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LOCAL UNITARITY

BUILDING A NUMERICAL COLLIDER

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CERN RAS

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OVERVIEW OF LOCAL UNITARITY



ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with \mathcal{J} a measurement function, over $x \in [0, 10]$

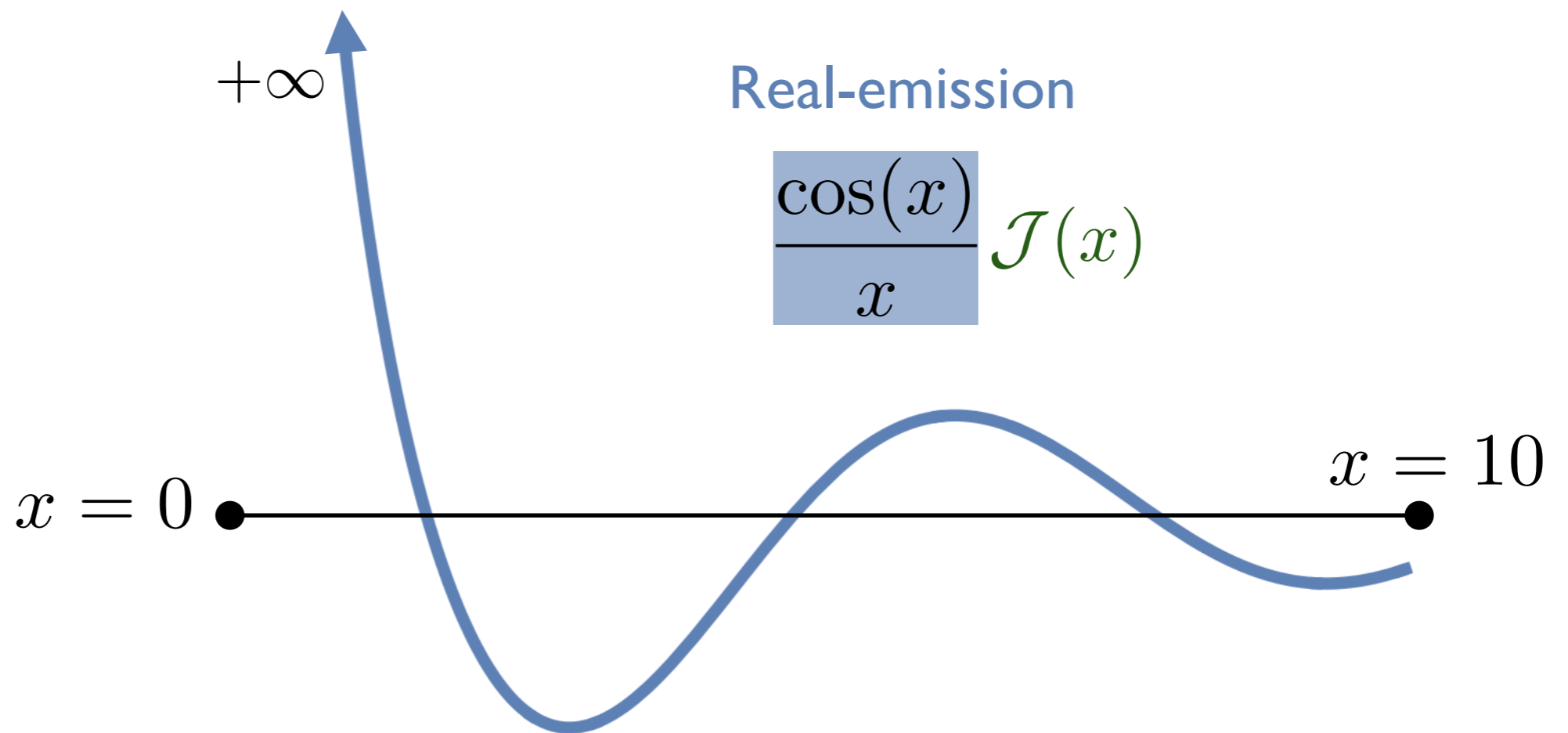
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with \mathcal{J} a measurement function, over $x \in [0, 10]$

