# Theoretical upper bounds on the DM-$e^-$ scattering rate in the generalized susceptibility formalism

Michał Iglicki
in collaboration with Riccardo Catena

CHALMERS
UNIVERSITY OF TECHNOLOGY

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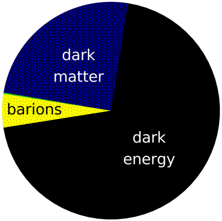
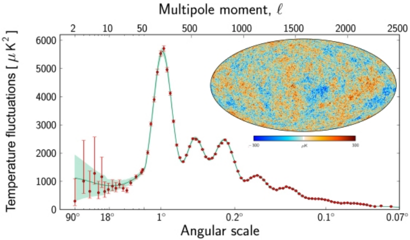
IFT Madrid, 11 June 2024

# Dark matter

https://apod.nasa.gov/apod

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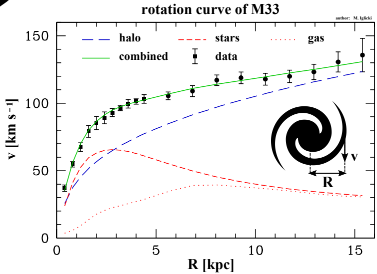
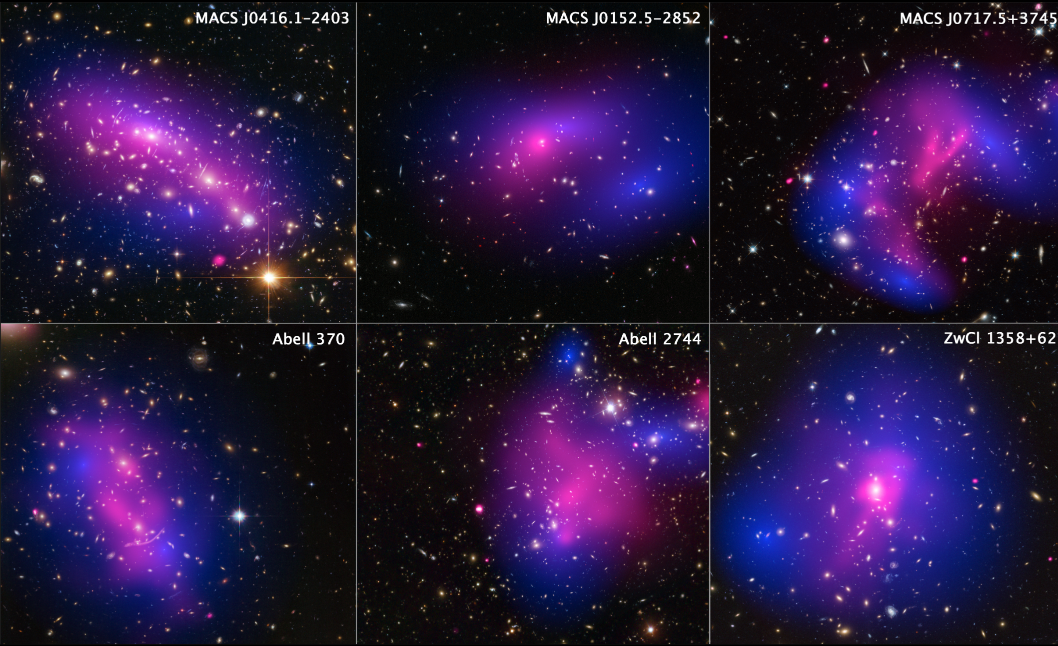
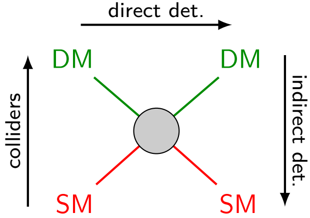


