_j^- \end{pmatrix} = 0, \tag{19}$$

where we introduced

$$\xi_j = \sqrt{2|qB|}\left(x - \frac{\sigma p_j^y}{|qB|}\right), \quad \sigma = \mathrm{sgn}(qB). \tag{20}$$

- With the above notation, $f_j^\pm(\xi_j)$ satisfy

$$\left(\frac{\partial^2}{\partial \xi_j^2} - \frac{1}{4}\xi_j^2 + \nu_j^\pm + \frac{1}{2}\right) f_j^\pm = 0, \qquad \nu_j^\pm = \frac{E_j^2 - M^2 - (p_j^z)^2}{2|qB|} - \frac{1 \mp \sigma}{2}. \tag{21}$$

- **The above eq. has regular solutions when $\nu_j^\pm$ is a non-negative integer:**

$$f_j^\pm = \mathcal{C}_j^\pm e^{-\xi_j^2/4} H_{\nu_j^\pm}(\xi_j), \quad E_j^2 = M^2 + (p_j^z)^2 + 2n_j|qB|, \quad \nu_j^\pm = n_j - \frac{1 \mp \sigma}{2}. \tag{22}$$

where $\mathcal{C}_j^\pm$ are integration constants and $H_{\nu_j^\pm}(\xi_j)$ the Hermite polynomials.

(23)

Lowest Landau level (LLL) solutions

- The constants $\mathcal{C}_j^{\pm}$ satisfy

$$\begin{pmatrix} 2\lambda_j\sqrt{E_j^2 - M^2} - \sigma p_j^z & -i\sqrt{2|qB|} \\ i n_j\sqrt{2|qB|} & 2\lambda_j\sqrt{E_j^2 - M^2} + \sigma p_j^z \end{pmatrix} \begin{pmatrix} \mathcal{C}_j^{\sigma} \\ \mathcal{C}_j^{-\sigma} \end{pmatrix} = 0. \tag{24}$$

- **The LLL corresponds to $n_j = 0$,** when $E_j^2 = M^2 + (p_j^z)^2$, leading to

$$\begin{pmatrix} 2\lambda_j|p_j^z| - \sigma p_j^z & -i\sqrt{2|qB|} \\ 0 & 2\lambda_j|p_j^z| + \sigma p_j^z \end{pmatrix} \begin{pmatrix} \mathcal{C}_j^{\sigma} \\ \mathcal{C}_j^{-\sigma} \end{pmatrix} = 0. \tag{25}$$

- When $\sigma = \pm 1$, $\nu_j^{\mp} = -1$; this requires $\mathcal{C}_j^{-\sigma} = 0$ for finite solutions.
- $\mathcal{C}_j^{\sigma} \neq 0$ **supported only when** $2\sigma\lambda_j = \mathrm{sgn}(p_j^z)$**.**
- The normalized LLL particle and anti-particle modes U_j and $V_j = i\gamma^2 U_j^* \rfloor_{q\to -q}$ are

$$U_j\Big|_{n_j=0} = \theta(\sigma\lambda_j p_j^z) e^{-\frac{\xi_j^2}{4}} \left(\frac{|qB|}{\pi}\right)^{1/4} \frac{e^{-iE_j t + i p_j^y y + i p_j^z z}}{\sqrt{8\pi^2}} \begin{pmatrix} \mathcal{K}_j^+ \\ 2\lambda_j \mathcal{K}_j^- \end{pmatrix} \otimes \begin{pmatrix} \frac{1+\sigma}{2} \\ \frac{1-\sigma}{2} \end{pmatrix},$$

$$V_j\Big|_{n_j=0} = \theta(-\sigma\lambda_j p_j^z) e^{-\frac{\xi_{j;c}^2}{4}} \left(\frac{|qB|}{\pi}\right)^{1/4} \frac{e^{iE_j t - i p_j^y y - i p_j^z z}}{\sqrt{8\pi^2}} \begin{pmatrix} 2\lambda_j \mathcal{K}_j^- \\ -\mathcal{K}_j^+ \end{pmatrix} \otimes \begin{pmatrix} \frac{1+\sigma}{2} \\ -\frac{1-\sigma}{2} \end{pmatrix}.$$

Higher Landau levels (HLL)

