Modified Homotopic approach for diffractive production in the saturation region

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Outline

- Introduction
- BK equation and Geometric Scaling $N(r_T, Y, b)$
- Evolution equation for $N^D(r_T, Y, b)$
- Homotopic approach
- Summary and Outlook

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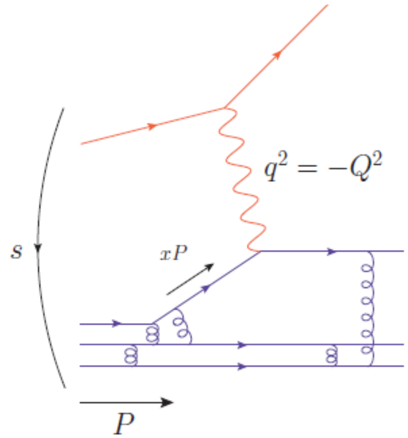
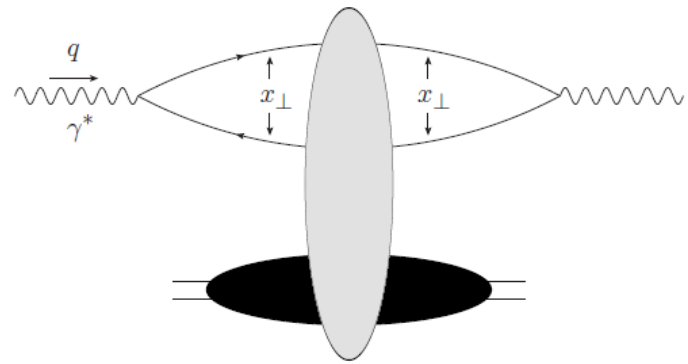
Eur. Phys. J. C **79 (2019) 10, 842**
arXiv 2406.11673

Dipole Approximation





- HERA data represented the most direct way of probing that the virtual photon fluctuates into a $\bar{q}q$ pair long before the scattering
- the $\bar{q}q$ color dipole acts as a probe of the gluon distribution at small x
- the dipole transverse size r is preserved by the scattering
- Photon wave function $\Psi^{\gamma}{}_{q\bar{q}}\,(\mathrm{r,z,Q^2})$ in dipole approximation (known to NLO light cone)
- $N(r_1, Y, b)$ is the dipole-Hadron scattering Amplitud

$$Scattering\ amplitud\ S(r_T, Y, b) = 1 - N(r_T, Y, b) \quad where\ Y = \ln\left(\frac{1}{x}\right) = \ln s \text{ rapidity}$$

- $\sigma_{T,L}(Y, Q^2) = 2\int d^2r \int d^2b \int dz|\,|\Psi_{\mathrm{NL,T}}(\mathrm{r,z,Q^2})|^2 N(r, Y, b)$

Quantum Chromodynamics at high Energy, Kovchegov and Levin Cambridge 2012

Mueller 2002

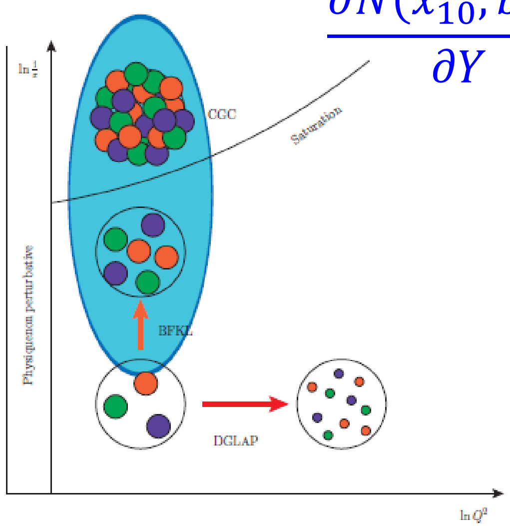
BK and BFKL equation

QCD at small $x_{Bj} \sim Q^2/s$

Regge limit: $Q^2 \ll s$

Evolution in small x variable is given by the Balitsky, Fadin, Kuraen and Lipatov BFKL (1977) equation ***N(r, Y, b)***

$$\frac{\partial N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2 \frac{x_{02}^2}{x_{12}^2\, x_{01}^2}\left(N(x_{12},b,Y) + N(x_{20},b,Y)\right.$$
$$\left.-(N(x_{10},b,Y)\right)$$



To high Rapidity Y (High density)

There is problem with this linear equation and one need to introduce non linear term: Saturation region (Balitsky, Kovchegov 1996)

$$\frac{\bar{\alpha}}{2\,\pi}\int d^2x_2 \frac{x_{02}^2}{x_{12}^2\, x_{01}^2} N(x_{10},b,Y) * N(x_{20},b,Y)$$

Rigorous treatment requires solution of b-dependent BK equation (or JIMWLK CGC).

Evalution in Q^2: $DGLAP\ equation$ Better resolution in the partons

Dokshitzer–Gribov–Lipatov–Altarelli–Parisi

Rezaeian and Schmidt *Phys. Rev.D* 88 (2013)

