Equations for particles with spin S = 0 and S = 1 in spinor representation

B. E. Grinyuk

Bogolyubov Institute for Theoretical Physics
of the National Academy of Sciences of Ukraine
Kyiv, Ukraine

Sfantu Gheorghe, Romania
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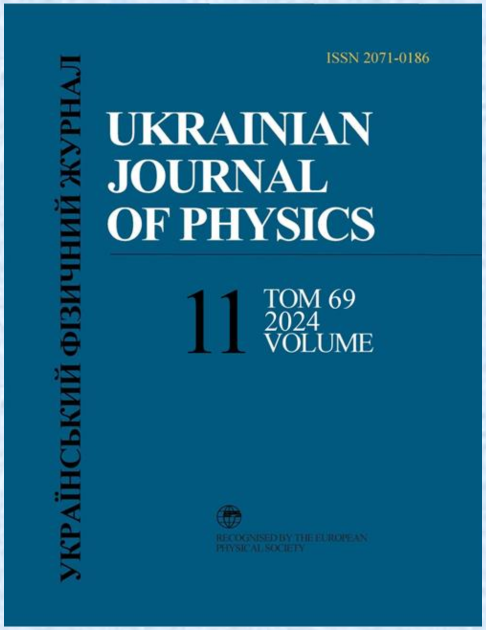

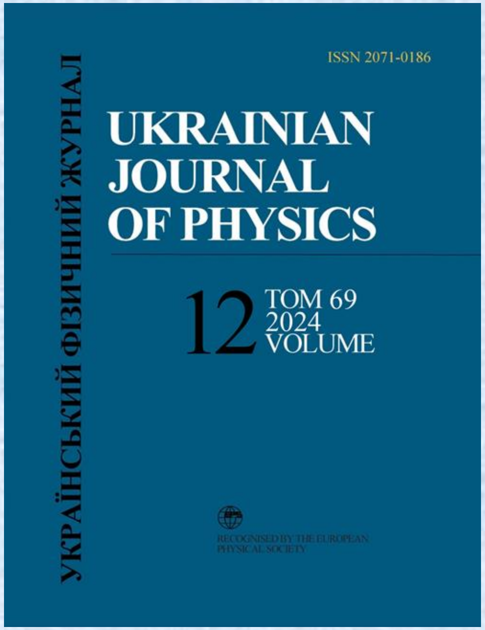

Duffin-Kemmer equations for particles with spin S=0

We remind that the well-known Duffin-Kemmer first-order differential equations [2] for a particle with spin $S = 0$ can be derived from the Klein-Gordon-Fock equation (here and further $\hbar = 1$ and $c = 1$)

$$\left(\Box - m^2\right)\varphi = 0 \qquad (1)$$

by introducing the new fields A_μ ($\mu = 0, 1, 2, 3$)

$$A_\mu \equiv \partial_\mu \varphi. \qquad (2)$$

Then instead of (1) one has the system of Duffin-Kemmer equations

$$\begin{cases} \partial_t \varphi = A_0 \,, \\ -\partial_t A_0 = (\nabla \cdot \mathbf{A}) + m^2 \varphi \,, \\ \mathbf{A} = -\nabla \varphi \,, \end{cases} \qquad (3)$$

for the five-component wave function $(\varphi, A_0, \mathbf{A})$.

The presented above Duffin-Kemmer equations (3) contain the second power of the mass of particle which is not natural for the first-order differential equations. More over, in the limit $m \to 0$, the system of equations (3) (after differentiating the last equation over time) splits into the system of four closed equations

$$\begin{cases} -\partial_t A_0 = (\nabla \cdot \mathbf{A}) \ , \\ \quad \partial_t \mathbf{A} \ = -\nabla A_0 \ , \end{cases} \tag{7}$$

with an additional condition

$$[\nabla \times \mathbf{A}] = 0 \tag{8}$$

following from the last equality of the system (3) due to identity $[\nabla \times \nabla\varphi] \equiv 0$, and a separate equation $\partial_t \varphi = A_0$, which becomes rather a definition of an additional field φ.

Now we are going to construct another system of equations having the regular limit at $m \to 0$. Let us introduce, instead of $(A_0, \mathbf{A})$ (2), the following four components $(B_0, \mathbf{B})$:

$$\begin{cases} B_0 \equiv (-\partial_t + im)\,\varphi, \\ \mathbf{B} \;\equiv\; -\nabla\varphi, \end{cases} \tag{10}$$

where φ obeys the equation (1). Then one has

$$\begin{cases} \partial_t B_0 = (\nabla \cdot \mathbf{B}) - imB_0\,, \\ \partial_t \mathbf{B} \;=\; \nabla B_0 + im\mathbf{B}\;, \end{cases} \tag{14}$$

with the additional condition

$$[\nabla \times \mathbf{B}] = 0. \tag{15}$$

The latter follows from the definition of $\mathbf{B}$ (the last equality from (10)) and the known identity $[\nabla \times \nabla\varphi] \equiv 0$.

