A novel method for calculating Bose-Einstein correlation functions with Coulomb final-state interaction

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Also see Aletta Purzsa's poster

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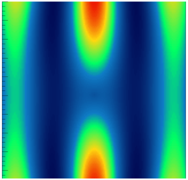


Outline

- Introduction
  - HBT correlations, Coulomb effect, basic formulas
  - Need for refinement: non-Gaussian sources, precision measurements
  - Lévy sources in heavy ion collisions
- New method for treatment of Coulomb interaction
  - Numerical & methodological motivation
  - Calculation of the Coulomb integral kernel
  - Rigorous mathematics needed
  - Spherically symmetric case: limiting functional expressed
  - Implementation; esp. for Lévy-type sources
- Outlook
  - Ready to use in experimental analyses
  - Generalizations: non-spherically symmetric case, strong interaction

Introduction

- Bose-Einstein-correlations of like particles ($\pi^+\pi^+$, $\pi^-\pi^-$, $K^+K^+\dots$): measure fm-scale space-time extent of particle emitting source
- Some definitions:

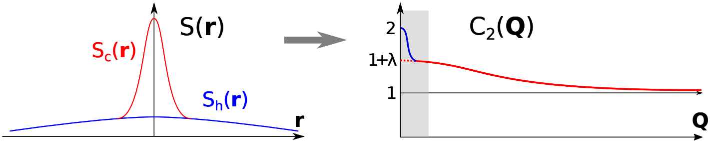
source function:	$S(x, \mathbf{p})$
single part. distr.:	$N_1(\mathbf{p}) = \int \mathrm{d}x\, S(x, p)$
pair wave function:	$\psi^{(2)}(x_1, x_2)$
pair mom. distr.:	$N_2(\mathbf{p}_1, \mathbf{p}_2) = \int \mathrm{d}x_1 dx_2\, S(x_1, p_1) S(x_2, p_2) \left\|\psi^{(2)}(x_1, x_2)\right\|^2$
corr. function:	$C(\mathbf{p}_1, \mathbf{p}_2) = \frac{N_2(\mathbf{p}_1, \mathbf{p}_2)}{N_1(\mathbf{p}_1) N_1(\mathbf{p}_2)}$
pair source:	$D(\mathbf{r}, \mathbf{K}) = \int \mathrm{d}^4\rho\, S(\rho + \frac{r}{2}, \mathbf{K}) S(\rho - \frac{r}{2}, \mathbf{K})$

- Approximately thus

$$C(\mathbf{k}, \mathbf{K}) = \frac{\int D(\mathbf{r}, \mathbf{K}) |\psi_{\mathbf{k}}(\mathbf{r})|^2 \mathrm{d}\mathbf{r}}{\int D(\mathbf{r}, \mathbf{K}) \mathrm{d}\mathbf{r}}, \qquad \mathbf{K} := \frac{\mathbf{p}_1 + \mathbf{p}_2}{2}, \quad \mathbf{k} := \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}.$$

Introduction

- Core-halo model for intercept parameter λ (Csörgő, Lörstad, Zimányi, *Z. Phys. C* 71, 491 (1996))



λ measures core fraction: $S = \sqrt{\lambda} S_c + (1-\sqrt{\lambda}) S_h \quad \Rightarrow \quad$ Bowler-Sinyukov formula:

$$\text{for ,,large'' } S_h,\ C(\mathbf{k},\mathbf{K}) = 1-\lambda+\lambda\frac{\int D_c(\mathbf{r},\mathbf{K})|\psi_\mathbf{k}(\mathbf{r})|^2 \mathrm{d}\mathbf{r}}{\int D_c(\mathbf{r},\mathbf{K})\mathrm{d}\mathbf{r}}.$$

- No final state interactions: $C(\mathbf{k}) \equiv C^{(0)}(\mathbf{k})$, Fourier transform of source

$$\left|\psi^{(0)}_\mathbf{k}(\mathbf{r})\right|^2 = 1+\cos(2\mathbf{kr}) \quad \Rightarrow \quad C^{(0)}(\mathbf{k}) = 1+\lambda\frac{\int D_c(\mathbf{r},\mathbf{K})\cos(2\mathbf{kr})\mathrm{d}\mathbf{r}}{\int D_c(\mathbf{r},\mathbf{K})\mathrm{d}\mathbf{r}}.$$

