

Modular Stabilization and Modular Inflation

arXiv:2405.06497 with Guijun Ding, Siyi Jiang

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Outline

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- 2 Modular Symmetry
- 3 Modular invariant inflation
- 4 Summary

Motivation

- **Modular symmetry** has been successfully used as a guiding principle to explain several puzzles in the SM:
 - Fermion mass hierarchy,
 - Flavor mixing,
 - CP violation,

where a scalar (**modulus**) field, determines the Yukawa coupling.

Motivation

- **Modular symmetry** has been successfully used as a guiding principle to explain several puzzles in the SM:
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where a scalar (**modulus**) field, determines the Yukawa coupling.

- The **vacuum** of the modulus potential is important but **dynamics** of modulus field is less so.
- The dynamics of modulus field can be used to realize **inflation**.

More on modular inflation: 1604.02995, 2208.10086, 2303.02947, 2405.08924

Modular Symmetry I

Modular Group $SL(2, \mathbb{Z})$

$$= \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \quad a, b, c, d \in \mathbb{Z}; \quad ad - bc = 1$$

Modular Transformation

$$z \mapsto \frac{az + b}{cz + d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}); \quad \text{Im } z > 0$$

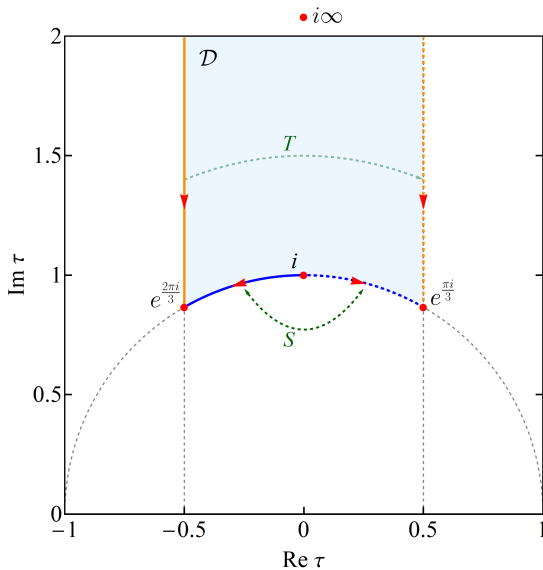
$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \det = +1$$

Modular Forms

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z); \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

where the weight k is a generic non-negative integer.

Modular Symmetry II: Fundamental domain



Modular Symmetry III

The derivative of a weight k modular form f satisfies:

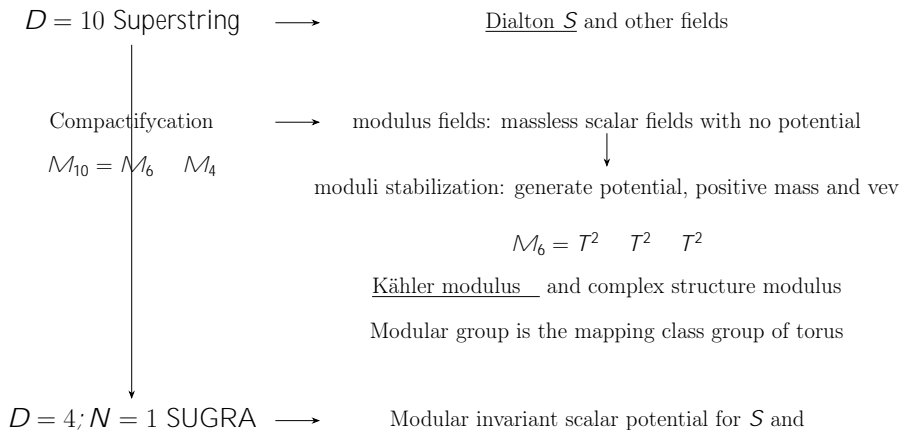
$$f'(z) = (c + dz)^{k+2} f'(z) + \frac{k}{2} c (c + dz)^{k+1} f(z); \quad 2 \leq k;$$

For a weight 0 modular form, it's derivative is a weight 2 modular form. There are 3 fixed points (under S or T or their combinations) in the fundamental domain:

$$i; ! = e^{\frac{2\pi i}{3}}; i1$$

Derivatives of weight 0 modular form have to vanish there. i and $!$ are natural candidates for **vacuum!**

