Quantum effects in the early stages of the expansion

Outline

- Introduction: hydrodynamics from relativistic kinematics in extreme conditions
- How large are the quantum corrections to the classical free streaming
- Generalized hydrodynamic expansion through regularized moments

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10.1103/PhysRevD.108.076022
10.1103/PhysRevD.108.036015

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What's wrong with the relativistic kinetic theory?

Probability density ρ in a 6N+1 dimensional space

$$\rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N; \boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_N; t)$$

integration over all particles, but one

$$f(\boldsymbol{x}, \boldsymbol{p}; t) = \sum_i \int \prod_{j \neq i} \frac{d^3 x_j d^3 p_j}{(2\pi\hbar)^3} \rho$$

normalized to N (by construction)

$$N = \int \frac{d^3 x d^3 p}{(2\pi\hbar)^3} f(\boldsymbol{x}, \boldsymbol{p}; t)$$

The distribution function $f(\boldsymbol{x}, \boldsymbol{p}; t)$

Then molecular chaos, and further approximations

$$\hbar c \simeq 200 \text{ MeV} \cdot \text{fm}$$

What's wrong with the relativistic kinetic theory?

The relativistic Boltzmann equation

$$p \cdot \partial f(x, \mathbf{p}) = -C[f, \bar{f}]$$
$$p \cdot \partial \bar{f}(x, \mathbf{p}) = -\bar{C}[f, \bar{f}]$$

Kinetic only contributions to $T^{\mu\nu}$ and to J^{μ}

$$T^{\mu\nu}(x) = \frac{g_S}{(2\pi\hbar)^3} \int \frac{d^3p}{E_{\mathbf{p}}} \; p^{\mu} p^{\nu} \left(f(x, \mathbf{p}) + \bar{f}(x, \mathbf{p}) \right)$$
$$J_B^{\mu}(x) = \frac{g_S}{(2\pi\hbar)^3} \int \frac{d^3p}{E_{\mathbf{p}}} \; p^{\mu} \left(f(x, \mathbf{p}) - \bar{f}(x, \mathbf{p}) \right)$$

It neglects spin, asymptotically small interactions only!

What if we could use it? (in heavy-ion collisions)

Relativistic Boltzmann equation

$$p \cdot \partial f = -\mathcal{C}[f]$$

$$\Rightarrow \int_{\boldsymbol{p}} p^{\nu} p \cdot \partial f = -\int_{\boldsymbol{p}} p^{\nu} C = 0$$

$$\partial_\mu T^{\mu\nu}$$

$$u \cdot \partial f = \dot{f} = -\frac{p \cdot \nabla f}{(p \cdot u)} - \frac{C[f]}{(p \cdot u)}$$

extra needed equations

$$\dot{T}^{\mu\nu} = \int_{\boldsymbol{p}} p^{\mu} p^{\nu} \dot{f}$$

covariant momentum integral

$$\int_{\mathbf{p}} = \int d^4p \, 2\Theta(p_0)\delta(p^2 - m^2)$$

G Denicol, J.Phys. G41 (2014) no.12, 124004

What if we could use it? (in heavy-ion collisions)

$$p \cdot \partial f = -\mathcal{C}[f]$$

convenient basis

$$\mathcal{F}_r^{\mu_1 \cdots \mu_s} = \int_{\mathbf{p}} (p \cdot u)^r \, p^{\mu_1} \cdots p^{\mu_s} f$$

$$\dot{\mathcal{F}}_r^{\mu_1 \cdots \mu_s} + C_{r-1}^{\mu_1 \cdots \mu_s} = r \, \dot{u}_\alpha \, \mathcal{F}_{r-1}^{\alpha \mu_1 \cdots \mu_s} - \nabla_\alpha \mathcal{F}_{r-1}^{\alpha \mu_1 \cdots \mu_s} + (r-1) \, \nabla_\alpha u_\beta \, \mathcal{F}_{r-2}^{\alpha \beta \mu_1 \cdots \mu_s}$$

special case: $T^{\mu\nu}$ (r=0,s=2)

$$\dot{T}^{\mu\nu} + C_{-1}^{\mu\nu} = -\nabla_\alpha \mathcal{F}_{-1}^{\alpha\mu\nu} - \nabla_\alpha u_\beta \, \mathcal{F}_{-2}^{\alpha\beta\mu\nu}$$

in particular

$$u_\nu \dot{T}^{\mu\nu} + C_0^\nu = -u_\nu \left(\nabla_\alpha \mathcal{F}_{-1}^{\alpha\mu\nu} + \nabla_\alpha u_\beta \mathcal{F}_{-2}^{\alpha\beta\mu\nu} \right) \Rightarrow \partial_\mu T^{\mu\nu} = 0$$

What if we could use it? (in heavy-ion collisions)

$\mathcal{O}^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} = \Delta^{\mu_1}_{\alpha_1}\cdots\Delta^{\mu_l}_{\alpha_l}\mathcal{O}^{\alpha_1\cdots\alpha_l}$ even more convenient basis $\mathfrak{f}^{\mu_1\cdots\mu_l}_r = \int_{\boldsymbol{p}} (p\cdot u)^r\, p^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} f$

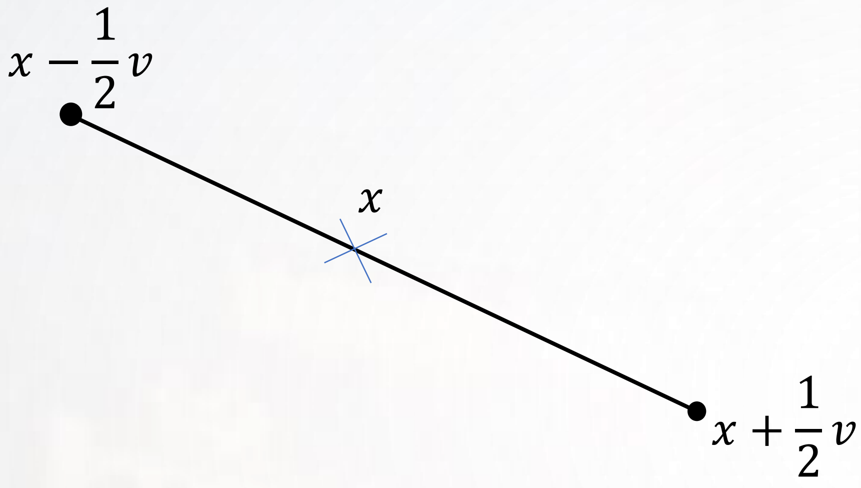
a popular decomposition of the degrees of freedom

$$\partial_\mu u_\nu = u_\mu \dot{u}_\nu + \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta\Delta_{\mu\nu}, \qquad T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}^{\mu\nu} = \mathcal{E}u^\mu u^\nu - (\mathcal{P}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

lots of self interactions in the exact evolution

$$\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} + C^{\langle\mu\rangle\langle\nu\rangle}_{-1} = 2(\mathcal{P}+\Pi)\sigma^{\mu\nu} + \frac{5}{3}\theta(\mathcal{P}+\Pi)\Delta^{\mu\nu} - \frac{5}{3}\theta\pi^{\mu\nu} - 2\pi^{(\mu}_{\alpha}\sigma^{\nu)\alpha} + 2\pi^{(\mu}_{\alpha}\omega^{\nu)\alpha}$$
$$-\nabla_\alpha \mathfrak{f}^{\alpha\langle\mu\rangle\langle\nu\rangle}_{-1} - \left(\sigma_{\alpha\beta} + \frac{1}{3}\theta\Delta_{\alpha\beta}\right)\mathfrak{f}^{\alpha\beta\mu\nu}_{-2}$$

The link between quantum fields and relativistic kinetic theory



$$W(x,k) \propto \int \frac{d^4v}{(2\pi)^4} e^{-ik\cdot v} \langle \Phi^\dagger (x + 1/2v)\Phi(x - 1/2v)\rangle$$

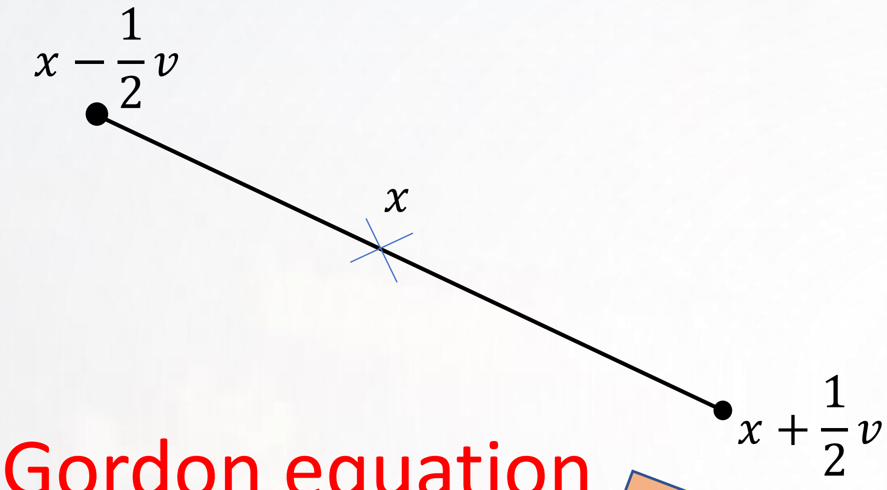
$$T^{\mu\nu}(x) = \int d^4k \;\; k^\mu k^\nu W(x,k)$$

$$W(x,k) \longrightarrow (2\pi)\delta(p^2 \; - m^2) f_{cl.}(x,p)$$

$$k \cdot \partial W(x,k) = \cdots \longrightarrow p \cdot \partial f(x,p) = \cdots$$

- Relativistic Kinetic Theory. Principles and Applications - De Groot, S.R. et al. Amsterdam, Netherlands: North-Holland (1980)

The link between quantum fields and relativistic kinetic theory



$$W(x,k) \propto \int \frac{d^4v}{(2\pi)^4} e^{-ik\cdot v} \langle \Phi^\dagger (x + 1/2v)\Phi(x - 1/2v)\rangle$$

$$T^{\mu\nu}(x) = \int d^4k \;\; k^\mu k^\nu W(x,k)$$

Klein-Gordon equation

$$\left[\frac{1}{4}\hbar^2\Box - \left(k^2 - m^2c^2\right) + i\hbar k\cdot\partial\right] W(x,k) = \cdots$$

- T. S. Biro and A. Jakovac, *Emergence of Temperature in Examples and Related Nuisances in Field Theory*, Springer Briefs in Physics (2019)

Simplest case: free streaming

Of-shell non-trivial solutions!

$$W(x,k) = \delta(k^2 - m^2) W_{\rm on}(x,k) = \delta(k^2 - m^2) \int \frac{d^4\xi}{(2\pi)^2} \, e^{ix\cdot\xi} \, \tilde{W}_{\rm on}(\xi,k)$$

$$\hbar^2 \Box W(x,k) = 4\left(k^2 - m^2c^2\right) W(x,k)$$

Incompatible constraints for the Fourier modes…

$$k \cdot \partial W(x,k) = 0$$

$$\xi^2 \, \tilde{W}_{\rm on}(\xi,k) = 0$$

$$k \cdot \xi \, \tilde{W}_{\rm on}(\xi,k) = 0$$

10.1103/PhysRevD.108.076022

also H T Elze, *J. Phys. G: Nucl. Part. Phys.* **28** 2235 (2002)

Simplest case: free streaming

Exact solutions in 1+1 dimensions

$w = zk^0 - tk^z$

$$W(t,z;k^0,k_T,k^z) = \delta(k^0)\delta(k^z)\int d\xi \left[e^{-i\left(t\sqrt{4m_T^2+\xi^2}-z\,\xi\right)}\mathcal{A}(\xi;k_T) + e^{i\left(t\sqrt{4m_T^2+\xi^2}-z\,\xi\right)}\mathcal{A}^*(\xi;k_T)\right]$$

$$+\cos\left(2w\sqrt{\frac{k^2-m^2}{(k^0)^2-(k^z)^2}}\right)\mathcal{F}_{\rm even}(k_0,k_T,k^z) + \sqrt{\frac{(k^0)^2-(k^z)^2}{k^2-m^2}}\,\sin\left(2w\sqrt{\frac{k^2-m^2}{(k^0)^2-(k^z)^2}}\right)\mathcal{F}_{\rm odd}(k_0,k_T,k^z)$$