- Final state Coulomb interaction: $\psi^{(0)}$ replaced by solution of two-body Coulomb Schr. eq.; NR case: well known formulas (see below)

$$C^{(0)}(\mathbf{k}) = \frac{C(\mathbf{k})}{K(\mathbf{k})}, \quad K(\mathbf{k}) \equiv \frac{\int D_c(\mathbf{r})|\psi_\mathbf{k}(\mathbf{r})|^2\mathrm{d}\mathbf{r}}{\int D_c(\mathbf{r})|\psi^{(0)}_\mathbf{k}(\mathbf{r})|^2\mathrm{d}\mathbf{r}} \quad \text{Coulomb correction}$$

Source types; Lévy source functions

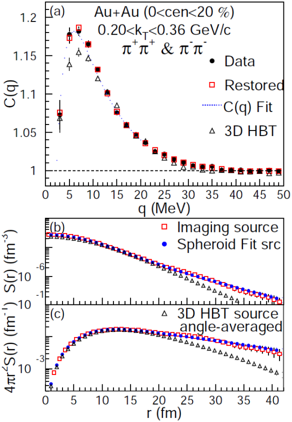
- Gaussian: usual choice; $D_{cc}(\mathbf{r}) \propto \exp\left(-r_k r_l (\mathbf{R}^2)^{-1}_{kl}\right)$.
  - Fit parameters: $R_{kl}(\mathbf{K})$ and $\lambda(\mathbf{K})$
  - A generalization: Edgeworth expansion of $C(k)$; in thisource: FT of $C^{(0)}(k)$
    see eg. Csörgő, Hegyi, *PLB* 489, 15 (2000)
- Cauchy sources ⇔ exponential $C(k)$; employed at CMS in pp collisions
- Lévy-type sources (Csörgő, Hegyi, Zajc, *EPJ C* 36, 67 (2004)): generalized Gaussian
  new parameter: $\alpha \in \mathbb{R}^+$ *stability index*; $\alpha \leq 2$. Expression with a Fourier transform:

$$D_{cc}(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, e^{i\mathbf{qr}} \exp\left(-|\mathbf{q}R|^\alpha\right) \quad \Leftrightarrow \quad C_2^{(0)}(\mathbf{Q}) = 1 + \lambda \exp\left(-|QR|^\alpha\right).$$

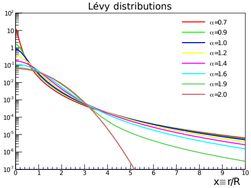
  - Generalization: Levy polynomials (same as Edgeworth for Gaussians)
    Novák et al., *Acta Phys. Polon. Supp.* 9, 289 (2016)
  - $\alpha = 2$: Gaussian; $\alpha = 1$: Cauchy distribution; $\alpha \neq 1, 2$: $D_{cc}(\mathbf{r})$ itself only numerically
- Possible reasons for of Lévy sources: all rest on *stability* (just as for Gaussian)
  - Fractal structure of jet fragmentation (Csörgő, Hegyi, Novák, Zajc, *Acta Phys. Polon.* B 36, 329 (2005))
  - Anomalous diffusion (Csanád, Csörgő, MN, *Braz. J. Phys.* 37, 1002 (2007))
  - Critical phenomena, closeness of CEP (Csörgő, Hegyi, Novák, Zajc, *AIP Conf. Proc.* 828, 525 (2006))
  - Event & directional averaging (Cimerman, Plumberg, Tomasik, *Phys. Part. Nucl.* 51, 282 (2020))
    However: event-by-event Lévy shape in EPOS simulation (Kincses, Stefaniak, Csanád, *Entropy* 24, 308 (2022))

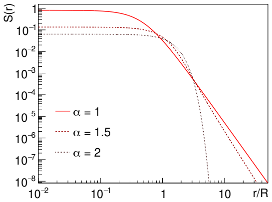
Lévy sources in heavy ion collisions

- Non-Gaussian behavior:
  - Source extraction („imaging”)
    Brown, Danielewicz, *PLB* 398, 252 (1997)
  - PHENIX, PRL 98 (2007) 132301