Modular Symmetry from String Theory



SuperGravity framework

In SUGRA, scalar potential is determined by Kähler potential K and superpotential W in a combined way:

$$G(\phi, \psi; S, \bar{S}) = K(\phi, \psi; S, \bar{S}) + \ln |W(\phi, \psi; S)|^2;$$

And the scalar potential reads:

$$\begin{aligned} V(\phi, \psi; S) &= e^K (K^{-1} D_i W \overline{D^i W} - 3|W|^2) \\ &= e^G (G_{i\bar{j}} G^{\bar{j}i} - 3) \end{aligned}$$

where the covariant derivative is defined by $D_i W = \partial_i W + W(\partial_i K)$ and K^{-1} is the inverse of the Kähler metric $K_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$. The total bosonic action:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{Pl}}^2}{2} R - g^{\mu\nu} K_{\mu\bar{\nu}} - V(\phi, \psi) \right];$$

Potential setup I

$$K(\tau; \tau; S; S) = K(S; S) - 3 \ln(\eta(\tau));$$

$$W(S; \tau) = \frac{3}{W} \frac{(S)H(\tau)}{6(\tau)};$$

- We assume dilaton S is stabilized.
- $\eta(\tau)$ is the Dedekind eta function with a modular *weight* $1=2$:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n); \quad q = e^{2\pi i \tau};$$

- Under Modular transformation, they reads:

$$3 \ln[\eta(\tau)] \rightarrow 3 \ln[\eta(\tau)] + 3 \ln(c + d) + 3 \ln(c - d);$$

$$W \rightarrow e^{i\pi(c+d)} (c + d)^3 W;$$

- $G(\tau; \tau; S; S)$ and potential are modular invariant.

Potential setup II

The most general form without singularity inside the fundamental domain:

$$H(\phi) = (j(\phi) - 1728)^{m-2} j(\phi)^{n-3} P(j(\phi)); \quad m, n \geq 2 \in \mathbb{N};$$

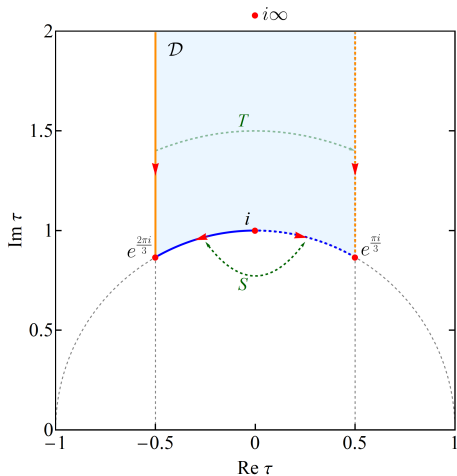
where j is called **Klein j invariant**.

$$j(i) = +1; \quad j(!) = 0; \quad j(i) = 1728 = 12^3;$$

m, n determine vacua of the potential and we choose:

- $m = 0; n \geq 2$, slow roll from i (saddle point) to $!$ (Minkowski minimum) **along the arc**.
- $m \geq 2; n \geq 2$, we consider slow roll from i to the fixed point $!$ (Minkowski minimum) **along the left boundary**.
- $m = n = 0$, slow roll from i (saddle point) to $!$ (dS minimum) **along the arc** (King, Wang, 2405.08924).

Inflation in the Fundamental domain



Modular symmetry + Reality of potential
stabilize the orthogonal direction of inflation!

Full potential

We choose the following polynomial:

$$P(j(\tau)) = 1 + \frac{j(\tau)}{1728} + \frac{j(\tau)^2}{1728};$$

and the full potential reads:

$$V(\tau) = \frac{S^4}{i(\tau)^3 j(\tau)^{12}} \left(A(S; S) - 3jH(\tau) \right)^2 + \hat{V}(\tau; S);$$

$$A(S; S) = \frac{K^{SS} D_S W D_S W}{j W j^2} = \frac{K^{SS} j S + K_S j^2}{j j^2};$$

$$\hat{V}(\tau; S) = \frac{(\tau)^2}{3} H(\tau) - \frac{3i}{2} H(\tau) \mathcal{G}_2(\tau; S);$$

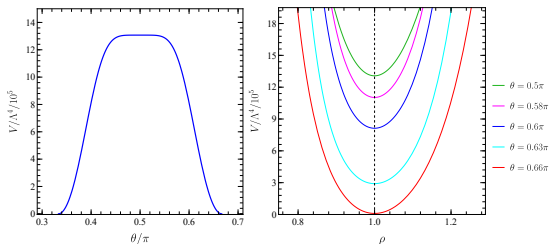
$$Z(\tau; S) = \frac{1}{i(\tau)^3 j(\tau)^{12}};$$