Proca equations for spin S=1

Let us recall that if the Klein–Gordon–Fock equation (here and below, $\hbar = 1$, and $c = 1$)

$$\left(\Box - m^2\right) A_\mu = 0, \tag{1}$$

supplemented by the condition

$$\partial^\mu A_\mu = 0, \tag{2}$$

is considered to be the initial equation for a vector massive field, then the Proca equations of the first order in derivatives follow from (1) and (2), if the antisymmetric tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3}$$

is introduced, and (1) is rewritten (with regard for (2)) as

$$\partial^\mu F_{\mu\nu} + m^2 A_\nu = 0. \tag{4}$$

Equations (4) together with (3) are the ten Proca equations for ten components of the field.

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{H} = \mathbf{rot}\,\vec{A},$$

$$\vec{u} = \frac{\partial \vec{H}}{\partial t} + im\vec{H}, \qquad \vec{v} = \mathbf{rot}\,\vec{H}.$$

The first-order differential equations for a particle with $S = 1$ can be written in the form [1]:

$$\begin{cases} \partial_t \mathbf{u} = -[\nabla \times \mathbf{v}] + im\mathbf{u}\,, \\ \partial_t \mathbf{v} = \quad [\nabla \times \mathbf{u}] - im\mathbf{v}\,, \end{cases} \tag{44}$$

with the additional conditions

$$(\nabla \cdot \mathbf{u}) = 0, \qquad (\nabla \cdot \mathbf{v}) = 0. \tag{45}$$

Quaternions and complex matrices

We remind that quaternions, or hypercomplex numbers of the form

$$\mathbf{q} = \mu_0 + \mu_1 \mathbf{e}_1 + \mu_2 \mathbf{e}_2 + \mu_3 \mathbf{e}_3, \qquad (28)$$

with real numbers μ_k, are elements of four-dimensional linear space with a certain multiplication rule, which is suitable to define by the following table of mutiplication of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1,$$
$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_2,$$
$$\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_3, \quad \mathbf{e}_3\mathbf{e}_2 = -\mathbf{e}_1, \quad \mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_2. \qquad (29)$$

For each quaternion $\mathbf{q}$, one may consider the conjugate quantity $\bar{\mathbf{q}}$:

$$\bar{\mathbf{q}} = \mu_0 - \mu_1 \mathbf{e}_1 - \mu_2 \mathbf{e}_2 - \mu_3 \mathbf{e}_3. \qquad (30)$$

Then the absolute value squared of the quaternion (28) is defined as

$$|\mathbf{q}|^2 = \bar{\mathbf{q}}\mathbf{q} = \mathbf{q}\bar{\mathbf{q}} = \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^3. \qquad (31)$$

$$\hat{\mathbf{q}} = \mu_0 \hat{I} - i\mu_1\hat{\sigma}_1 - i\mu_2\hat{\sigma}_2 - i\mu_3\hat{\sigma}_3, \tag{32}$$

where the Pauli matrices are commonly used in the form

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{33}$$

In representation (32), the absolute value squared can be calculated as

$$|\hat{\mathbf{q}}|^2 = \frac{1}{2} tr\left(\bar{\hat{\mathbf{q}}}\hat{\mathbf{q}}\right) = \mu_0^2 + \mu_1^2 + \mu_2^2 + \mu_3^3, \tag{34}$$

where $\bar{\hat{\mathbf{q}}}$ is a conjugate matrix to $\hat{\mathbf{q}}$ (compare with (32)):

$$\bar{\hat{\mathbf{q}}} = \mu_0 \hat{I} + i\mu_1\hat{\sigma}_1 + i\mu_2\hat{\sigma}_2 + i\mu_3\hat{\sigma}_3. \tag{35}$$

A system of two Dirac equations with constraints for a particle with spin S = 0

Now we are going to generalize (32) and assume μ_k to be complex numbers. This means that, instead of quaternions with real μ_k, we now consider general complex matrices $\hat{\mathbf{q}}$ of 2×2 dimension parameterized by complex numbers μ_k according to (32) with the use of Pauli matrices. The absolute value squared for the new "numbers" is $\frac{1}{2}tr(\hat{\mathbf{q}}\dagger\hat{\mathbf{q}}) = |\mu_0|^2 + |\mu_1|^2 + |\mu_1|^2 + |\mu_3|^2$, where

$$\hat{\mathbf{q}}\dagger = \mu_0^* \hat{I} + i\mu_1^* \hat{\sigma}_1 + i\mu_2^* \hat{\sigma}_2 + i\mu_3^* \hat{\sigma}_3. \quad (36)$$

