Analytic solutions of the complex diffusion equation

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Outline

- ***Introduction*** *various PDEs, basic solutions, self-similar and travelling wave and other Anzätze, our fomer activity*

- ***The real and the complex diffusion equation*** *and the derived analytic solutions*

- ***Summary & Outlook*** *possible generalizations future plans*

The very basics of the idea

Initially: a (nonlinear) partial differential
equation (system) for u(x,t) is given

Ansatz: combination from space and time variable x,t
to a new variable η(x,t) so u(η) called reduction technics

Result: a (nonlinear) ordinary differential
equation (system) u' (η)

+ with some tricks analytic solutions can be
found which depend on some free physical
parameter(s), general global properties can be studied
like: non-continous solutions, compact supports,
oscillations, asymptotic power-law behaviour etc. etc.

A way of interpretation

there are the two basic time-dependent linear PDEs which describe propagation in space and time

each has it's natural Ansatz with physical interpretation

$$u_t = a u_{xx}$$

$$u = t^{-\alpha} \cdot f\left(\frac{x}{t^\beta}\right)$$

$$u_{xx} = \frac{1}{c^2} u_{tt}$$

$$u = f(x \pm ct)$$

we may attack any kind of non-linear PDE with self-similar or traveling wave Ansatz ☺ asking how diffusive/dissipative or how wave-like it is

(the connection between the two Ansätze is nontivial !! Sometimes can be seen

How these solutions look like

- **Self-similar** in 1D Sedov, Zeldovich, Barenblatt

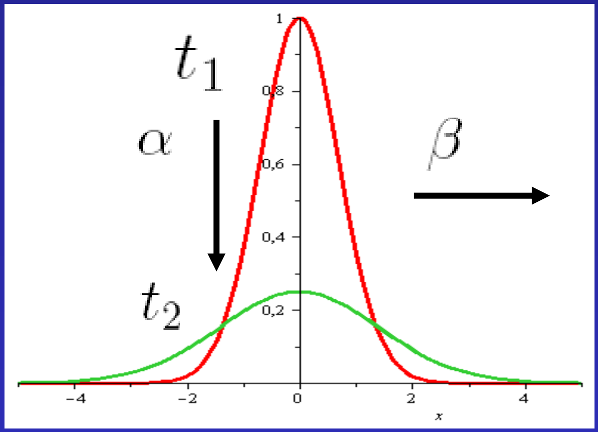
$$u(x,t) = t^{-\alpha} f(x/t^{\beta})$$

α and β are of primary physical importance

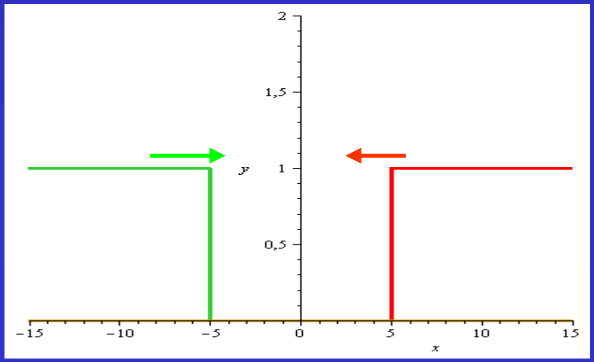
α represents the rate of decay

β is the rate of spread (or contraction if $\beta < 0$)

$t_1 < t_2$ *in Fourier heat-conduction the Gaussian*



- **Traveling waves:**
  arbitrary wave fronts
  u(x,t) ~ g(x-ct), g(x+ct)



Additional relevant solutions for PDEs

We go for diffusion now so u(x,t) is changed to C(x,t)

$$C(x,t) = a(t) \cdot h\left(\frac{x}{b(t)}\right) = a(t) \cdot h(\omega)$$

The „generalized self-similar Ansatz"

$$C(x,t) = a(t) \cdot h\left(\frac{x-b(t)}{c(t)}\right) = a(t) \cdot h(\omega)$$

Traveling profile Ansatz interpolates between self-similar and traveling wave

Behamioduce 2008 the unknown functions a(t), b(t), c(t) have to be deteminded

$$C(x,t) = t^{-\alpha} f\left(\frac{x-ct}{t^{\beta}}\right)$$

It is also a kind of interpolating Ansatz

$$C(x,t) = at^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) + bt^{-\alpha} f\left(\frac{x}{t^{\beta}}\right)^2 = at^{-\alpha} f(\eta) + bt^{-\alpha} f(\eta)^2$$

generalization with a finite series (we applied to diffusion)

$$C(x,t) = \beta(x,t) \cdot W(z[x,t])$$

Clarkson-Kruskál 1989 a non-classical symmetry, also a kind of generalization

The regular diffusion/heat conduction equation

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2}$$

Written in one Cartesian coordinate
The diffusion/heat conduction coefficient is a positive $D > 0$ real number, and independent of the concentration/temperature

