Triple (and quadruple) soft-parton radiation in QCD hard scattering

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Outline

- Review of tree-level soft current for 1 or 2 soft gluons

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  - Colour structure with dipoles and quadrupoles
  - Kinematical coefficients in strong energy ordering
  - Collinear singularities

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  - c-number factorization
  - Casimir scaling (violation)
  - Multi-eikonal formula in strong energy ordering
- Squared currents with 2 hard partons
  - Same analysis as for 3 hard partons
  - Extension to 4 soft gluons: colour monster, quartic Casimir,. . .
  - Prediction of N^{3}L gluon splitting factors at small-x

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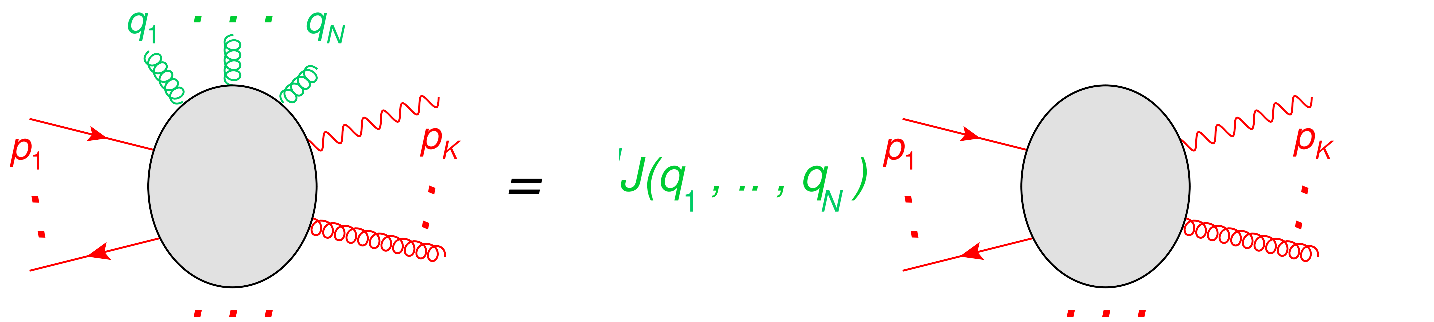
- (Squared) currents for soft $g q \bar{q}$

Motivations

- High-precision LHC data $\leftrightarrow$ high precision in theoretical predictions
  - to test our present understanding of the Standard Model
  - to discover (probably tiny) signals of new physics
- Explicit knowledge of soft/collinear factorization of scattering amplitudes necessary in resummed calculations at N^3LL
- Calculation of large logarithmic terms can be used to obtain approximated fixed-order results
- Soft/collinear factorization provides the theoretical basis of parton shower algorithms for Monte Carlo event generators

- Behaviour of scattering amplitude $\mathcal{M}(\{p_k\},\{q_i\})$ when some external gluons $q_i = \xi \bar{q}_i$ become soft ($\xi \to 0$)



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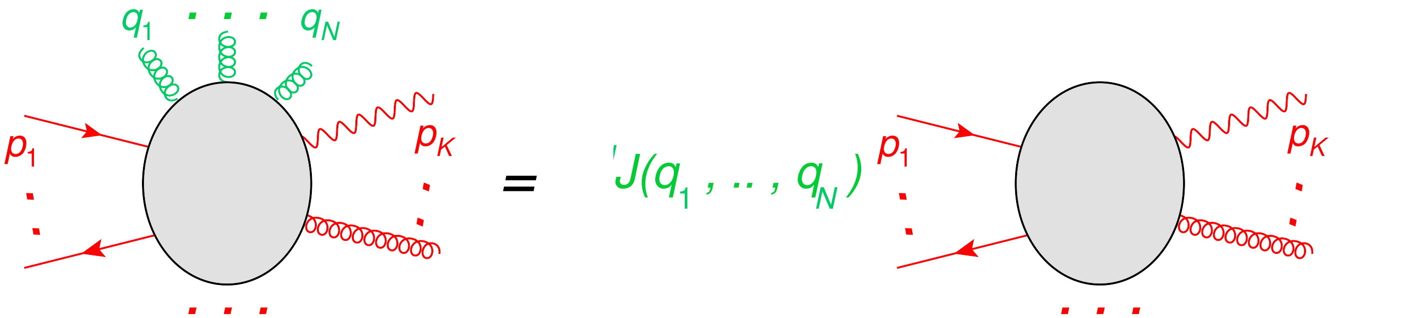
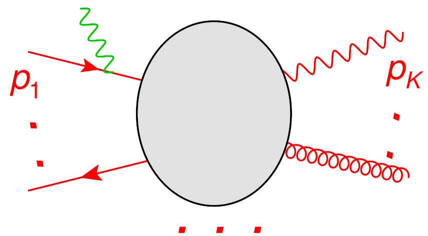


- Leading $1/\xi$: only gluon insertions on the external hard legs



QED: $$J(q) = J^\mu(q)\varepsilon_\mu(q) = \sum_{k=1}^{K} \frac{e_k p_k^\mu}{p_k \cdot q}\varepsilon_\mu(q)$$

QCD: $$J^a(q) = J^{a,\mu}(q)\varepsilon_\mu(q) = \sum_{k=1}^{K} \frac{g T_k^a p_k^\mu}{p_k \cdot q}\varepsilon_\mu(q)$$

$$\begin{cases} (T_q^a)_{bc} = t_{bc}^a \\ (T_{\bar{q}}^a)_{bc} = -t_{cb}^a \\ (T_g^a)_{bc} = \mathrm{i} f^{abc} \end{cases}$$

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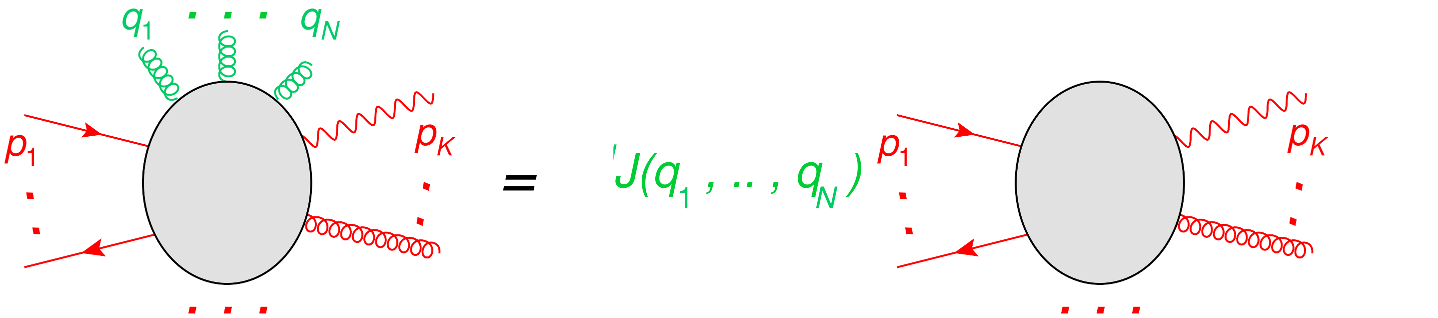