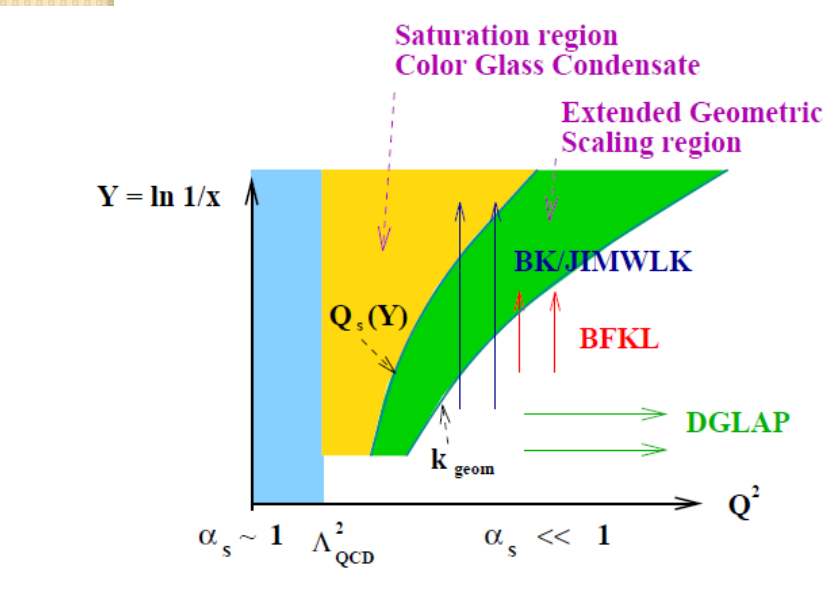
BFKL and BK equation: Transition to saturation and Geometric Scaling

Stasto, Golec-Biernat and Kwiecinski Phys. Rev. Lett. 86 (2001) 596.

have shown that the HERA data on DIS at low x , functions of two independent variables — the photon virtuality Q^2 and the Bjorken variable x , are consistent with scaling in terms of the variable $\tau = r^2 Q^2_{sat}(Y)$

Saturation momentum:

$$Q_s^2(Y) = Q_{s0}^2 e^{\lambda Y} = Q_{s0}^2 \left(\frac{x_0}{x}\right)^{\lambda} \rightarrow \tau = Q^2/Q_0^2 \left(\frac{x}{x_0}\right)^{\lambda} \qquad \lambda = 0.3 - 0.4$$



is to show that the scaling region for the various distribution functions is in fact much larger than the saturation region

Balistky Phys. Rev. D75, 014001 (2007)
I. Balitsky, Nucl. Phys. B463, 99 (1996)
Kovchegov, Iancu, Itakura, and Larry McLerran arXiv 0203.137
Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK)

BK evolution

$$\frac{\partial(N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2 \frac{x_{02}^2}{x_{12}^2\,x_{01}^2}(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) - N(x_{12},b,Y) * N(x_{20},b,Y))$$

in principle need to solve the fully impact parameter dependent Balitsky Kovchegov (BK) equation

this work: local approximation → b becomes an external parameter

Initial condition at Y_0: McLerran -Venugopalan model MV:

$$\mathrm{N}(r_\perp, b, Y) = 1 - \exp\left[-\frac{r_\perp^2 Q_{s0}^2}{4}\ln\left(\frac{1}{r_\perp\Lambda} + e\right)\right]$$

Kowalski, Lappi, Marquet, Venugopalan, PRC 78 (2008) 045201
Lappi, M¨antysaari, PRD 88 (2013) 114020

where initial saturation scales:

$$Q_{s0}^2 = \begin{cases} Q_{s0}^2 & \text{for proton} \\ \mathrm{A}^{1/3} \times Q_{s0}^2 & \text{for nucleus} \end{cases}$$

Analytical Solutions to the BK equation

- satisfying the initial condition given by the BFKL Pomeron
- satisfying the Geometric Scaling inside the saturation region, Non-perturbative

Unfortunately finding an exact analytical solution seems difficult task, its is non–linear. In the saturation regime the series diverges, but allows us to construct an asymptotic solution by analytical continuation

Perturbations approach

- First BFKL Pomeron Lipatov solution (1986)

Eigenfunctions of the Casimir operators of conformal algebra

$$\bar{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}.$$

$$E^{n,\nu}(\rho_0,\rho_1) = \left(\frac{\rho_{01}}{\rho_0\,\rho_1}\right)^{\frac{1+n}{2}+i\nu} \left(\frac{\rho_{01}^*}{\rho_0^*\,\rho_1^*}\right)^{\frac{1-n}{2}+i\nu}$$

$$\chi(n,\nu) = 2\psi(1) - \psi\left(\frac{1+|n|}{2} + i\,\nu\right) - \psi\left(\frac{1+|n|}{2} - i\,\nu\right);$$

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz}$$ Digamma Function

$$N(x_T, Y) = \int_{-\infty}^{\infty} d\nu\; e^{2\,\bar{\alpha}_s\,\chi(0,\nu)\,Y}\,(x_T Q_{s0})^{1+2\,i\,\nu}\,C_\nu$$

$\gamma = \frac{1}{2} + i\,\nu$ Dominant contribution to high Y is

$$C_\nu \equiv \tilde{C}_\nu\,2^{-2\,i\,\nu}\,\frac{\Gamma\left(\frac{1-2\,i\,\nu}{2}\right)}{\Gamma\left(\frac{1+2\,i\,\nu}{2}\right)}.$$

$$\chi(0,\gamma) = 2\psi(1) - \psi(\gamma) - \psi(\gamma - 1) = \chi(\gamma)$$

Perturbations approach

Y. Kovchegov arXiv 9905214

Perturbative solution and it is convergent outside of the saturation region.