Proper classical limit

$$T^{\mu\nu}(x) = \int d^4k\; k^\mu k^\nu W(x,k) \xrightarrow{\text{small } \hbar} \int \frac{d^3p}{(2\pi\hbar)^3 E_{\mathbf{p}}/c}\, p^\mu p^\nu \left[f(x,\mathbf{p}) + \bar{f}(x,\mathbf{p})\right],$$

$$J^\mu = \int d^4k\; k^\mu W(x,k) \xrightarrow{\text{small } \hbar} \int \frac{d^3p}{(2\pi\hbar)^3 E_{\mathbf{p}}/c}\, p^\mu \left[f(x,\mathbf{p}) - \bar{f}(x,\mathbf{p})\right].$$

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Simplest case: free streaming

Numerical results

$$\chi = 2\sqrt{\frac{k^2 - m^2}{k^2 - m_T^2}}\,,\quad A = T_0\tau_0\,,\quad \varepsilon = \frac{\hbar}{A},\qquad \tilde{w} = \frac{w}{A}$$

$$f(w, k_T) = \frac{\pi^4}{30}\,\exp\left\{-\frac{k_T^2}{2T_0^2} - \frac{w^2}{2T_0^2\tau_0^2}\right\}$$

$$\tilde{f}_{\rm even} = 2\sqrt{2\pi}\,\frac{\pi^4}{30}\,\exp\left\{-\frac{k_T^2}{2T_0^2}\frac{4}{4-\chi^2} - \frac{\chi^2}{2\varepsilon^2}\right\}$$

$$\mathcal{P}_L =$$

$$= \frac{T_0^4}{(2\pi\hbar)^3}\frac{\pi^5}{15}\left(\frac{\tau_0}{\tau}\right)^3\int_{-\infty}^{\infty} d\tilde{w}\,\sqrt{\frac{\pi}{2}}\,\exp\left\{\frac{\tilde{w}^2\tau_0^2}{2\tau^2}\right\}\,\mathrm{Erfc}\left(\frac{\tilde{w}^2\tau_0^2}{2\tau^2}\right)\,\tilde{w}^2\,\left\{\left[1 + \frac{1}{4}\frac{\partial^2}{\partial\tilde{w}^2}\right]\left(\exp\left\{\frac{-\tilde{w}^2}{2}\right\}\mathrm{Re}\left[\mathrm{Erf}\left(\frac{2 - i\varepsilon\tilde{w}}{\varepsilon\sqrt{2}}\right)\right]\right)\right\}$$

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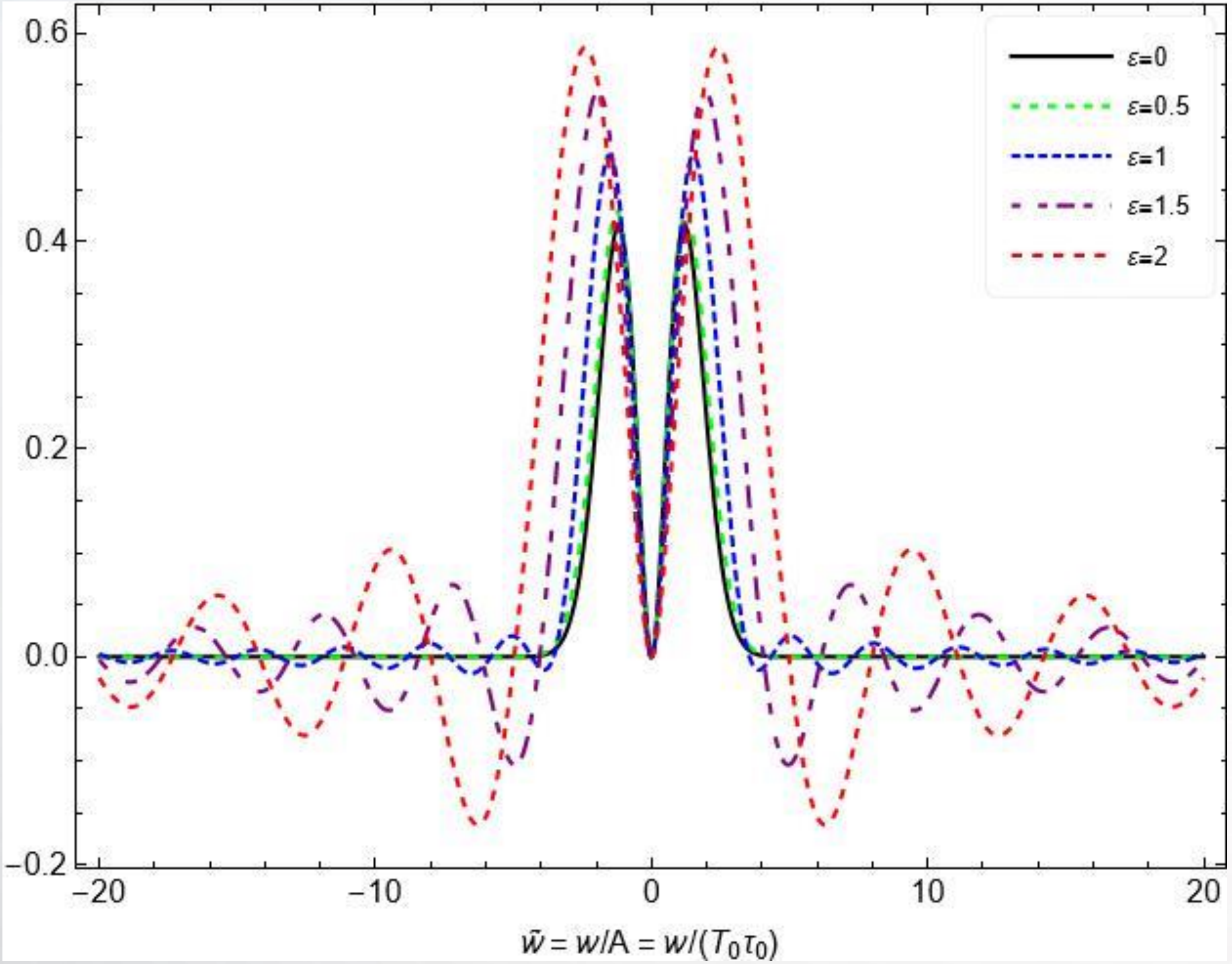
Simplest case: free streaming

Numerical results



The (non-trivial part of the) integrand of $\mathcal{P}_L$

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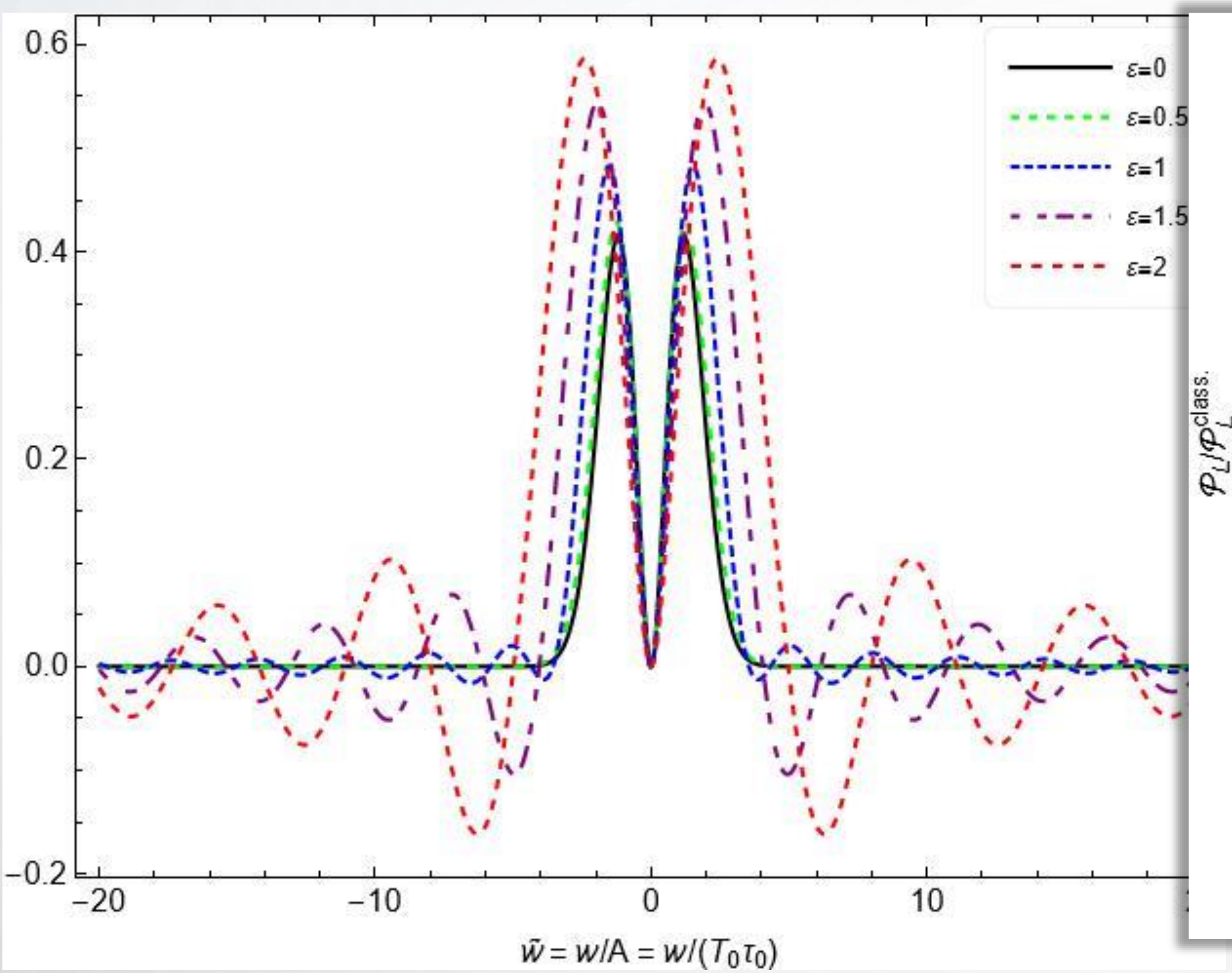
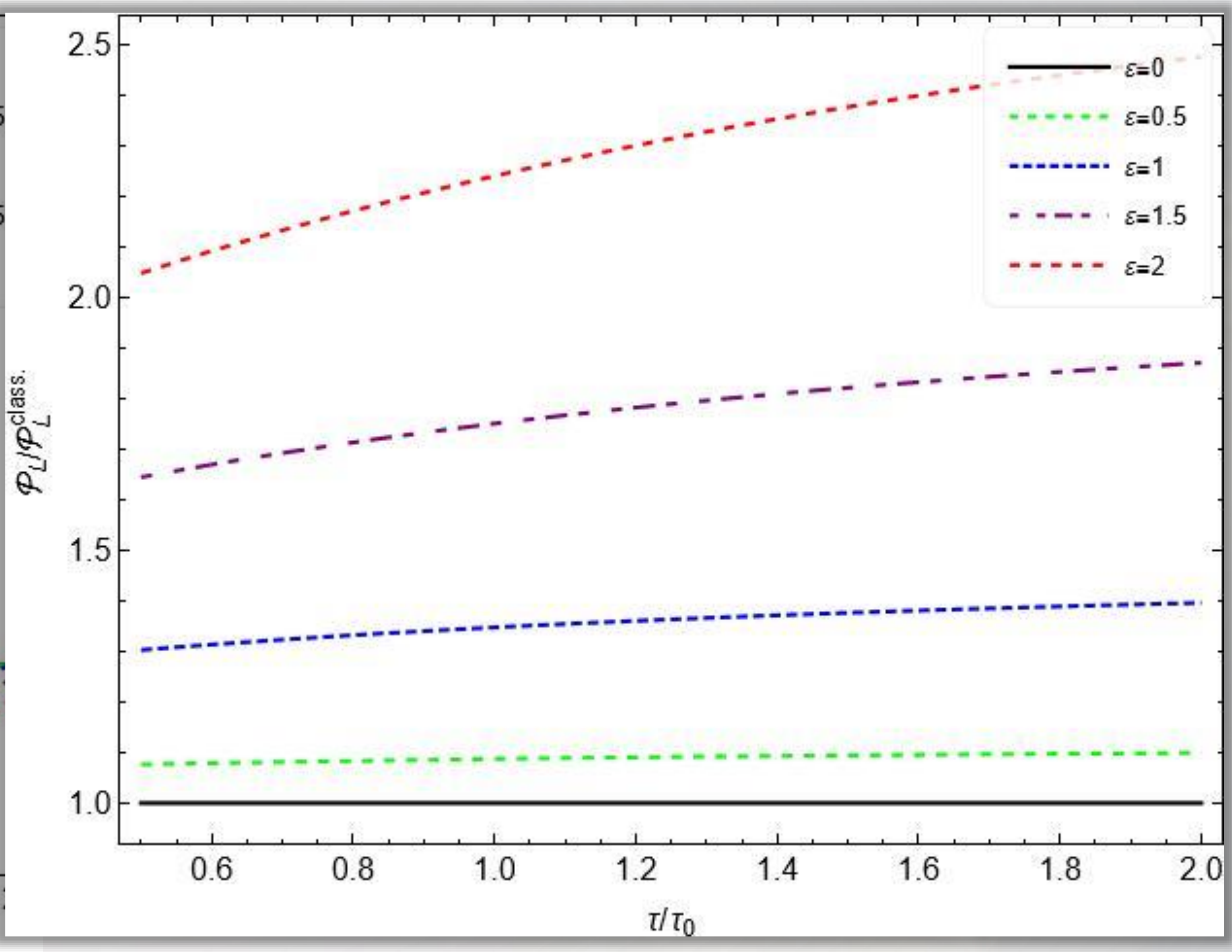
Simplest case: free streaming