- For $n_j > 0$, both helicities are allowed and the particle/anti-particle modes are

$$U_j(x) = \frac{e^{-iE_j t}}{\sqrt{2}} \begin{pmatrix} \mathcal{K}_j^+ \\ 2\lambda_j \mathcal{K}_j^- \end{pmatrix} \otimes \phi_j^\sigma(\mathbf{x}), \quad V_j(x) = \frac{e^{iE_j t}}{\sqrt{2}} \begin{pmatrix} 2\lambda_j \mathcal{K}_j^- \\ -\mathcal{K}_j^+ \end{pmatrix} \otimes \phi_j^{c;\sigma}(\mathbf{x}), \quad (26)$$

  where $V_j = i\gamma^2 U_j^* \rfloor_{q\to -q}$.
- The Pauli particle two-spinors are given by $[\mathfrak{p}_j^\pm = [1 \pm p_j^z/\sqrt{E_j^2 - M^2}]^{1/2}]$

$$\phi_j^+(\mathbf{x}) = \frac{e^{ip_j^y y + ip_j^z z}}{2\pi\sqrt{n_j!}} e^{-\xi_j^2/4} \left(\frac{|qB|}{4\pi}\right)^{1/4} \begin{pmatrix} \mathfrak{p}_j^{2\lambda_j} H_{n_j}(\xi_j) \\ -2i\lambda_j \mathfrak{p}_j^{-2\lambda_j} \sqrt{n_j} H_{n_j-1}(\xi_j) \end{pmatrix},$$
$$\phi_j^-(\mathbf{x}) = \frac{e^{ip_j^y y + ip_j^z z}}{2\pi\sqrt{n_j!}} e^{-\xi_j^2/4} \left(\frac{|qB|}{4\pi}\right)^{1/4} \begin{pmatrix} \mathfrak{p}_j^{2\lambda_j} \sqrt{n_j} H_{n_j-1}(\xi_j) \\ 2i\lambda_j \mathfrak{p}_j^{-2\lambda_j} H_{n_j}(\xi_j) \end{pmatrix}. \quad (27)$$

- The Pauli anti-particle two-spinors $\phi_j^c = i\sigma^2 \phi_j^*$ are

$$\phi_j^{c;+}(\mathbf{x}) = \frac{e^{-ip_j^y y - ip_j^z z}}{2\pi\sqrt{n_j!}} e^{-\xi_{j;c}^2/4} \left(\frac{|qB|}{4\pi}\right)^{1/4} \begin{pmatrix} -2i\lambda_j \mathfrak{p}_j^{-2\lambda_j} H_{n_j}(\xi_{j;c}) \\ -\mathfrak{p}_j^{2\lambda_j} \sqrt{n_j} H_{n_j-1}(\xi_{j;c}) \end{pmatrix},$$
$$\phi_j^{c;-}(\mathbf{x}) = \frac{e^{-ip_j^y y - ip_j^z z}}{2\pi\sqrt{n_j!}} e^{-\xi_{j;c}^2/4} \left(\frac{|qB|}{4\pi}\right)^{1/4} \begin{pmatrix} 2i\lambda_j \mathfrak{p}_j^{-2\lambda_j} \sqrt{n_j} H_{n_j-1}(\xi_{j;c}) \\ -\mathfrak{p}_j^{2\lambda_j} H_{n_j}(\xi_{j;c}) \end{pmatrix}. \quad (28)$$

- It can be seen that $\phi^{c;\sigma}_{n,p^y,p^z,\lambda}(\mathbf{x}) = -2i\lambda\sigma\phi^\sigma_{n,-p^y,-p^z,\lambda}(\mathbf{x})$.