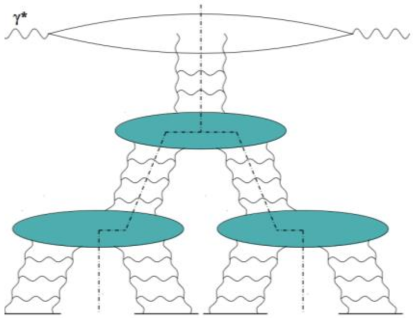
$$\frac{\partial \tilde{N}_1(k,Y)}{\partial Y} = \frac{2\alpha N_c}{\pi}\,\chi\left(-\frac{\partial}{\partial \ln k}\right)\tilde{N}_1(k,Y), \qquad \tilde{N}_1(k,Y) = \int \frac{d\lambda}{2\pi i}\,\exp\left[\frac{2\alpha N_c}{\pi}\,Y\,\chi(-\lambda)\right]\left(\frac{k}{\Lambda}\right)^{\lambda} C_\lambda.$$

He showed that as energy increases the scattering cross section of the quark–antiquark pair of a fixed transverse separation on a hadron or nucleus given by the solution of BK equation inside of the saturation region unitarizes

We need to construct a solution **BK in coordinate space, N(x⊥,Y),** and the corresponding structure function F2 one has to have a better knowledge of the momentum space solution inside the saturation region.

A solution of BK would probably be very helpful in determining exact values of N(x⊥, Y) and F2 at intermediately large rapidity

$$\frac{\partial \tilde{N}_2(k,Y)}{\partial Y} = \frac{2\alpha N_c}{\pi}\,\chi\left(-\frac{\partial}{\partial \ln k}\right)\tilde{N}_2(k,Y) - \frac{\alpha N_c}{\pi}\,\tilde{N}_1(k,Y)^2,$$



Motika and Sadzikowski arXiv 230602118 Hight Twist corrections

Analytical Solutions to the BK equation

Non-Perturbations approach

Levin-Tuchin Solutions inside the saturation region GS (2000)

$$N(r_{\perp}, b, Y) = 1 - \Delta(r_{\perp}, Y, b)$$

consider that the side of the dipole are: $x_{21} \approx x_{10}$; $x_{20} < x_{10}$; $\mathrm{x_{ij}} \gg \frac{1}{Q_s(Y)}$

we can find

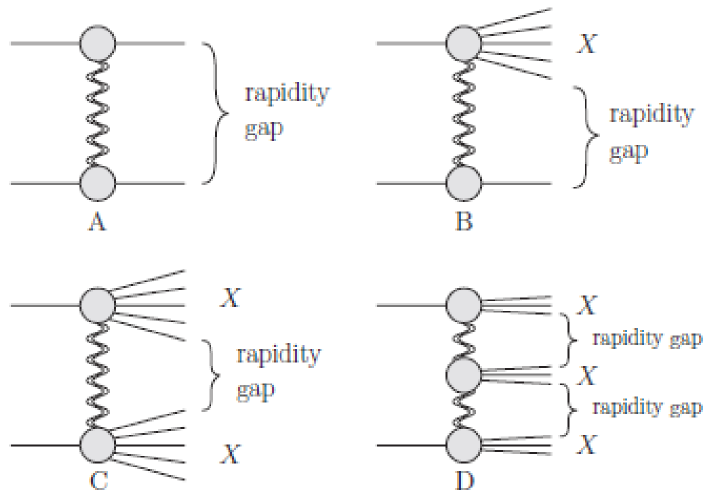
$$\frac{\partial \Delta(r_{\perp}, Y, b)}{\partial Y} = -\bar{\alpha}\, z\, \Delta(r_{\perp}, Y, b)$$

where $z = \ln r^2 Q_s^2(Y) = \bar{\alpha}\, \kappa\, Y + \ln r^2 Q_s^2(Y_0) = \bar{\alpha}\, \kappa\, Y + \xi_0$

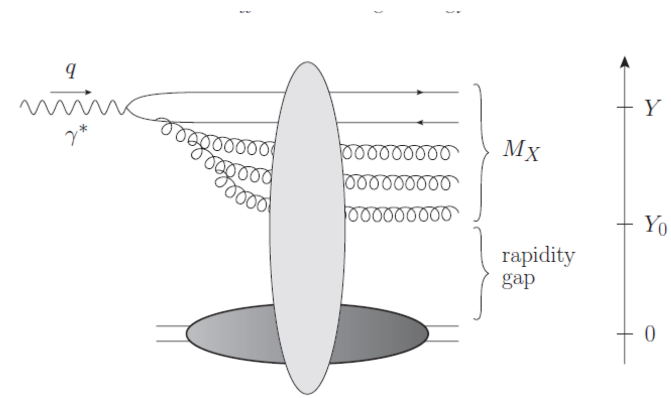
The solution es

$$N(r, b, Y) = 1 - \Delta_0 e^{-\frac{z^2}{2\kappa}}$$

Diffraction at high energy



Single Diffraction



Kovchegov-Levin (KL) equation (2000)

cross section of diffractive production with the rapidity gap larger than Y_0

$$\sigma^{\rm diff}(Y, Y_0, Q^2) \; = \int d^2 r_\perp \int dz \; |\Psi^{\gamma^*}(Q^2; r_\perp, z)|^2 \; \sigma^{diff}_{\rm dipole}(r_\perp, Y, Y_0)$$

where

$$\sigma^{diff}_{dipole}(r_\perp, Y, Y_o) = \int d^2 b \, N^D(r_\perp, Y, Y_0, b)$$