Numerical results





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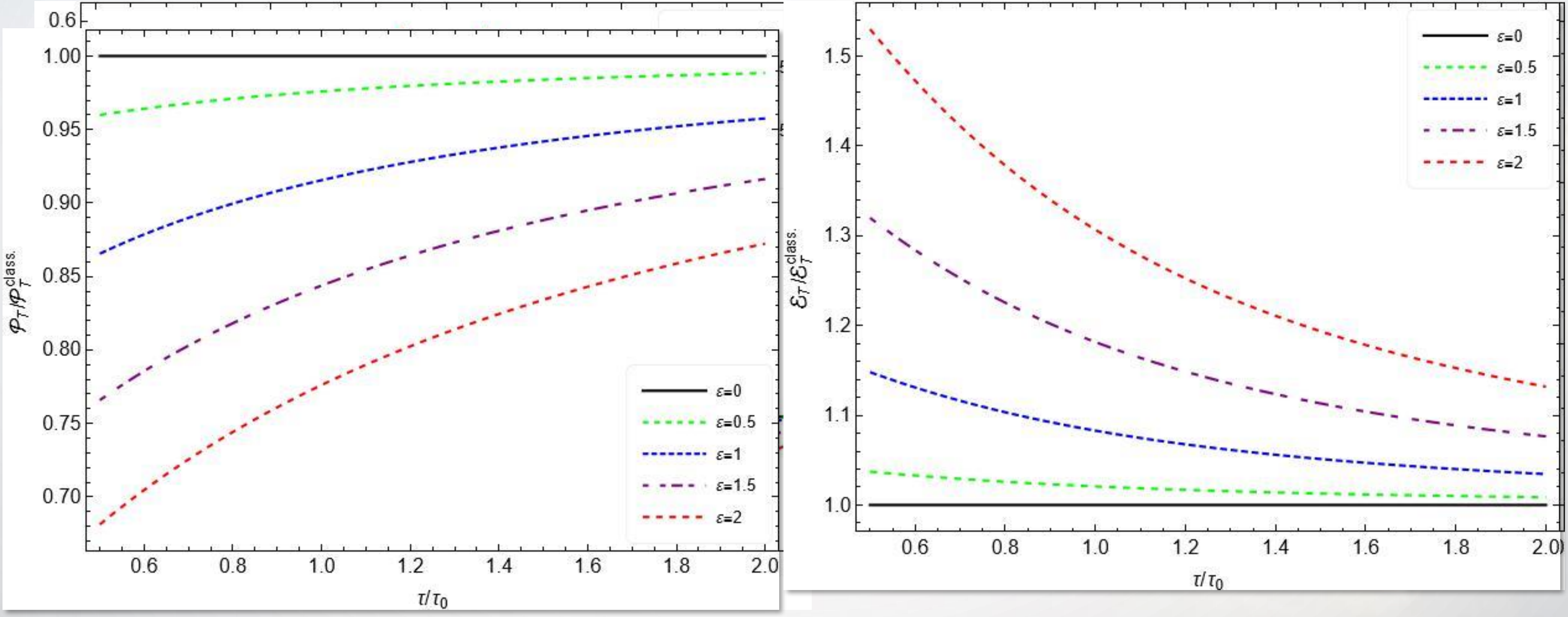
Simplest case: free streaming

Numerical results



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Problem extending to the Wigner distribution

$$\delta(p^2 - m^2) f \to W$$

$$p \cdot \partial f \to k \cdot \partial W$$

different physical situations
very similar to kinetic theory but…

…ill defined from the start

$$\dot{T}^{\mu\nu} + C^{\mu\nu}_{-1} \;=\; -\nabla_\alpha \mathcal{F}^{\alpha\mu\nu}_{-1} - \nabla_\alpha u_\beta\, \mathcal{F}^{\alpha\beta\mu\nu}_{-2}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{k^\alpha k^\mu k^\nu}{(k\cdot u)} W = \int \frac{d^4k}{(2\pi)^4} \frac{k^{\langle\alpha\rangle} k^{\langle\mu\rangle} k^{\langle\nu\rangle}}{(k\cdot u)} W + 3u^{(\alpha}\, T^{\mu\nu)} - 2\mathcal{E} u^\alpha u^\mu u^\nu$$

…similar situation for the rank four tensor…

10.1103/PhysRevD.108.036015

Resummed moments

Making use of regularized moments

$$\phi_n^{\mu_1\cdots\mu_s}(x,\zeta) = \int \frac{d^4k}{(2\pi)^4}\,(k\cdot u)^n\, e^{-\zeta(k\cdot u)^2} k^{\langle\mu_1\rangle}\dots k^{\langle\mu_s\rangle} W(x,k)$$

⇒ **well defined set of equations**

Particularly convenient, their version in the Bjorken (0+1)-d symmetric expansion, with RTA $k\cdot\partial\, W = -(k\cdot u)/\tau_R\ \delta W$

$$\hat{\mathcal{L}}\,[f] = 2\zeta\, f(\zeta) - \int_\zeta^\infty d\zeta' f(\zeta')$$

$$L_n = \phi_2^{\mu_1\cdots\mu_{2n}} z_{\mu_1}\cdots z_{\mu_{2n}}, \qquad T_n = \phi_2^{\mu_1\cdots\mu_{2n}\alpha\beta} z_{\mu_1}\cdots z_{\mu_{2n}}\, x_\alpha x_\beta$$

$$\dot{L}_n + \frac{1}{\tau_R}(L_n - L_n^{eq.}) = -\frac{2n+1}{\tau} L_n + \frac{1}{\tau}\hat{\mathcal{L}} L_{n+1}$$

$$\dot{T}_n + \frac{1}{\tau_R}(T_n - T_n^{eq.}) = -\frac{2n+1}{\tau} T_n + \frac{1}{\tau}\hat{\mathcal{L}} T_{n+1}$$

one can integrate the equations in ζ

10.1103/PhysRevD.108.036015

Hydrodynamic expansion

$$\hat{\mathcal{L}}\,[f] = 2\zeta\, f(\zeta) - \int_{\zeta}^{\infty} d\zeta' f(\zeta')$$

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}(\mathcal{P}_L - \mathcal{P}) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}(\mathcal{P}_T - \mathcal{P}) = -\frac{1}{\tau}\mathcal{P}_T + \frac{1}{\tau}\mathcal{R}_T^{(1)}$$

systematically improvable set of scalar equations…

$$\mathcal{E} = L_0(\tau, \zeta = 0)$$

$$\mathcal{P}_L = \int_{\zeta}^{\infty} d\zeta'\, L_1(\tau, \zeta')$$

$$\mathcal{P}_T = \int_{\zeta}^{\infty} d\zeta'\, T_0(\tau, \zeta')$$

…to test against the exact solutions

$$\mathcal{R}_T^{(n)} = \int_0^{\infty} d\zeta\, \left(\hat{\mathcal{L}}\right)^n T_n\,, \qquad \mathcal{R}_L^{(n)} = \int_0^{\infty} d\zeta\, \left(\hat{\mathcal{L}}\right)^n L_{n+1}$$

$$\dot{\mathcal{R}}_T^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+1}{\tau}\mathcal{R}_T^{(n)} + \frac{1}{\tau}\mathcal{R}_T^{(n+1)}$$

$$\dot{\mathcal{R}}_L^{(n)} + \frac{1}{\tau_R}\delta\mathcal{R}_T^{(n)} = -\frac{2n+3}{\tau}\mathcal{R}_L^{(n)} + \frac{1}{\tau}\mathcal{R}_L^{(n+1)}$$

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Exact solutions for the Wigner distribution

- Conformal equation of state (equilibrium), $W_{eq.} = \frac{2\delta(k^2)}{(2\pi)^3} e^{-\frac{1}{T(\tau)}\sqrt{k_T^2+\frac{w^2}{\tau^2}}}$
- Constant shear-viscosity over entropy ratio: $\tau_R = 5\bar{\eta}/T$
- $\bar{\eta} = 3/(4\pi)$
- $\tau_0 = 1/4$ fm/c, $T_0 = 0.6$ GeV, two possible initial conditions:

$$W_0^{iso} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{v^2}{2\tau_0^2\sigma}} e^{-\frac{1}{T_0}\sqrt{\sigma=k_T^2+\frac{w^2}{\tau_0^2}}} \longrightarrow \mathcal{P}_0 = \mathcal{P}_{\mathrm{eq.}} = \frac{1}{3}\mathcal{E}$$

$$W_0^{a} = \frac{2}{(2\pi)^3\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{v^2}{2\tau_0^2\sigma}} e^{-\frac{1}{T_0}\sqrt{\sigma=k_T^2+\frac{w^2}{\tau_0^2}}} \left[1 - 3P_2\left(\frac{w}{\tau_0\sqrt{\sigma}}\right)\right] \longrightarrow \mathcal{P}_T^0 = \frac{8}{5}\mathcal{P}_{\mathrm{eq.}}, \quad \mathcal{P}_L^0 = -\frac{1}{5}\mathcal{P}_{\mathrm{eq.}}$$

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Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E}+\mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\left(\mathcal{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}\Big|_{eq}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\left(\mathcal{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathcal{P}_T + \frac{1}{\tau}\mathcal{R}_T^{(1)}\Big|_{eq}$$

What can we say for the isotropic case

$$\mathrm{R}_L^{eq.} = \frac{1}{5}\mathcal{E} \qquad \mathrm{R}_L^{0} = -\frac{1}{5}\mathcal{E}$$

$$\mathrm{R}_T^{eq.} = \frac{1}{15}\mathcal{E} \qquad \mathrm{R}_T^{0} = -\frac{1}{15}\mathcal{E}$$

$$\frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}\Big|_0 = -\frac{1}{3}$$