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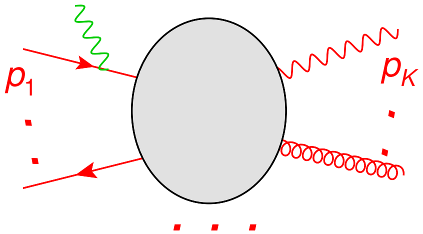

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$$\sum_{k=1}^{K} e_k = 0 \,, \qquad \sum_{k=1}^{K} T_k^a \stackrel{\text{cs}}{=} 0 \quad \Longrightarrow \quad q_\mu J^{a,\mu} \stackrel{\text{cs}}{=} 0$$

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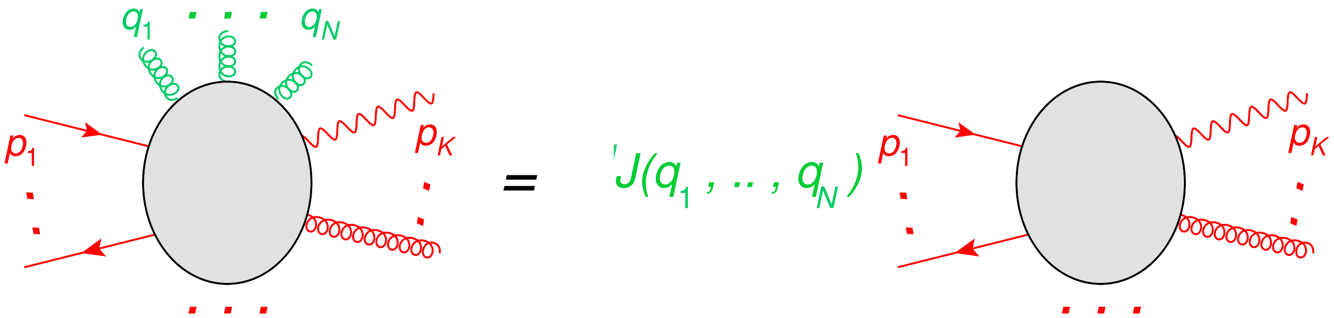

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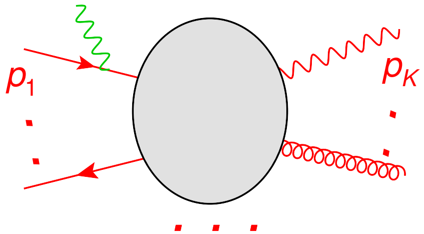

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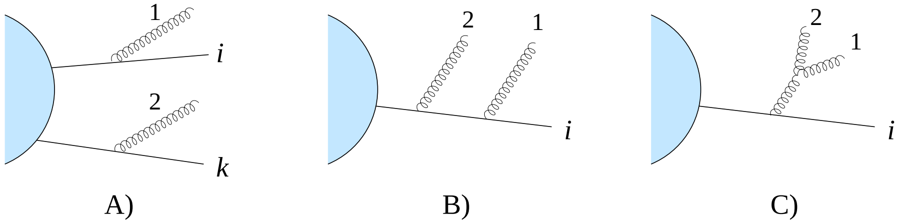
GRAV: $$J(q) = J^{\mu\nu}(q)\varepsilon_{\mu\nu}(q) = \sum_{k=1}^{K} \frac{\kappa p_k^\nu p_k^\mu}{p_k \cdot q}\varepsilon_{\mu\nu}(q)$$

2 soft gluon emission



[Catani, Grazzini '00] "Independent" + max. non-abelian correlation

$$J^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) = J^{a_1}_{\mu_1}(q_1) * J^{a_2}_{\mu_2}(q_2) + \Gamma^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2)$$

$$A * B \equiv \frac{1}{2}(AB + BA)$$

$$\Gamma^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) = i f^{a_1 a_2\, b} \sum_{k \in \text{hard}} T^b_k \, \gamma^{\mu_1 \mu_2}_k(q_1, q_2)$$

$$\gamma^{\mu_1 \mu_2}_k(q_1, q_2) = \frac{1}{p_k \cdot (q_1 + q_2)} \left\{ \frac{p^{\mu_1}_k p^{\mu_2}_k}{2\, p_k \cdot q_1} + \frac{1}{q_1 \cdot q_2} \left(p^{\mu_1}_k q^{\mu_2}_1 + \frac{1}{2} g^{\mu_1 \mu_2} p_k \cdot q_2 \right) \right\} - (1 \leftrightarrow 2)$$

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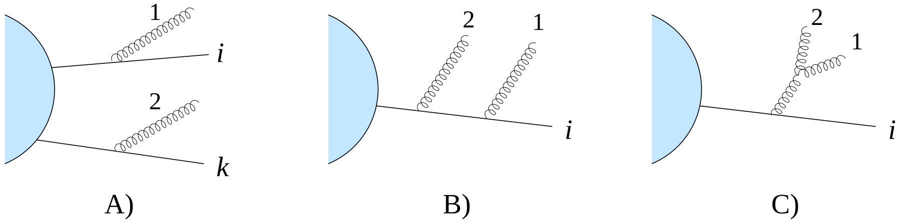