Diffraction at high energy

Kovchegov-Levin (KL) equation (2000)

$$\frac{\partial N^D(Y, Y_0, r_{10}; b)}{\partial Y_M} =$$
$$\frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2 \, K(r_{10}|r_{12}, r_{02}) \left\{ N^D(Y, Y_0, r_{12}; b) + N^D(Y, Y_0, r_{20}; b) - N^D(Y, Y_0, r_{10}; b) \right.$$
$$+ N^D(Y; Y_0, r_{12}; b) N^D(Y; Y_0, r_{20}; b) - 2\, N^D(Y; Y_0, r_{12}; b)\, N(Y; r_{20}; b)$$
$$\left. - 2\, N(Y; r_{12}; b) N^D(Y; Y_0, r_{20}; b) + 2 N(Y; r_{12}; b) N(Y; r_{20}; b) \right\}$$

$$\bar{\alpha}_s = \frac{\alpha_s N_c}{\pi}$$

- This is a nonlinear evolution equation with the initial condition specified at $Y = Y_0$
- To solve it one has first to solve the BK equation to find the dipole amplitude *N*
- For this equation for N^D and the BK equation, no analytic solution exists.
- Numerical solution: Levin and Lublinsky 2001-2002 geometrical scaling in a broad kinematic region for rapidity

Diffraction at high energy

Introducing a new variables:

$$\mathcal{N}(z, \delta\tilde{Y}, \delta Y_0) = 2\,N(z, \delta\tilde{Y}) - N^D(z, \delta\tilde{Y}, \delta Y_0)$$

with the Initial Condition

$$N^D(r, Y = Y_0, Y_0) = N^2(r, Y = Y_0)$$

$$\frac{\partial \mathcal{N}_{01}}{\partial Y} = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x_{01}^2}{x_{02}^2\, x_{12}^2} \left\{ \mathcal{N}_{02} + \mathcal{N}_{12} - \mathcal{N}_{02}\mathcal{N}_{12} - \mathcal{N}_{01} \right\}$$

$$\boldsymbol{N(r_{10}, Y) \to N(\xi, Y) \to N(z, Y)}$$

,

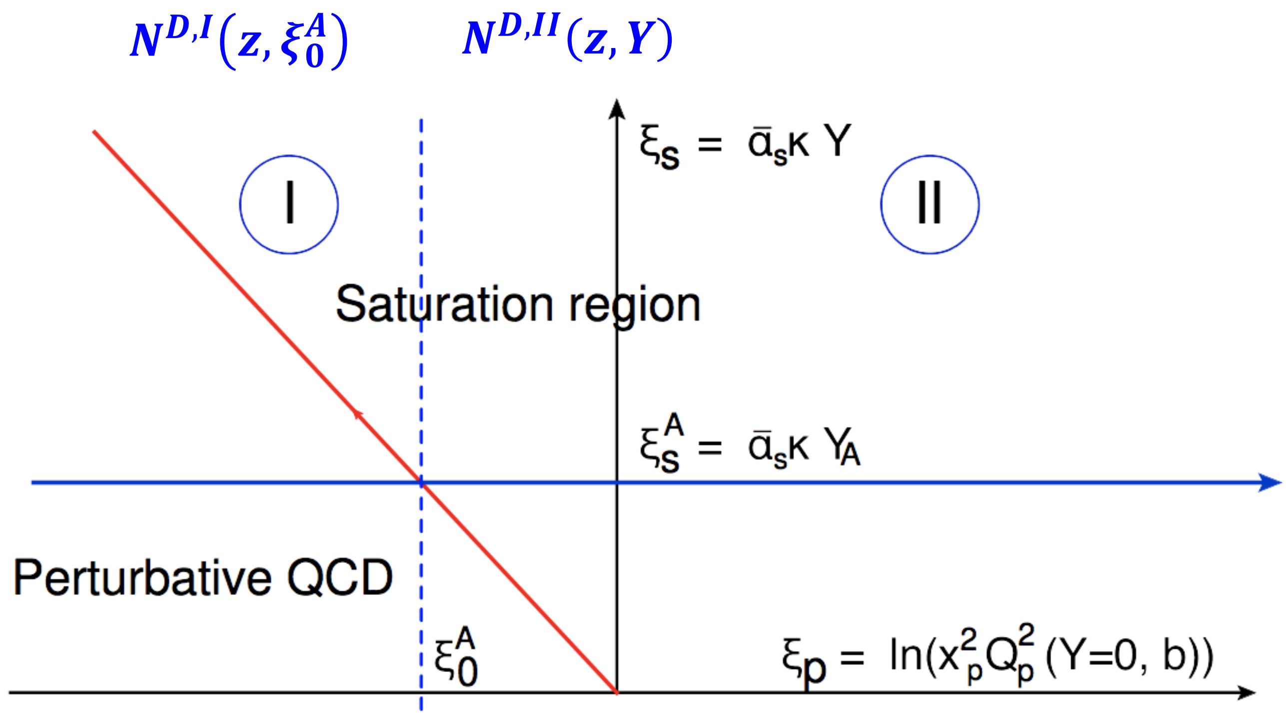
$$z = \ln\left(r^2 Q_s^2\left(\delta\tilde{Y}, b\right)\right) = \bar{\alpha}_S\, \kappa\, (Y - Y_A) + \xi \qquad \xi = \ln\left(Q_s^2\left(\delta\tilde{Y} = 0\right) r^2\right)$$

,

$$Q_s^2\,(Y, b) = Q_s^2\,(Y = Y_A, b)\; e^{\bar{\alpha}_S\, \kappa (Y - Y_A)}$$

This equation takes the form of the Balitsky-Kovchegov (BK) equation

$$z = \ln\left(r^2 Q_s^2\left(\delta\tilde{Y}, b\right)\right) = \bar{\alpha}_S \, \kappa \, (Y - Y_A) + \xi$$



Saturation region of QCD for the scattering amplitude. The critical line (z=0) is shown in red.