- ⇒ experimental motivation for Lévy sources
  - For $\alpha \neq 2$, power law like $r \to \infty$ decrease ($\sim r^{-3-\alpha}$); no finite variance

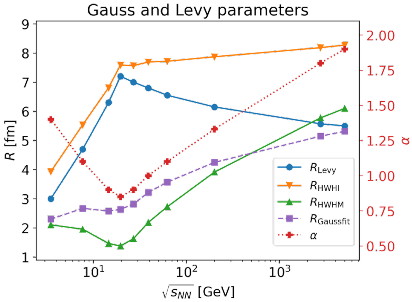




- Meaning of Lévy R: through FWHM, FWHI (full width at half maximum/integral)

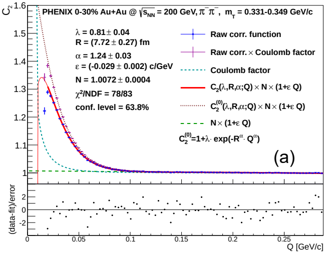
Lévy sources in heavy ion collisions

- Need for precision investigations. . .
- Illustration (courtesy of M. Csanád, see talk at ISMD2023): ,,simulating" Lévy $C(Q)$ with toy R and α, fit with Gaussian



- Interplay of α and R can hide interesting details, eg. non-monotonicity w.r.t. $\sqrt{s_{NN}}$, when fitting w/ Gaussians

- First Lévy HBT in heavy-ion collisions (PHENIX, *PRC* 97, 064911 (2018))



- $\alpha \neq 2$ confirmed m_t-independently
- Since then, done at several experiments
  - STAR (talk by D. Kincses)
  - CMS (talk by M. Csanád)
  - NA61 (talk by B. Pórfy')

Coulomb interaction

- *An essential ingredient for precision HBT measurements*
- Non-relativistic treatment: valid in Pair Co-Moving System (PCMS).
- $\mathbf{p}=\hbar\mathbf{k}$: relative momentum, $E=\frac{p^2}{2m}$, m: reduced mass
- Sommerfeld parameter (Coulomb parameter) η: ratio of classical closest distance $r_0\equiv\frac{q_e^2}{4\pi\varepsilon_0}\frac{1}{E}$ to wavelength $\lambda\equiv\frac{2\pi\hbar}{p}$:

$$\eta\equiv\alpha_{\rm em}\frac{mc}{\hbar k}=\frac{\pi r_0}{\lambda},\quad \text{with}\quad \alpha_{\rm em}\equiv\frac{q_e^2}{4\pi\varepsilon_0}\frac{1}{\hbar c}\approx\frac{1}{137}.$$

- Two-particle wave function: symmetrized scattering ,,out" state
  - ,,out" states asymptotically plane wave + *incoming* spherical wave
  - alternate ,,in" state (plane wave + outgoing spherical wave) yields same results

$$\psi^{(C)}=e^{i\mathbf{KR}}\times\frac{\mathcal{N}^*}{\sqrt{2}}e^{-ikr}\Big\{M\big(1-i\eta,1,i(kr+\mathbf{kr})\big)+(\mathbf{k}\leftrightarrow-\mathbf{k})\Big\}.$$

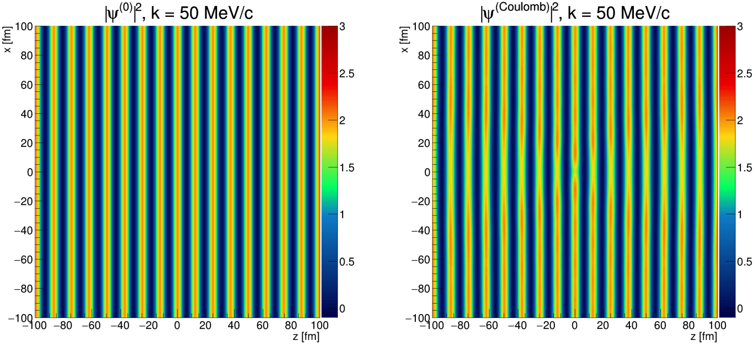
Making use of the $M(a,b,z)$ confluent hypergeometric function