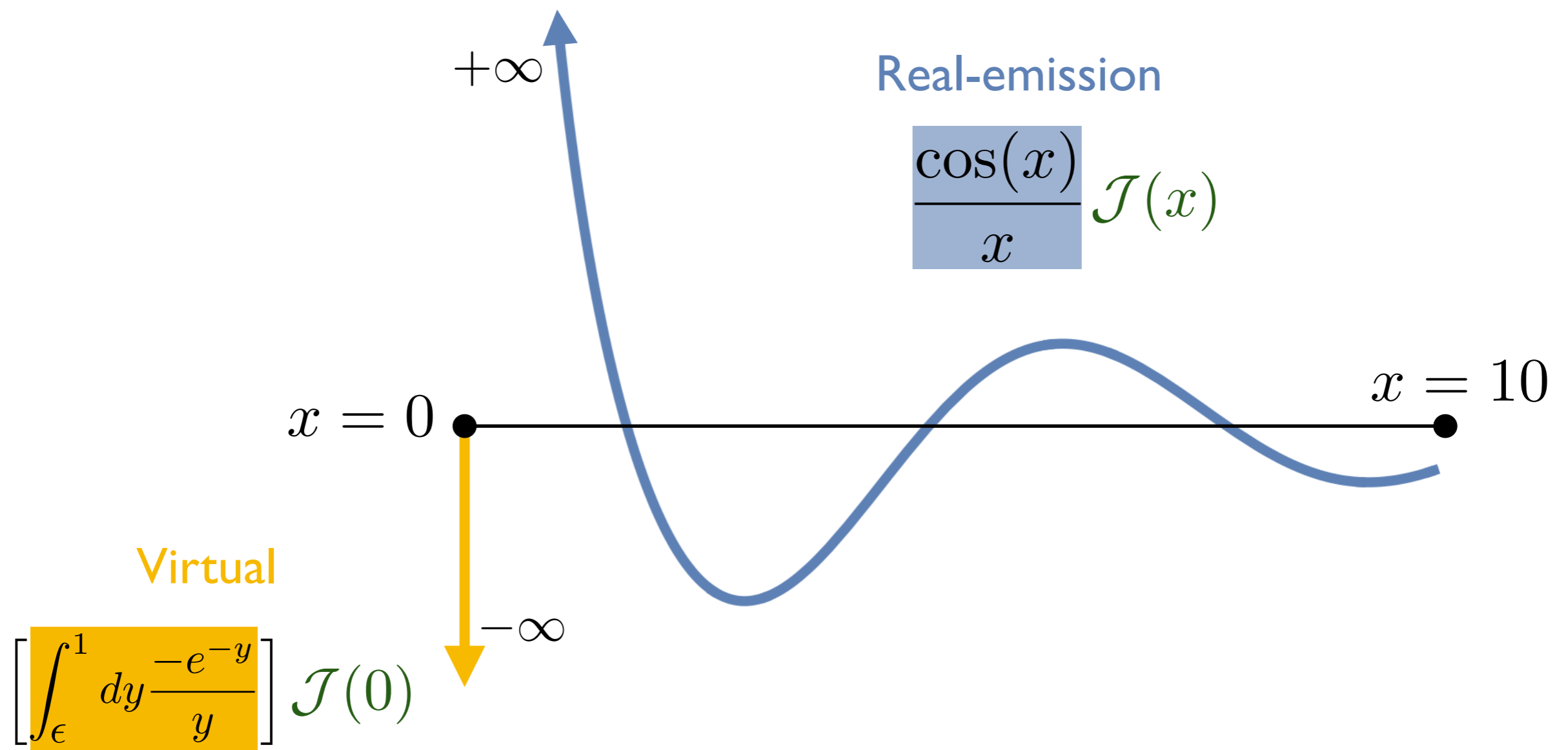
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