The self-similar Ansatz

$$C(x,t) = t^{-\alpha} f\left(\frac{x}{t^{\beta}}\right) = t^{-\alpha} f(\eta)$$

Calculating the derivatives and pluging back to the diffusion equation

$$-\alpha t^{-\alpha-1} f(\eta) - \beta t^{-\alpha-1} \eta \frac{df(\eta)}{d\eta} = D t^{-\alpha-2\beta} \frac{d^2 f(\eta)}{d\eta^2}$$

This object is obtained

This must not depend on time if want to get an ODE so the $\alpha+1 = \alpha+2\beta$ constraint has to be fulfilled which means that α = arb. real number and $\beta = 1/2$

$$-\alpha f - \frac{1}{2}\eta f' = D f''$$

if $\alpha = 1/2$ it is a total derivative and can be integrated once

$$-\frac{1}{2}\eta f + c_1 = D f'$$

c_1 integral constant

and we can get the Gaussian solution

The general solution

What happens if $\alpha \neq 1/2$ the general case never,examined

$$-\alpha f - \frac{1}{2}\eta f' = D f''$$

The solutions contain the Kummer's M and U functions where α is a real number

$$f(\eta) = \eta \cdot e^{-\frac{\eta^2}{4D}} \left(c_1 M\left[1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right] + c_2 U\left[1-\alpha, \frac{3}{2}, \frac{\eta^2}{4D}\right]\right)$$

Other interesting point, that positive interger α gives

$$f(\eta) = e^{-\frac{\eta^2}{4D}} \left(\tilde{c}_1 H_{2\alpha-1}\left[\frac{\eta}{2\sqrt{D}}\right] + \tilde{c}_2 \cdot {}_1F_1\left[\frac{1-2\alpha}{2}, \frac{1}{2}; \frac{\eta^2}{4D}\right]\right)$$

Hermite polynom and the a confluent hypergeometric function

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The general solution

defintion of Kummer's M function via the hypergeometric series:

$$M(a,b,z)=1+\frac{az}{b}+\frac{(a)_2 z^2}{(b)_2 2!}+\ldots+\frac{(a)_n z^n}{(b)_n n!},$$

Where: $(a)_n=a(a+1)(a+2)\ldots(a+n-1), (a)_0=1$
Rising-factorial or Pochammer symbol if „a" is a negative integer it is reduced to a finite series

Kummer's U can be expressed with Kummer's M:

$$U(a,b,z)=\frac{\pi}{sin(\pi b)}\left(\frac{M[a,b,z]}{\Gamma[1+a-b]\Gamma[b]}-z^{1-b}\frac{M[1+a-b,2-b,z]}{\Gamma[a]\Gamma[2-b]}\right)$$