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Diffraction at high energy

replacing $\mathcal{N}(z, \delta Y_0)$ by $\mathcal{N}(z, \delta Y_0) = 1 - \Delta^D(z, \delta Y_0)$

we get

$$\frac{\partial \Delta^D_{01}}{\partial Y} = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x^2_{01}}{x^2_{02}\, x^2_{12}} \left\{ \Delta^D_{02} \Delta^D_{12} - \Delta^D_{01} \right\}$$

with initial conditions

$$\mathcal{N}(z \to z_0, \delta\tilde{Y} = \delta Y_0, \delta Y_0) = 2\, N(z_0, \delta Y_0) - N^2(z_0, \delta Y_0) = 1 - N^2(z_o, \delta Y_0)$$

$$\text{Region I}: \quad \Delta^D_{01}(z \to z_0, \delta Y_0) = C^2 \exp\left(-\frac{(z_0 - \tilde{z})^2}{\kappa} \right)$$

$$\text{Region II}: \quad \Delta^D_{01}\left(z \to z_0, \delta\tilde{Y} \to \delta Y_0, \delta Y_0\right) = 1 - N_{in} = G^2(\xi) \exp\left(-\frac{(z_0 - \tilde{z})^2}{\kappa} \right)$$

with $\quad G(\xi) = \exp\left(\frac{(\xi - \tilde{z})^2}{2\,\kappa} - \frac{1}{4} e^{\xi} \right)$ and $\tilde{z} = 2(\ln 2 + \psi(1)) - \zeta$

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Modified homotopy approach

$$\frac{\partial \Delta^D_{01}}{\partial Y} = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x^2_{01}}{x^2_{02}\, x^2_{12}} \left\{ \Delta^D_{02}\Delta^D_{12} - \Delta^D_{01} \right\}$$

$$\mathscr{L}[u] + \mathscr{N}_{\mathscr{L}}[u] = 0$$

The homotopy method we can use for the general equation:

$$\mathscr{H}(p,u) = \mathscr{L}[u_p] + p\, \mathscr{N}_{\mathscr{L}}[u_p] = 0$$

Solving, we reconstruct the function

$$u_p(Y, \boldsymbol{x}_{10}, \boldsymbol{b}) = u_0(Y, \boldsymbol{x}_{10}, \boldsymbol{b}) + p\, u_1(Y, \boldsymbol{x}_{10}, \boldsymbol{b}) + p^2\, u_2(Y, \boldsymbol{x}_{10}, \boldsymbol{b}) + \dots$$

We suggest to simplify the non-linear term replacing it by

$$\bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x^2_{01}}{x^2_{02}\, x^2_{12}} \Delta^D_{02}\Delta^D_{12} \to \Delta^D_{01} \int_0^z dz' \Delta^D_{02} = \Delta^D_{01} \left(\zeta - \int_z^\infty dz' \Delta^D_{02} \right) \text{ with } \zeta = \int_0^\infty dz' \Delta^D_{02}$$

$$\text{with } \zeta = \int_0^\infty dz' \Delta^D_{02}$$

We modify the homotopic approach considering:

$$\mathscr{L}\left(\Delta_0^D\right) = \left(\frac{\partial}{\partial \tilde{Y}} + z - \zeta\right)\Delta_0^D + \Delta_0^D\left(z, \tilde{Y}, z_0\right)\int_z^\infty dz' \Delta_0^D\left(z', \delta\tilde{Y}, z_0\right)$$

$$\mathscr{N}_{\mathscr{L}}[\Delta^D] = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi}\frac{x_{01}^2}{x_{02}^2\, x_{12}^2}\Delta_0^D\left(x_{02}\right)\Delta_0^D\left(x_{12}\right) - \Delta_0^D \int^{x_{01}^2} \frac{dx_{02}^2}{x_{02}^2}\Delta_{02}^D$$

The first iteration (p = 0) gives

$$\mathscr{L}\left(\Delta_0^D\right) = 0; \quad \left(\frac{\partial}{\partial \tilde{Y}} + z - \zeta\right)\Delta_0^D + \Delta_0^D\left(z, \tilde{Y}, z_0\right)\int_z^\infty dz' \Delta_0^D\left(z', \delta\tilde{Y}, \boldsymbol{z}_0\right) = 0;$$

We need information about $\boldsymbol{\Delta_0^D}$

Traveling Wave solution

Munier and Peschanski 2003

$$\kappa \frac{d\mathcal{N}_{01}(z,\xi_s)}{d\xi_s} = (1-\mathcal{N}_{01}(\xi,\xi_s)) \int_{-\xi_s}^{\xi} d\xi' \mathcal{N}_{02}(\xi',\xi_s); \quad \text{with} \quad \xi_s = \kappa\,\delta\tilde{Y}$$

Introducing $\Delta^{(0)}(z,\xi_s) = 1 - \mathcal{N}_{01}(z,\xi_s) = \exp\left(-\Omega^{(0)}(z,\xi_s)\right)$ we obtain:

$$\frac{\partial^2\Omega^{(0)}(z,\xi_s)}{\partial z^2} - \frac{\partial^2\Omega^{(0)}(z,\xi_s)}{\partial t^2} = \frac{1}{\kappa}\left(1 - e^{-\Omega^{(0)}(z',\xi_s)}\right)$$

where $z = \xi_s + \xi$ and $t = \xi_s - \xi$.