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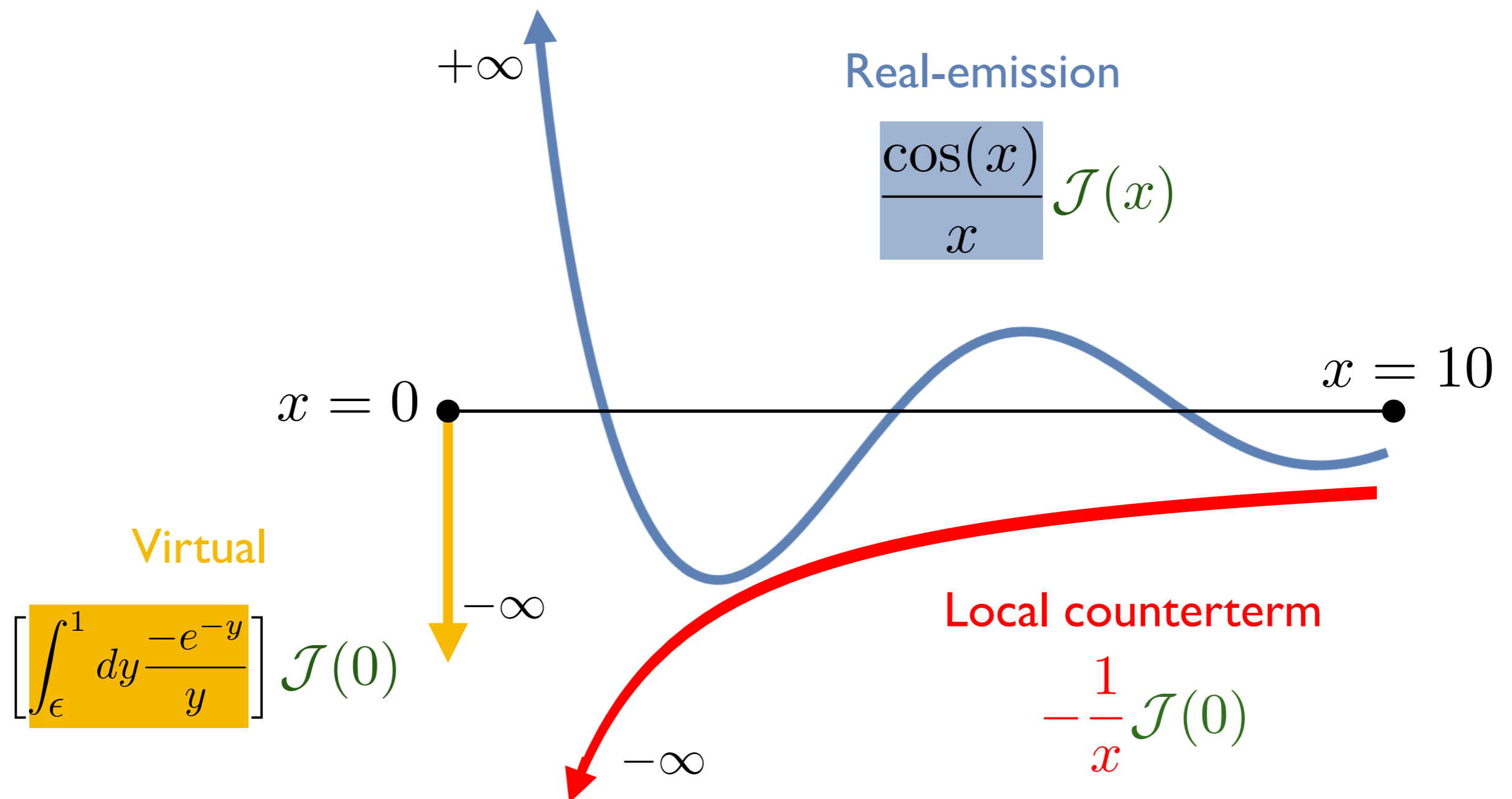
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with \mathcal{J} a measurement function, over $x \in [0, 10]$