Second quantization: Conserved operators

- Consider a full set of particle/anti-particle modes $\{U_j, V_j\}$.
- The general solution ψ of the Dirac operator is promoted to the field operator:

$$\psi \to \hat{\psi} = \sum_j (U_j \hat{a}_j + V_j \hat{b}_j^\dagger), \quad \sum_j \equiv \sum_{\lambda_j = \pm\frac{1}{2}} \int_{-\infty}^{\infty} dp_j^y \int_{-\infty}^{\infty} dp_j^z \sum_{n_j=0}^{\infty}. \qquad (29)$$

- The Hamiltonian $\widehat{H}$ and charge operators $\widehat{Q}_{V/A/H}$ are **diagonal**:

$$\begin{aligned}
&\text{Energy eigenmodes:} && :\widehat{H}: = \sum_j E_j (\hat{a}_j^\dagger \hat{a}_j + \hat{b}_j^\dagger \hat{b}_j),\\
&\text{Particle/anti-particle:} && :\widehat{Q}_V: = \sum_j (\hat{a}_j^\dagger \hat{a}_j - \hat{b}_j^\dagger \hat{b}_j),\\
&\text{Helicity eigenmodes:} && :\widehat{Q}_H: = \sum_j 2\lambda_j (\hat{a}_j^\dagger \hat{a}_j - \hat{b}_j^\dagger \hat{b}_j),\\
&\text{Only for } M = 0\text{:} && :\widehat{Q}_A: = \sum_j 2\lambda_j (\hat{a}_j^\dagger \hat{a}_j + \hat{b}_j^\dagger \hat{b}_j). \qquad (30)
\end{aligned}$$

where $:\widehat{A}: \equiv \widehat{A} - \langle 0|\widehat{A}|0\rangle$ denotes Wick ordering.

Thermal expectation values

- We now consider the **grand canonical ensemble** [Canuto, Chiu, Phys. Rev. **173** (1968) 1210; 1220; 1229]

$$\hat{\rho} = e^{-\beta(\widehat{H} - \boldsymbol{\mu}\cdot\widehat{\mathbf{Q}})}, \quad \boldsymbol{\mu}\cdot\widehat{\mathbf{Q}} = \mu_V \widehat{Q}_V + \mu_A \widehat{Q}_A + \mu_H \widehat{Q}_H\,. \tag{31}$$

- Thermal expectation values are computed via

$$\langle \widehat{A} \rangle \equiv \mathcal{Z}^{-1} \mathrm{tr}(\hat{\rho}\widehat{A}), \qquad \mathcal{Z} = \mathrm{tr}(\hat{\rho}). \tag{32}$$

- We now consider that $\widehat{A} \equiv \mathcal{A}(\hat{\psi}, \hat{\psi})$ is quadratic in $\hat{\psi}$:

$$\widehat{A} = \sum_{j,j'} \Big[\mathcal{A}(U_j, U_{j'}) \hat{a}_j^\dagger \hat{a}_{j'} + \mathcal{A}(V_j, V_{j'}) \hat{b}_j \hat{b}_{j'}^\dagger \\ + \mathcal{A}(U_j, V_{j'}) \hat{a}_j^\dagger \hat{b}_{j'}^\dagger + \mathcal{A}(V_j, U_{j'}) \hat{b}_j \hat{a}_{j'}\Big],$$

  where $\mathcal{A}(\psi, \chi)$ is some sesquilinear form.
- $\langle \widehat{A} \rangle$ requires the t.e.v. of products of 2 one-p. operators, $\langle \hat{a}_j^\dagger \hat{a}_{j'} \rangle$ etc.

Building blocks: $\langle \hat{a}_j^\dagger \hat{a}_{j'} \rangle$ and $\langle \hat{b}_j^\dagger \hat{b}_{j'} \rangle$

- In order to compute $\mathcal{E}_j^\pm$, we start from the decompositions (30), implying

	Particles	Anti-particles
Energy eigenmodes:	$[\widehat{H}, \hat{a}_j^\dagger] = E_j \hat{a}_j^\dagger,$	$[\widehat{H}, \hat{b}_j^\dagger] = E_j \hat{b}_J^\dagger,$
Particle/anti-particle:	$[\widehat{Q}_V, \hat{a}_j^\dagger] = \hat{a}_j^\dagger,$	$[\widehat{Q}_V, \hat{b}_j^\dagger] = -\hat{b}_j^\dagger,$
Helicity eigenmodes:	$[\widehat{Q}_H, \hat{a}_j^\dagger] = 2\lambda_j \hat{a}_j^\dagger,$	$[\widehat{Q}_H, \hat{b}_j^\dagger] = -2\lambda_j \hat{b}_j^\dagger,$
For $M = 0$:	$[\widehat{Q}_A, \hat{a}_j^\dagger] = 2\lambda_j \hat{a}_j^\dagger,$	$[\widehat{Q}_A, \hat{a}_j^\dagger] = 2\lambda_j \hat{b}_j^\dagger.$ (33)