Instead of $\hat{\Phi}$ in the form (16), we now consider the wave function for a particle in the form

$$\hat{\Phi} = B_0 \hat{I} - iB_1\hat{\sigma}_1 - iB_2\hat{\sigma}_2 - iB_3\hat{\sigma}_3 \equiv B_0\hat{I} - i(\hat{\boldsymbol{\sigma}} \cdot \mathbf{B}), \quad (37)$$

where B_0 and B_k $(k = 1, 2, 3)$ are the same as in (10).

In these notations, the system of equations (14) can be written in the form

$$i\partial_t \hat{\Phi} = \left(i\left(\hat{\boldsymbol{\sigma}} \cdot \mathbf{p}\right) + m\right)\bar{\hat{\Phi}}, \qquad (38)$$

where $\mathbf{p} \equiv -i\nabla$ is the momentum operator, and notation $\bar{\hat{\Phi}}$ means a conjugate value (see (35)). It is suitable to accomplish (38) with the equation for $\bar{\hat{\Phi}}$:

$$i\partial_t \bar{\hat{\Phi}} = \left(-i\left(\hat{\boldsymbol{\sigma}} \cdot \mathbf{p}\right) + m\right)\hat{\Phi}. \qquad (39)$$

Each of the two equations (38), (39) is equivalent to the system of equations (14). Now we unite the functions $\hat{\Phi}$ and $\bar{\hat{\Phi}}$ into one matrix $\hat{\psi}$

$$\hat{\psi} \equiv \begin{pmatrix} \bar{\hat{\Phi}} \\ \hat{\Phi} \end{pmatrix} = \begin{pmatrix} B_0 \hat{I} + i\left(\hat{\boldsymbol{\sigma}} \cdot \mathbf{B}\right) \\ B_0 \hat{I} - i\left(\hat{\boldsymbol{\sigma}} \cdot \mathbf{B}\right) \end{pmatrix} = \begin{pmatrix} B_0 + iB_3 & iB_1 + B_2 \\ iB_1 - B_2 & B_0 - iB_3 \\ B_0 - iB_3 & -iB_1 - B_2 \\ -iB_1 + B_2 & B_0 + iB_3 \end{pmatrix} \qquad (40)$$

with the absolute value squared

$$|B_0|^2 + |B_1|^2 + |B_2|^2 + |B_3|^2 = \frac{1}{4} tr\left(\hat{\psi}\dagger\hat{\psi}\right). \qquad (41)$$

It is obvious that the equation for $\hat{\psi}$ has the form

$$i\partial_t\hat{\psi} = (\hat{\boldsymbol{\alpha}}\cdot\mathbf{p})\,\hat{\psi} + m\hat{\beta}\hat{\psi}, \tag{42}$$

where $\hat{\alpha}_k$ and $\hat{\beta}$ are the Dirac matrices in the following representation:

$$\hat{\alpha}_k \equiv \hat{\sigma}_2 \otimes \hat{\sigma}_k, \quad \hat{\beta} \equiv \hat{\sigma}_1 \otimes \hat{I}. \tag{43}$$

Thus we have the Dirac equation (42) for two four-component columns, or (written for each column separately) the system of two Dirac equations for ordinary four-component wave functions. But it is essential to take into account additional condition (15) or (20). It is also important to take into account that the both four-component wave functions are bounded between themselves (see (40)), since we have only four independent fields B_μ, where $\mu = 0, 1, 2, 3$.

A system of two Dirac equations with constraints for a particle with spin S = 1

The first-order differential equations for a particle with $S=1$ can be written in the form [1]:

$$\begin{cases} \partial_t \mathbf{u} = -[\nabla \times \mathbf{v}] + im\mathbf{u}\,, \\ \partial_t \mathbf{v} = [\nabla \times \mathbf{u}] - im\mathbf{v}\,, \end{cases} \tag{44}$$

with the additional conditions

$$(\nabla \cdot \mathbf{u}) = 0, \qquad (\nabla \cdot \mathbf{v}) = 0. \tag{45}$$