- Normalization ($\mathcal{N}$) and Gamow factor ($|\mathcal{N}|^2$):

$$\mathcal{N}=e^{-\pi\eta/2}\Gamma(1+i\eta),\qquad |\mathcal{N}|^2=e^{-\pi\eta}|\Gamma(1+i\eta)|^2=\frac{2\pi\eta}{e^{2\pi\eta}-1}.$$

Coulomb interaction

- Coulomb wave function: distorted plane wave, asymptotically logarithmic corrections



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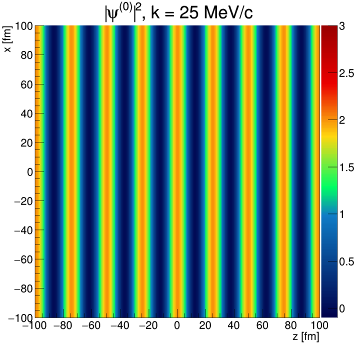

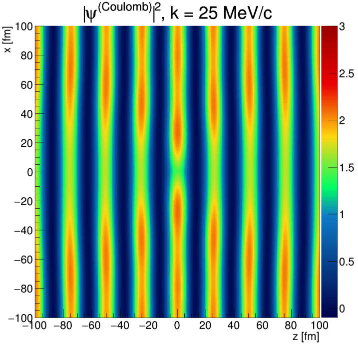

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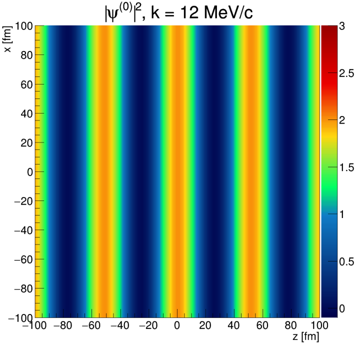

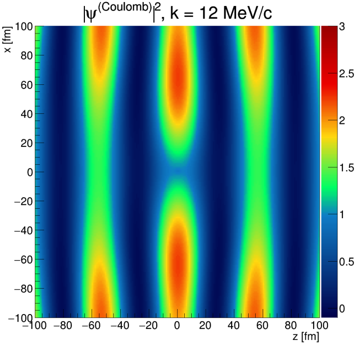

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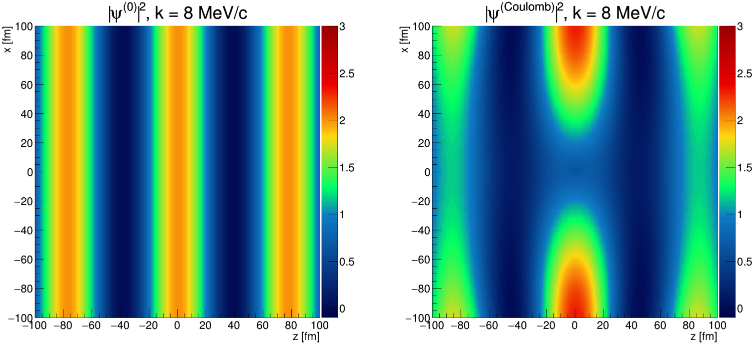

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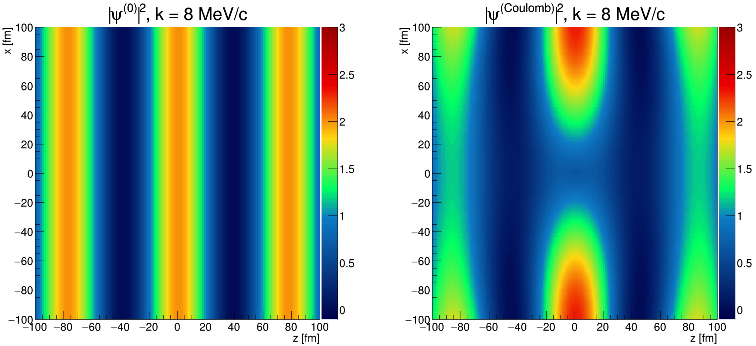