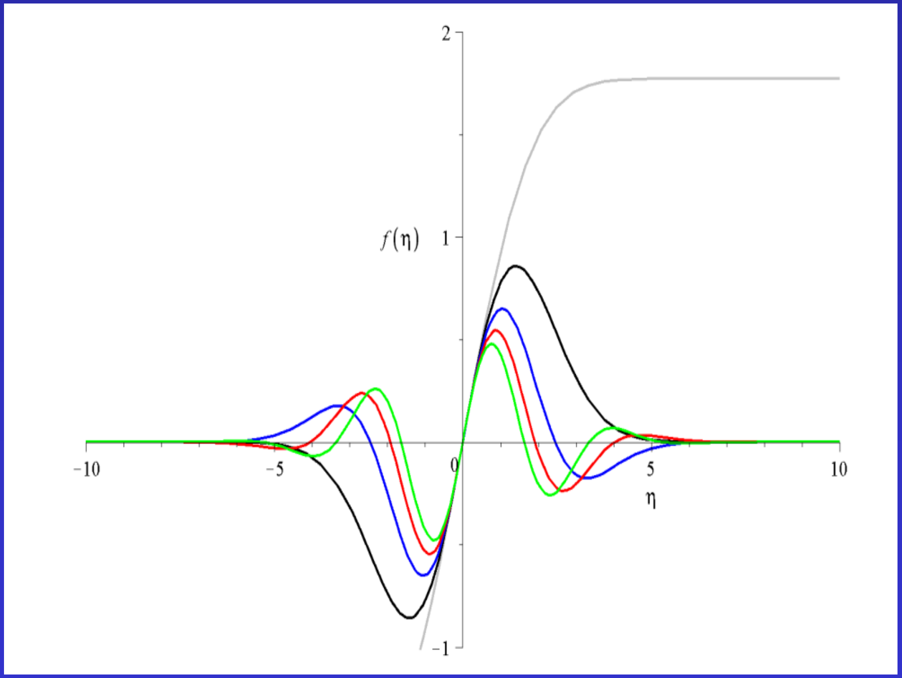
General properties of the solutions

there are four regimes for different α values

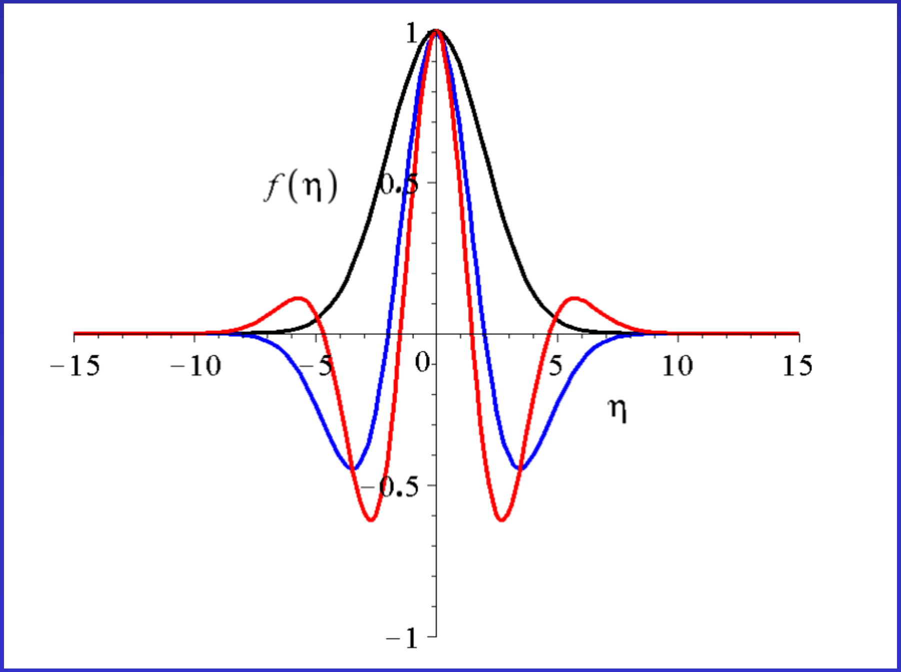
$\alpha < 0$ divergent for large argument, $\alpha = 0$ goes to finite assymptotics

$0 < \alpha < 1$ have a global maxima than goes to zero, $\alpha > 1$ shows some oscillations and decay to zero

most of the solution are odd but some are even, which have interesting properties "are kind of higher harmonics of Gaussian"



α = 0, 1, 2, 3, 4



α = 1/2, 3/2, 5/2

Analysis of the complex diffusion equation

the whole derivation is very similar,
we consider the Cartesian case first
trivially true:

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2}$$

$$\frac{\partial \Psi(x,t)}{\partial t} = i\hat{D} \frac{\partial^2 \Psi(x,t)}{\partial x^2}$$

with $\hat{D} = \frac{\hbar}{2m} > 0$

the Ansatz:

$$\Psi(x,t) = t^{-\alpha} g\left(\frac{x}{t^{\beta}}\right) = t^{-\alpha} g(\eta)$$

the derived ordinary differential equation:

$$-\alpha g - \frac{1}{2}\eta g' = i\hat{D} g''$$

the solution:

$$g(\eta) = \tilde{c}_1 M\left[\alpha + \frac{1}{2}, \frac{3}{2}, \frac{i\eta^2}{4\hat{D}}\right]\eta + \tilde{c}_2 U\left[\alpha + \frac{1}{2}, \frac{3}{2}, \frac{i\eta^2}{4\hat{D}}\right]\eta$$

main difference to classical case: no Gaussian factor + complex argument

Results

we should check the first some Taylor expansion terms, and make a parameter study for real, complex and absolut value of the Kummer's M and U functions for the shape function g(η)
finally investigate the $|\psi(x,t)|^2$

Kummer's M

$$g(\eta) = \tilde{c}_1 \left[\eta + \frac{i \cdot \left(\alpha + \frac{1}{2}\right)}{6\hat{D}} \eta^3 + \frac{\left(\alpha + \frac{1}{2}\right) \cdot \left(\alpha + \frac{3}{2}\right)}{120\hat{D}^2} \eta^5 + \frac{i \cdot \left(\alpha + \frac{1}{2}\right) \cdot \left(\alpha + \frac{3}{2}\right) \cdot \left(\alpha + \frac{5}{2}\right)}{5040\hat{D}^3} \eta^7 - \ldots \right] +$$

Kummer's U

$$\tilde{c}_2 \left[\frac{2\sqrt{\pi}}{\sqrt{\frac{i}{\hat{D}}}\Gamma\left(\alpha + \frac{1}{2}\right)} - \frac{2\sqrt{\pi}}{\Gamma(\alpha)}\eta + \frac{i\alpha\sqrt{\pi}}{\sqrt{i \cdot \hat{D}}\Gamma\left(\alpha + \frac{1}{2}\right)}\eta^2 - \frac{i\sqrt{\pi}\left(\alpha + \frac{1}{2}\right)}{3\hat{D}\Gamma(\alpha)}\eta^3 - .. \right]$$

Results

Real, complex and absolute value of the Kummer's M function

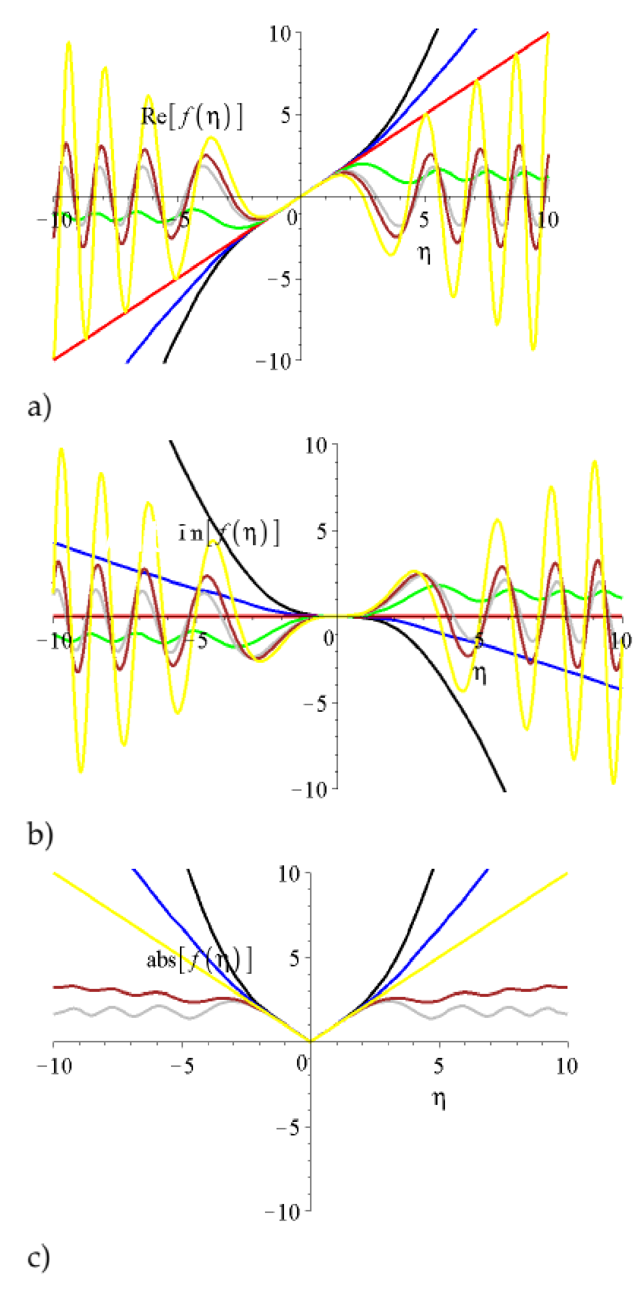
α = -1,-2/3, -1/2, 0,1/2, 2/3, 1
black, blue,red, green, gray, brown yellow

