Modular Stabilization and Modular Inflation

arXiv:2405.06497 with Guijun Ding, Siyi Jiang

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Outline

Motivation

- Modular symmetry has been successfully used as a guiding principle to explain several puzzles in the SM:
  - Fermion mass hierarchy,
  - Flavor mixing,
  - CP violation,

  where a scalar (modulus) field, determines the Yukawa coupling.

Motivation

- Modular symmetry has been successfully used as a guiding principle to explain several puzzles in the SM:
  - Fermion mass hierarchy,
  - Flavor mixing,
  - CP violation,

  where a scalar (modulus) field, determines the Yukawa coupling.
- The vacuum of the modulus potential is important but dynamics of modulus field is less so.
- The dynamics of modulus field can be used to realize inflation.

More on modular inflation: 1604.02995, 2208.10086, 2303.02947, 2405.08924

Modular Symmetry I

Modular Group SL(2,$\mathbb{Z}$)

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}.$$

Modular Transformation

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \;\; \mathrm{Im}\tau > 0\,.$$

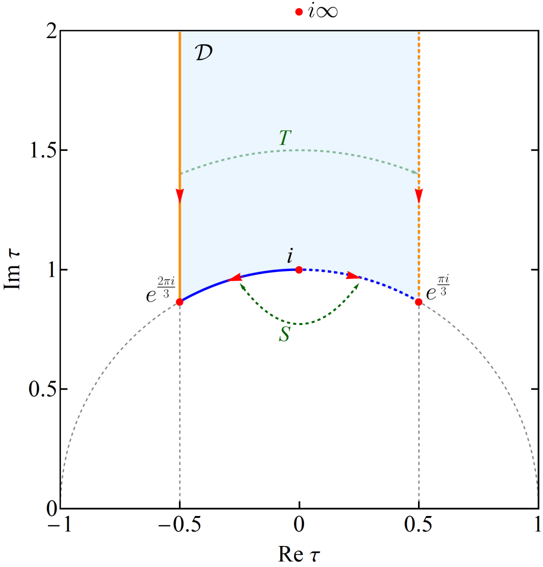
$$\mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \tau \to -\frac{1}{\tau}\,, \quad \mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \tau \to \tau + 1\,,$$

Modular Forms

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma\,,$$

where the weight k is a generic non-negative integer.

Modular Symmetry II:Fundamental domain



Modular Symmetry III

The derivative of a weight k modular form f satisfies:

$$f'(\gamma\tau) = (c\tau + d)^{k+2} f'(\tau) + \frac{k}{2\pi i} c(c\tau + d)^{k+1} f(\tau), \quad \gamma \in \Gamma\,,$$

For a weight 0 modular form, it's derivative is a weight 2 modular form. There are 3 fixed points (under $\mathcal{S}$ or $\mathcal{T}$ or their combinations) in the fundamental domain:

$$i, \omega = e^{\frac{2\pi i}{3}}, i\infty$$

Derivatives of wight 0 modular form have to vanishes there.
i and ω are natural candidates for vacuum!

Modular Symmetry from String Theory

$D = 10$ Superstring $\longrightarrow$ Dialton S and other fields

$\downarrow$

Compactifycation $\longrightarrow$ modulus fields: massless scalar fields with no potential

$\mathcal{M}_{10} = \mathcal{M}_6 \times \mathcal{M}_4$

$\downarrow$

moduli stabilization: generate potential, positive mass and vev

$$\mathcal{M}_6 = T^2 \times T^2 \times T^2$$

Kähler modulus τ and complex structure modulus

Modular group is the mapping class group of torus

$\downarrow$

$D = 4, N = 1$ SUGRA $\longrightarrow$ Modular invariant scalar potential for S and τ

SuperGravity framework

In SUGRA, scalar potential is determined by Kähler potential $\mathcal{K}$ and superpotential $\mathcal{W}$ in a combined way:

$$\mathcal{G}(\tau,\bar{\tau},S,\bar{S}) = \mathcal{K}(\tau,\bar{\tau},S,\bar{S}) + \ln|\mathcal{W}(\tau,S)|^2\,,$$