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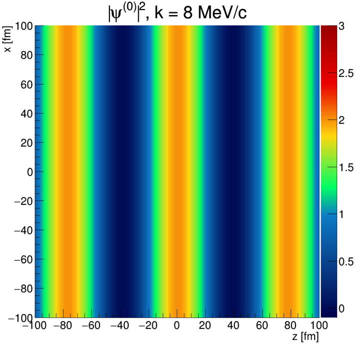

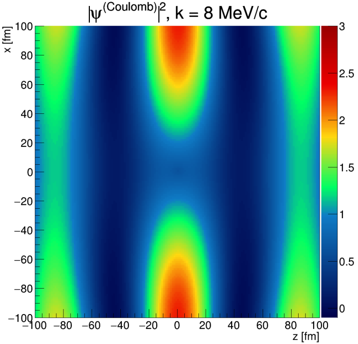

- Gamow correction captures only the value at the origin
- Calculational methods:
  - Direct integrating $D(\mathbf{r})|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2$ during fit: time-consuming, even nowadays
  - Pre-calculate a ,,Coulomb correction" with fix parameters (say, $R = 5\,\mathrm{fm}$ Gaussian): fast but inconsistent
  - Use iterative method, use memory lookup table...

New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r})\dots$
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
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New method needed

- Many (if not all) interesting source types defined as Fourier transforms

$$D(\mathbf{r}) := \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

- In many cases (eg. Lévy sources), even this is possible only numerically
- Direct numerical calculation of $C_2(\mathbf{Q})$ thus (although used) **very** problematic
  - Slow decrease of $D(\mathbf{r})$, oscillating asymptotic $\psi_{\mathbf{k}}^{(2)}(\mathbf{r})$...
  - Awkward: Fourier transform, then ,,almost inverse'' Fourier transform, *numerically*...
- Natural idea: ,,interchange order of ,,integrals''

$$C_2(\mathbf{Q}) = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{q}\, f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \overset{??}{=}$$

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- Not working in this form

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  - Workaround: regularization, $\lambda \in \mathbb{R}^+$, then $\lambda \to 0$.

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- Not working in this form: Fourier transform $\neq$ integral (Lebesgue)
  - Workaround: regularization, $\lambda{\in}\mathbb{R}^+$, then $\lambda{\to}0$.
  - Careful math needed (once in a physicist's lifetime...)

Details of derivation

- Interchanging our integrals in a careful way:

$$C_2(\mathbf{Q})$$

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$$
\begin{aligned}
\underline{\underline{C_2(\mathbf{Q})}} &= \int \mathrm{d}^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 D(\mathbf{r}) = \int \mathrm{d}^3\mathbf{r} \lim_{\lambda\to 0} e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, D(\mathbf{r}) = \\
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&= \lim_{\lambda\to 0} \int \mathrm{d}^3\mathbf{r} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q})\, e^{-\lambda r} |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2\, e^{i\mathbf{q}\mathbf{r}} = \\
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\end{aligned}
$$

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  - Conditions of (Lebesgue) integrability

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- ,,Ingredients":
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  - *Fubini's theorem* (for interchanging repeated integrals)

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- In last step, *cannot* interchange $\int \mathrm{d}^3\mathbf{q}$ and $\lim_{\lambda\to 0}$.
- As of now, continuing only in the spherically symmetric case: $f(\mathbf{q})\equiv f_s(q)$, $D_{cc}(r)=2\pi\int_0^\infty \mathrm{d}q\,q^2\sin(qr)f_s(q)$.

Details of derivation (cont'd)

- After substituting $\psi^{(2)}_{\mathbf{k}}(\mathbf{r})$, „master" formula thus reads as

$$C_2(Q)=\frac{|\mathcal{N}|^2}{2\pi^2}\lim_{\lambda\to 0}\int_0^\infty \mathrm{d}q\, q^2 f_s(q)\Big[\mathcal{D}_{1\lambda s}(q)+\mathcal{D}_{2\lambda s}(q)\Big], \qquad \text{where}$$

$$\mathcal{D}_{1\lambda s}(q)=\int \mathrm{d}^3\mathbf{r}\,\frac{\sin(qr)}{qr}e^{-\lambda r}M\big(1+i\eta,1,-i(kr+\mathbf{kr})\big)M\big(1-i\eta,1,i(kr+\mathbf{kr})\big),$$

$$\mathcal{D}_{2\lambda s}(q)=\int \mathrm{d}^3\mathbf{r}\,\frac{\sin(qr)}{qr}e^{-\lambda r}M\big(1+i\eta,1,-i(kr-\mathbf{kr})\big)M\big(1-i\eta,1,i(kr+\mathbf{kr})\big).$$