Due to the transformation relation:

$$M(a,b,z) = e^{z} M(b-a,b,z)$$

$$\left| M\left(1, \frac{3}{2}, \frac{i\eta^2}{4\hat{D}}\right) \right| = \left| M\left(\frac{1}{2}, \frac{3}{2}, \frac{i\eta^2}{4\hat{D}}\right) \right|$$

some curves coincide,
Main difference:
shape functions do not go to zero

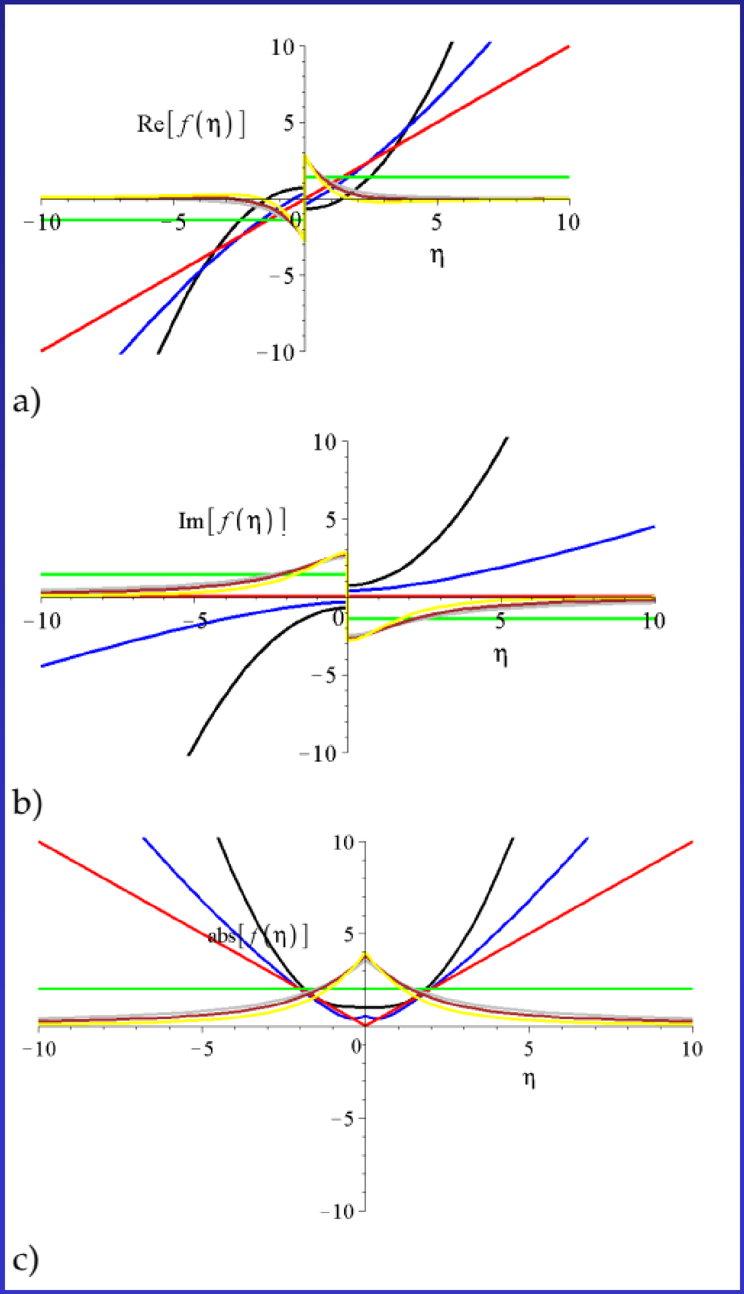


Results