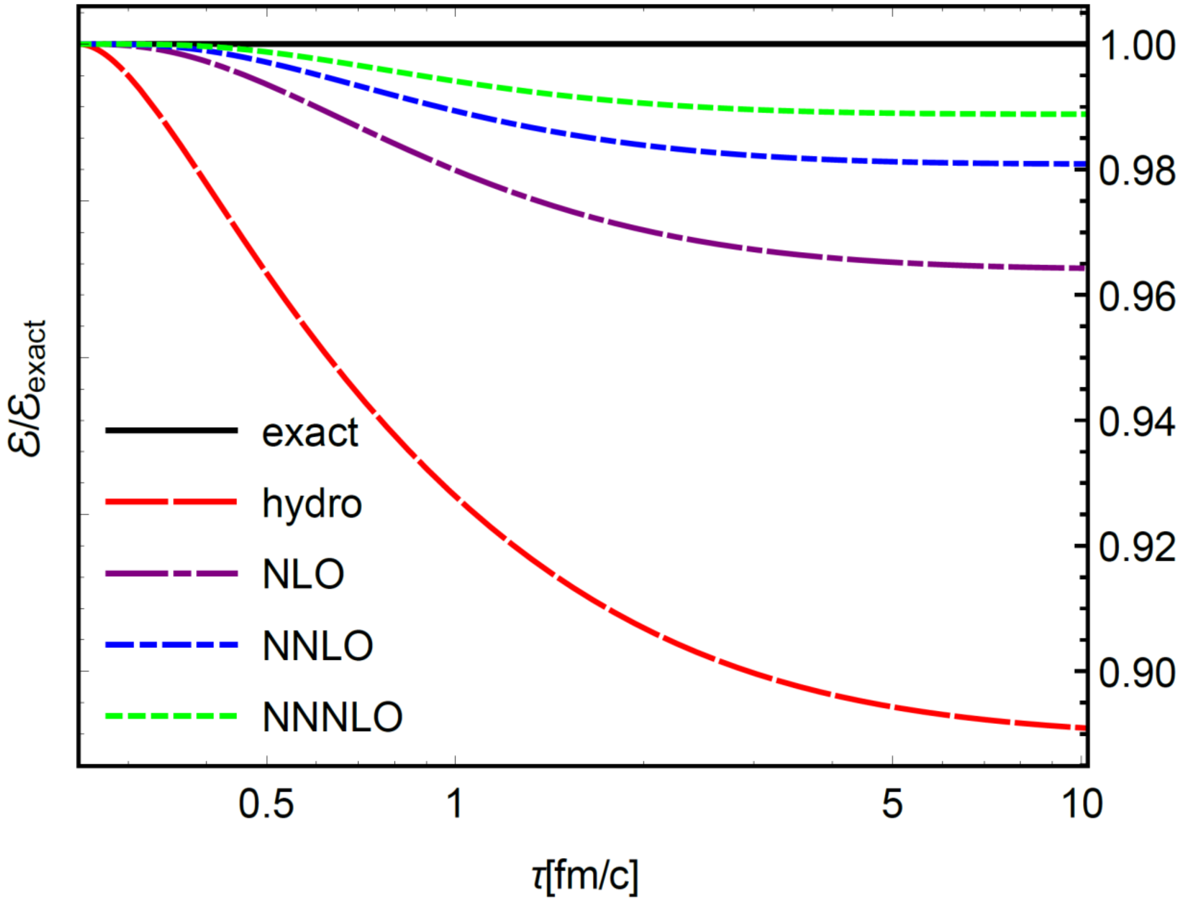
$$\frac{\delta\dot{\mathcal{P}}_T}{\dot{\mathcal{P}}_T}\Big|_0 = -\frac{1}{3}$$

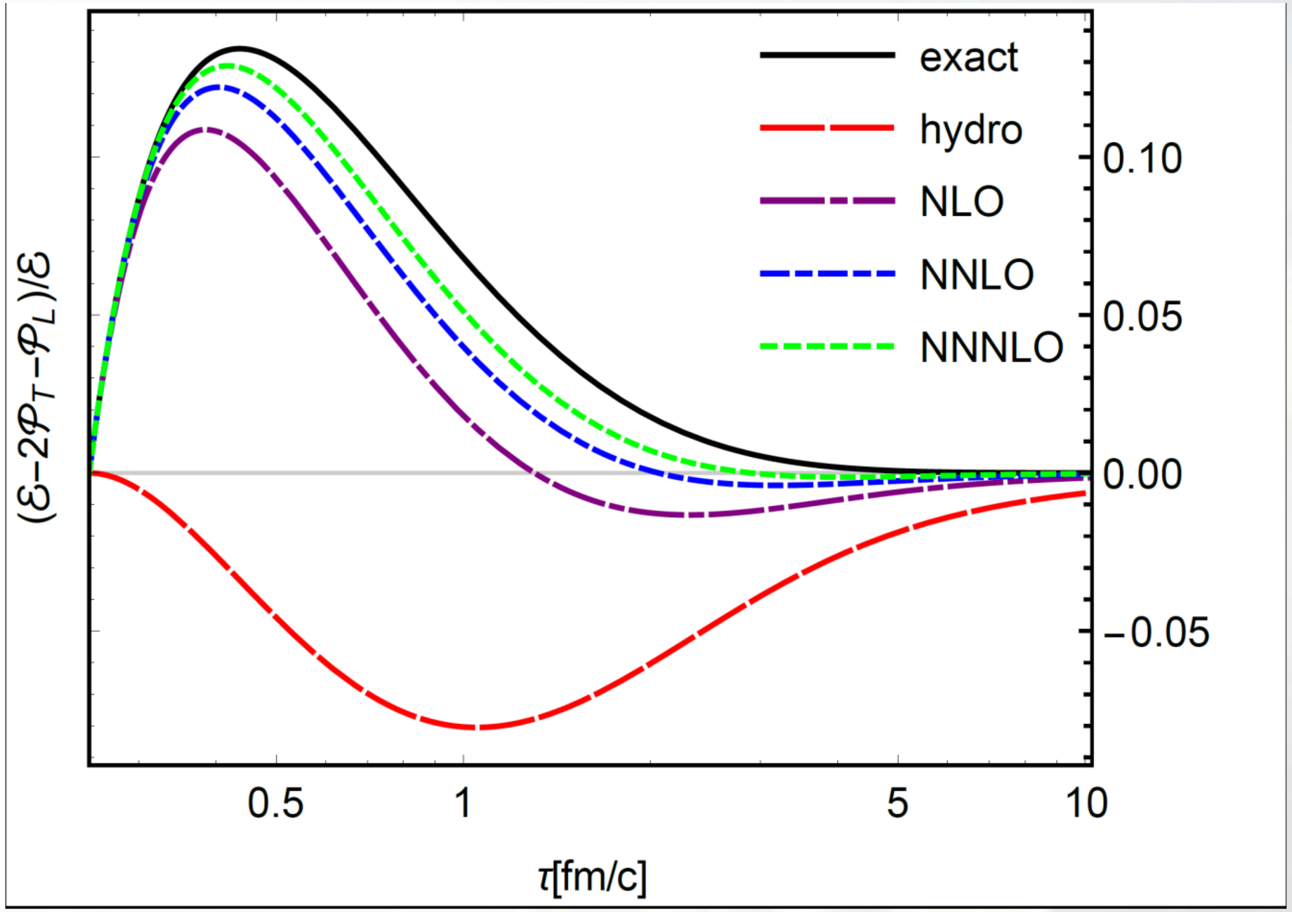
$$\delta\mathcal{P}_L = \int_{\tau_0}^{\tau} ds\, \delta\dot{\mathcal{P}}_L \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\int \delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \text{Maximum if } 0 = \partial_\tau\left(\frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\right) = \frac{\delta\dot{\mathcal{P}}_L}{\mathcal{P}_L} - \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}\frac{\dot{\mathcal{P}}_L}{\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{P}_L}{\mathcal{P}_L} = \frac{\delta\dot{\mathcal{P}}_L}{\dot{\mathcal{P}}_L}$$

$$\frac{\delta\mathcal{E}}{\mathcal{E}} = \frac{\delta\dot{\mathcal{E}}}{\dot{\mathcal{E}}} = \frac{\delta\mathcal{E}+\delta\mathcal{P}_L}{\mathcal{E}+\mathcal{P}_L} \Rightarrow \frac{\delta\mathcal{E}}{\mathcal{E}} \simeq \frac{\delta\mathcal{P}_L}{\mathcal{P}_L}$$

…but for the trace anomaly $\mathcal{E} - 2\mathcal{P}_T - \mathcal{P}_L = -3\Pi$ $\qquad \frac{\delta\dot{\Pi}}{\dot{\Pi}} = -1$