- Using Baker-Campbell-Hausdorff, $e^X Y e^{-X} = \sum_{n=0}^\infty \frac{1}{n!}[X^n, Y]$, one can compute

$$\hat{\rho} \hat{a}_j^\dagger \hat{\rho}^{-1} = e^{-\beta \mathcal{E}_j^+} \hat{a}_j^\dagger, \quad \hat{\rho} \hat{b}_j^\dagger \hat{\rho}^{-1} = e^{-\beta \mathcal{E}_j^-} \hat{b}_j^\dagger, \tag{34}$$

with

$$[\widehat{H} - \boldsymbol{\mu} \cdot \widehat{\mathbf{Q}}, \hat{a}_j^\dagger] = \mathcal{E}_j^+ \hat{a}_j^\dagger, \qquad [\widehat{H} - \boldsymbol{\mu} \cdot \widehat{\mathbf{Q}}, \hat{b}_j^\dagger] = \mathcal{E}_j^- \hat{b}_j^\dagger, \tag{35}$$

from where

$$\mathcal{E}_j^{\varsigma_j} = E_j - \varsigma_j \mu_V - 2\lambda_j \mu_A - 2\varsigma_j \lambda_j \mu_H, \qquad \varsigma_j = \begin{cases} +1, & \text{for particles,} \\ -1, & \text{for antiparticles.} \end{cases} \tag{36}$$

- Then, $\langle \hat{a}_j^\dagger \hat{a}_{j'} \rangle = \mathcal{Z}^{-1} \mathrm{tr}(\hat{\rho} \hat{a}_j^\dagger \hat{a}_{j'}) = e^{-\beta \mathcal{E}_j^+} \mathcal{Z}^{-1} \mathrm{tr}(\hat{a}_j^\dagger \hat{\rho} \hat{a}_{j'}) = \delta(j, j') / (e^{\beta \mathcal{E}_j^+} + 1)$.

Expectation values for charge currents

- The t.e.v.s of the charge currents can be put in the form

$$J^\mu_\ell = Q_\ell u^\mu + \sigma^B_\ell B^\mu, \quad u^\mu = \delta^\mu_0, \quad B^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta} = B\delta^\mu_z, \tag{37}$$

where $F_{\mu\nu} = -B(g_{\mu x} g_{\nu y} - g_{\mu y} g_{\nu x})$.
- $Q_\ell u^\mu$ is the classical transport of the charge density Q_ℓ.
- $\sigma^B_\ell B^\mu$ gives the anomalous transport component.
- Both Q_ℓ and the magnetic conductivities σ^B_ℓ must be obtained using TFT:

$$\begin{pmatrix} Q_V \\ Q_A \\ Q_H \end{pmatrix} = \frac{|qB|}{4\pi^2} \sum_{n_j=0}^{\infty} \sum_{\lambda_j, \varsigma_j} \int_0^\infty \frac{dp^z_j \, g_j}{e^{\beta \mathcal{E}_j} + 1} \begin{pmatrix} \varsigma_j \\ 2\lambda_j \mathcal{K}^+_j \mathcal{K}^-_j \\ 2\lambda_j \varsigma_j \end{pmatrix},$$

$$\begin{pmatrix} \sigma^B_V \\ \sigma^B_A \\ \sigma^B_H \end{pmatrix} = \frac{q}{4\pi^2} \sum_{n_j=0}^{\infty} \delta_{n_j,0} \sum_{\lambda_j, \varsigma_j} \int_0^\infty \frac{dp^z_j \, g_j}{E_j(e^{\beta \mathcal{E}_j} + 1)} \begin{pmatrix} 2\lambda_j p^z_j \\ \varsigma_j E_j \\ p^z_j \end{pmatrix}, \tag{38}$$

with $g_j = 1$ for $n_j = 0$ and $g_j = 2$ when $n_j > 0$.
- **The anomalous transport part** is fully determined by the **LLL** **($n_j = 0$)!**