Real, complex and absolute value of the Kummer's U function

α = -1,-2/3, -1/2, 0,1/2, 2/3 1
black, blue,red, green, gray, brown yellow

have a cusp in the origin

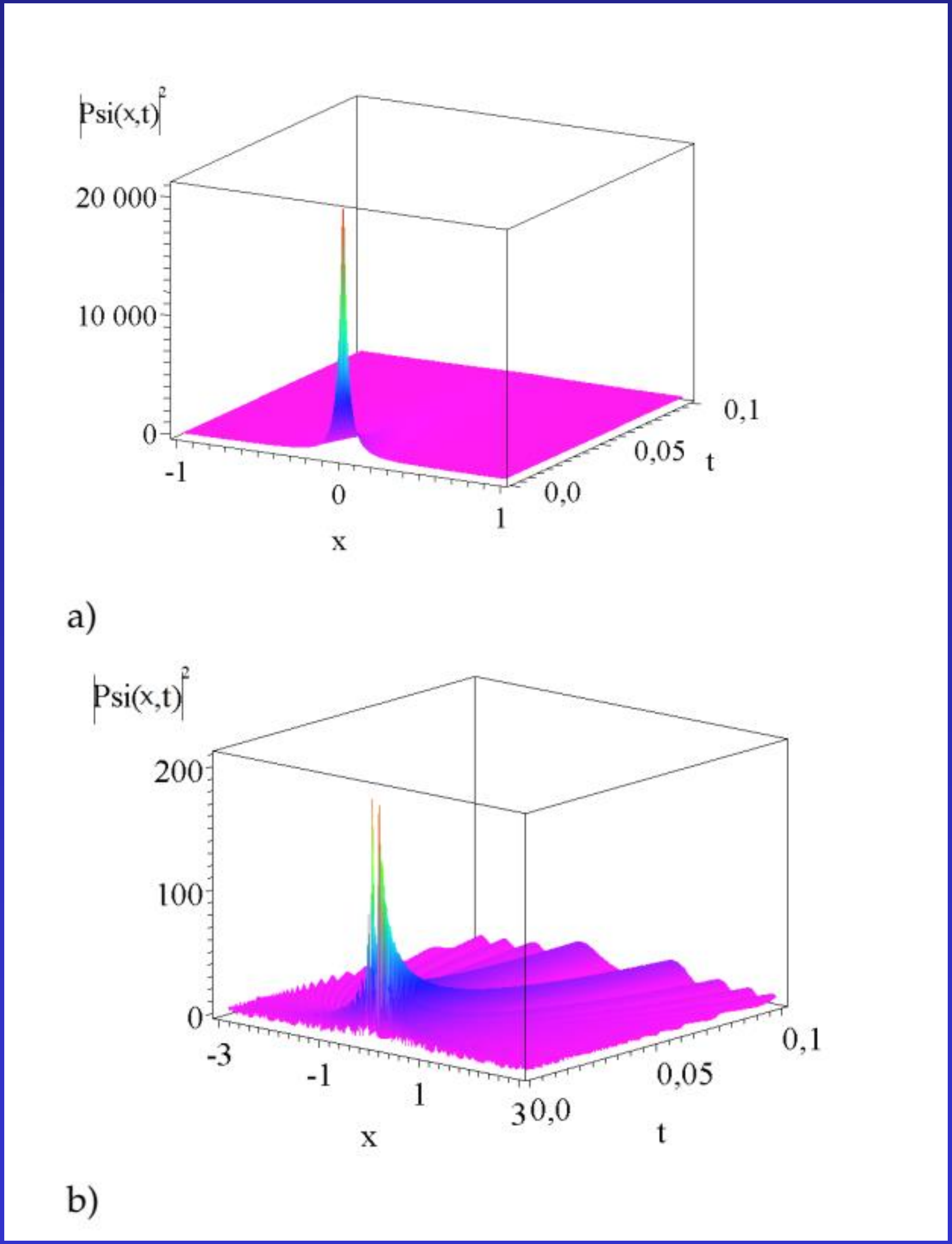


Results

The $|\psi(x,t)|^2$ for the Kummer's U function
$\alpha = 1/4$
The integral is finite ☺

The $|\psi(x,t)|^2$ for the Kummer's M function
$\alpha = 1/4$
The intergal is finite ☺