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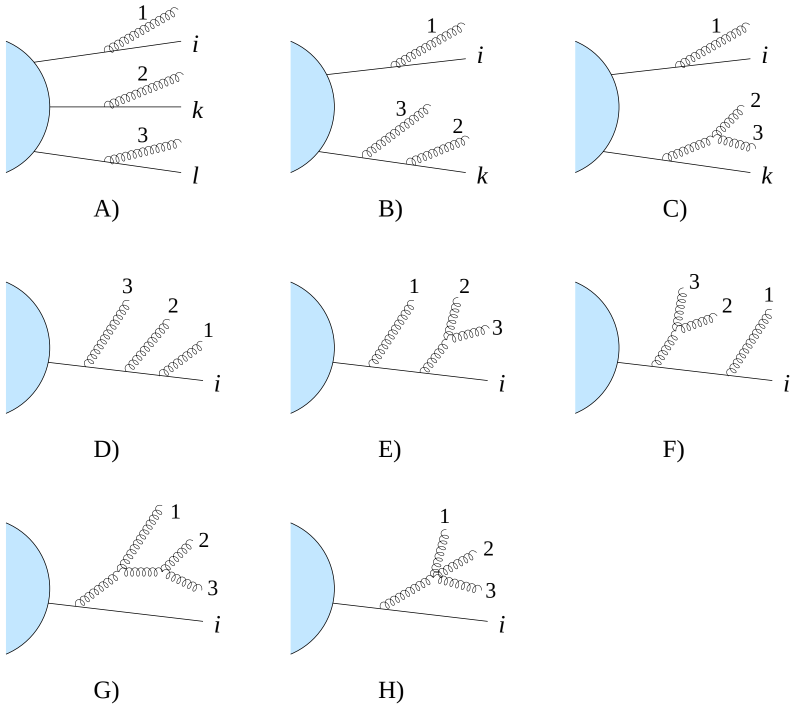
- conservation of current: $q_1^{\mu_1} J^{a_1 a_2}_{\mu_1 \mu_2}(q_1, q_2) \, \varepsilon^{\mu_2}(q_2) \stackrel{\text{cs}}{=} 0$
- Abelian case ($f^{abc} = 0$): only independent emission
- QCD: $J(1) * J(2) \implies$ colour-correlations with hard partons (furthermore currents do not commute)

3 soft gluon emission



[Catani, DC, Torrini '20]

- Eikonal vertices on hard lines
- Exact vertices elsewhere
- Exact propagators of soft gluons (light-cone / covariant) gauges

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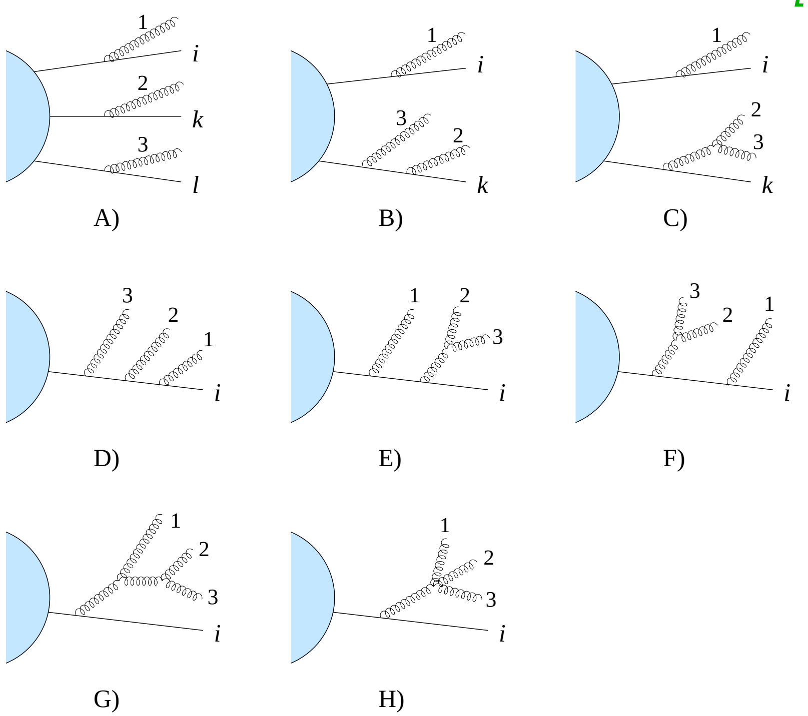

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- Eikonal vertices on hard lines
- Exact vertices elsewhere
- Exact propagators of soft gluons (light-cone / covariant) gauges
- Current is conserved on colour-singlet states
- Irreducible term has colour structure $T^b \sum_s f^{a_1 a_2 s} f^{s a_3 b}$

$$J(1,2,3) = J(1) * J(2) * J(3) + \left[\sum_{\text{cyc.123}} J(1) * \Gamma(2,3) \right] + \Gamma(1,2,3)$$

$$\Gamma(1,2,3) = \sum_{k \in \text{hard}} T_k^b \sum_{\text{cyc.123}} f^{a_1 a_2,\, a_3 b} \, \gamma_k^{\mu_1 \mu_2 \mu_3}(q_1, q_2; q_3)$$

Kinematical coefficient of maximally non-abelian term

$$\gamma_k(1,2;3) = \frac{1}{p_k \cdot q_{123}} \left\{ \frac{1}{12} \frac{p_k^{\mu_1} p_k^{\mu_2} p_k^{\mu_3} \, p_k \cdot (3q_3 - q_{12})}{p_k \cdot q_2 \; p_k \cdot q_3 \; p_k \cdot q_{12}} \right.$$

$$+ \frac{p_k^{\mu_3} \, p_k \cdot (q_3 - q_{12})}{p_k \cdot q_3 \; p_k \cdot q_{12} \; q_{12}^2} \left(\frac{1}{2} g^{\mu_1 \mu_2} \, p_k \cdot q_1 + p_k^{\mu_2} \, q_2^{\mu_1} \right)$$

$$+ \frac{1}{q_{123}^2 \; q_{12}^2} \left[q_{12}^2 \, p_k^{\mu_1} \, g^{\mu_2 \mu_3} + 2 q_2^{\mu_1} g^{\mu_2 \mu_3} \, p_k \cdot (q_3 - q_{12}) \right.$$

$$+ 4 \, q_3^{\mu_1} \, q_1^{\mu_2} \, p_k^{\mu_3} + 4 \, q_2^{\mu_1} \, p_k^{\mu_2} \, q_{12}^{\mu_3}$$

$$\left. \left. + g^{\mu_1 \mu_2} \left(q_{23}^2 \, p_k^{\mu_3} + q_1^{\mu_3} \, p_k \cdot (q_{13} - 3 q_2) \right) \right] \right\} - \left(1 \leftrightarrow 2 \right)$$

Notation: $q_{ij} = q_i + q_j$, $\qquad q_{123} = q_1 + q_2 + q_3$

Soft factors for colour-ordered subamplitudes

Colour-ordered subamplitudes decomposition *[Berends,Giele '89]*

$$M(1,\cdots,n) = \sum_{\mathrm{perm}(1,\cdots,n-1)} \mathrm{tr}(t^{a_1}\cdots t^{a_n})\, C(1,\cdots,n)$$

In the soft limit $k_2,\ldots k_m \to 0$

$$C(1,\underline{2},\cdots,\underline{m},m+1,\cdots,n) = s_{1,\underline{2},\cdots,\underline{m},m+1}\, C(1,m+1,\cdots,n)$$