And the scalar potential reads:

$$\begin{aligned} V(\tau,S) &= e^{\mathcal{K}}(\mathcal{K}^{\alpha\overline{\beta}}D_\alpha\mathcal{W}\overline{D_\beta\mathcal{W}} - 3|\mathcal{W}|^2) \\ &= e^{\mathcal{G}}(\mathcal{G}_\alpha\mathcal{G}^{\alpha\bar{\beta}}\mathcal{G}_{\bar{\beta}} - 3) \end{aligned}$$

where the covariant derivative is defined by $D_\alpha\mathcal{W} \equiv \partial_\alpha\mathcal{W} + \mathcal{W}(\partial_\alpha\mathcal{K})$ and $\mathcal{K}^{\alpha\overline{\beta}}$ is the inverse of the Kähler metric $\mathcal{K}_{\alpha\overline{\beta}} = \partial_\alpha\partial_{\overline{\beta}}\mathcal{K}$. The total bosonic action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}\mathcal{R} - g^{\mu\nu}\mathcal{K}_{\alpha\overline{\beta}}\partial_\mu\phi^\alpha\partial_\nu\overline{\phi^\beta} - V(\phi)\right]\,,$$

Potential setup I

$$\mathcal{K}(\tau,\bar{\tau},S,\bar{S}) = K(S,\bar{S}) - 3\ln(-i(\tau-\bar{\tau}))\,,$$
$$\mathcal{W}(S,\tau) = \Lambda_W^3\,\frac{\Omega(S)H(\tau)}{\eta^6(\tau)}\,,$$

- We assume dialton S is stabilized.
- η is the Dedekind eta function with a modular *weight* 1/2:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q \equiv e^{2\pi i\tau}\,,$$

- Under Modular transformation, they reads:

$$-3\ln\left[-i(\tau-\bar{\tau})\right] \to -3\ln\left[-i(\tau-\bar{\tau})\right] + 3\ln(c\tau+d) + 3\ln(c\bar{\tau}+d)\,.$$
$$\mathcal{W} \to e^{i\delta(\gamma)}(c\tau+d)^{-3}\mathcal{W}\,,$$

- $\mathcal{G}(\tau,\bar{\tau},S,\bar{S})$ and potential are modular invariant.

Potential setup II

The most general form without singularity inside the fundamental domain:

$$H(\tau) = (j(\tau) - 1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau)) \,, \quad m, n \in \mathbb{N} \,,$$

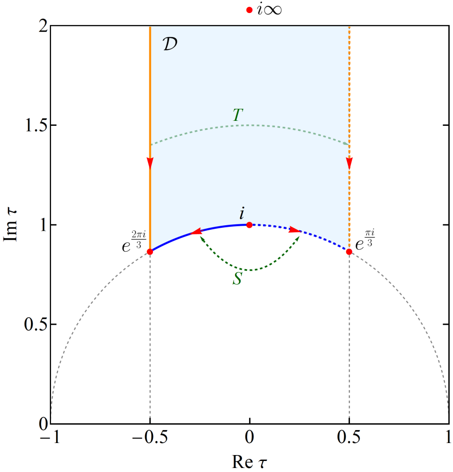
where j is called Klein j invariant.

$$j(i\infty) = +\infty \,, \qquad j(\omega) = 0 \,, \qquad j(i) = 1728 = 12^3 \,.$$

m, n determine vacua of the potential and we choose:

- $m = 0, n \geq 2$, slow roll from i (saddle point) to ω (Minkowski minimum) along the arc.
- $m \geq 2, n \geq 2$, we consider slow roll from $i\infty$ to the fixed point ω (Minkowski minimum) along the left boundary.
- $m = n = 0$, slow roll from i (saddle point) to ω (dS minimum) along the arc (King, Wang, 2405.08924).

Inflation in the Fundamental domain



Modular symmetry + Reality of potential
stabilize the orthogonal direction of inflation!

Full potential

We choose the following polynomial:

$$\mathcal{P}(j(\tau)) = 1 + \beta \left(1 - \frac{j(\tau)}{1728}\right) + \gamma \left(1 - \frac{j(\tau)}{1728}\right)^2 ,$$

and the full potential reads:

$$V(\tau) = \frac{\Lambda_S^4}{i(\tau-\bar{\tau})^3 |\eta(\tau)|^{12}} \left[(A(S,\bar{S}) - 3)|H(\tau)|^2 + \hat{V}(\tau,\bar{\tau}) \right] ,$$

$$A(S,\bar{S}) = \frac{K^{S\bar{S}} D_S W D_{\bar{S}} \bar{W}}{|W|^2} = \frac{K^{S\bar{S}} |\Omega_S + K_S \Omega|^2}{|\Omega|^2} ,$$

$$\hat{V}(\tau,\bar{\tau}) = \frac{-(\tau-\bar{\tau})^2}{3} \left| H_\tau(\tau) - \frac{3i}{2\pi} H(\tau) \widehat{G}_2(\tau,\bar{\tau}) \right|^2 ,$$