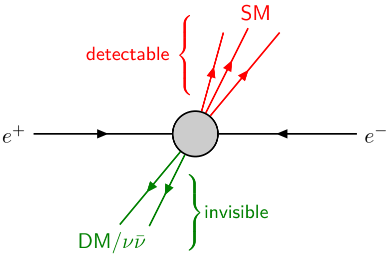

data from: arXiv:astro-ph/9909252




https://hubblesite.org

# How to detect particle dark matter?

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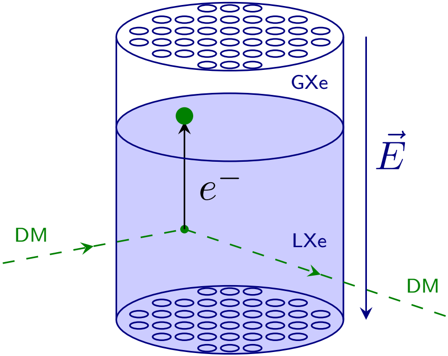
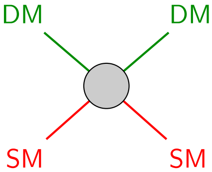


# Direct detection experiments




- dark matter is non-relativistic
- nuclear vs. electronic recoil

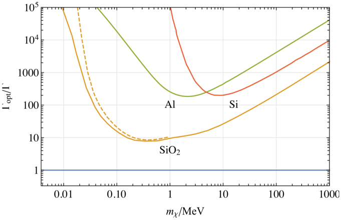
$$\Delta E_{\mathsf{SM}} \leqslant \frac{4\,\mu}{(1+\mu)^2} E^{\mathsf{in}}_{\mathsf{DM}} \qquad \leftarrow \text{ maximized for } \mu \equiv m_{\mathsf{SM}}/m_{\mathsf{DM}} = 1$$

$\Rightarrow m_{\mathsf{SM}}$ should be as close to $m_{\mathsf{DM}}$ as possible!

$\Rightarrow$ electrons preferable for light DM

- liquid/gaseous target: no success so far

- solid target: what material to use?




Lasenby & Prabhu, 2110.01587

# Outline

- effective approach to non-relativistic DM-$e^-$ interactions
- linear response theory

$$[\text{interaction rate}] = \int [\text{DM model}] \times [\text{material properties of the detector}]$$

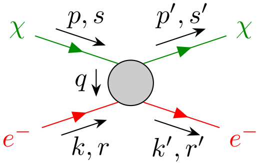
- generalized susceptibilities $\chi_{a^\dagger b}(\omega, q)$
- Kramers-Kronig relations $\Rightarrow$ theoretical upper bound on the interaction rate
- application of Kramers-Kronig relations to generalized susceptibilities calculated within the linear response theory **[new!]**

# Effective non-relativistic theory for spin-$1/2$ DM

Catena et al., 2105.02233

$$\boldsymbol{v}^{\perp} \equiv \frac{\boldsymbol{p}+\boldsymbol{p}'}{2\,m_\chi} - \frac{\boldsymbol{k}+\boldsymbol{k}'}{2\,m_e}$$

$$\boldsymbol{q}\cdot\boldsymbol{v}^{\perp} \xrightarrow{\text{en. cons.}} 0$$




Assumptions:

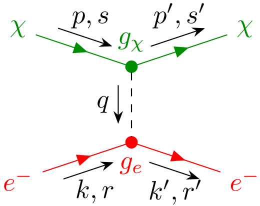
- non-relativistic limit
- Lorentz (Galilean) invariance

$$\Rightarrow \quad \mathcal{M} = \sum_i c_i\,\mathcal{O}_i \qquad i = 1, \cancel{2}, 3, 4, \ldots, 15$$

# Possible operators for a spin-$^1/_2$ dark particle

Catena et al., 2105.02233

$$\mathcal{M} = \sum_i c_i\, \mathcal{O}_i \qquad i = 1, \cancel{2}, 3, 4, \ldots, 15$$