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Soft factors in terms of kinematical coefficients

$$s_{i\underline{1}k} = j_k(1) - j_i(1) \qquad \left(j_k^\mu(1) \equiv \frac{p_k^\mu}{p_k\cdot q_1}\right)$$

$$s_{i\underline{12}k} = \gamma_i(1,2) + \tfrac{1}{2}[j_i(1)j_i(2) - j_i(1)j_k(2)] + \binom{1\leftrightarrow 2}{i\leftrightarrow k}$$

$$s_{i\underline{123}k} = \gamma_i(1,2;3) + \gamma_i(3,2;1) - \tfrac{1}{2}\left[\gamma_i(1,2)\,j_i(3) + j_i(1)\,\gamma_i(2,3)\right] + \gamma_i(1,2)\,j_k(3) + \tfrac{1}{2}\,j_i(1)\,j_i(2)\,j_k(3) - \tfrac{1}{6}\,j_i(1)\,j_i(2)\,j_i(3) - \binom{1\leftrightarrow 3}{i\leftrightarrow k}$$

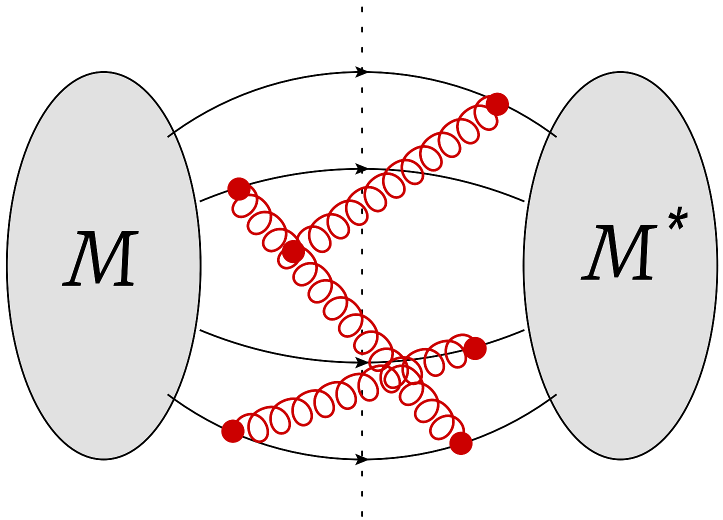
Square of soft current

Dirac's notation in colour $\otimes$ helicity space

$$\mathcal{M}^{c_1 c_2 \dots}_{\lambda_1 \lambda_2 \dots} =: \langle c_1, c_2 \dots | \otimes \langle \lambda_1, \lambda_2 \dots | \mathcal{M} \rangle$$

$$|\mathcal{M}(p_k, q_i)\rangle \simeq \boldsymbol{J}(q_i)|\mathcal{M}(p_k)\rangle$$

$$|\mathcal{M}(p_k, q_i)|^2 \simeq \langle \mathcal{M}(p_k)| \, |\boldsymbol{J}(q_i)|^2 \, |\mathcal{M}(p_k)\rangle$$



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$$|\mathcal{M}(p_k, q_i)|^2 \simeq \langle \mathcal{M}(p_k)| \, |\boldsymbol{J}(q_i)|^2 \, |\mathcal{M}(p_k)\rangle$$

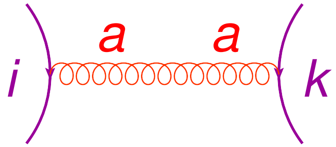
Current conservation on colour-singlets $q_\ell^{\mu_\ell} J^{..a_\ell..}_{..\mu_\ell..}(..q_\ell..) \overset{\text{cs}}{=} 0$ $\Downarrow$

$$|\boldsymbol{J}(q_1 \cdots q_N)|^2 \overset{\text{cs}}{=} \left[\prod_{\ell=1}^{N} -g^{\mu_\ell \nu_\ell} \right] J^{a_1 \dots a_N}_{\mu_1 \dots \mu_N}(q_1 \cdots q_N)^\dagger J^{a_1 \dots a_N}_{\nu_1 \dots \nu_N}(q_1 \cdots q_N)$$

- Explicitly gauge invariant
- Still a colour operator that depends on the colour charges of the hard partons in $\mathcal{M}(p_k)$
- Important simplifications for 2 or 3 hard partons

Square of soft currents for 1 or 2 gluon emission

- Single gluon emission



$$|\boldsymbol{J}(q)|^2 \stackrel{\text{cs}}{=} - \sum_{i,k \in hard} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \, \mathcal{S}_{ik}(q) =: W(q)$$

$$\left(\boldsymbol{T}_i \cdot \boldsymbol{T}_k \equiv \sum_a T_i^a T_k^a \right)$$

$$\mathcal{S}_{ik}(q) = \frac{p_i \cdot p_k}{p_i \cdot q \; p_k \cdot q}$$

colour dipoles

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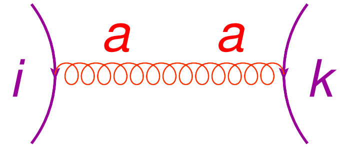

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colour dipoles

- Double gluon emission
  involves just colour dipoles in irreducible correlation

$$|\boldsymbol{J}(q_1,q_2)|^2 \overset{\text{cs}}{=} W(q_1)*W(q_2) + W(q_1,q_2)$$

$$W(q_1,q_2) = -C_A \sum_{i,k\in hard} \boldsymbol{T}_i\cdot\boldsymbol{T}_k\,\mathcal{S}_{ik}(q_1,q_2)$$

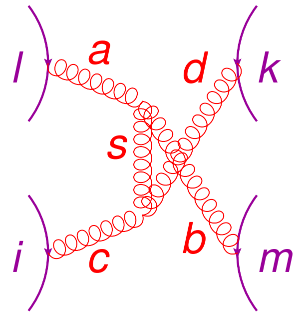
Square of soft current for 3 gluon emission

$$|\boldsymbol{J}(q_1,q_2,q_3)|^2 \stackrel{\text{cs}}{=} W(q_1)*W(q_2)*W(q_3) + \Big[\sum_{\text{cyc.123}} W(q_1)*W(q_2,q_3)\Big] + W(q_1,q_2,q_3)$$

Each W is gauge-invariant

The irreducible correlation involves dipoles and **quadrupoles**

$$W(q_1,q_2,q_3) = -C_A^2 \sum_{i,k} \boldsymbol{T}_i\cdot\boldsymbol{T}_k\, \mathcal{S}_{ik}(q_1,q_2,q_3) + \sum_{iklm} Q_{iklm}\, \mathcal{S}_{iklm}(q_1,q_2,q_3)$$



Irred. quadrupoles $Q_{iklm} \equiv \frac{1}{2} f^{ab,cd}\left(T_l^a\{T_i^c,T_k^d\}T_m^b + \text{h.c.}\right)$