- These can be calculated (using complex analysis; method pioneered by Nordsieck in the theory of bremsstrahlung & pair creation)

A. Nordsieck, *Phys. Rev.* 93, 785 (1954).

$$\mathcal{D}_{1\lambda s}(q)=\frac{4\pi}{q}\mathrm{Im}\left[\frac{1}{(\lambda-iq)^2}\left(1+\tfrac{2k}{q+i\lambda}\right)^{2i\eta}\mathcal{F}_+\left(\tfrac{4k^2}{(q+i\lambda)^2}\right)\right],$$

$$\mathcal{D}_{2\lambda s}(q)=\frac{4\pi}{q}\mathrm{Im}\left[\frac{(\lambda-iq-2ik)^{i\eta}(\lambda-iq+2ik)^{-i\eta}}{(\lambda-iq)^2+4k^2}\right],$$

Here $\mathcal{F}_+(x)\equiv {}_2F_1(i\eta,1+i\eta,1,x)$ is the hypergeometric function

- For $\lim_{\lambda\to 0}$, function forms of $\mathcal{D}_{1\lambda s}$, $\mathcal{D}_{2\lambda s}$ become „ill-behaved"
- Need to calculate & simplify $\lambda\to 0$ limit (numerical limit-taking... $\Rightarrow$ ↯)

The main result

- *A remark:* approximation of $\delta(x)$ Dirac delta: a well known simpler similar case

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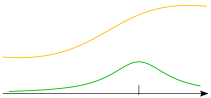

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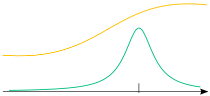

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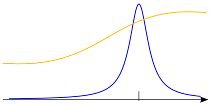

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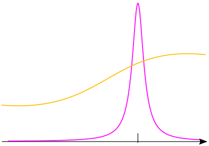

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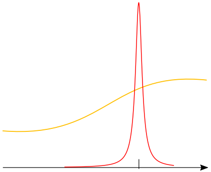

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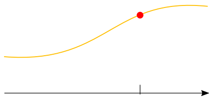

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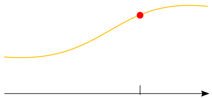

- In our case, result of the calculation is

$$C_2(Q)=|\mathcal{N}|^2\left(1+f_s(2k)+\frac{\eta}{\pi}\left[\mathcal{A}_{1s}+\mathcal{A}_{2s}\right]\right),\quad \text{where}$$

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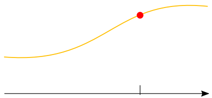

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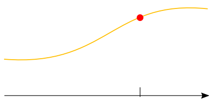

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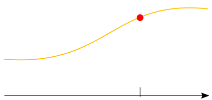

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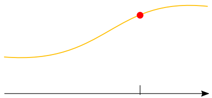

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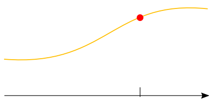

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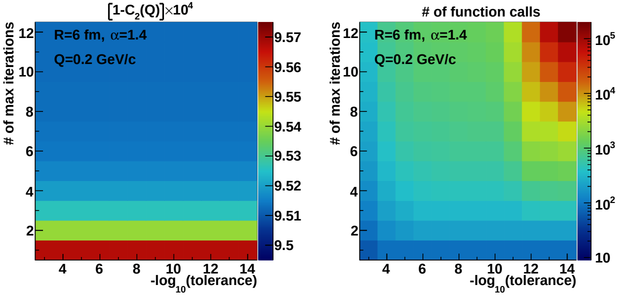
  - Care needed about branch cuts ($\pm i0$ terms) of $\mathcal{F}_+(x)$ and complex powers