- simple example: scalar coupling

$$\mathcal{M} = g_\chi\, \bar{u}_\chi^{s'}(p')\, u_\chi^{s}(p)\, \frac{i}{q^\mu q_\mu - M^2}\, g_e\, \bar{u}_e^{r'}(k')\, u_e^{r}(k)$$

$$\simeq \underbrace{-i\, \frac{g_\chi\, g_e}{q^2 + M^2}\, 4\, m_\chi m_e}_{c_1}\ \underbrace{\delta^{ss'} \delta^{rr'}}_{\mathcal{O}_1}$$

- other examples:

$$\mathcal{O}_1^{rr'ss'} = \delta^{rr'} \delta^{ss'}\,, \qquad \mathcal{O}_4^{rr'ss'} = \frac{\boldsymbol{\sigma}^{rr'}}{2} \cdot \frac{\boldsymbol{\sigma}^{ss'}}{2}$$

$$\mathcal{O}_9^{rr'ss'} = i\left(\frac{\boldsymbol{\sigma}^{rr'}}{2} \times \frac{\boldsymbol{q}}{m_e}\right) \cdot \frac{\boldsymbol{\sigma}^{ss'}}{2}\,, \qquad \mathcal{O}_{12}^{rr'ss'} = \left(\frac{\boldsymbol{\sigma}^{rr'}}{2} \times \boldsymbol{v}^\perp\right) \cdot \frac{\boldsymbol{\sigma}^{ss'}}{2}$$

$$\mathcal{O}_{15}^{rr'ss'} = \left[\left(\frac{\boldsymbol{\sigma}^{rr'}}{2} \times \frac{\boldsymbol{q}}{m_e}\right) \cdot \boldsymbol{v}^\perp\right] \left(\frac{\boldsymbol{\sigma}^{ss'}}{2} \cdot \frac{\boldsymbol{q}}{m_e}\right)$$

# Linear response theory

Catena & Spaldin, 2402.06817

- the electronic part of each operator can be factored out, giving

$$\mathcal{M} = \sum_i c_i\,\mathcal{O}_i = \sum_a F_a^{ss'}(\boldsymbol{q},\boldsymbol{v})\,J_a^{rr'}(\boldsymbol{v}_e^\perp)\,, \qquad \boldsymbol{v} \equiv \frac{\boldsymbol{p}}{m_\chi}\,, \qquad \boldsymbol{v}_e^\perp \equiv \frac{\boldsymbol{k}+\boldsymbol{k}'}{2m_e}$$

where

$$J_0^{rr'} \equiv \delta^{rr'}\,, \quad J_A^{rr'} \equiv \boldsymbol{v}_e^\perp \cdot \sigma^{rr'}\,,$$
$$\boldsymbol{J}_5^{rr'} \equiv \boldsymbol{\sigma}^{rr'}\,, \quad \boldsymbol{J}_M^{rr'} \equiv \boldsymbol{v}_e^\perp\,\delta^{rr'}\,, \quad \boldsymbol{J}_E^{rr'} \equiv -i\,\boldsymbol{v}_e^\perp \times \boldsymbol{\sigma}^{rr'}\,,$$

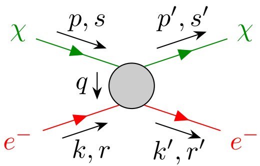
and $F_a^{ss'}(\boldsymbol{q},\boldsymbol{v})$ contain $c_1,\ldots,c_{15}$.

- scalar coupling:

$$\mathcal{M} \simeq \overbrace{\underbrace{-i\,\frac{g_\chi\,g_e}{q^2+M^2}\,4\,m_\chi m_e}_{c_1}\,\delta^{ss'}}^{F_0}\;\underbrace{\overbrace{\delta^{rr'}}^{J_0}}_{\mathcal{O}_1}\,.$$

# Interaction rate for bounded electrons

Catena et al., 1912.08204




- electron-averaged matrix element

$$|\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2 \equiv \frac{1}{4}\sum_{\text{sp.}}\left|\int \frac{d^3l}{(2\pi)^3}\psi^*_{i'\boldsymbol{k}'}(\boldsymbol{l}+\boldsymbol{q})\mathcal{M}(\boldsymbol{l},\boldsymbol{p},\boldsymbol{q})\psi_{i\boldsymbol{k}}(\boldsymbol{l})\right|^2$$