Region I: Geometric Scaling $\Omega(z,\xi) \to \Omega(z)$

$$\kappa \frac{\partial^2\Omega^{(0)}(z,\xi_s)}{\partial \xi_s\,\partial z} = 1 - e^{-\Omega^{(0)}(z,\xi_s)}$$

$$\frac{d\Omega^{(0)}(z)}{dz} = p\left(\Omega^{(0)}\right) \qquad \frac{1}{2}\kappa\frac{dp^2}{d\Omega^{(0)}} = 1 - e^{-\Omega^{(0)}(z)}$$

(region I)

$$p = \frac{d\Omega^{(0)}}{dz} = \sqrt{\frac{2}{\kappa}\left(\Omega^{(0)} + \exp\left(-\Omega^{(0)}\right) - 1\right) + C_1}$$

we solve this, getting the following implicit solution

$$\int_{\Omega_0^{(0)}}^{\Omega^{(0)}} \frac{d\Omega'}{\sqrt{\Omega' + \exp(\Omega') - \Omega_0}} = \sqrt{\frac{2}{\kappa}}(z - \tilde{z})$$

with

$$\Omega_0^{(0)} = (z_0 - \tilde{z})^2/(2\kappa) - 2\ln(C).$$

We define

$$\mathscr{U}\left(\Omega^{(0,I)}, \Omega_0^{(0)}\right) = \sqrt{\frac{2}{\kappa}}(z - \tilde{z})$$

For finding $\Omega^{(0,I)}(z)$ in a general case we have to find the inverse function for $\mathscr{U}$

Then we obtain the following result

$$\Delta_0^{(0,I)}(z) \;=\; \Delta_{LT}(z)\exp\left(-a+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2k-1)!!}{2^k k\,k!}\left(\frac{2\kappa}{(z-\tilde{z})^2}\right)^{k+\frac{1}{2}}\Delta_{LT}^{k}(z)\right)$$

$$\text{where } \Delta_{LT}(z) \;=\; \exp\left(-\frac{(z-\tilde{z})^2}{2\,\kappa}\right)$$

Region II: General solution $\Omega(z, \xi)$

$$\frac{\partial^2 \Omega^{(0)}\left(z, \xi_s\right)}{\partial z^2} - \frac{\partial^2 \Omega^{(0)}\left(z, \xi_s\right)}{\partial t^2} = \frac{1}{\kappa}\left(1 - e^{-\Omega^{(0)}\left(z', \xi_s\right)}\right)$$

where $z = \xi_s + \xi$ and $t = \xi_s - \xi$.

Solving this equation, we obtain

$$\int_{\Omega_0}^{\Omega^{(0)}(z,\xi_s)} \frac{d\Omega'}{\sqrt{-\Omega_0 + \Omega' + \exp\left(-\Omega'\right)}} = \sqrt{\frac{2}{\kappa}}\left(\left(1+\nu\right) z + \nu t - \tilde{z} - 2\nu\, \xi_{0,s}\right)$$

where $\xi_{0,s} = \kappa \delta Y_0$ and

$$\Omega_0 = \frac{\left(z_0 - \tilde{z}\right)^2}{2\kappa} - \frac{\left(z_0 - \xi_{0,s} - \tilde{z}\right)^2}{\kappa} + \frac{1}{2}\exp\left(z_0 - \xi_{0,s}\right)$$

$$\mathscr{U}\left(\Omega^{(0,II)},\Omega_0\right) = \sqrt{\frac{2}{\kappa}}\left(\left(1+\nu\right)z + \nu t - \hat{z}\right)$$

where $\hat{z} = \tilde{z} + 2\nu\xi_{0,s}$

We repeat the same steps of before an obtain:

$$\Delta_0^{(0,II)}\left(z,\delta\tilde{Y}\right) = \tilde{\Delta}_{LT}\left(z,\xi_s\right)\exp\left(-\Omega_0 + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2k-1)!!}{2^k k\,k!}\left(\frac{2\kappa}{\left((1+\nu)z+\nu t-\hat{z}\right)^2}\right)^{k+\frac{1}{2}}\tilde{\Delta}^k_{LT}\left(z,\xi_s\right)\right)$$

where

$$\tilde{\Delta}_{LT}\left(z,\xi_s\right) = \exp\left(-\frac{\left((1+\nu)z+\nu t-\hat{z}\right)^2}{2\,\kappa}\right)$$

$$\Delta_0^{(0,II)}\left(z,\xi_s\right) = \exp\left(-\frac{\left((1+\nu)z+\nu t-\tilde{z}\right)^2}{2\,\kappa} - \frac{\left(z_0-\tilde{z}\right)^2}{2\,\kappa} + \frac{\left(z_0-\xi_{0,s}-\tilde{z}\right)^2}{\kappa} - \frac{1}{2}\exp\left(z_0-\xi_{0,s}\right)\right.$$

$$\nu = 0$$

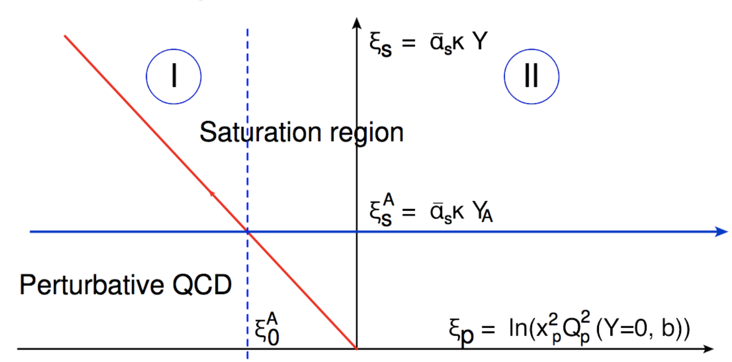
Matching on the line $\boldsymbol{\xi} = \boldsymbol{\xi_0^A}$

$$\Delta_0^{(0,II)}\left(z_A,\delta\tilde{Y}\right) = \underbrace{\exp\left(-\frac{\left((1+\nu)z_A+\nu t_A-\tilde{z}-2\nu\,\xi_{0,s}\right)^2}{2\,\kappa}-\frac{\left(z_{0,A}-\tilde{z}\right)^2}{2\,\kappa}+\frac{\left(z_{0,A}-\xi_{0,s}-\tilde{z}\right)^2}{\kappa}-\frac{1}{2}\exp\left(z_{0,A}-\xi_{0,s}\right)\right)}_{region\quad II}$$

$$= \underbrace{\Delta^{(0,I)}(\eta(z_A)) = C^2\exp\left(-\frac{(z_A-\tilde{z})^2}{2\,\kappa}-\Omega_0^{0,I}\right)}_{region\quad I}$$

we find $\nu = 0$

and $$C^2 = \exp\left(\frac{\left(\xi_0^A-\tilde{z}\right)^2}{\kappa}-\frac{1}{2}e^{\xi_0^A}\right)$$