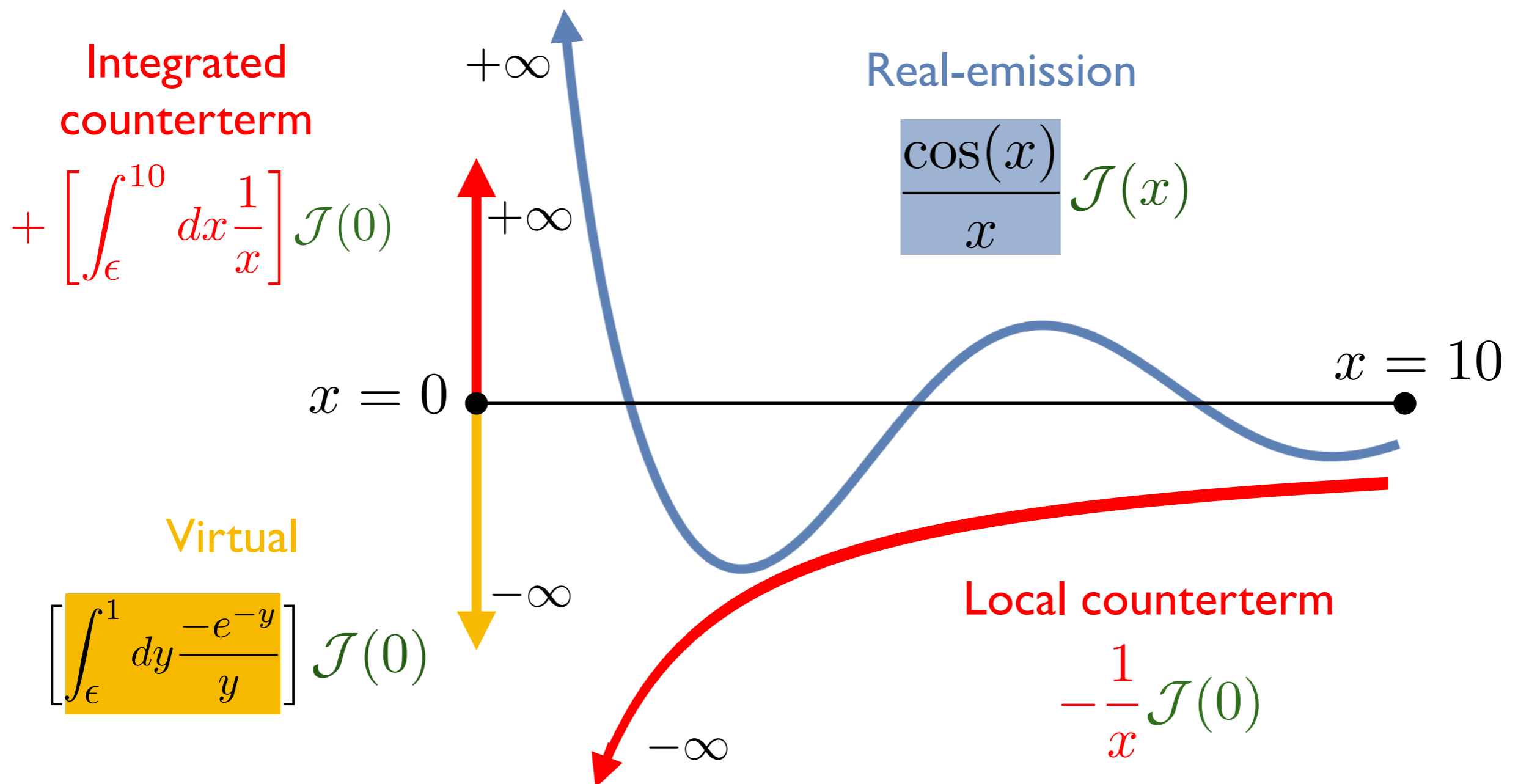
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{1}{x} \mathcal{J}(0) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



ONE-DIMENSIONAL TOY EXAMPLE

- Toy expression with \mathcal{J} a measurement function, over $x \in [0, 10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{1}{x} \mathcal{J}(0) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0) + \left[\int_0^{10} dx \frac{1}{x} \right] \mathcal{J}(0)$$