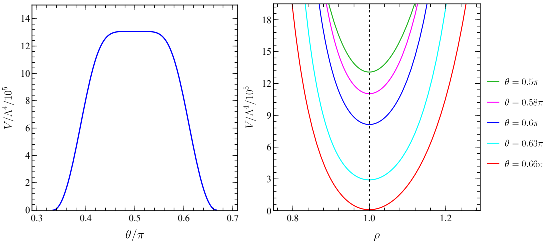
$$Z(\tau,\bar{\tau}) = \frac{1}{i(\tau-\bar{\tau})^3 |\eta(\tau)|^{12}} ,$$

In short, 3 parameter sets: (m,n), (β,γ), $A(S,\bar{S})$

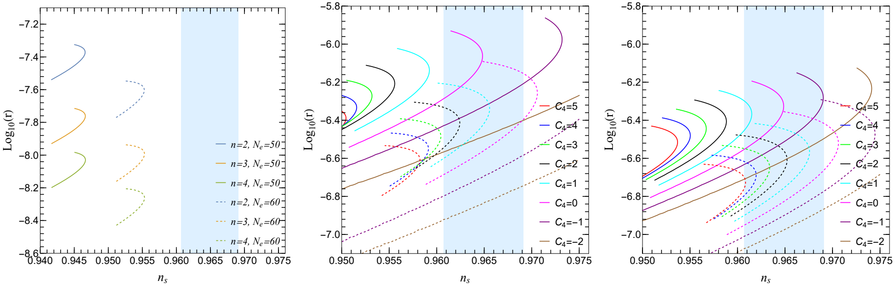
Slow roll along the unit arc

$m=0, n\geq 2$: $\tau=\rho e^{i\theta}$ and $\tau=i$ is the start point of inflation:

$$\begin{aligned} V>0 \quad &\Rightarrow \quad A(S,\bar{S})>3\,, \\ \varepsilon_V=\frac{1}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \quad &\Rightarrow \quad \text{modular symmetry}\,, \\ \eta_V=\frac{V''}{V}\ll 1 \quad &\Rightarrow \quad (\beta,\gamma)\,. \end{aligned}$$



Example: $m=0$, $n=2$, $A=24.3091$ and $\beta=0.126425, \gamma=0$.



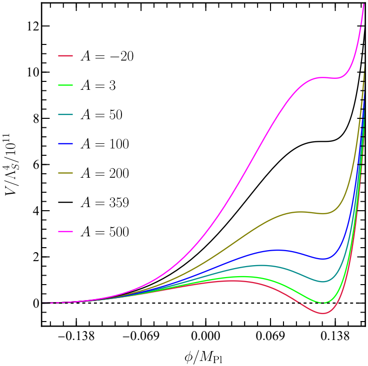
(a) $\mathcal{P}(j) = 1$.   (b) with β.   (c) with γ.

- Taylor expansion: $V(\phi) = V_0(1 - \sum_{k=1}^{\infty} C_{2k}\phi^{2k})$,
- The simplest case, $P(j) = 1$ gives too small spectral index.
- The rest: $r < 10^{-6}$, $\alpha \approx -10^{-4}$.

Slow roll in the left boundary

$m \geq 2, n \geq 2$: $\tau = \mathrm{Re}(\tau) + i\,\mathrm{Im}(\tau)$.

Accidental inflation: up-lifting of adjacent minimum leads to inflation.



$m = 2$, $n = 2$ and $\beta = -0.633431$

A narrow region for slow roll + ultra slow roll:

$$357.85 < A < 358.75$$

Summary

- It is interesting to combine modular symmetry with inflation.
- Modular symmetry is a strong constraint as well as a useful handle.
- Three parameter sets: $A(S, \bar{S})$, (m, n), (β, γ).
- Two inflationary trajectories: Along the arc or left boundary.
- Outlook:
  - Maybe fine-tuned. A more natural way?
  - Dynamics of dilaton field?
  - Non-single field inflation?
  - Post-inflation: preheating, reheating?

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Thanks for your attention!

Eisenstein series

The Eisenstein series $G_{2k}(\tau)$ of weight $2k$ for integer $k > 1$ is defined as:

$$G_{2k}(\tau) = \sum_{\substack{n_1,n_2\in\mathbb{Z} \\ n_1,n_2\neq(0,0)}} (n_1 + n_2\,\tau)^{-2k}\,,$$

and the Fourier series of Eisenstein series read:

$$G_{2k}(q) = 2\zeta(2k)\left(1 + c_{2k}\sum_{i=1}^{\infty}\sigma_{2k-1}(i)q^i\right)\,,$$

where the coefficients c_{2k} are given by

$$c_{2k} = \frac{(2\pi i)^{2k}}{(2k-1)!\zeta(2k)} = -\frac{-4k}{B_{2k}} = \frac{2}{\zeta(1-2k)}\,. \tag{1}$$