First order correction in the modified Homotopic Approach

$$\left(\kappa\frac{\partial}{\partial z} + z - \zeta\right)\Delta_1^D(z,z_0) + \underbrace{\Delta_0^D(z,z_0)\int\limits_z^\infty dz'\Delta_1^D(z',\boldsymbol{z}_0) + \Delta_1^D(z,z_0)\int\limits_z^\infty dz'\Delta_0^D(z',\boldsymbol{z}_0)}_{\sim(\Delta^0)^3} = -\underbrace{\mathscr{N}_{\mathscr{L}}[\Delta_0^D(z)}_{\sim(\Delta^0)^2}$$

Taking into account only terms of the order of $\left(\Delta^0\right)^2$

$$\left(\kappa\frac{\partial}{\partial z} + z - \zeta\right)\Delta_1^D(z,z_0) = -\mathscr{N}_{\mathscr{L}}[\Delta_0^D(z)$$

The particular solution can be written as follows

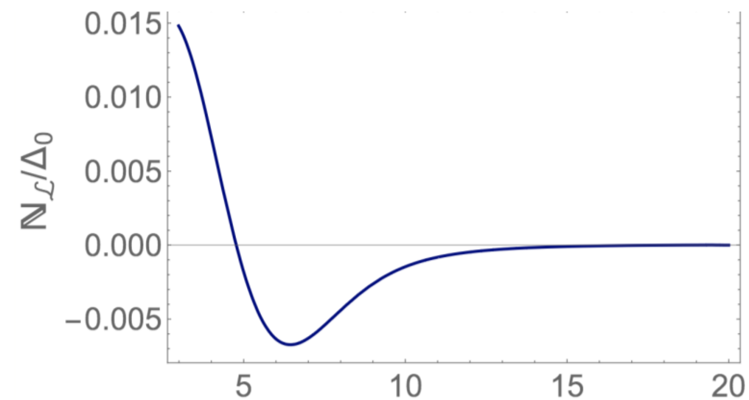
$$\Delta_1^D(z,z_0) = -\Delta_0^D(z,z_0)\int\limits_z^\infty dz'\frac{1}{\Delta_0^D(z',z_0)}\mathscr{N}_{\mathscr{L}}[\Delta_0^D(z')]$$

$$\Delta_0^D(z,\xi) = \exp\left(-\frac{(z-\tilde{z})^2}{2\kappa} + \phi^I(\zeta,z_0)\right)$$

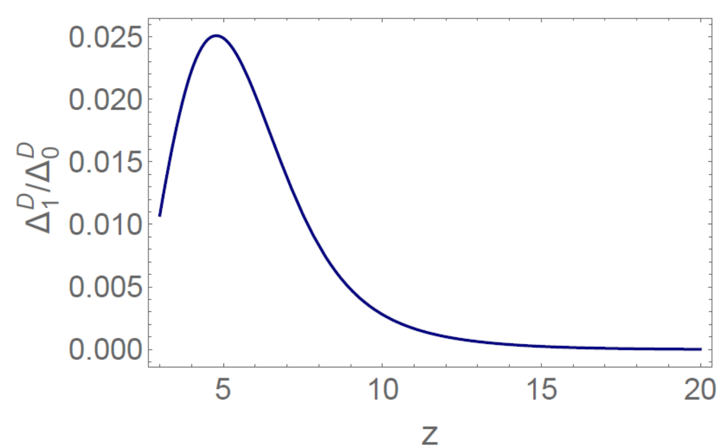
numerical estimation

Indeed, this correction turns to be small for region I.

$$\left(\kappa\frac{\partial}{\partial z} + z - \zeta\right)\Delta_1^D(z,z_0) = -\mathscr{N}_{\mathscr{L}}[\Delta_0^D(z)$$



$$\Delta_1^D(z,z_0) = -\Delta_0^D(z,z_0)\int_z^\infty dz' \frac{1}{\Delta_0^D(z',z_0)}\mathscr{N}_{\mathscr{L}}[\Delta_0^D(z')]$$



For region II.

numerical estimation

$$\Delta_0^D(z,\xi,z_0) = \exp\left(-\frac{z^2}{2\kappa} + \phi^{II}(\xi,z_0)\right)$$

$$\exp\left(\phi^{II}(\xi,z_0)\right) = \exp\left(\frac{z_0^2}{2\kappa}\right) G^2(\xi) \exp\left(-\frac{(z_0-\tilde{z})^2}{\kappa}\right)$$

$$\phi^{II}(\xi,z_0) = \frac{z_0^2}{2\kappa} - \frac{(z_0-\tilde{z})^2}{\kappa} + \frac{(\xi-\tilde{z})^2}{\kappa} - \frac{1}{2}e^{\xi}.$$

$$\mathcal{N}_{\mathcal{L}}[\Delta_0^{(0,II)}] = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \Delta_0^{(0,II)}(x_{02}) \Delta_0^{(0,II)}(x_{12}) - \Delta_0^{(0,II)} \int^{x_{01}^2} \frac{dx_{02}^2}{x_{02}^2} \Delta_0^{(0,II)}$$

$$\Delta_0^D(z,\xi,z_0) = \exp\left(-\Sigma(z(\xi),\xi,z_0(\xi)) - \frac{1}{2}e^{\xi}\right) \quad \text{with} \quad \Sigma(z,\xi,z_0) = \frac{z^2}{2\kappa} - \frac{z_0^2}{2\kappa} + \frac{(z_0-\tilde{z})^2}{\kappa} - \frac{(\xi-\tilde{z})^2}{\kappa}$$