Square of soft current for 3 gluon emission

- In strong energy ordering $E_1 \ll E_2 \ll E_3$ and massless hard partons

$$
\begin{aligned}
S_{ik}^{\text{seo}} = & \frac{2(p_i \cdot p_k)^3}{3(p_i \cdot q_1)(p_k \cdot q_1)(p_i \cdot q_2)(p_k \cdot q_2)(p_i \cdot q_3)(p_k \cdot q_3)} \\
& - \frac{2(p_i \cdot p_k)^2}{(q_1 \cdot q_2)(p_i \cdot q_1)(p_k \cdot q_2)(p_i \cdot q_3)(p_k \cdot q_3)} \\
& + \frac{2 p_i \cdot p_k}{(q_1 \cdot q_3)(q_2 \cdot q_3)(p_i \cdot q_1)(p_k \cdot q_2)} \ + \text{perms. } \{1, 2, 3\}
\end{aligned}
$$

symmetric (!) under exchange of soft gluons momenta q_1, q_2, q_3

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\end{aligned}
$$

symmetric (!) under exchange of soft gluons momenta q_1, q_2, q_3

- Quadrupole kinematical coefficient

$$
\begin{aligned}
S_{iklm}^{\text{seo}} = & \frac{p_i \cdot p_k}{(p_i \cdot q_3)(p_k \cdot q_3)(p_l \cdot q_1)(p_m \cdot q_2)} \left\{ \frac{(p_m \cdot p_l)(p_l \cdot p_i)}{3(p_l \cdot q_2)(p_i \cdot q_1)} \right. \\
& \left. - \frac{p_m \cdot p_i}{3p_i \cdot q_2} \left[\frac{p_l \cdot p_i}{p_i \cdot q_1} + \frac{p_l \cdot p_m}{p_m \cdot q_1} + \frac{2p_l \cdot p_k}{p_k \cdot q_1} \right] + \frac{p_l \cdot q_3}{q_3 \cdot q_1} \left[\frac{2p_m \cdot p_i}{p_i \cdot q_2} - \frac{p_m \cdot q_3}{q_3 \cdot q_2} \right] \right\} + (1 \leftrightarrow 2)
\end{aligned}
$$

symmetric (!) in the exchange $q_1 \leftrightarrow q_2$.

Collinear singularities of squared currents

- $|\mathcal{M}|^2$ singular (not integrable) when momenta of two or more of external *massless* legs become collinear

Collinear singularities of squared currents

- $|\mathcal{M}|^2$ singular (not integrable) when momenta of two or more of external *massless* legs become collinear
- 1 gluon emission: singular when $q \simeq z\, p_B$

$$|\boldsymbol{J}(q)|^2 = -\sum_{i \neq k} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \frac{p_i \cdot p_k}{p_i \cdot q\; p_k \cdot q} \simeq -\sum_{i \neq k} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \frac{\delta_{iB} + \delta_{kB}}{z\, p_B \cdot q}$$

$$= \frac{2}{z\, p_B \cdot q} \boldsymbol{T}_B \cdot \Big(-\sum_{k \neq B} \boldsymbol{T}_k \Big) \overset{\text{cs}}{=} \frac{1}{p_B \cdot q} \frac{2 C_B}{z} \rightarrow \begin{cases} C_F & (B = \mathrm{q}) \\ C_A & (B = \mathrm{g}) \end{cases}$$

- absence of colour correlations $\Rightarrow$ colour coherence

Collinear singularities of squared currents

recall $|\boldsymbol{J}(1,2,3)|^2 \overset{\text{cs}}{=} W(1) * W(2) * W(3) + \Big[\sum_{\text{cyc.123}} W(1) * W(2,3) \Big] + W(1,2,3)$

Expansion in irreducible correlations reduces collinear singularities of W's

- $W(2,3)$
  - c_1 double-collinear limit of the 2 soft gluons (exact $P^{\mu\nu}_{g_1 g_2}$)
  - c_2 triple-collinear limit of the 2 soft gluons and a hard parton
- $W(1,2,3)_{\text{dipole}}$
  - c_3 double-collinear limit of 2 soft gluons (exact)
  - c_4 triple-collinear limit of the 3 soft gluons (exact $P^{\mu\nu}_{g_1 g_2 g_3}$)
  - c_5 quadruple-collinear limit of 3 soft gluons and a hard parton C
    $\Longrightarrow$ soft limit of $P^{ss'}_{g_1 g_2 g_3 C}$ (new!)
- $W(1,2,3)_{\text{quadrupole}}$ has no collinear singularity!

3 hard partons

Consider amplitudes with 3 hard partons A, B, C plus soft gluons (and colourless objs)

- $\{A, B, C\} = \{\mathrm{g}, \mathrm{q}, \bar{\mathrm{q}}\}$ or $\{A, B, C\} = \{\mathrm{g}, \mathrm{g}, \mathrm{g}\}$ (flavour cons.)

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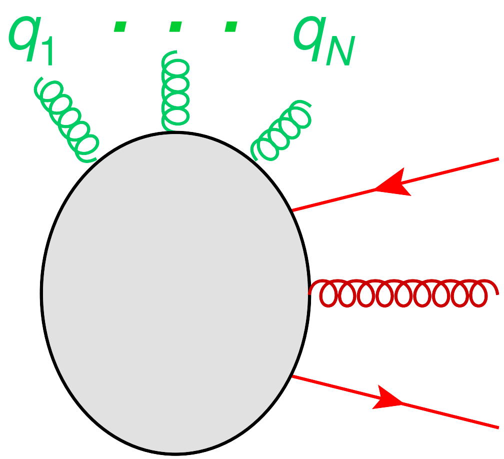

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- $|J|^2$ is colour conserving, thus

c-number

$$|\boldsymbol{J}(q_1 \cdots q_N)|^2 |\mathrm{g q \bar{q}}\rangle = |\mathrm{g q \bar{q}}\rangle \, |\boldsymbol{J}(q_1 \cdots q_N)|^2_{\mathrm{g q \bar{q}}}$$

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$$|\boldsymbol{J}(q_1 \cdots q_N)|^2 |gq\bar{q}\rangle = |gq\bar{q}\rangle \; |\boldsymbol{J}(q_1 \cdots q_N)|^2_{gq\bar{q}}$$

$$|\mathcal{M}(p_k, q_1 \cdots q_N)|^2 \simeq |\mathcal{M}(p_k)|^2 \; |\boldsymbol{J}(q_1 \cdots q_N)|^2_{gq\bar{q}}$$