  $i$ – energy band, $\boldsymbol{k}$ – momentum in the $1^{\text{st}}$ Brillouin zone

- interaction rate per dark particle

$$\Gamma(\boldsymbol{v}) \sim \int \frac{d^3q}{(2\pi)^3}\sum_{ii'}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\,|\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2\,\delta(\text{cons.})$$

- total interaction rate

$$\mathcal{R} = n_\chi V \int d^3v\,\rho(\boldsymbol{v})\,\Gamma(\boldsymbol{v})$$

# Linear response theory, cont.

Catena & Spaldin, 2402.06817

- we have

$$|\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2 \equiv \frac{1}{4}\sum_{\text{sp.}}\left|\int \frac{d^3l}{(2\pi)^3}\psi^*_{i'\boldsymbol{k}'}(\boldsymbol{l}+\boldsymbol{q})\mathcal{M}(\boldsymbol{l},\boldsymbol{p},\boldsymbol{q})\psi_{i\boldsymbol{k}}(\boldsymbol{l})\right|^2$$

- non-relativistic limit

$$\mathcal{M}(\boldsymbol{l},\boldsymbol{p},\boldsymbol{q}) = \sum_a F_a(\boldsymbol{p},\boldsymbol{q})\, J_a(\boldsymbol{v}_e^\perp)$$

$$\simeq \sum_a F_a(\boldsymbol{q},\boldsymbol{v})\left(J_a^0(\boldsymbol{q}) + \frac{\boldsymbol{l}}{m_e}\cdot \boldsymbol{J}_a^1(\boldsymbol{q})\right)$$

$$\Rightarrow \quad |\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2 \simeq \sum_{ab}\overbrace{\mathcal{F}_{ab}(\boldsymbol{q},\boldsymbol{v})}^{\text{DM physics}}\underbrace{\Big[\; |f_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})|^2\,\mathcal{J}_{ab}(\boldsymbol{q}) + f_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})\,\boldsymbol{f}^*_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})\cdot\boldsymbol{\mathcal{J}}_{ab}(\boldsymbol{q}) + \text{c.c.} + \boldsymbol{f}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})^*\cdot\hat{\mathcal{J}}_{ab}(\boldsymbol{q})\cdot\boldsymbol{f}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})\Big]}_{\text{material response}}$$

# Generalized susceptibilities

Catena & Spaldin, 2402.06817

- total interaction rate

$$\mathcal{R} = n_\chi V \int d^3v\, \rho(\boldsymbol{v})\, \Gamma(\boldsymbol{v})$$

$$\Gamma(\boldsymbol{v}) \sim \int \frac{d^3q}{(2\pi)^3} \sum_{ii'} \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} |\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2\, \delta(\text{cons.})$$

$$|\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2 \simeq \sum_{ab} \overbrace{\mathcal{F}_{ab}(\boldsymbol{q},\boldsymbol{v})}^{\text{DM physics}} \times [\text{material response}]$$

- the **material response** part gives the generalized susceptibilities

$$\Gamma(\boldsymbol{v}) \sim \int \frac{d^3q}{(2\pi)^3} \sum_{ab} \mathcal{F}_{ab}(\boldsymbol{q},\boldsymbol{v})\, (\chi_{a^\dagger b} - \chi^*_{b^\dagger a})(\boldsymbol{q}, \omega_{\boldsymbol{v},\boldsymbol{q}}) \ ,$$

- for $a = b = n_0$,

$$\chi_{00} = \chi = 1 - \frac{4\pi\alpha}{q^2}\varepsilon^{-1} \quad \leftarrow \quad \text{„standard'' susceptibility}$$