Comparisons with the exact solutions

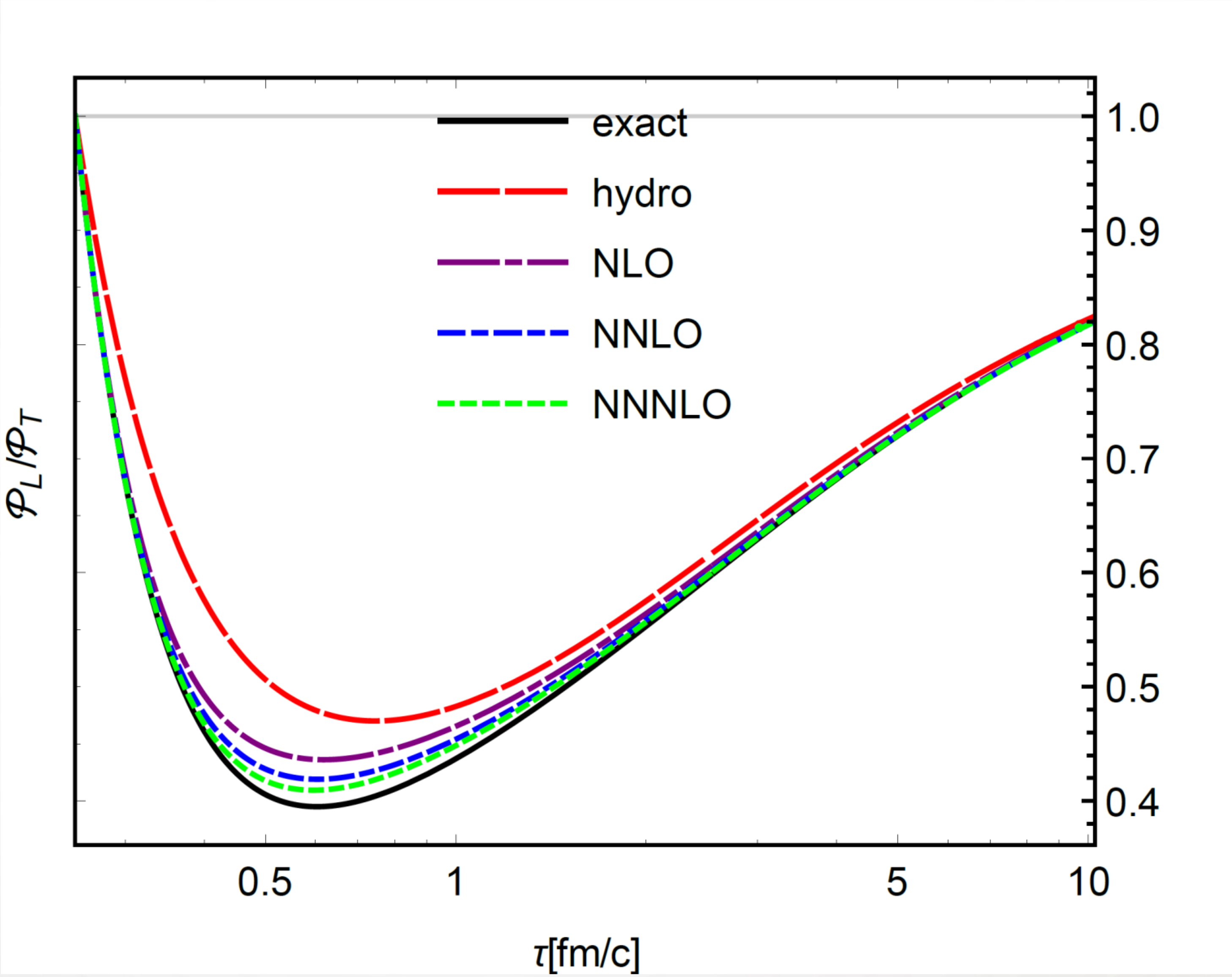




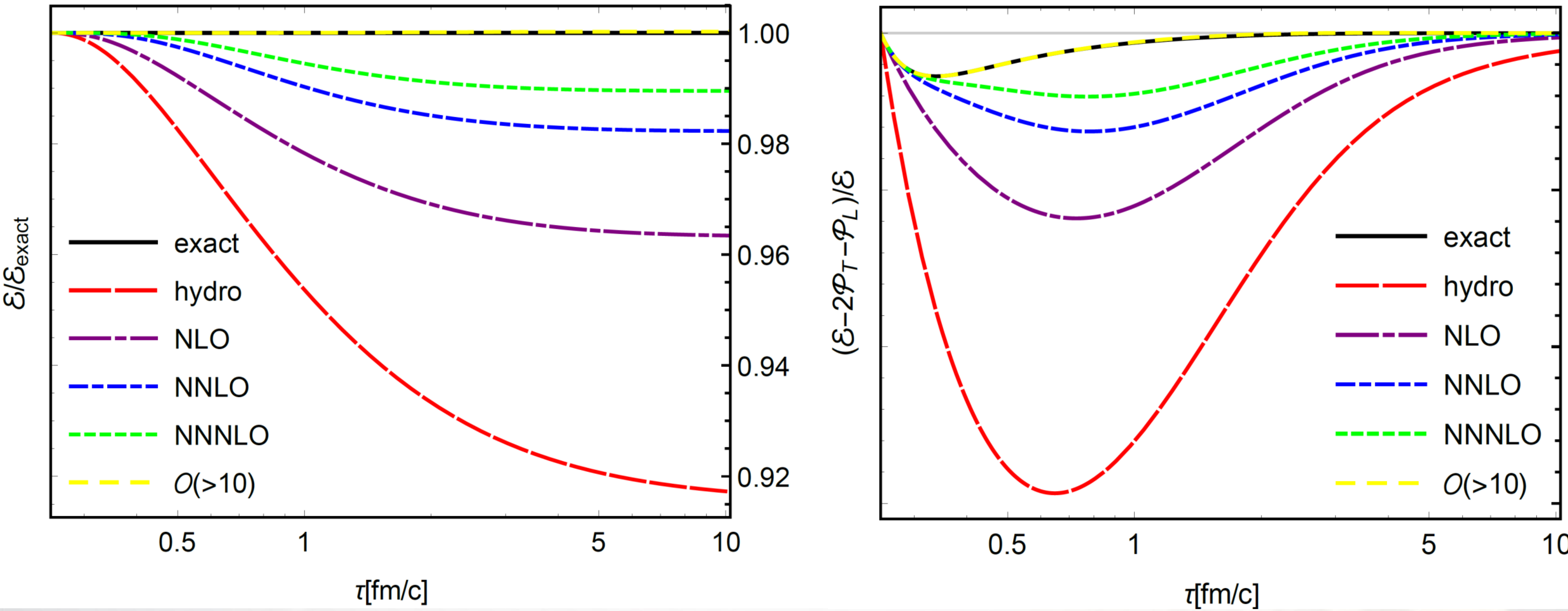
$$(\mathcal{E} - 2\mathcal{P}_T - \mathcal{P}_L)/\mathcal{E} = -\frac{3\Pi}{\mathcal{E}} = -\frac{\Pi}{\mathcal{P}}$$

Comparisons with the exact solutions

fast convergence for the pressure anisotropy too

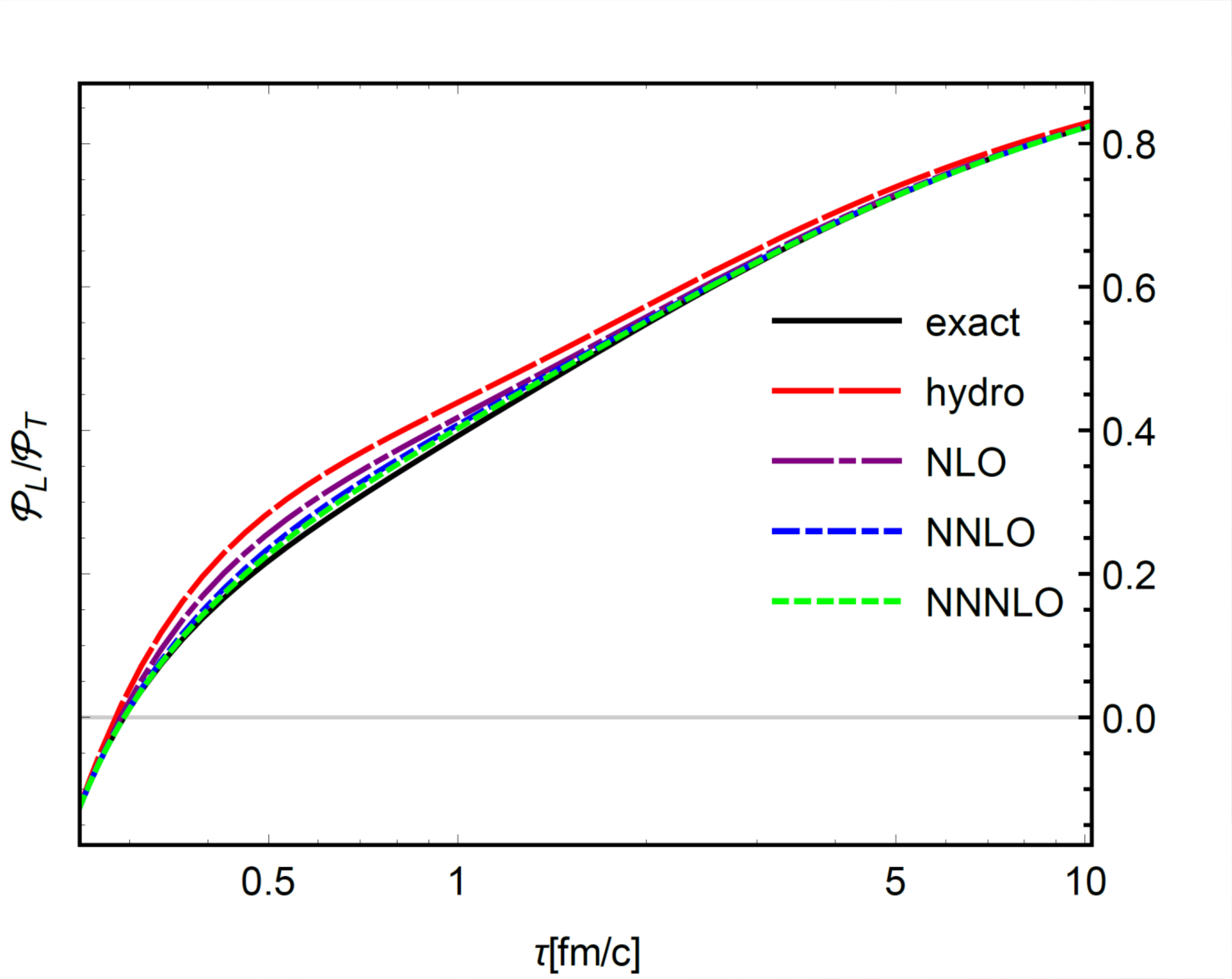


Comparisons for the anisotropic initial conditions



similar conclusions

Comparisons for the anisotropic initial conditions



reasonable approximation for the pressure anisotropy from the start

similar conclusions

Conclusions and outlook

- Large quantum corrections (especially initial stages)
- Qualitative behavior constrained by the symmetry
- Generalized hydrodynamic expansion

Thank you for your attention!

Back up slides

$$\int [g(x)+h(x)]\,dx \neq \int g(x)dx + \int h(x)dx$$

$$\int \lim_{\varepsilon\to 0} f(\varepsilon,x)dx \neq \lim_{\varepsilon\to 0}\int f(\varepsilon,x)dx$$

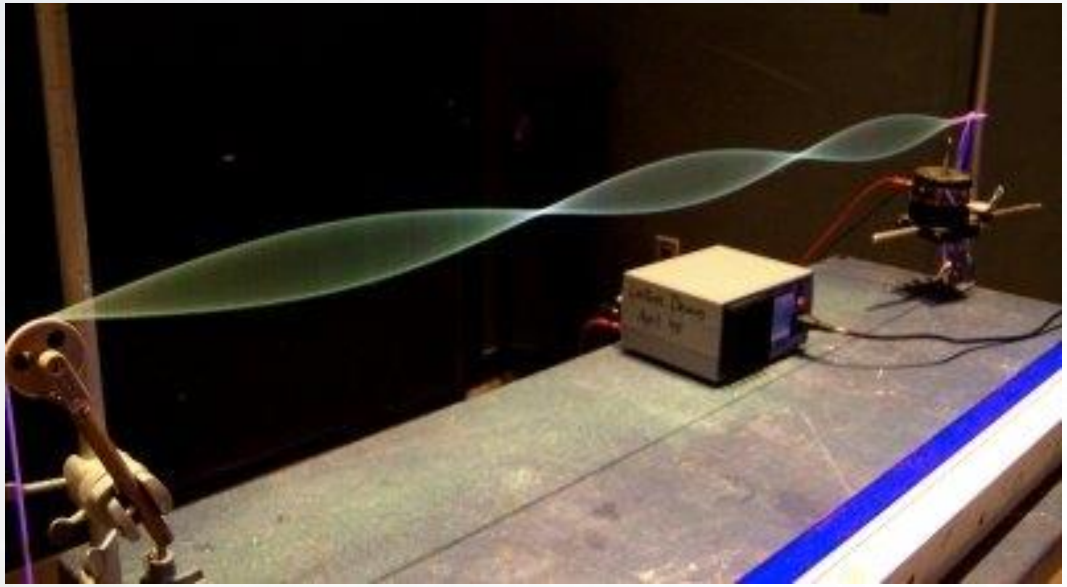
$$\frac{1}{\beta} = \int_0^\infty \left[-\partial_\beta\left(\frac{e^{-\beta x}}{x}\right)\right] dx \neq -\partial_\beta\left(\int_0^\infty \frac{e^{-\beta x}}{x} dx \equiv \infty\right)$$