- c-number soft-gluon factorization formula is valid at arbitrary loop orders

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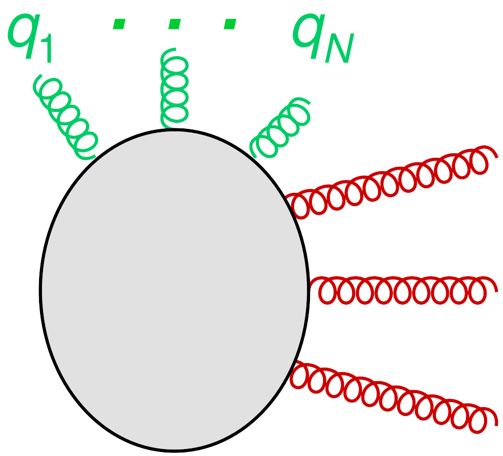
- $\{A, B, C\} = \{\mathrm{g}, \mathrm{q}, \bar{\mathrm{q}}\}$ or $\{A, B, C\} = \{\mathrm{g}, \mathrm{g}, \mathrm{g}\}$ (flavour cons.)
- Two colour-singlet states, with ± 1 charge conjugation

$$\langle abc | (\mathrm{ggg})_f \rangle = \mathrm{i} f^{abc} \qquad \langle abc | (\mathrm{ggg})_d \rangle = d^{abc}$$



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$|J|^2$ is a 2×2 colour-matrix, diagonal in the $\{f, d\}$ basis

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$|J|^2$ is a 2×2 colour-matrix, diagonal in the $\{f, d\}$ basis

- $|\boldsymbol{J}(q_1 \cdots q_N)|^2_f \neq |\boldsymbol{J}(q_1 \cdots q_N)|^2_d$ for $N \geq 3$
  Quadrupole colour correlations (vanishing on d-state) remove degeneracy

3 hard partons: Radiation at tree level of 1 or 2 gluons

Dipoles $\propto$ Casimirs: $\boldsymbol{T}_A \cdot \boldsymbol{T}_B \, |ABC\rangle = \frac{1}{2}(C_C - C_A - C_B) \, |ABC\rangle$

Rewrite $W(q_1 \cdots q_N) \overset{\text{cs}}{=} -\frac{1}{2} C_A^{n-1} \sum_{i,k} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \, w_{ik}(q_1 \cdots q_N)$

$$w_{AB} = \mathcal{S}_{AB} + \mathcal{S}_{BA} - \mathcal{S}_{AA} - \mathcal{S}_{BB}$$

$$w_{ABC} = w_{AB} + w_{AC} - w_{BC}$$

- $|\boldsymbol{J}(q)|^2_{ABC} = C_B \, w_{BC}(q) + C_A \, w_{ABC}(q)$

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$$w_{AB} = \mathcal{S}_{AB} + \mathcal{S}_{BA} - \mathcal{S}_{AA} - \mathcal{S}_{BB}$$

$$w_{ABC} = w_{AB} + w_{AC} - w_{BC}$$

- $|\boldsymbol{J}(q)|^2_{ABC} = C_B \, w_{BC}(q) + C_A \, w_{ABC}(q)$
- $|\boldsymbol{J}(q_1, q_2)|^2_{ABC} = C_B^2 \, w_{BC}(q_1) \, w_{BC}(q_2) + C_B C_A \, [w_{BC}(q_1, q_2) + w_{BC}(q_1) \, w_{ABC}(q_2) + w_{BC}(q_2) \, w_{ABC}(q_1)] + C_A^2 \, [w_{ABC}(q_1, q_2) + w_{ABC}(q_1) \, w_{ABC}(q_2)]$
- dependence on colour state of hard-partons entirely expressed through the Casimir coefficients C_A e C_B
- Casimir scaling:

ABC :	$gq\bar{q}$	ggg
C_B :	C_F	C_A

3 hard partons: Radiation at tree level of 3 gluons

- Dipole term: Casimir scaling

$$
\begin{aligned}
&|\boldsymbol{J}(q_1,q_2,q_3)|^{2,\mathrm{dip}}_{ABC} = C_B^3 \; w_{BC}(q_1) w_{BC}(q_2)\; w_{BC}(q_3) \\
&+ C_B^2 C_A \; [w_{BC}(q_1)\, w_{BC}(q_2,q_3) + \cdots] \\
&+ C_B C_A^2 \; [w_{BC}(q_1,q_2,q_3) + \cdots] \\
&+ C_A^3 \left[w_{ABC}(q_1,q_2,q_3) + w_{ABC}(q_1)\; w_{ABC}(q_2)\; w_{ABC}(q_3) + \cdots\right]
\end{aligned}
$$

3 hard partons: Radiation at tree level of 3 gluons

- Dipole term: Casimir scaling

$$|\boldsymbol{J}(q_1,q_2,q_3)|^{2,\mathrm{dip}}_{ABC} = C_B^3\, w_{BC}(q_1) w_{BC}(q_2)\, w_{BC}(q_3)$$
$$+ C_B^2 C_A\, [w_{BC}(q_1)\, w_{BC}(q_2,q_3) + \cdots]$$
$$+ C_B C_A^2\, [w_{BC}(q_1,q_2,q_3) + \cdots]$$
$$+ C_A^3 \left[w_{ABC}(q_1,q_2,q_3) + w_{ABC}(q_1)\, w_{ABC}(q_2)\, w_{ABC}(q_3) + \cdots\right]$$

- Quadrupole term: (sizeable) Casimir scaling violation $\mathcal{O}\left(N_c^{-2}\right)$

$$|\boldsymbol{J}(q_1,q_2,q_3)|^{2,\mathrm{quad}}_{ABC} = \lambda_B N_c\, w^{\mathrm{quad}}_{ABC}(q_1,q_2,q_3)$$
$$|\mathrm{g q\bar{q}}\rangle : \lambda_F = 1/2\,, \qquad |f\rangle : \lambda_A = 3\,, \qquad |d\rangle : \lambda = 0$$

$$w^{\mathrm{quad}}_{ABC} \equiv [\mathcal{S}_{ABAB} - \mathcal{S}_{ABBA} + \mathcal{S}_{ABCA} - \mathcal{S}_{ABAC} + \mathcal{S}_{BAAC} - \mathcal{S}_{BACA}] + \mathrm{perm}\{A,B,C\}$$

collinear safe w.r.t angular integration over soft-gluon momenta

3 hard *gluons*: radiation at tree level of N gluons

- Strong energy ordering: $E_1 \ll E_2 \ll \cdots \ll E_N$

$$|\mathcal{M}(p_k, q_i)|^2 \simeq |\mathcal{M}(p_k)|^2 \; |\boldsymbol{J}(q_i)|^{2,\text{seo}}_{\text{ggg}_f} \left\{1 + \mathcal{O}\left(N_c^{-2}\right)\right\} \qquad \text{(new)}$$