Numerical implementation

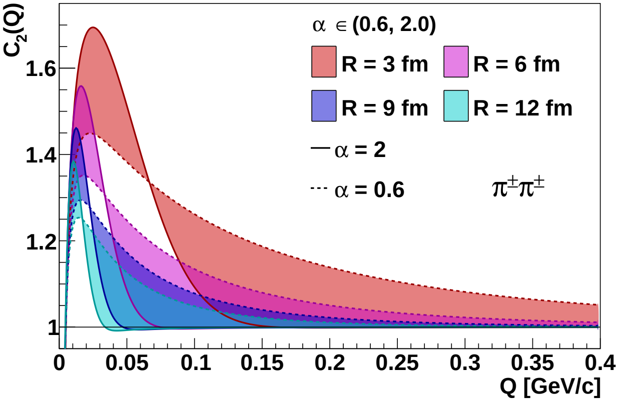
- Final numerical integrals needed: in $\mathcal{A}_{1\lambda_s}$ and $\mathcal{A}_{2\lambda_s}$
- Transform integral to $x \in [0,1] \Rightarrow$ smooth, ,,beautiful" integrands
- Gauss-Krohnrod quadrature (from `C++ boost` library) used:
  - Main parameters: # of max iterations & tolerance
  - Investigated; optimal value found:
    few hundred integrand evaluations (instead of many 10000-s)



- Real-time calculation (during fit procedure) possible!
- Codes archived at: `github.com/csanadm/CoulCorrLevyIntegral`

Example calculations: illustrations

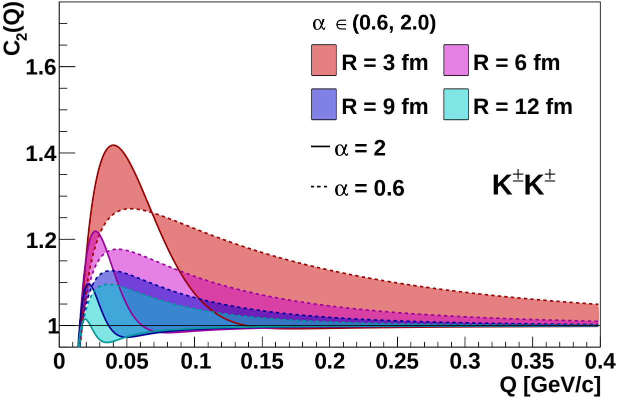
- For Lévy sources, for pion ($\pi^+\pi^+$, $\pi^-\pi^-$) pairs:



- most frequent target of HBT measurements
- Shaded region ,,swept" over by $C_2(Q)$ as α changes
- Apparent ,,nodes" disappear with increased zooming in

Example calculations: illustrations

- For Lévy sources, for kaon (K^+K^+, K^-K^-) pairs:



- Similarly to the case of pions; ,,nodes" are only apparent
- Coulomb effect stronger ($m_K > m_\pi$; η increases)
- Considerable interplay of experimentally measurable λ, R, α

Summary and outlook

- Efficient new method for Coulomb interacting HBT correlation function calculation
  - Calculations directly in momentum (Fourier) space
  - Distribution theory motivated careful mathematical methods invoked
  - Cross-checked with previous direct calculations
  - Numerical implementation done, ready for use in data analysis
- As of now, implementation only for spherically symmetric sources
  - Prospective generalization I: go beyond spherical symmetry
    This is where efficiency becomes crucial...
  - Prospective generalization II: short-range final state strong interactions
    Usual treatment: only s-wave (1 parameter: strong scattering length f_0)
    For a direct calculation for Lévy sources, see: Kincses, MN, Csanád, *PRC* 102, 064912 (2020)
  - Prospective generalization (in fact, simplification) for non-identical particle correlations: only $\mathcal{D}_{1\lambda_s}$ (ie. $\mathcal{A}_{1s}$) term needed

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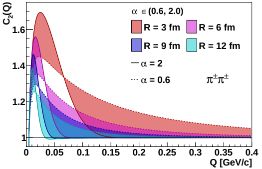
New exact analytic formulas for QM Coulomb problem! ☺



Thank you for your attention!