# Kramers-Kronig relations and the dielectric function

Lasenby & Prabhu, 2110.01587

- if function $f : \mathbb{C} \to \mathbb{C}$
  - is analytic in the upper half-plane
  - satisfies $f(z) \xrightarrow{|z| \to \infty} 0$,

then

$$\begin{cases} \Re f(0) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Im f(x)}{x} \\ \Im f(0) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Re f(x)}{x} \end{cases}$$

- the dielectric function $\varepsilon^{-1}$
  - is causal, so $\varepsilon^{-1}(\omega)$ – well-defined for $\Im\omega \geqslant 0$;
  - is real in the time-domain, so $\Re\varepsilon^{-1}(\omega)$ – even, $\Im\varepsilon^{-1}(\omega)$ – odd;
  - satisfies $\varepsilon^{-1}(\omega) \xrightarrow{\omega \to \infty} 1$.

Hence,

$$\int_0^{\infty} \frac{d\omega}{\omega} \Im \left[1 - \varepsilon^{-1}(\omega, k)\right] = \frac{\pi}{2} \left[1 - \varepsilon^{-1}(0, k)\right]$$

# Kramers-Kronig relations and the generalized susceptibilities

- KK relation for $\chi_{a^\dagger a}$

$$\int_0^\infty \frac{d\omega}{\omega} \Im \left[1-\varepsilon^{-1}(\omega,k)\right] = \frac{\pi}{2}\left[1-\varepsilon^{-1}(0,k)\right]$$

$$\downarrow$$

$$\int_0^\infty \frac{d\omega}{\omega} \Im \left[\frac{4\pi\alpha}{q^2}\,\chi_{a^\dagger a}(\omega,\boldsymbol{q})\right] = \frac{\pi}{2}\left[\frac{4\pi\alpha}{q^2}\,\chi_{a^\dagger a}(0,\boldsymbol{q})\right]$$

- interaction rate

$$\mathcal{R} \sim \int d^3v\,\rho(\boldsymbol{v}) \int \frac{d^3q}{(2\pi)^3} \sum_{ab} \mathcal{F}_{ab}(\boldsymbol{q},\boldsymbol{v})\,(\chi_{a^\dagger b} - \chi^*_{b^\dagger a})(\boldsymbol{q},\omega_{\boldsymbol{v},\boldsymbol{q}})$$

- diagonal terms ($a=b$) for $\mathcal{F}_{aa} = \frac{\text{const}}{q^4}$

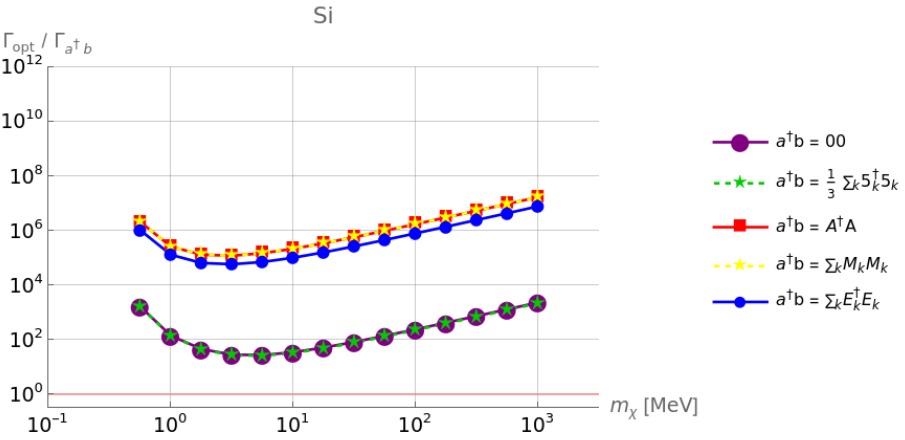
$$\begin{aligned}
\mathcal{R}_{aa} &\sim \int dq \int_0^\infty d\omega\rho(\omega,q)\,\Im \frac{4\pi\alpha}{q^2}\,\chi_{a^\dagger a}(\boldsymbol{q},\omega_{\boldsymbol{v},\boldsymbol{q}}) \\
&< \frac{\pi}{2}\int dq\,\max_\omega[\omega\,\rho(\omega,q)]\,\frac{4\pi\alpha}{q^2}\,\chi_{a^\dagger a}(0,\boldsymbol{q}) \\
&< \frac{\pi}{2}\underbrace{\max_q\left[\frac{4\pi\alpha}{q^2}\,\chi_{a^\dagger a}(0,\boldsymbol{q})\right]}_{\text{material response (but } \lesssim 1)}\underbrace{\int dq\,\max_\omega[\omega\,\rho(\omega,q)]}_{\text{calculable for a given dark halo model}}
\end{aligned}$$