In short, 3 parameter sets: $(m; n); (\tau; S); A(S; S)$

Slow roll along the unit arc

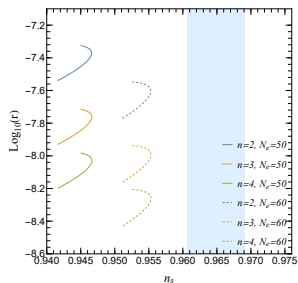
$m = 0; n = 2$: $V = e^i$ and $\rho = i$ is the start point of inflation:

$$\begin{aligned}
 & V > 0 \quad) \quad A(S; S) > 3; \\
 \text{"} V &= \frac{1}{2} \frac{V^{\theta}}{V} \quad) \quad \text{modular symmetry}; \\
 v &= \frac{V^{\theta\theta}}{V} \quad) \quad (;):
 \end{aligned}$$

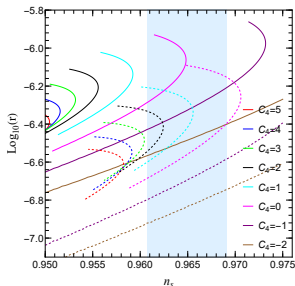


Example: $m = 0, n = 2, A = 24.3091$ and $\theta = 0.126425\pi; \rho = 0$.

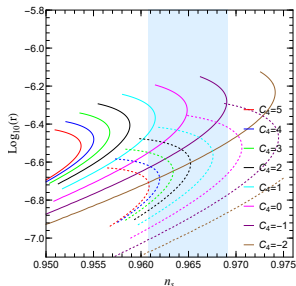
Slow roll along the unit arc



(a) $\mathcal{P}(j) = 1$.



(b) with β .



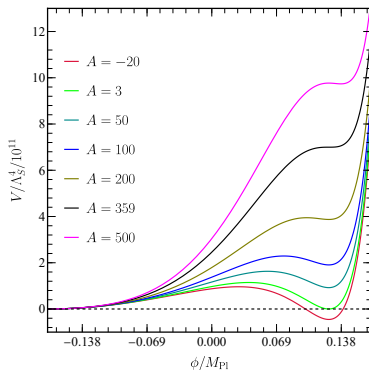
(c) with γ .

- Taylor expansion: $V(\phi) = V_0 \left(1 + \sum_{k=1}^{\mathcal{P}} C_{2k} \phi^{2k} \right)$;
- The simplest case, $\mathcal{P}(j) = 1$ gives too small spectral index.
- The rest: $r < 10^{-6}$, $n_s < 10^{-4}$.

Slow roll in the left boundary

$$m = 2; n = 2: \quad = \text{Re}(\) + i \text{Im}(\).$$

Accidental inflation: up-lifting of adjacent minimum leads to inflation.



$$m = 2, n = 2 \text{ and } = -0.633431$$

A narrow region for slow roll + ultra slow roll:

$$357.85 < A < 358.75$$

Summary

- It is interesting to combine modular symmetry with inflation.
- Modular symmetry is a strong constraint as well as a useful handle.
- Three parameter sets: $A(S; S)$, $(m; n)$, $(\ ; \)$.
- Two inflationary trajectories: Along the arc or left boundary.
- Outlook:
 - Maybe fine-tuned. A more natural way?
 - Dynamics of dilaton field?
 - Non-single field inflation?
 - Post-inflation: preheating, reheating?

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Thanks for your attention!

Eisenstein series

The Eisenstein series $G_{2k}(\tau)$ of weight $2k$ for integer $k > 1$ is defined as:

$$G_{2k}(\tau) = \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1, n_2 \neq (0,0)}} (n_1 + n_2 \tau)^{-2k};$$

and the Fourier series of Eisenstein series read:

$$G_{2k}(\tau) = 2(2k-1) + c_{2k} \sum_{i=1}^{\infty} \sigma_{2k-1}(i) q^i;$$

where the coefficients c_{2k} are given by

$$c_{2k} = \frac{(2k-1)^{2k}}{(2k-1)!(2k)} = \frac{4k}{B_{2k}} = \frac{2}{(1-2k)}; \quad (1)$$