Let us introduce the wave function $\hat{\varphi}$ in the form of matrix (having two four-component columns):

$$\hat{\varphi} \equiv \begin{pmatrix} (\hat{\boldsymbol{\sigma}} \cdot \mathbf{u}) \\ (\hat{\boldsymbol{\sigma}} \cdot \mathbf{v}) \end{pmatrix} = \begin{pmatrix} u_3 & u_1 - iu_2 \\ u_1 + iu_2 & -u_3 \\ v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} \tag{46}$$

with the absolute value squared

$$\sum_{k=1}^{3} \left(|v|_k^2 + |u|_k^2 \right) = \frac{1}{2} tr\left(\hat{\varphi} \dagger \hat{\varphi}\right). \tag{47}$$

In these notations, the system of equations (44) takes the form of the Dirac equation

$$i\partial_t \hat{\varphi} = (\hat{\boldsymbol{\alpha}} \cdot \mathbf{p})\,\hat{\varphi} + m\tilde{\hat{\beta}}\hat{\varphi}. \tag{48}$$

(or the system of two Dirac equations for each of the two columns of the wave function $\hat{\varphi}$).

$$\tilde{\hat{\beta}} = -\hat{\sigma}_3 \otimes \hat{I}. \tag{49}$$

Obviously, one can carry out a similarity transformation and obtain the equation (48) in the same representation as the one of (42). It is necessary to keep in mind that the components of the wave function (45) obey the conditions (45).

Thus we have an important conclusion that the equations for both the particles with spin $S = 0$ and $S = 1$ formally obey the same system of two Dirac equations, but also are constrained with different additional conditions: (15) for $S = 0$, and (45) for $S = 1$. In essence, the additional conditions "construct" particles with even spin (0 or 1) from two particles with spin $\frac{1}{2}$.

High energy limit – new degrees of freedom

Let us assume that the both particles have the same mass m. If we introduce the wave function

$$\hat{F} \equiv \begin{pmatrix} \hat{I}\cdot f + i\,(\hat{\boldsymbol{\sigma}}\cdot \mathbf{u}) \\ \hat{I}\cdot g + i\,(\hat{\boldsymbol{\sigma}}\cdot \mathbf{v}) \end{pmatrix} = \begin{pmatrix} f+iu_3 & iu_1+u_2 \\ iu_1-u_2 & f-iu_3 \\ g+iv_3 & iv_1+v_2 \\ iv_1-v_2 & g-iv_3 \end{pmatrix}, \qquad (50)$$

then the Dirac equation

$$i\partial_t \hat{F} = (\hat{\boldsymbol{\alpha}}\cdot\mathbf{p})\,\hat{F} + m\hat{\beta}\hat{F} \qquad (51)$$

combines the both systems of equations for the case $S = 0$ (see (14), (15)) and for $S = 1$ (see (44), (45)). In order to make this fact obvious, we rewrite the system (50) in an explicit form (without use of matrices):

$$\begin{cases} \partial_t f &= -\,(\nabla\cdot\mathbf{v}) + imf, \\ \partial_t \mathbf{v} &= -\,\nabla f + [\nabla\times\mathbf{u}] - im\mathbf{v}\,, \\ \partial_t g &= (\nabla\cdot\mathbf{u}) - img\,, \\ \partial_t \mathbf{u} &= \nabla g - [\nabla\times\mathbf{v}] + im\mathbf{u}\,. \end{cases} \qquad (52)$$

$$i\partial_t \hat{F} = (\hat{\boldsymbol{\alpha}} \cdot \mathbf{p})\, \hat{F} + m\hat{\beta}\hat{F} \tag{51}$$

At the same time, if we make a transition from eight independent components f, g, $\mathbf{u}$, and $\mathbf{v}$ of the wave function $\hat{F}$ (50) to eight components ξ_μ and η_μ ($\mu = 1, 2, 3, 4$) according to

$$\hat{F} = \begin{pmatrix} f+iu_3 & iu_1+u_2 \\ iu_1-u_2 & f-iu_3 \\ g+iv_3 & iv_1+v_2 \\ iv_1-v_2 & g-iv_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \\ \xi_3 & \eta_3 \\ \xi_4 & \eta_4 \end{pmatrix}, \tag{53}$$

we find that the system of equations is nothing else but the set of two Dirac equations for two independent particles with spin $S = \frac{1}{2}$.

Conclusions

To summarize, we would like to notice the following conclusions and make the following generalizations:

* Equations for a particle with spin S can be formulated (or reformulated) in the form of a system of Dirac equations with additional conditions (constraints).

** The equations for $S = 0$ and $S = 1$ in the limit of high energies reveal new degrees of freedom being particles with spin $S = \frac{1}{2}$ without any preliminary assumptions about their existence.

THANK YOU!