# Preliminary results

## for the lowest-order approximation of $\chi_{a^\dagger a}$

$$\Gamma_{a^\dagger a} = \int dq \int_0^\infty d\omega\, \rho(\omega, q) \Im \frac{4\pi\alpha}{q^2}\, \chi_{a^\dagger a}(\boldsymbol{q}, \omega_{\boldsymbol{v},\boldsymbol{q}}) \,, \qquad \Gamma_{\text{opt}} = \frac{\pi}{2} \int dq \max_\omega [\omega\, \rho(\omega, q)]$$

(truncated thermal distribution assumed)




numerical data for $\Im\chi$ based on Catena et al., 2105.02233, 2210.07305

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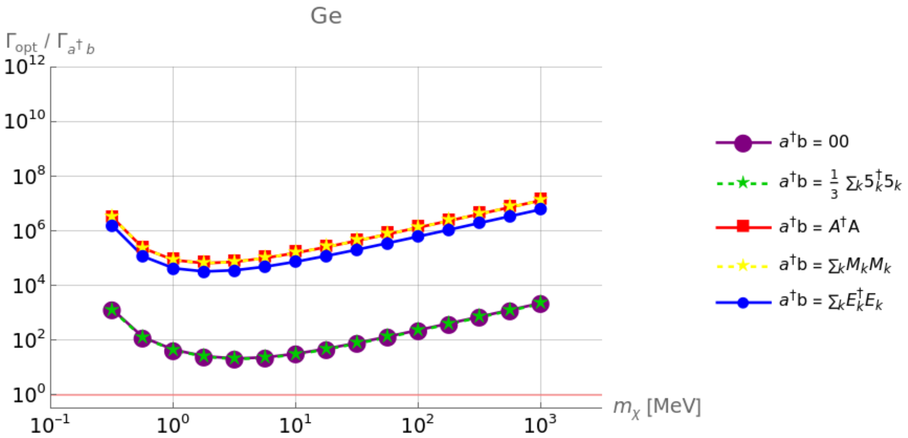


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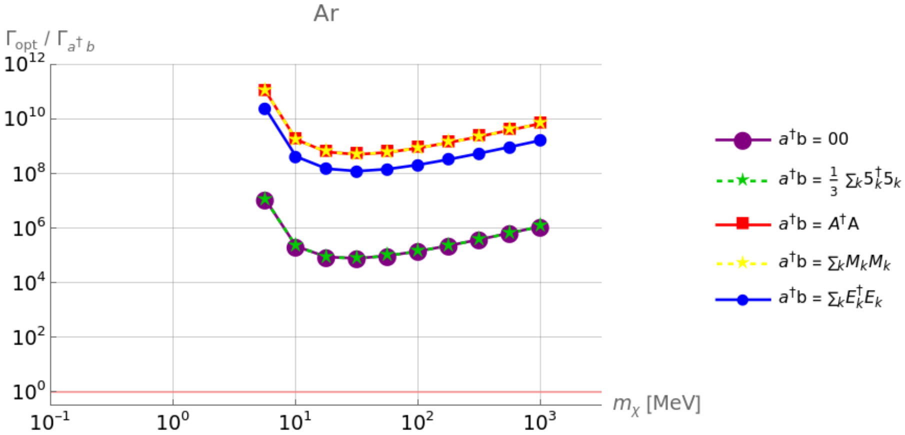