Expectation values for the energy-momentum tensor

- The general decomposition of $\Theta^{\mu\nu}$ w.r.t. u^μ reads

$$\Theta^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} + j_\epsilon^\mu u^\nu + u^\mu j_\epsilon^\nu, \qquad (39)$$

with $j_\epsilon^\mu = \sigma_\epsilon^B B^\mu$ the **anomalous heat flux.**

- The **dynamical pressure** Π and **shear stress** $\pi^{\mu\nu} = \pi_B \times \mathrm{diag}(0, -\frac{1}{2}, -\frac{1}{2}, 1)$ represent deviations from the ideal form.
- We find

$$\epsilon = \frac{|qB|}{4\pi^2} \sum_{n_j=0}^{\infty} \sum_{\lambda_j, \varsigma_j} \int_0^\infty \frac{dp_j^z \, g_j E_j}{e^{\beta \mathcal{E}_j} + 1},$$

$$P + \Pi = \frac{|qB|}{12\pi^2} \sum_{n_j=0}^{\infty} \sum_{\lambda_j, \varsigma_j} \int_0^\infty \frac{dp_j^z \, g_j (E_j^2 - M^2)}{E_j (e^{\beta \mathcal{E}_j} + 1)},$$

$$\pi_B = \frac{|qB|}{6\pi^2} \sum_{n_j=0}^{\infty} \sum_{\lambda_j, \varsigma_j} \int_0^\infty \frac{dp_j^z \, g_j (p_{z;j}^2 - n_j |qB|)}{E_j (e^{\beta \mathcal{E}_j} + 1)},$$

$$\sigma_B^\epsilon = \frac{q}{4\pi^2} \sum_{n_j=0}^{\infty} \delta_{n_j,0} \sum_{\lambda_j, \varsigma_j} 2\varsigma_j \lambda_j \int_0^\infty \frac{dp_j^z p_j^z}{e^{\beta \mathcal{E}_j} + 1}. \qquad (40)$$

- Again, anomalous transport (σ_ϵ^B) comes from the LLL.

Massless limit

- In the massless limit, the anomalous conductivities $\sigma^B_{V/A/H}$ and σ^B_ϵ are given by

$$\begin{aligned}
\sigma^B_V &= \frac{q}{4\pi^2\beta}\sum_{\lambda_j,\varsigma_j} 2\lambda_j \ln(1+e^{\beta \mathbf{q}_j\cdot\boldsymbol{\mu}}) \simeq \frac{q\mu_A}{2\pi^2} + \frac{q\beta\mu_V\mu_H}{4\pi^2}[1+O(\beta^2)],\\
\sigma^B_A &= \frac{q}{4\pi^2\beta}\sum_{\lambda_j,\varsigma_j} 2\lambda_j\varsigma_j q^\ell_j \ln(1+e^{\beta \mathbf{q}_j\cdot\boldsymbol{\mu}}) \simeq \frac{q\mu_V}{2\pi^2} + \frac{q\beta\mu_A\mu_H}{4\pi^2}[1+O(\beta^2)],\\
\sigma^B_H &= \frac{q}{4\pi^2\beta}\sum_{\lambda_j,\varsigma_j} 2\lambda_j\varsigma_j q^\ell_j \ln(1+e^{\beta \mathbf{q}_j\cdot\boldsymbol{\mu}}) \simeq \frac{q}{\pi^2\beta}\ln 2 + \frac{q\beta\boldsymbol{\mu}^2}{8\pi^2} + O(\beta^3),\\
\sigma^B_\epsilon &= -\frac{q}{4\pi^2\beta^2}\sum_{\lambda_j,\varsigma_j} 2\lambda_j\varsigma_j \mathrm{Li}_2(-e^{\beta \mathbf{q}_j\cdot\boldsymbol{\mu}}) \simeq \frac{q\mu_H}{\pi^2\beta}\ln 2 + \frac{q\mu_V\mu_A}{2\pi^2} + O(\beta).
\end{aligned} \tag{41}$$

- The other quantit's (Q_ℓ, ϵ, P, Π, π_B) can be evaluated on the LLL (i.e., for $|qB| \gg T$):

$$Q_\ell \simeq \frac{|qB|\mu_\ell}{2\pi^2} + O(\beta), \quad \epsilon = 3(P+\Pi) = \frac{3}{2}\pi_B = \frac{|qB|}{12\beta^2} + \frac{|qB|\boldsymbol{\mu}^2}{4\pi^2} + O(\beta^{-1}). \tag{42}$$