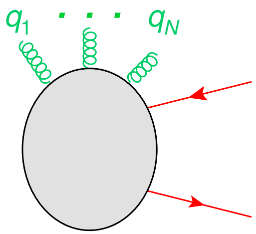
$$|\boldsymbol{J}(q_i)|^{2,\text{seo}}_{\text{ggg}} = C_A^N \; p_A \cdot p_B \;\; p_B \cdot p_C \;\; p_C \cdot p_A \; F_{\text{eik}}(p_A, p_B, p_C, q_1, \cdots, q_N)$$

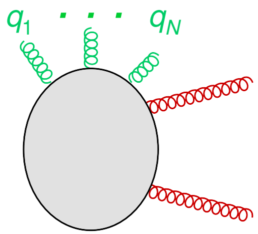
$$F_{\text{eik}}(k_1, \cdots, k_M) \equiv \left[(k_1 \cdot k_2)(k_2 \cdot k_3) \ldots (k_{M-1} \cdot k_M)(k_M \cdot k_1)\right]^{-1} + \text{ineq. perm}\{k_1, \cdots, k_M\}$$

  in terms of multi-eikonal F_{eik} *[Bassetto Ciafaloni Marchesini 83]*

- The 2 eigenvalues of squared current become degenerate, up to colour suppressed terms

2 hard partons





- $\{B, C\} = \{\mathrm{q}, \bar{\mathrm{q}}\}$ or $\{B, C\} = \{\mathrm{g}, \mathrm{g}\}$ (flavour cons.)
- 1-dimensional colour spaces $\Longrightarrow$ c-number factorization
- Non-abelian effects are in $SU(N_c)$ colour coefficients

$$
\begin{aligned}
|\boldsymbol{J}(q_1, q_2, q_3)|^2_{BC} = {} & C_B^3\, w_{BC}(q_1)\, w_{BC}(q_2)\, w_{BC}(q_3) \\
& + C_B^2 C_A [w_{BC}(q_1)\, w_{BC}(q_2, q_3) + \text{cyc.perm.}(123)] \\
& + C_B C_A^2\, w_{BC}(q_1, q_2, q_3)
\end{aligned}
$$

- Casimir scaling valid up to 3 soft gluons

2 hard partons

- Strong energy ordering with 2 hard gluons: check of *[BCM]* [link a $F_{\rm eik}$]

$$\left|\boldsymbol{J}(q_1 \cdots q_N)\right|^{2,\text{seo}}_{BC} = 2\, C_A^N \,(p_B \cdot p_C)^2 F_{eik}(p_B, p_C, q_1 \cdots q_N) + \mathcal{O}\left(N_c^{N-2}\right)$$

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- If one soft gluon is much harder than the others $(E_N \gg E_1, \cdots, E_{N-1})$

$$|J(q_1 \cdots q_N)|^2_{BC} \simeq |J(q_N)|^2_{BC}\, |J(q_1 \cdots q_{N-1})|^2_{ABC}$$

$p_A \equiv q_N$

4 soft gluon sq.current constrained by 3 soft gluon sq.current

2 hard partons, 4 soft gluons

Expansion in irreducible correlations

$$
\begin{aligned}
|\boldsymbol{J}(1,2,3,4)|^2 \overset{\text{cs}}{=} & \; W(q_1) * W(q_2) * W(q_3) * W(q_4) \\
& + [W(q_1,q_2) * W(q_3) * W(q_4) + \text{ineq. perms. } \{1,2,3,4\}] \\
& + [W(q_1,q_2) * W(q_3,q_4) + \text{ ineq. perms. } \{1,2,3,4\}] \\
& + [W(q_1) * W(q_2,q_3,q_4) + \ (1 \leftrightarrow 2) + (1 \leftrightarrow 3) + (1 \leftrightarrow 4)] \\
& + W(q_1,q_2,q_3,q_4)
\end{aligned}
$$

Action on a colour-singlet $|BC\rangle$ state when $E_4 \gg E_{1,2,3}$

$$
|\boldsymbol{J}(1,2,3,4)|^2_{BC} = C_B \, w_{BC}(q_4) \left[|\boldsymbol{J}(1,2,3)|^{2,\text{dip}}_{4BC} + \lambda_B N_c \, w^{\text{quad}}_{4BC}(q_1,q_2,q_3) \right]
$$

New irreducible correlation: leading + subleading colour

$$
W(q_1 \cdots q_4)|_{BC} = C_B \left[C_A^3 \, w^{(L)}_{BC}(q_1 \cdots q_4) + \lambda_B N_c \, w^{(S)}_{BC}(q_1 \cdots q_4) \right]
$$

2 hard partons, 4 soft gluons, energy ordering

$$W(q_1 \cdots q_4)|_{BC} = C_B \left[C_A^3 \, w_{BC}^{(L)}(q_1 \cdots q_4) + \lambda_B N_c \, w_{BC}^{(S)}(q_1 \cdots q_4) \right]$$

- Exact colour structure for massive hard particles and arbitrary soft energies; Kinematical coeffs. known in energy ordering
- Last term is related to quartic Casimir

$$C_B \, \lambda_B \, N_c = 2 \, \frac{d_{AB}^{(4)}}{D_B} - \frac{1}{12} \, C_B \, C_A^3$$

D_B dimension of B representation

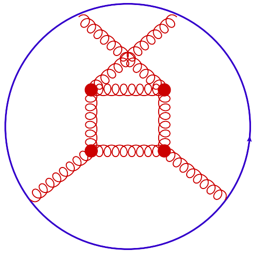
- Generalized Casimir scaling: $C_F \to C_A \, , \quad d_{AF}^{(4)} \to d_{AA}^{(4)}$
- First correction to multi-eikonal BCM formula $\quad \{B, C\} = \{g, g\}$

$$|\boldsymbol{J}(1, \cdots, 4)|_{BC}^{2,\text{seo}} = 2 \, C_A^4 (p_B \cdot p_C)^2 \, F_{eik}^{(6)}(p_B, p_C, q_1, q_2, q_3, q_4) + 3 \, N_c^2 \, w_{BC}^{(S,\text{seo})}(q_1, q_2, q_3, q_4)$$

2 hard partons, 4 soft gluons, strong ordering

$$W(q_1 \cdots q_4)|_{BC} = C_B \left[C_A^3 \, w_{BC}^{(L)}(q_1 \cdots q_4) + \lambda_B N_c \, w_{BC}^{(S)}(q_1 \cdots q_4) \right]$$

- *[Dok Kho Mue Tro 91]* examined 4 soft gluon radiation from 2 massless hard partons in strong energy ordering . . .
- . . . and found a contribution $\sim C_B N_c$ from colour monster