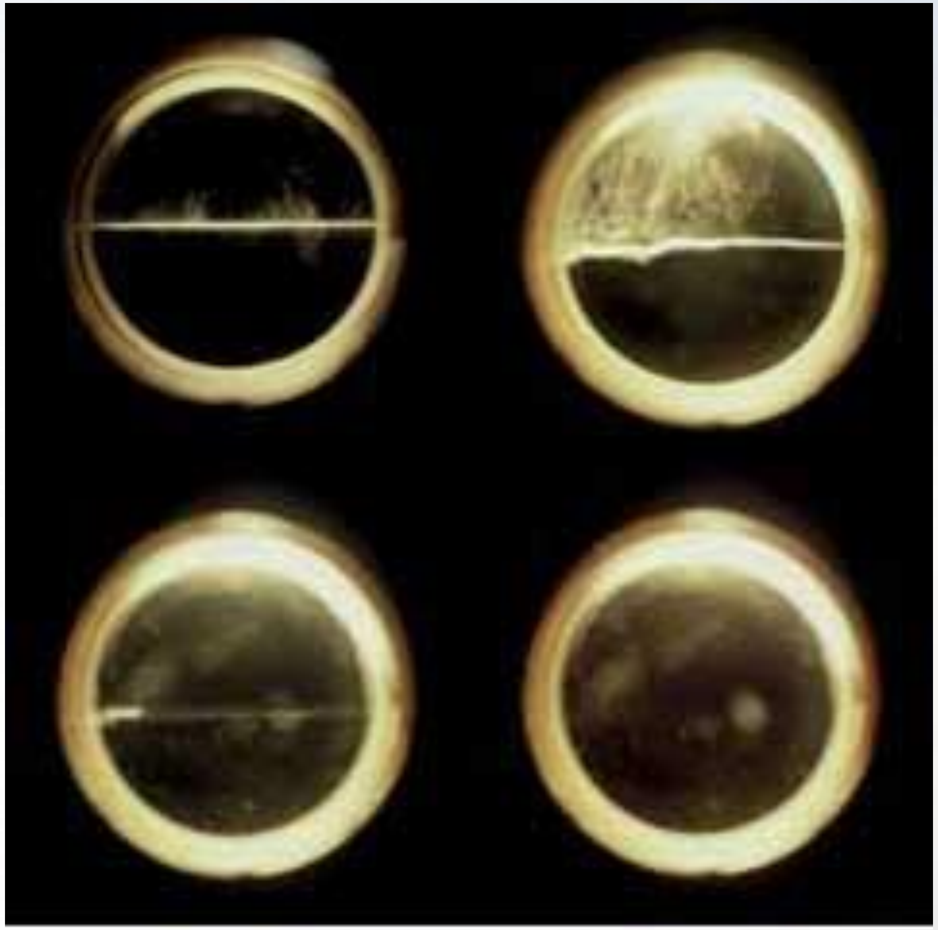
$$\frac{1}{x} = \int_0^\infty e^{-\alpha x} d\alpha$$

$$\frac{1}{(\alpha+\beta)^2} = \int_0^\infty dx\left[-\partial_\beta\left(e^{-(\alpha+\beta)x}\right)\right] = -\partial_\beta\left(\int_0^\infty dx\, e^{-(\alpha+\beta)x} = \frac{1}{\alpha+\beta}\right),$$

$$\int_0^\infty d\alpha\left[\frac{1}{(\alpha+\beta)^2} = \partial_\alpha\left(-\frac{1}{\alpha+\beta}\right)\right] = \frac{1}{\beta}$$

Hydrodynamics

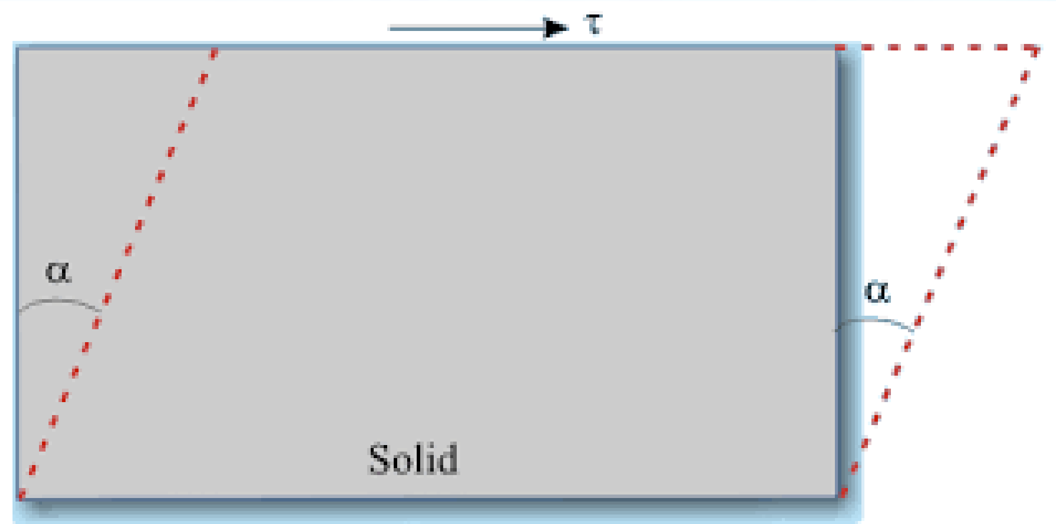




Hydrodynamics is the low-energy, long wave-length limit of a theory

Hydrodynamics require small gradients/deviations from equilibrium

Hydrodynamics



$$\partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho\, \boldsymbol{v}) = 0$$

$$\rho \boldsymbol{a} = -\nabla_{\mathbf{x}} \mathrm{P}$$

generalizes to

$$\rho \boldsymbol{a} = \nabla_x \cdot T$$

the Cauchy Tensor T

definition of a fluid:

$$T_{ij}\Big|_{eq} = -\mathrm{P}\, \delta_{ij}$$

For an incompressible fluid $\nabla_x \cdot \boldsymbol{v} = \mathbf{0}$

$$T_{ij} \simeq -\mathrm{P}\, \delta_{ij} + \eta \left(\partial_i v_j + \partial_j v_i\right) + \cdots$$

Relativistic hydrodynamics

$$\boldsymbol{v} \to u = \begin{pmatrix} \gamma \\ \gamma \boldsymbol{v}/c \end{pmatrix}$$

$$\rho \to \mathcal{E}$$

relativistic degrees of freedom

projector: $\Delta^{\mu\nu} = u^\mu u^\nu - g^{\mu\nu}$

$$T^{\mu\nu} = \mathcal{E} u^\mu u^\nu - \mathrm{P}\Delta^{\mu\nu} + \cdots$$

$$\partial_\mu \mathrm{T}^{\mu\nu} = 0$$

local four-momentum conservation

which implies

$$\begin{cases} 0 = u_\nu \partial_\mu T^{\mu\nu} \xrightarrow{c\to\infty} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho\, \boldsymbol{v}) \\ 0 = \Delta_{i\nu} \partial_\mu T^{\mu\nu} \xrightarrow{c\to\infty} (\rho \boldsymbol{a} + \nabla_{\mathbf{x}} \mathrm{P})\Big|_i + \cdots \end{cases}$$

Relativistic hydrodynamics

$$\left.\begin{aligned} \partial_\mu \hat{T}^{\mu\nu} &= 0 \\ T^{\mu\nu} &= tr(\hat{\rho}\,\hat{T}^{\mu\nu}) \end{aligned}\right\} \rightarrow \partial_\mu T^{\mu\nu} = 0$$

Hydro

$$T^{\mu\nu} = \mathcal{E}\, u^\mu\, u^\nu - \mathcal{P}\Delta^{\mu\nu} + \delta T^{\mu\nu}$$

From quantum field theory, but at least ten degrees of freedom and only four equations

Gradient expansion

- Requires small gradients
- Unstable (even in the non-relativistic limit)
- Not converging

A Buchel, M P Heller, J Noronha, arXiv:1603.05344

G Denicol, J Noronha, arXiv:1608.07869

$$\delta T^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \cdots$$

transport coefficients times gradients

What if we could use it? (in heavy-ion collisions)

$$\mathcal{O}^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} = \Delta^{\mu_1}_{\alpha_1}\cdots\Delta^{\mu_l}_{\alpha_l}\mathcal{O}^{\alpha_1\cdots\alpha_l}$$

even more convenient basis

$$\mathfrak{f}_r^{\mu_1\cdots\mu_l} = \int_{\boldsymbol{p}} (p\cdot u)^r\, p^{\langle\mu_1\rangle\cdots\langle\mu_l\rangle} f$$

$$\partial_\mu u_\nu = u_\mu \dot{u}_\nu + \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta\Delta_{\mu\nu}, \qquad T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}^{\mu\nu} = \mathcal{E}u^\mu u^\nu - (\mathcal{P}+\Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\begin{cases} u_\nu \partial_\mu T^{\mu\nu} = 0 \\ \partial_\mu T^{\mu\langle\nu\rangle} = 0 \end{cases} \Rightarrow \begin{cases} \dot{\mathcal{E}} = -\theta(\mathcal{E}+\mathcal{P}+\Pi) + \pi^{\mu\nu}\sigma_{\mu\nu} \\ (\mathcal{E}+\mathcal{P}+\Pi)\,\dot{u}^\nu = \nabla^\nu(\mathcal{P}+\Pi) - \nabla_\mu \pi^{\mu\langle\nu\rangle} + \pi^{\nu\alpha}\dot{u}_\alpha \end{cases}$$

$$\dot{\mathcal{P}}^{\langle\mu\rangle\langle\nu\rangle} + C_{-1}^{\langle\mu\rangle\langle\nu\rangle} = 2(\mathcal{P}+\Pi)\sigma^{\mu\nu} + \frac{5}{3}\theta(\mathcal{P}+\Pi)\Delta^{\mu\nu} - \frac{5}{3}\theta\pi^{\mu\nu} - 2\pi_\alpha^{(\mu}\sigma^{\nu)\alpha} + 2\pi_\alpha^{(\mu}\omega^{\nu)\alpha}$$
$$-\nabla_\alpha \mathfrak{f}_{-1}^{\alpha\langle\mu\rangle\langle\nu\rangle} - \left(\sigma_{\alpha\beta} + \frac{1}{3}\theta\Delta_{\alpha\beta}\right)\mathfrak{f}_{-2}^{\alpha\beta\mu\nu}$$

Simplest case: free streaming

Classical limit of the exact solutions

$$\lim_{\hbar\to 0}\left[(2\pi\hbar)^3\, W(x,k)\right] \propto \delta(k^2-m^2)$$

$$\chi = 2\sqrt{\frac{k^2-m^2}{k^2-m_T^2}}$$

Particles

(similar for the antiparticles)

$$(2\pi\hbar)^3\, W^+ = \theta(k^0)\theta(k^2-m^2c^2)\frac{(4-\chi^2)^2}{4m_T^2\chi}\left[\cos\left(\frac{w\chi}{\hbar}\right)\tilde{f}_{\rm even}(k^0,k_T,k^z)+\sin\left(\frac{w\chi}{\hbar}\right)\tilde{f}_{\rm odd}(k^0,k_T,k^z)\right]\frac{A}{2\pi\hbar}$$

$$\varepsilon = \frac{\hbar}{A}$$

$$\int\frac{dx}{\varepsilon}\, g\left(\frac{x}{\varepsilon};x,p_1\cdots\right)\psi(x) = \int dy\, g\left(y;y\varepsilon,p_1\cdots\right)\psi(y\varepsilon) \xrightarrow{\varepsilon\to 0} \psi(0)\int dy\, g(y;0,p_1,\cdots),$$

$$\Rightarrow \qquad \frac{1}{\varepsilon}\, g\left(\frac{x}{\varepsilon};x,p_1\cdots\right) \xrightarrow{\varepsilon\to 0} \delta(x)\int dy\, g(y;0,p_1,\cdots).$$

10.1103/PhysRevD.108.076022

Simplest case: free streaming

Classical limit of the exact solutions

$\varepsilon = \frac{\hbar}{A} \qquad \tilde{w} = \frac{w}{A}$

$$\frac{\theta(\chi)\theta(2-\chi)}{(2\pi\varepsilon)} \cos\left(\tilde{w}\frac{\chi}{\varepsilon}\right) \tilde{f}_{\text{even}}\left(\frac{\chi}{\varepsilon}; k^0, k_T, k^z\right) \xrightarrow{\varepsilon\to 0^+} \frac{1}{2}\delta(\chi) \int \frac{d\chi'}{(2\pi)} \cos\left(\tilde{w}\chi'\right) \tilde{f}_{\text{even}}\left(\chi'; \sqrt{m_T^2 + (k^z)^2}, k_T, k^z\right)$$

$$\frac{\theta(\chi)\theta(2-\chi)}{(2\pi\varepsilon)} \sin\left(\tilde{w}\frac{\chi}{\varepsilon}\right) \tilde{f}_{\text{odd}}\left(\frac{\chi}{\varepsilon}; k^0, k_T, k^z\right) \xrightarrow{\varepsilon\to 0^+} \frac{1}{2}\delta(\chi) \int \frac{d\chi'}{(2\pi)} \sin\left(\tilde{w}\chi'\right) \tilde{f}_{\text{odd}}\left(\chi'; \sqrt{m_T^2 + (k^z)^2}, k_T, k^z\right)$$