- **Question: How to disentangle Π and P?**

Thermodynamics

- The grand canonical potential of the system reads

$$\Phi = \phi V = -\frac{|qB|V}{4\pi^2\beta}\sum_{n_j=0}^{\infty}\sum_{\lambda_j,\varsigma_j}\int_0^\infty dp_j^z\, g_j \ln(1+e^{-\beta\mathcal{E}_j}). \tag{43}$$

- An integration by parts shows that

$$P = -\phi = \frac{|qB|}{4\pi^2}\sum_{\lambda_j,\varsigma_j}\int_0^\infty \frac{dp_j^z\, g_j\, p_{z;j}^2}{E_j(e^{\beta\mathcal{E}_j}+1)} \equiv \Theta^{zz}, \tag{44}$$

where $\Theta^{zz} = P + \Pi + \pi_B$ is a component of $\Theta^{\mu\nu}$.

- This implies that $\Pi + \pi_B = 0$. Using $\pi_B = \frac{1}{3}(2\Theta^{zz} - \Theta^{xx} - \Theta^{yy})$ gives

$$\Pi = -\pi_B = \frac{|qB|}{6\pi^2}\sum_{n_j=0}^{\infty}\sum_{\lambda_j,\varsigma_j}\int_0^\infty \frac{dp_j^z\, g_j(n_j|qB| - p_{z;j}^2)}{E_j(e^{\beta\mathcal{E}_j}+1)}. \tag{45}$$

- On the LLL and for massless fermions, the relation $\Pi = -\pi_B$ can be used in conjunction with Eq. (42) to obtain:

$$\epsilon = P, \quad \pi_B = -\Pi = \frac{2}{3}P. \tag{46}$$

New excitations?

- The rationale behind the chiral magnetic wave (CMW) comes from

$$J_V^\mu = Q_V u^\mu + \sigma_V^B B^\mu, \qquad Q_V = \frac{|qB|\mu_V}{2\pi^2}, \qquad \sigma_V^B \simeq \frac{q\mu_A}{2\pi^2},$$
$$J_A^\mu = Q_A u^\mu + \sigma_A^B B^\mu, \qquad Q_A = \frac{|qB|\mu_A}{2\pi^2}, \qquad \sigma_A^B \simeq \frac{q\mu_V}{2\pi^2}. \tag{47}$$

- In a neutral plasma, small oscillations in μ_V, μ_A propagate according to $\partial_\mu J_V^\mu = \partial_\mu J_A^\mu = 0$:

$$\partial_t \delta\mu_V + \sigma \partial_z \delta\mu_A = 0,$$
$$\partial_t \delta\mu_A + \sigma \partial_z \delta\mu_V = 0, \tag{48}$$

  with $\sigma = \mathrm{sgn}(qB)$.
- Combining the above equations leads to

$$\partial_t^2 \delta\mu_{V/A} - \partial_z^2 \delta\mu_{V/A} = 0. \tag{49}$$

- The above eq. describes the CMW, propagating w. the speed of light, $c_{\rm CMW} = 1$.
- **Question:** will the relations

$$\sigma_H^B \simeq \frac{q \ln 2}{\beta\pi^2}, \qquad \sigma_\epsilon^B \simeq \frac{q\mu_H \ln 2}{\beta\pi^2} \tag{50}$$

  lead to a new excitation, **the helical heat wave (HHW)?**

Small fluctuations over LLL

- Consider a small perturbation propagating along B^μ in a quiescent, neutral fluid.
- Splitting observables as $a = \bar{a} + \delta a$ into background $\bar{a}$ and fluctuations δa, we have

$$\begin{aligned}
\overline{Q}_\ell &= 0, & \bar{\sigma}^B_{V/A} &= 0, & \bar{\sigma}^B_H &= \frac{q\overline{T}\ln 2}{\pi^2}, & \bar{\sigma}^B_\epsilon &= 0, \\
\delta Q_\ell &= \frac{|qB|}{2\pi^2}\delta\mu_\ell, & \delta\sigma^B_{V/A} &= \frac{q\delta\mu_{A/V}}{2\pi^2}, & \delta\sigma^B_H &= \frac{q\ln 2}{\pi^2}\delta T, & \delta\sigma^B_\epsilon &= \frac{q\overline{T}\ln 2}{\pi^2}\delta\mu_H.
\end{aligned} \tag{51}$$