ONE-DIMENSIONAL TOY EXAMPLE

- Local unitarity would align the measure between real and virtual

$$\left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \quad \sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \right]$$



ONE-DIMENSIONAL TOY EXAMPLE

- Local unitarity would align the measure between real and virtual

$$\left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \qquad \sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{e^{-x/10}}{x} \mathcal{J}(0) \right]$$

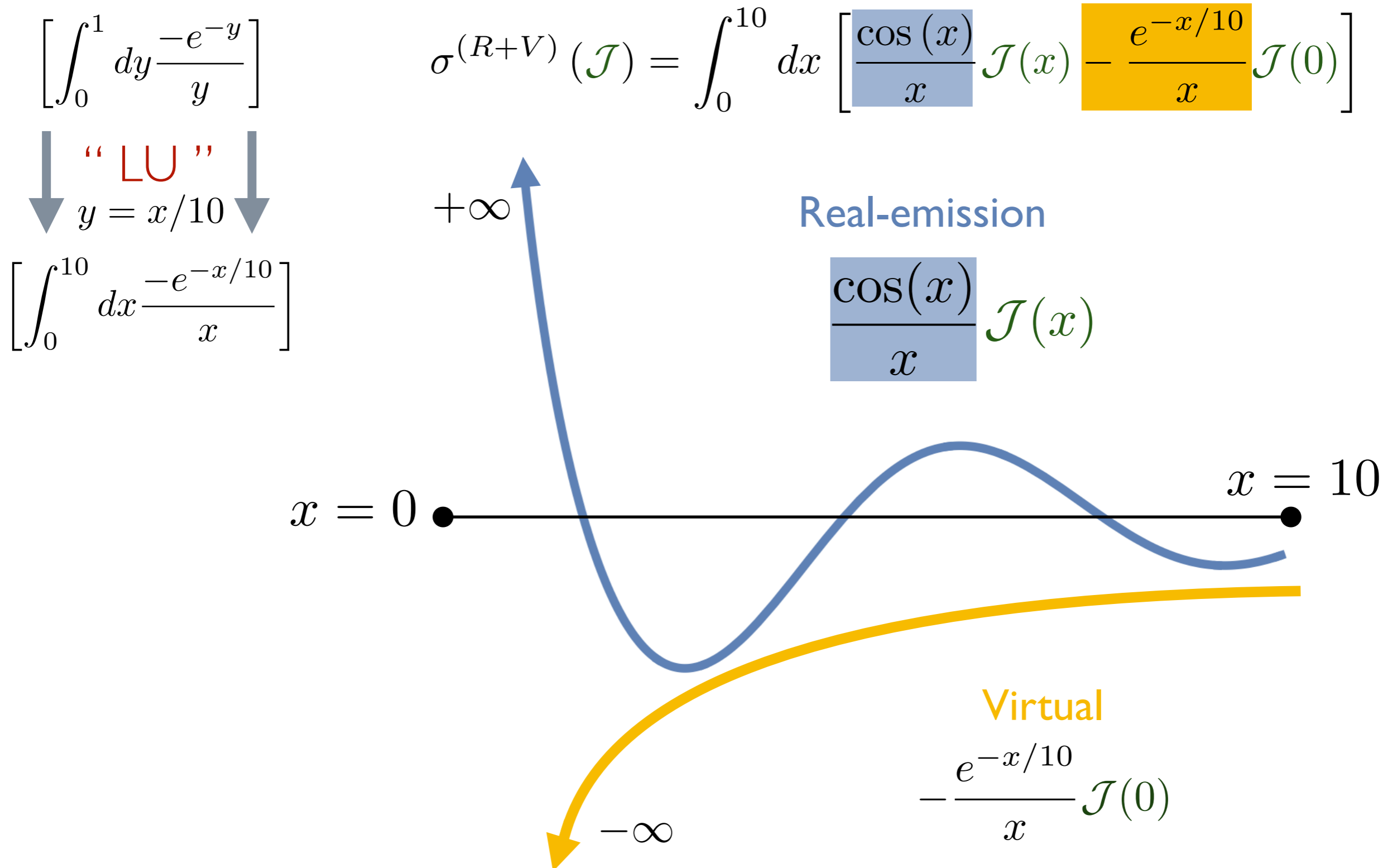
“LU”
↓ $y = x/10$ ↓

$$\left[\int_0^{10} dx \frac{-e^{-x/10}}{x} \right]$$



ONE-DIMENSIONAL TOY EXAMPLE

- Local unitarity would align the measure between real and virtual



REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The diagrams are Feynman diagrams for the process $\gamma^* \rightarrow d\bar{d}$. Each diagram consists of a wavy line (photon) and a fermion line (quark). The first two diagrams in each row show the photon splitting into a quark-antiquark pair, with the quark and antiquark lines forming a loop. The first diagram in each row has a vertical wavy line (gluon) connecting the quark and antiquark lines. The second diagram in each row has a diagonal wavy line (gluon) connecting the quark and antiquark lines. The diagrams are summed and then multiplied by their complex conjugates to give the cross-section.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} =$$

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 &+ \left(\text{Diagram 5} + \text{Diagram 6} \right) \times \left(\text{Diagram 7} + \text{Diagram 8} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 9}
 \end{aligned}$$

The diagrams are as follows:

- Diagram 1: A blue square on the left, a wavy line, and a semi-circle with a vertical dashed line.
- Diagram 2: A wavy line and a semi-circle with a vertical dashed line.
- Diagram 3: A green square on the left, a wavy line, and a semi-circle with a vertical dashed line.
- Diagram 4: A wavy line and a semi-circle with a vertical dashed line.
- Diagram 5: A wavy line and a semi-circle with a diagonal dashed line.
- Diagram 6: A wavy line and a semi-circle with a diagonal dashed line.
- Diagram 7: A wavy line and a semi-circle with a diagonal dashed line.
- Diagram 8: A wavy line and a semi-circle with a diagonal dashed line.