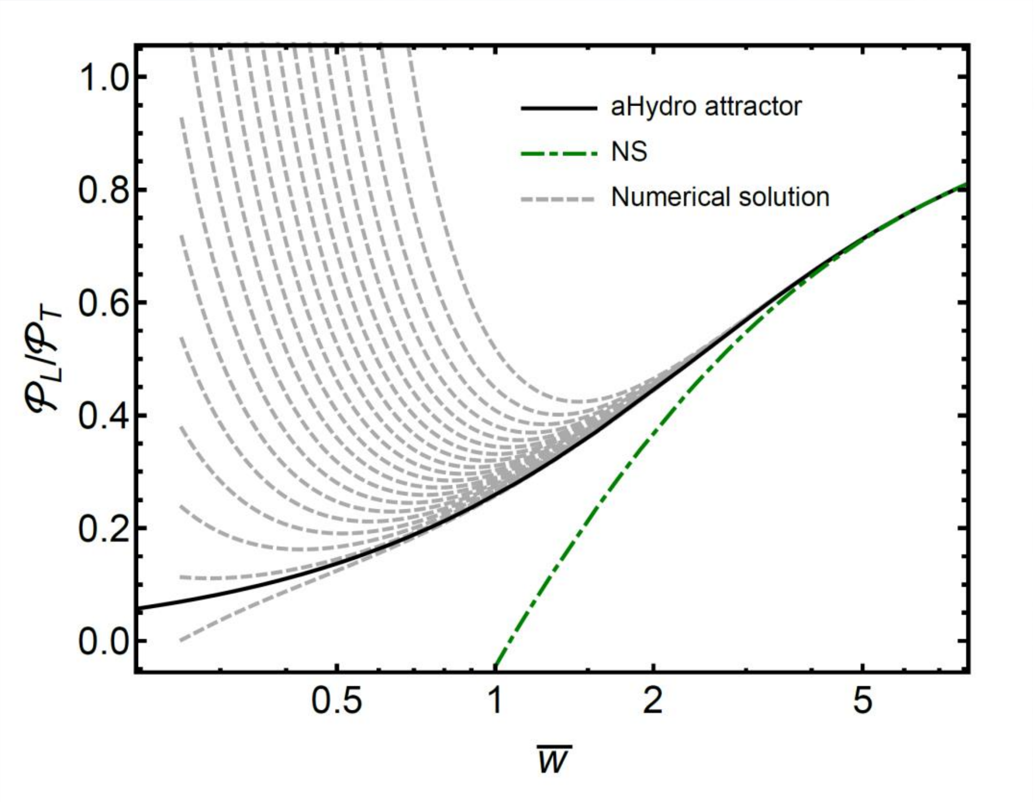
Proportional to the real (hence even in $\tilde{w}$) and imaginary (odd) part of the Fourier transform

$$\tilde{f}_{\text{even}}\left(\frac{\chi}{\varepsilon}; k_T, k^z\right) = 2\text{Re}\left[\int d\tilde{w}'\ f\left(\tilde{w}'; k_T, k^z\right)\ e^{-i\tilde{w}'\frac{\chi}{\varepsilon}}\right]$$

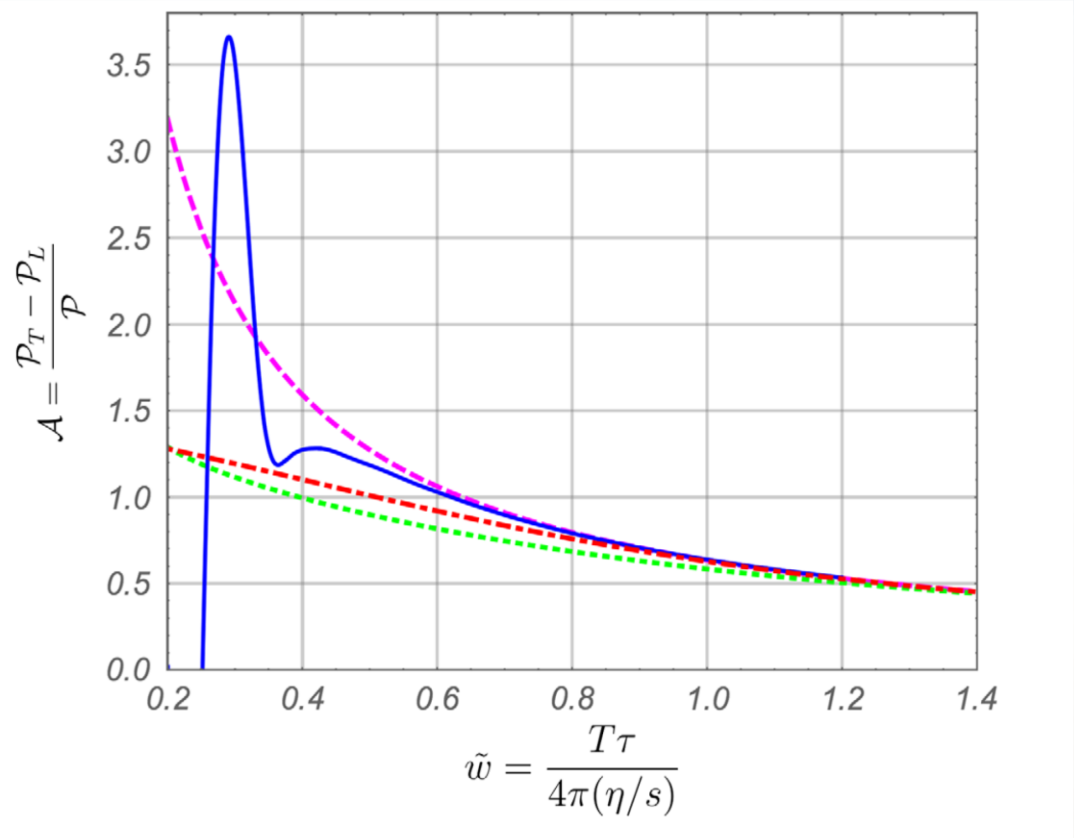
$$\tilde{f}_{\text{odd}}\left(\frac{\chi}{\varepsilon}; k_T, k^z\right) = 2\text{Im}\left[\int d\tilde{w}'\ f\left(\tilde{w}'; k_T, k^z\right)\ e^{-i\tilde{w}'\frac{\chi}{\varepsilon}}\right]$$

10.1103/PhysRevD.108.076022

If the gradient expansion diverges, can hydrodynamics make sense?

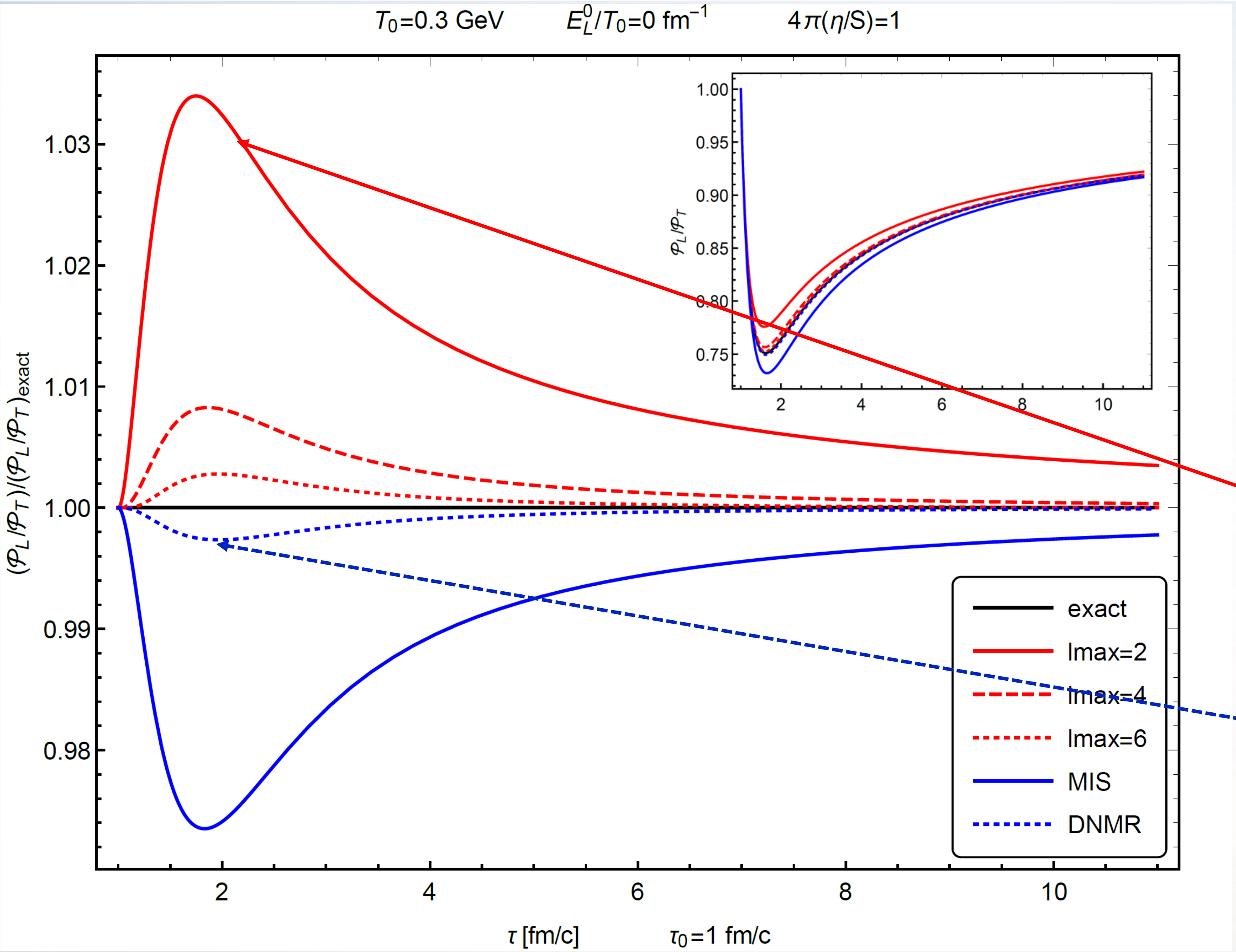


M Strickland, J Noronha, G Denicol
arXiv:1709.06644



M P. Heller, A Kurkela, M Spalinski, V Svensson
arXiv:1609.04803

Attractor behavior! …but A Behtash, C N Cruz-Camacho, M Martinez arXiv:1711.01745



The method of moments converges fast

different treatment of the residual moments

$$\rho_{-2}^{\mu_1\cdots\mu_4} \to \rho_{-2}^{\mu_1\cdots\mu_4}\Big|_{eq.}$$

$$\rho_{-1}^{\mu_1\cdots\mu_4} \neq \rho_{-2}^{\mu_1\cdots\mu_4}\Big|_{eq.}$$

L T, G Vujnovich, J Noronha, U Heinz **arXiv:1808.06436**

L T, G Vujnovich **(WIP)**

Generalization?