- Similarly, $\overline{B}^\mu = B\delta^\mu_z$ and $\delta B^\mu = B\delta u^z \delta^\mu_z$ for $u^\mu = \delta^\mu_0 + \delta u^\mu$.
- The fluctuations evolve according to the hydro equations:

$$\begin{aligned}
\text{Charge cons.:} &\quad \dot{Q}_\ell + Q_\ell\theta + \partial_\mu j^\mu_\ell = 0, \\
\text{Energy cons.:} &\quad \dot{\epsilon} + (\epsilon + P + \Pi)\theta - \pi^{\mu\nu}\sigma_{\mu\nu} + \partial_\mu j^\mu_\epsilon - j^\nu_\epsilon \dot{u}_\nu = 0, \\
\text{Mom. cons.:} &\quad (\epsilon + P + \Pi)\dot{u}^\mu - \nabla^\mu(P + \Pi) + \Delta^\mu_\lambda \partial_\nu \pi^{\nu\lambda} \\
&\quad + j^\lambda_\epsilon \nabla_\lambda u^\mu + D j^{\langle\mu\rangle}_\epsilon + j^\mu_\epsilon \theta = 0,
\end{aligned} \tag{52}$$

where $j^\mu_\ell = \sigma^B_\ell B^\mu$ and $j^\mu_\epsilon = \sigma^B_\epsilon B^\mu$, while the red terms are quadratic in fluctuations and vanish.

Fourier modes

- We now take the Fourier modes $\delta a = \int d^3k \widetilde{\delta a}_{\mathbf{k}}(\omega) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$.
- This leads to $\mathbb{M}\mathbb{U} = 0$ with

$$\mathbb{M} = \begin{pmatrix} \omega & -\sigma k^z & 0 & 0 & 0 & 0 \\ -\sigma k^z & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma\omega}{2\ln 2} & -k^z & \omega & 0 \\ 0 & 0 & -\frac{6\ln 2}{\pi^2}\sigma k^z & \omega & -k^z & -\frac{1}{2}\mathbf{k}_\perp \\ 0 & 0 & \frac{6\ln 2}{\pi^2}\sigma\omega & -k^z & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega\mathbb{I} \end{pmatrix}, \quad \mathbb{U} = \begin{pmatrix} \widetilde{\delta\mu}_V \\ \widetilde{\delta\mu}_A \\ \widetilde{\delta\mu}_H \\ \widetilde{\delta T} \\ \widetilde{\delta u}^z \\ \widetilde{\delta \mathbf{u}}_\perp \end{pmatrix} = 0. \quad (53)$$

- The nontrivial solutions correspond to $\det\mathbb{M} = 0$:

$$(\omega^2 - k_z^2) \times (\omega^2 - k_z^2) \times \left(\frac{\sigma}{2\ln 2} - \frac{6\ln 2}{\pi^2}\sigma\right)\omega \times \omega^2. \quad (54)$$

- The Chiral Magnetic Wave (CMW) and sound waves propagate along the magnetic field with the speed of light ($\omega = \pm k^z$).
- The transverse modes $\widetilde{\delta\mathbf{u}}_\perp$ and the helical mode $\widetilde{\delta\mu}_H$ do not propagate ($\omega = 0$).

Conclusion

- Fermions in a background magnetic field exhibit the *helical separation effect* (HSE).
- CSE: $\mathbf{J}_A$ generated at finite μ_V, even when $\mu_A = \mu_H = 0$.
- HSE: $\mathbf{J}_H$ generated at finite T, even when $\mu_V = \mu_A = \mu_H = 0$.
- J_H^μ is conserved even for massive fermions.
- Fun fact: Thermodynamic analysis reveals non-ideal LLL quantum corrections: $\pi_B = \Pi = -2P/3$, when $P = \epsilon$.
- Helical fluctuations $\delta\mu_H$ do not propagate as perturbations on the LLL.
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