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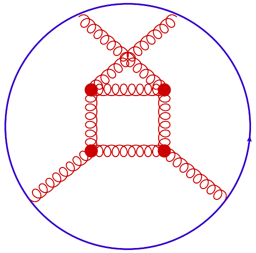
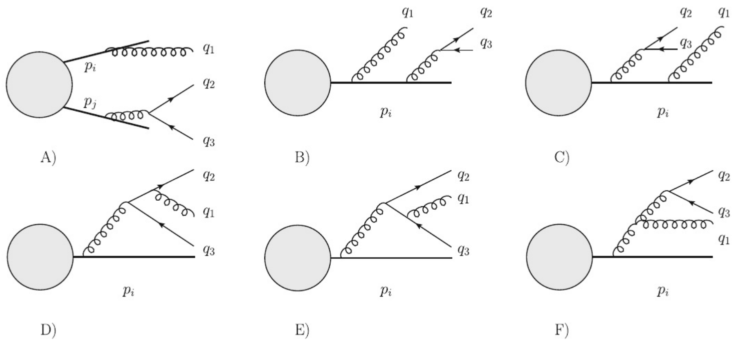


- Our results are fully consistent with the colour monster ...
- ... related to quartic Casimir $C_B \lambda_B N_c = 2 \frac{d_{AB}^{(4)}}{D_B} - \frac{1}{12} C_B C_A^3$
- $w_{BC}^{(S)}$ has collinear singularities $\Rightarrow \alpha_S^4 \, d_{AB}^{(4)} / \epsilon$ in coll.ev.kernels contribute to $\Gamma_{\rm cusp}^{(4)}$ term that violates Casimir scaling

Soft gluon-quark-antiquark: dipoles and tripoles



$$J(1,2,3) = J(1) * J(2,3) + \Gamma(1,2,3)$$

$$\Gamma(1,2,3) = \sum_{k \in \text{hard}} T_k^b \Big(\{\boldsymbol{t}^a, \boldsymbol{t}^a\} \, \gamma_k^{(\mathrm{ab})}(q_1, q_2, q_3) + [\boldsymbol{t}^a, \boldsymbol{t}^a] \, \gamma_k^{(\mathrm{na})}(q_1, q_2, q_3) \Big)$$

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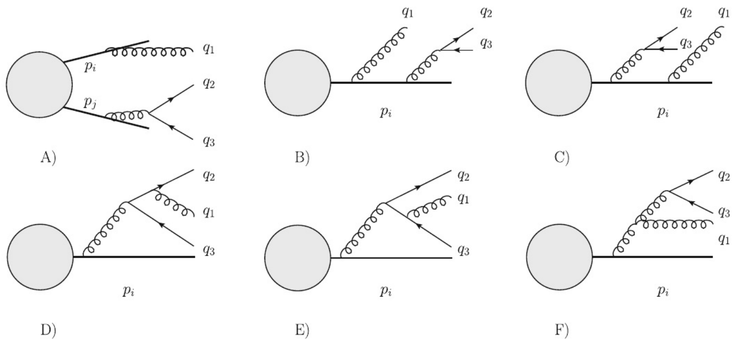

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$$|J(1,2,3)|^2 = |J(1)|^2 * |J(2,3)|^2 + W(1,2,3)$$

$$W(1,2,3) = T_R \Big\{ \sum_{ik \in \text{hard}} \boldsymbol{T}_1 \cdot \boldsymbol{T}_k \left[C_A S_{ik}^{(A)}(q_1,q_2,q_3) + C_F S_{ik}^{(F)}(q_1,q_2,q_3) \right] + \sum_{ikm \in \text{hard}} \mathcal{T}_{ikm}^{(d)} S_{ikm}(q_1,q_2,q_3) \Big\}$$

$$\mathcal{T}_{ikm}^{(d)} \equiv \sum_{a,b,c} d^{abc} \, T_i^a T_k^b T_m^c$$

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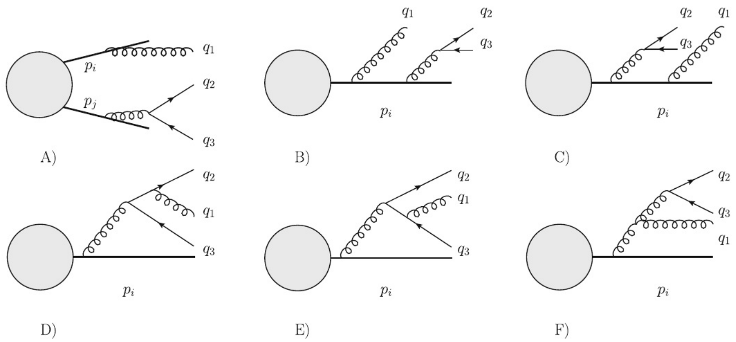

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- $S_{ik}^{(A)}$ purely non-abelian. $S_{ik}^{(F)}$ and S_{ikm} are present for g/γ-lepton-antilepton, too
- $|J(1,2,3)|^2$ becomes a c-number on amplitudes with 2 or 3 hard partons
- Agreement with *[Del Duca, Duhr, Haindl, Liu '22]*

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- With two hard partons: extended analysis to 4 soft gluons
  - Presented the full colour structure; kin.coeffs in energy ordering
  - Found col-monster contrib. ($\sim N_c^{-2}$) related to quartic Casimir
  - Generalization of Casimir scaling ($C_F \to C_A$, $d_{AF}^{(4)} \to d_{AA}^{(4)}$)
  - Colour-monster term has collinear singularities and contributes to the cusp anomalous dimension at α_S^4
  - Computed first correction $\mathcal{O}\left(1/N_c^2\right)$ to multi-eikonal BCM

Quartic Casimir

Here we are interested in contributions with quartic color factors which we abbreviate as

$$d_{xy}^{(4)} \equiv d_x^{abcd} d_y^{abcd} \,, \tag{2.6}$$

where x, y labels the representations with generators T_r^a and

$$d_r^{abcd} = \frac{1}{6} \mathrm{Tr}\,(\, T_r^a T_r^b T_r^c T_r^d + \text{ five } bcd \text{ permutations}\,) \,. \tag{2.7}$$

In SU(n_c), for fermions in the fundamental representation (trace-normalized with $T_F = \frac{1}{2}$),

$$d_{AA}^{(4)}/n_A = \frac{1}{24} n_c^2 (n_c^2 + 36) \,, \tag{2.8}$$

$$d_{FA}^{(4)}/n_A = \frac{1}{48} n_c (n_c^2 + 6) \,, \tag{2.9}$$

$$d_{FF}^{(4)}/n_A = \frac{1}{96} (n_c^2 - 6 + 18 n_c^{-2}) \,. \tag{2.10}$$

The dimension of the adjoint representation is related to $n_F = n_c = C_A$ by $n_A = (n_c^2 - 1) = 2 n_c C_F$.

Terms with quartic Casimir invariants occur for the first time at four loops, the order considered