So can be interpreted as quantum mechanical wave function



Analysis of the complex diffusion equation

Spherial case
the whole derivation is very similar:

$$i\hbar\frac{\partial\Psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{2}{r}\frac{\partial\Psi(r,t)}{\partial x} + \frac{\partial^2\Psi(r,t)}{\partial r^2}\right)$$

$$\Psi(r,t) = t^{-\alpha}g\left(\frac{r}{t^{\beta}}\right) = t^{-\alpha}g(\omega)$$

The derived ODE

$$i\left(-\alpha g - \frac{\omega g'}{2}\right) = -D\left(\frac{2g'}{\omega} + g''\right),$$

The solutions:

$$g(\omega) = c_1 M\left(\alpha, \frac{3}{2}, \frac{i\omega^2}{4D}\right) + c_2 U\left(\alpha, \frac{3}{2}, \frac{i\omega^2}{4D}\right).$$

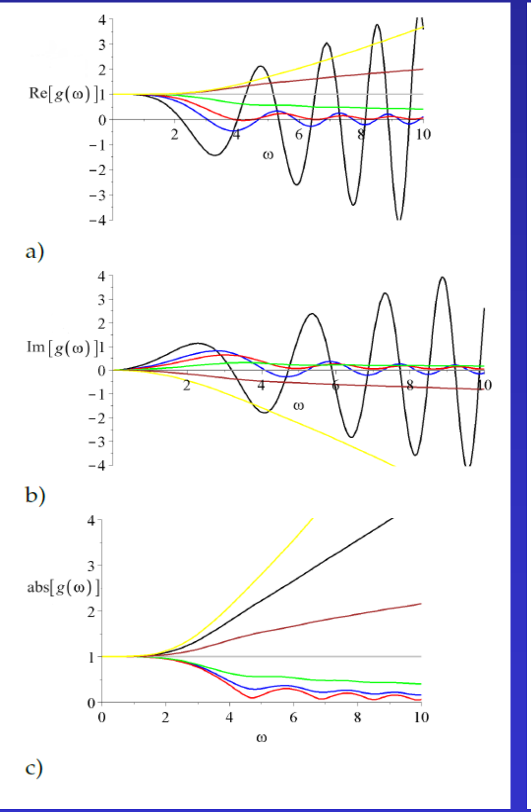
The parameters are a bit shifted

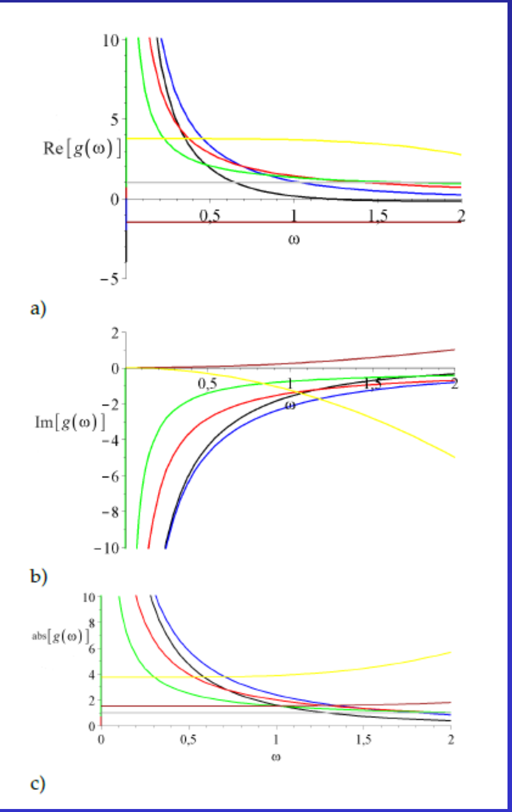
Results

Real, complex and absolute value of the Kummer's M and Kummer's U functions

α = 2, 1, 2/3, 1/4, 0, -1/4, -2/3

black, blue,red, green, gray, brown yellow





differences to the Cartesian case:
both g(ω) shape functions to to zero for some α values