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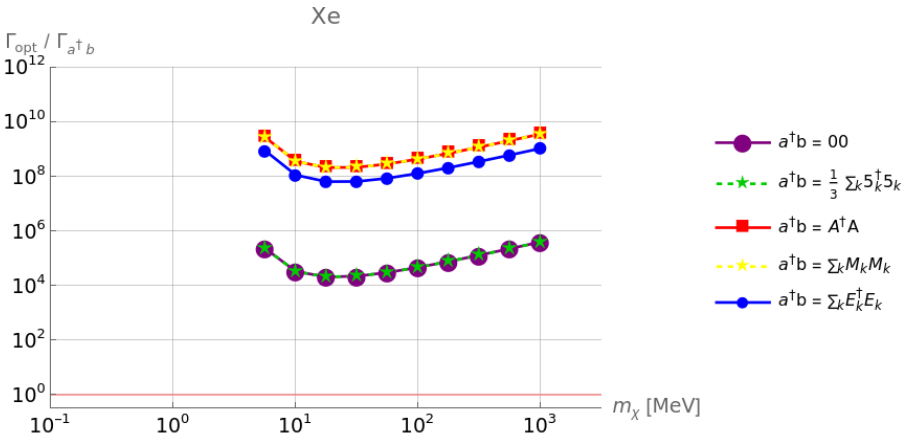


numerical data for $\Im\chi$ based on Catena et al., 2105.02233, 2210.07305

# Summary

- effective approach to non-relativistic DM-$e^-$ interactions
  - 14 operators in the leading order
- linear response theory

$$[\text{interaction rate}] = \int [\text{DM model}] \times [\text{material response of the detector}]$$

- material response $\longrightarrow$ generalized susceptibilities $\chi_{a^\dagger b}(\omega, q)$
- Kramers-Kronig relations

$$f \text{ causal, analytic} \qquad \Rightarrow \qquad \int_0^\infty \frac{d\omega}{\omega} \Im f(\omega) = \frac{\pi}{2} f(0)$$

- theoretical upper bound on the interaction rate

$$\Gamma = \int dq \int_0^\infty d\omega\, \rho(\omega, q) \Im \chi_{a^\dagger a}(\boldsymbol{q}, \omega_{\boldsymbol{v},\boldsymbol{q}})\ , \qquad \Gamma^{\text{opt}} = \frac{\pi}{2} \int dq \max_\omega [\omega\, \rho(\omega, q)]$$

- application of Kramers-Kronig relations to generalized susceptibilities calculated within the linear response theory **[new]**

# Summary

thank you!

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- application of Kramers-Kronig relations to generalized susceptibilities calculated within the linear response theory **[new]**

# BACKUP SLIDES

# Truncated thermal distribution

Baxter et al., 2105.00599

$$\rho(\boldsymbol{v}) = \mathcal{N}\,\exp\left[-\frac{(\boldsymbol{v}+\boldsymbol{v}_\oplus)^2}{v_0^2}\right]\theta(v_{\mathsf{esc}} - |\boldsymbol{v}+\boldsymbol{v}_\oplus|)$$

$$\begin{aligned} v_{\mathsf{esc}} &= 544\,\mathsf{km/s} \\ v_\oplus &= 250.5\,\mathsf{km/s} \\ v_0 &= 238\,\mathsf{km/s} \end{aligned}$$

$$\mathcal{N} = \frac{1}{2\pi\, v_0^3}\left(\frac{\sqrt{\pi}}{2}\,\mathsf{erf}\left[\frac{v_{\mathsf{esc}}}{v_0}\right] - \frac{v_{\mathsf{esc}}}{v_0}\,\exp\left[-\frac{v_{\mathsf{esc}}^2}{v_0^2}\right]\right)^{-1}$$