Modified homotopy approach

Introducing $\boldsymbol{x}_{02} = \frac{1}{2}\boldsymbol{r} + \boldsymbol{x}\,, \quad \boldsymbol{x}_{12} = \frac{1}{2}\boldsymbol{r} - \boldsymbol{x}$

$$\begin{aligned}
\mathscr{N}_{\mathscr{L}} &= 2\,\bar{\alpha}_S e^{-\xi_r} \exp\left(-2\Sigma\left(z(\xi_r), \xi_r, z_0(\xi_r)\right) - e^{\xi_r}\right) \\
&= 2\,\bar{\alpha}_S e^{-\xi_r} \exp\left(-2\left(\frac{z(\xi_r)^2}{2\kappa} - \frac{z_0(\xi_r)^2}{2\kappa} + \frac{(z_0(\xi_r) - \tilde{z})^2}{\kappa} - \frac{(\xi_r - \tilde{z})^2}{\kappa}\right) - e^{\xi_r}\right) \\
&= \bar{\alpha}_S \exp\left(-\frac{(z - 2\ln 2)^2}{\kappa} + \tilde{\phi}\left(\xi_r, z_0\right)\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
\kappa\,\Delta^D_1\left(z,\xi,z_0\right) &= \Delta^D_0\left(z,\xi,z_0\right) \int_z^\infty dz' \frac{\mathscr{N}_{\mathscr{L}}[\Delta^D_0\left(z'\right)]}{\Delta^D_0\left(z,\xi,z_0\right)} \\
&= \Delta^D_0\left(z,\xi,z_0\right) \int_z^\infty dz' \exp\left(-\frac{(z' - 2\ln 2)^2}{\kappa} + \frac{\boldsymbol{z}'^2}{2\,\kappa} + \left(\tilde{\phi}\left(\xi_r, z_0\right) - \phi^{II}\left(\xi_r, z_0\right)\right)\right) \\
&= \Delta^D_0\left(z,\xi,z_0\right) \sqrt{\frac{\pi\kappa}{2}} e^{\frac{4\ln^2 2}{\kappa}} \operatorname{erfc}\left(\frac{z - 4\ln 2}{\sqrt{2\kappa}}\right) e^{\left(\tilde{\phi}(\xi_r,z_0) - \phi^{II}(\xi_r,z_0)\right)} \xrightarrow{z \gg 1} \frac{\kappa}{z}\left(\Delta^D_0\left(z,\xi,z_0\right)\right)^2
\end{aligned}$$

Homotopy approach allows us to solve analytically and asymptotically. Numerical leads to small corrections.

Conclusiona and outlook

we developed the homotopy approach for solving the non-linear evolution for the diffraction production in DIS

First, we solved the linearized version of this equation in the coordinate space deep in the saturation region. We found that this solution has the geometric scaling behavior for $\xi < \xi_0^A$

For $\xi > \xi_0^A$ we observe the violation of the geometric scaling behavior in the saturation region.

This solution satisfies the boundary and initial conditions which are given perturbative QCD approach for $r\,Q_s < 1$ and by McLerran-Venugopalan for $Y = Y_A$.

Finally in our approach, we have to take into account the remaining part of the non-linear correction that have not been included in the linearized form of the $N^D -$ equation. It turns out that these corrections are rather small indicating that our procedure gives a self consistent way to account them

We believe that this method of finding solution, which allow us to treat the most essential part of the scattering amplitude analytically.

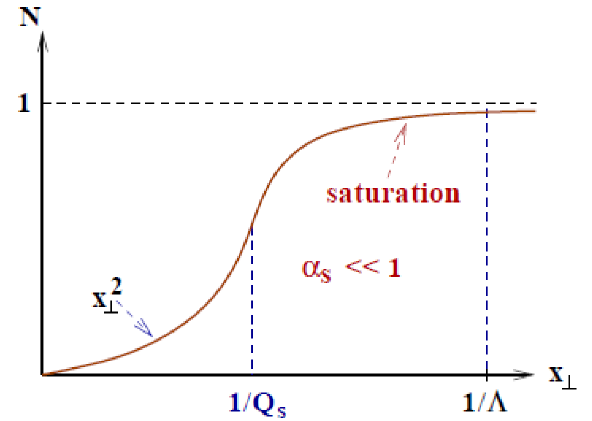
The numerical part of the calculations is expressed through well converged integrals and can be easily estimated.

Thank you for your attention!

In memory of my sister Sandra

Preliminary results:

The dipole-nucleus scattering amplitude N is plotted (schematically) as a function of x⊥.



- One can see that, at small $x_\perp \ll \frac{1}{Q_s}$, we have N ~ 0
- This result is natural, since in the zero-size dipole the color charges of the quark and the anti-quark cancel each other, leading to disappearance of the interactions with the target. This effect is known as color transparency
- at large dipole sizes $x_\perp > \frac{1}{Q_s}$, the growth stops and the amplitude levels off (saturates) at N = 1.
- (Kopeliovich et al 1981)
- The transition happens at around $x_\perp \sim \frac{1}{Q_s}$.
- Numerical solution