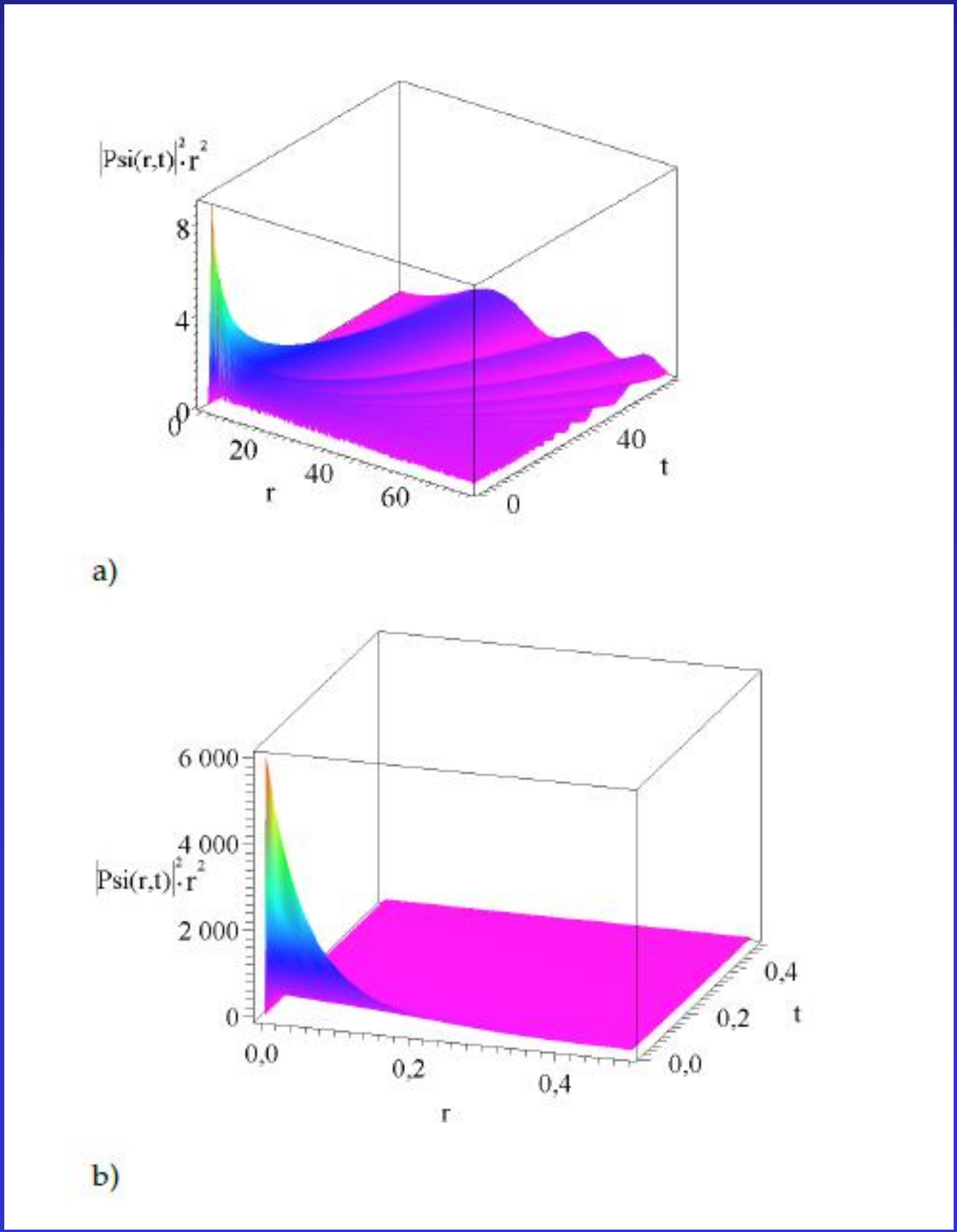
Kummer's U is singular in the origin that is ok

Results

The $|\psi(r,t)|^2$ for the
Kummer's U function
$\alpha = 1$
The integral is finite ☺

Kummer's M function
$\alpha = 1$
The integral is finite ☺

So can be interpreted as
a quantum mechanical wave
function



Summary & Outlook

Defined the self-similar Ansatz and other trial functions which can help us to find analytic solutions of PDEs or PDE systems

We investigated the classical and the complex diffusion equation (equivalent to the free Schrödinger equation)
and presented analytic results

Derivation can be generalized adding a potential term to the free Schrödinger equation, and even the role of the complex angular momenta can be investigated to $1/r^n$ potential which might be interested to Regge theory

Thank you for your attention!

Questions, Remarks, Comments?…