Here B_n are the Bernoulli numbers, $\zeta(z)$ is the Riemann's zeta function and $\rho(n)$ is the divisor sum function,

$$\rho(n) = \sum_{d|n} d^{\rho}; \quad (2)$$

j invariant

The Klein j -invariant function is a modular form of weight zero, defined in terms of Dedekind eta function and Eisenstein series as follows:

$$j(\tau) = \frac{3^6 5^3 G_4^3(\tau)}{12 \eta^{24}(\tau)} = \frac{3^6 5^3 G_4^3(\tau)}{12 \eta^{24}(\tau)}; \quad \eta(\tau) = 24 \eta(\tau);$$

For convenience, the q -expansion of j -function is given by

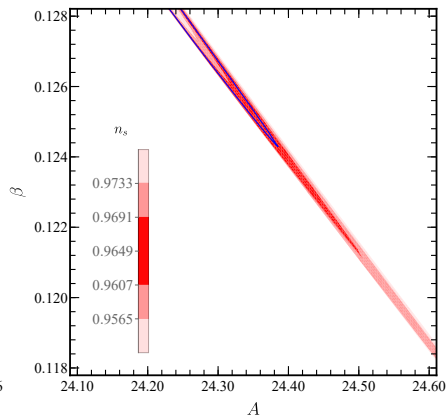
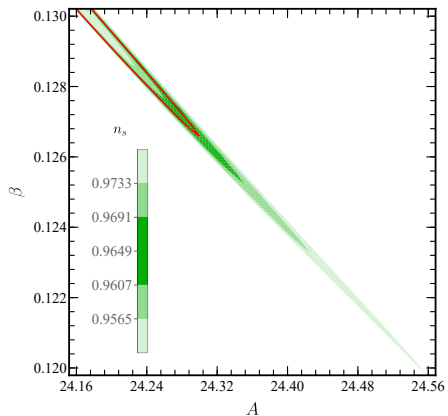
$$\begin{aligned} j(\tau) = & 744 + \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 \\ & + 20245856256q^4 + 333202640600q^5 + 4252023300096q^6 \\ & + 44656994071935q^7 + O(q^8): \end{aligned}$$

Vacuum structure of the potential

The vacuum structure of this potential at $\phi = i$ and at $\phi = ! = e^{i2} = 3$ has been extensively studied in 2212.03876, where they find the following results based on the choice of $(m; n)$:

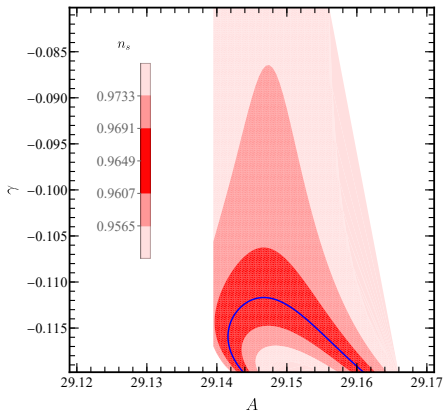
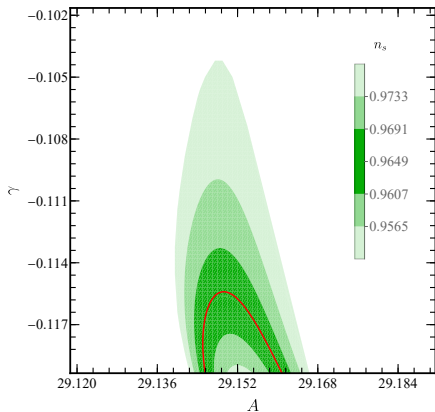
- If $m = n = 0$, then both fixed points can have a de Sitter (dS) vacuum.
- If $m > 1; n = 0$, then $\phi = !$ is a dS minimum, while $\phi = i$ is Minkowski minimum.
- If $m = 0; n > 1$, then $\phi = i$ is a conditional dS minimum, which depends on the value of $A(S; S)$. $\phi = !$ is always a Minkowski minimum.
- If $m = 1; n > 0$ or $n = 1; m > 0$, the vacuum is unstable.
- If $m > 1; n > 1$, then we always have Minkowski extrema in these two fixed points.

Slow roll along the unit arc



$$P(j) = 1 + (1 \quad j=1728).$$

Slow roll along the unit arc



$$P(j) = 1 + (1 - j/1728).$$