# Electronic couplings

## — scalar —

symmetry:

$$\mathcal{J}_{ba} = \mathcal{J}^*_{ab}$$

values:

$$\mathcal{J}_{00} = 1$$

$$\mathcal{J}_{AA} = \frac{q^2}{4m_e}$$

$$\mathcal{J}_{5_k 5_l} = \delta_{kl}$$

$$\mathcal{J}_{M_k M_l} = \frac{q_k q_l}{4m_e^2}$$

$$\mathcal{J}_{E_k E_l} = \frac{\delta_{kl} q^2 - q_k q_l}{2m_e^2}$$

$$\mathcal{J}_{0M_k} = \mathcal{J}_{A5_k} = \frac{q_k}{2m_e}$$

$$\mathcal{J}_{5_k E_l} = -i\varepsilon_{klm} \frac{q_m}{2m_e}$$

others 0
(up to the symmetry)

## — vector —

symmetry:

$$\boldsymbol{\mathcal{J}}_{ab} + \boldsymbol{\mathcal{J}}^*_{ba} = \boldsymbol{C}_{ab}$$

$$\equiv \frac{1}{2} \sum_{rr'} \nabla_{\boldsymbol{v}_e^\perp} \left( J_a^{rr'} J_b^{rr'} \right) \Big|_{\boldsymbol{v}_e^\perp = \frac{\boldsymbol{q}}{2m_e}}$$

values:

$$\boldsymbol{\mathcal{J}}_{0\bullet} = \boldsymbol{\mathcal{J}}_{5_k\bullet}$$

$$= \boldsymbol{\mathcal{J}}_{AM_k} = \boldsymbol{\mathcal{J}}_{M_k E_k} = 0$$

$$\boldsymbol{\mathcal{J}}_{AA} = \frac{\boldsymbol{q}}{2m_e}$$

$$\boldsymbol{\mathcal{J}}_{M_k M_l} = \frac{q_l}{2m_e} \boldsymbol{e}_k$$

$$\boldsymbol{\mathcal{J}}_{E_k E_l} = \frac{\delta_{kl} \boldsymbol{q} - q_k \boldsymbol{e}_l}{2m_e}$$

$$\boldsymbol{\mathcal{J}}_{AE_k} = -i\boldsymbol{e}_k \times \frac{\boldsymbol{q}}{2m_e}$$

others 0
(up to the symmetry)

$$\boldsymbol{v} \cdot \boldsymbol{e}_i \equiv v_i$$

## — tensor —

symmetry:

$$\hat{\mathcal{J}}_{ba} = \hat{\mathcal{J}}^\dagger_{ab}$$

values:

$$\hat{\mathcal{J}}_{AA} = \mathbb{1}$$

$$\hat{\mathcal{J}}_{M_k M_l} = \hat{e}_{kl}$$

$$\hat{\mathcal{J}}_{E_k E_l} = \delta_{kl} \mathbb{1} - \hat{e}_{lk}$$

$$\hat{\mathcal{J}}_{AE_k} = i\varepsilon_{kij} \hat{e}_{ij}$$

$$\boldsymbol{v} \cdot \hat{e}_{ij} \cdot \boldsymbol{u} \equiv v_i u_j$$

others 0
(up to the symmetry)

$$\boldsymbol{C}_{AA} = \frac{\boldsymbol{q}}{m_e}$$

$$\boldsymbol{C}_{M_k M_l} = \frac{q_k \boldsymbol{e}_l + q_l \boldsymbol{e}_k}{2m_e}$$

$$\boldsymbol{C}_{E_k E_l} = \frac{2\delta_{kl} \boldsymbol{q} - q_k \boldsymbol{e}_l - q_l \boldsymbol{e}_k}{2m_e}$$

$$\boldsymbol{C}_{0M_k} = C_{a5_k} = \boldsymbol{e}_k$$

$$\boldsymbol{C}_{5_k E_l} = -i\varepsilon_{klm} \boldsymbol{e}_m$$

others 0 (up to $\boldsymbol{C}_{ba} = \boldsymbol{C}^*_{ab}$)