Here B_n are the Bernoulli numbers, $\zeta(z)$ is the Riemann's zeta function and $\sigma_p(n)$ is the divisor sum function,

$$\sigma_p(n) = \sum_{d|n} d^p\,. \tag{2}$$

j invariant

The Klein j-invariant function is a modular form of weight zero, defined in terms of Dedekind eta function and Eisenstein series as follows:

$$j(\tau) \equiv \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\eta^{24}(\tau)} = \frac{3^6 5^3}{\pi^{12}} \frac{G_4^3(\tau)}{\Delta(\tau)}\,, \quad \Delta(\tau) \equiv \eta^{24}(\tau)\,,$$

For convenience, the q-expansion of j-function is given by

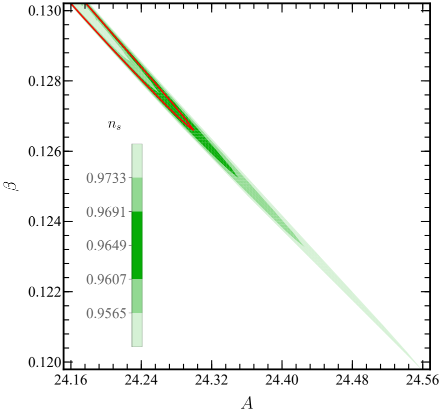
$$\begin{aligned} j(\tau) \;\; = \;\; & 744 + \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 \\ & + 20245856256q^4 + 333202640600q^5 + 4252023300096q^6 \\ & + 44656994071935q^7 + \mathcal{O}(q^8)\,. \end{aligned}$$

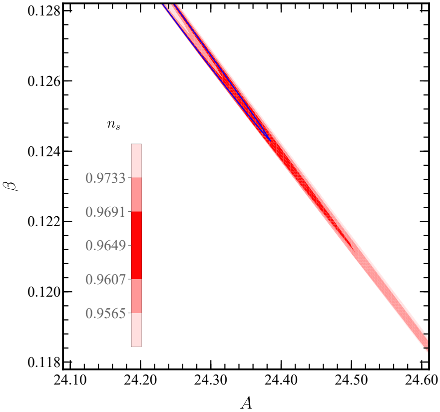
Vacuum structure of the potential

The vacuum structure of this potential at $\tau = i$ and at $\tau = \omega = e^{i2\pi/3}$ has been extensively studied in 2212.03876, where they find the following results based on the choice of (m, n):

- If $m = n = 0$, then both fixed points can have a de Sitter (dS) vacuum.
- If $m > 1, n = 0$, then $\tau = \omega$ is a dS minimum, while $\tau = i$ is Minkowski minimum.
- If $m = 0, n > 1$, then $\tau = i$ is a conditional dS minimum, which depends on the value of $A(S, \bar{S})$. $\tau = \omega$ is always a Minkowski minimum.
- If $m = 1, n > 0$ or $n = 1, m > 0$, the vacuum is unstable.
- If $m > 1, n > 1$, then we always have Minkowski extrema in these two fixed points.

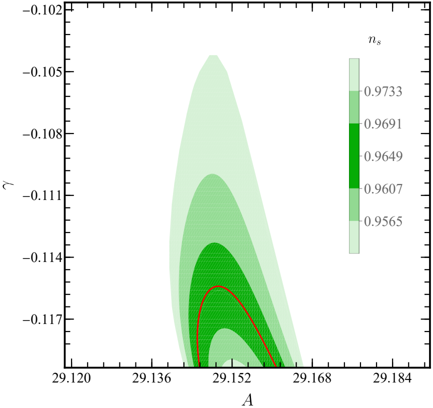
Slow roll along the unit arc

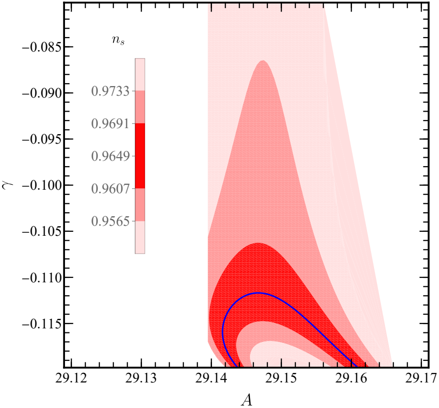




$$\mathcal{P}(j) = 1 + \beta(1 - j/1728)\,.$$

Slow roll along the unit arc





$$\mathcal{P}(j) = 1 + \gamma(1 - j/1728)\,.$$