multiple particle species

$$\Theta(p_0)\delta(p^2 - m^2)f \to \sum_i \Theta(p_0)\delta(p^2 - m_i^2)f_i$$

$$C[f] \to \sum_i C_i[f_1, \cdots, f_n]$$

long range interactions (not-immediate)

$$p \cdot \partial f \to p \cdot \partial f + F \cdot \partial_{(p)} f$$

Mild divergencies at higher orders, due to the coupling to external fields

L T, G Vujnovich, J Noronha, U Heinz **arXiv:1808.06436**

Wigner distribution (quantum)

$$\Theta(p_0)\delta(p^2 - m^2)f \to W$$

$$p \cdot \partial f \to k \cdot \partial W$$

Needs regularization from the start

L T, **arXiv:2003.09268**

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

$$f_r^{\mu_1\cdots\mu_s} = \mathcal{F}_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}$$
$$\phi_r^{\mu_1\cdots\mu_s} = \Phi_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle}$$

$$\begin{aligned}
\dot{f}_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + (\mathcal{F}_{\text{coll.}})_r^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = & -q\,s\,\varepsilon^{\rho\sigma\alpha(\mu_1} f_{r-1}^{\mu_2\cdots\mu_s)\beta} g_{\alpha\beta} u_\rho B_\sigma - q(r-1)\,E_\alpha\, f_{r-2}^{\alpha\mu_1\cdots\mu_s} - q\,s\,E^{(\mu_1} f_r^{\mu_2\cdots\mu_s)} \\
& + m\dot{m}\,(r-1)\, f_{r-2}^{\mu_1\cdots\mu_s} + s\,m\nabla^{(\mu_1} m\, f_{r-1}^{\mu_2\cdots\mu_s)} \\
& + r\,\dot{u}_\alpha f_{r-1}^{\alpha\mu_1\cdots\mu_s} - s\dot{u}^{(\mu_1} f_{r+1}^{\mu_2\cdots\mu_s)} \\
& - \nabla_\alpha f_{r-1}^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} - \theta\, f_r^{\mu_1\cdots\mu_s} - s\,\nabla_\alpha u^{(\mu_1} f_r^{\mu_2\cdots\mu_s)\alpha} \\
& + (r-1)\nabla_\alpha u_\beta\, f_{r-2}^{\alpha\beta\mu_1\cdots\mu_s},
\end{aligned}$$

$$\begin{aligned}
\dot{\phi}_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + (\Phi_{\text{coll.}})_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = & -q\left[s\,E^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)} - 2\xi^2\left(E_\alpha\,\phi_1^{\alpha\mu_1\cdots\mu_s} + m\dot{m}\,\phi_1^{\mu_1\cdots\mu_s}\right)\right] \\
& + s\frac{1}{\sqrt{\pi}}\int_{\xi^2}^{\infty}\frac{d\upsilon}{\sqrt{\upsilon-\xi^2}}\left[m\nabla^{(\mu_1} m\,\phi_1^{\mu_2\cdots\mu_s)} - q\,\varepsilon^{\rho\sigma\alpha(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\beta} g_{\alpha\beta} u_\rho B_\sigma\right] \\
& + \frac{1}{\sqrt{\pi}}\int_{\xi^2}^{\infty}\frac{d\upsilon}{\sqrt{\upsilon-\xi^2}}\left[\dot{u}_\alpha\phi_1^{\alpha\mu_1\cdots\mu_s} + s\,\dot{u}^{(\mu_1}\partial_\upsilon\phi_1^{\mu_2\cdots\mu_s)} + 2\xi^2\,\dot{u}_\alpha\,\partial_\upsilon\phi_1^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle}\right] \\
& - \theta\,\phi_1^{\mu_1\cdots\mu_s} - s\,\nabla_\alpha u^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\alpha} - 2\xi^2\nabla_\alpha u_\beta\,\phi_1^{\alpha\beta\mu_1\cdots\mu_s}.
\end{aligned}$$

Particles interacting with external fields

Boltzmann-Vlasov equation

$$p \cdot \partial f + m\partial_\alpha m\, \partial^\alpha_{(p)} f + qF_{\alpha\beta}p^\beta \partial^\alpha_{(p)} f = -\mathcal{C}[f]$$

Immediate (but problematic) generalization

$$\begin{aligned}\dot{\mathcal{F}}_r^{\mu_1\cdots\mu_s} + C_{r-1}^{\mu_1\cdots\mu_s} \;=\;& r\,\dot{u}_\alpha\,\mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} - \nabla_\alpha \mathcal{F}_{r-1}^{\alpha\mu_1\cdots\mu_s} + (r-1)\,\nabla_\alpha u_\beta\,\mathcal{F}_{r-2}^{\alpha\beta\mu_1\cdots\mu_s} \\ & + m\dot{m}\,(r-1)\,\mathcal{F}_{r-2}^{\mu_1\cdots\mu_s} + s\, m\partial^{(\mu_1} m\,\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)} \\ & - q(r-1)\,E_\alpha\,\mathcal{F}_{r-2}^{\alpha\mu_1\cdots\mu_s} - q\,s\,g_{\alpha\beta}F^{\alpha(\mu_1}\mathcal{F}_{r-1}^{\mu_2\cdots\mu_s)\beta}\end{aligned}$$

$$F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \varepsilon_{\mu\nu\rho\sigma}u^\rho B^\sigma$$

Moments with large negative r needed, infrared catastrophe!

arXiv:1808.06436

Exactly solvable case

Bjorken symmetry

$$\tau = \sqrt{t^2 - z^2}, \qquad v = k^0 t - z\,k^z, \qquad u = (\cosh\eta, 0,0, \sinh\eta)$$

$$\eta = \frac{1}{2}\ln\left(\frac{t+z}{t-z}\right), \qquad w = zk^0 - t\,k^z, \qquad z = (\sinh\eta, 0,0, \cosh\eta)$$

(as a consequence)

$$T^{\mu\nu} = \mathcal{E}(\tau)u^\mu u^\nu + \mathcal{P}_T(\tau)\,(x^\mu x^\nu + y^\mu y^\nu) + \mathcal{P}_L(\tau) z^\mu z^\nu$$

$$\pi^{\mu\nu} = -\frac{1}{2}\pi(\tau)\,(x^\mu x^\nu + y^\mu y^\nu) + \pi(\tau) z^\mu z^\nu$$

$$\mathcal{P}_T = \mathcal{P} + \Pi - \frac{1}{2}\pi, \qquad \mathcal{P}_L = \mathcal{P} + \Pi + \pi$$

RTA

$$k\cdot\partial W = -\frac{k\cdot u}{\tau_R}(W - W_{eq}) = -\frac{k\cdot u}{\tau_R}\left(W - \frac{2\delta(k^2)}{(2\pi)^3} e^{-\frac{1}{T(\tau)}\sqrt{k_T^2 + \frac{w^2}{\tau^2}}}\right) \Rightarrow \partial_\tau W + 2\frac{v^2 - w^2}{\tau}\partial v^2 W = \frac{1}{\tau_R}\delta W$$

in addition $W(\tau, v^2, k_T, w^2)$

10.1103/PhysRevD.108.036015

Resummed moments

Making use of regularized moments

$$\phi_n^{\mu_1\cdots\mu_s}(x,\zeta) = \int \frac{d^4k}{(2\pi)^4}\,(k\cdot u)^n\, e^{-\zeta(k\cdot u)^2} k^{\langle\mu_1\rangle}\dots k^{\langle\mu_s\rangle} W(x,k)$$

$$\partial_\zeta \phi_n^{\mu_1\cdots\mu_s} = -\phi_{n+2}^{\mu_1\cdots\mu_s}$$

$$\int_\zeta^\infty dv\, \phi_{n+2}^{\mu_1\cdots\mu_s} = \phi_n^{\mu_1\cdots\mu_s}$$

$$\phi_n^{\mu_1\cdots\mu_s}(x,0) = \Delta_{\boldsymbol{\alpha_1}}^{\boldsymbol{\mu_1}}\cdots\Delta_{\boldsymbol{\alpha_s}}^{\boldsymbol{\mu_s}}\mathcal{F}_{\mathrm{n}}^{\alpha_1\cdots\alpha_s} = \mathfrak{f}_n^{\alpha_1\cdots\alpha_s}$$

All (well-defined) previous moments recovered from the resumed ones, including $T^{\mu\nu}$

2 generations of dynamical moments needed

$$\dot{\phi}_2^{\langle\mu_1\rangle\cdots\langle\mu_1\rangle} + \tilde{C}_1^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = -\theta\phi_2^{\mu_1\cdots\mu_s} - s\nabla_\alpha u^{(\mu_1}\phi_2^{\mu_2\cdots\mu_s)\alpha} - \nabla_\alpha\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \dot{u}_\alpha\left[2\phi_1^{\alpha\mu_1\cdots\mu_s} + 2\zeta\partial_\zeta\phi_1^{\alpha\mu_1\cdots\mu_s}\right]$$
$$-s\dot{u}^{(\mu_1}\partial_\zeta\phi_1^{\mu_2\cdots\mu_s)} + \nabla_\alpha u_\beta\left[\int_\zeta^\infty dv\,\phi_2^{\alpha\mu_1\cdots\mu_s} - 2\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}\right]$$

$$\dot{\phi}_1^{\langle\mu_1\rangle\cdots\langle\mu_1\rangle} + \tilde{C}_0^{\langle\mu_1\rangle\cdots\langle\mu_s\rangle} = -\theta\phi_1^{\mu_1\cdots\mu_s} - s\nabla_\alpha u^{(\mu_1}\phi_1^{\mu_2\cdots\mu_s)\alpha} - \nabla_\alpha\int_\zeta^\infty dv\phi_1^{\alpha\langle\mu_1\rangle\cdots\langle\mu_s\rangle} + \dot{u}_\alpha\left[\int_\zeta^\infty dv\,\phi_2^{\alpha\mu_1\cdots\mu_s} - 2\zeta\phi_2^{\alpha\mu_1\cdots\mu_s}\right]$$
$$+s\dot{u}^{(\mu_1}\phi_2^{\mu_2\cdots\mu_s)} - 2\zeta\nabla_\alpha u_\beta\phi_1^{\alpha\beta\mu_1\cdots\mu_s}$$

10.1103/PhysRevD.108.036015

Hydrodynamic expansion

$$L_n = \phi_2^{\mu_1 \cdots \mu_{2n}} z_{\mu_1} \cdots z_{\mu_{2n}}, \qquad T_n = \phi_2^{\mu_1 \cdots \mu_{2n}\alpha\beta} z_{\mu_1} \cdots z_{\mu_{2n}} x_\alpha x_\beta$$

$$\dot{L}_n + \frac{1}{\tau_R}(L_n - L_n^{eq.}) = -\frac{2n+1}{\tau} L_n + \frac{1}{\tau}\hat{\mathcal{L}} L_{n+1}$$

$$\dot{T}_n + \frac{1}{\tau_R}(T_n - T_n^{eq.}) = -\frac{2n+1}{\tau} T_n + \frac{1}{\tau}\hat{\mathcal{L}} T_{n+1}$$

$$\mathcal{E} = L_0(\tau, \zeta = 0)$$

$$\mathcal{P}_L = \int_\zeta^\infty d\zeta' \, L_1(\tau, \zeta')$$

$$\mathcal{P}_T = \int_\zeta^\infty d\zeta' \, T_0(\tau, \zeta')$$

$$\hat{\mathcal{L}}[f] = 2\zeta \, f(\zeta) - \int_\zeta^\infty d\zeta' f(\zeta')$$

Hydrodynamics

$$\dot{\mathcal{E}} = -\frac{\mathcal{E} + \mathcal{P}_L}{\tau}$$

$$\dot{\mathcal{P}}_L + \frac{1}{\tau_R}\left(\mathcal{P}_L - \frac{1}{3}\mathcal{E}\right) = -\frac{3}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_L^{(1)}$$

$$\dot{\mathcal{P}}_T + \frac{1}{\tau_R}\left(\mathcal{P}_T - \frac{1}{3}\mathcal{E}\right) = -\frac{1}{\tau}\mathcal{P}_L + \frac{1}{\tau}\mathcal{R}_T^{(1)}$$

one can integrate the equations in ζ

…the same for the sources and their equations…

10.1103/PhysRevD.108.036015