- Diagram 9: A circle with a wavy line on the left and a wavy line on the right, split by a vertical red line. The left half is blue and the right half is green.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 &+ \left(\text{Diagram 5} + \text{Diagram 6} \right) \times \left(\text{Diagram 7} + \text{Diagram 8} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 9} + \text{Diagram 10}
 \end{aligned}$$

The diagrams are Feynman-like graphs for the process $\gamma^* \rightarrow d\bar{d}$.

 - Diagram 1: A wavy line (photon) enters from the left and splits into a quark and an antiquark.

 - Diagram 2: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a blue square.

 - Diagram 3: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square.

 - Diagram 4: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square.

 - Diagram 5: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square.

 - Diagram 6: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square.

 - Diagram 7: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square.

 - Diagram 8: Similar to Diagram 1, but the quark and antiquark lines are highlighted in a green square.

 - Diagram 9: A circular loop diagram with a wavy line entering from the left and exiting to the right. A red vertical line is drawn through the loop.

 - Diagram 10: Similar to Diagram 9, but the loop is highlighted in a blue square and a green square, with a red vertical line drawn through it.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 &+ \left(\text{Diagram 5} + \text{Diagram 6} \right) \times \left(\text{Diagram 7} + \text{Diagram 8} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11}
 \end{aligned}$$

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^* \\
 &+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 &+ \text{Diagram 8}
 \end{aligned}$$

The diagrams are Feynman-like graphs for the process $\gamma^* \rightarrow d\bar{d}$.
 - Diagram 1: A wavy line (photon) enters from the left and splits into a quark and an antiquark.
 - Diagram 2: A wavy line enters from the left, splits into a quark and an antiquark, with a vertical wavy line (gluon) connecting them.
 - Diagram 3: Similar to Diagram 1, but with a blue shaded region on the left.
 - Diagram 4: Similar to Diagram 2, but with a blue shaded region on the left.
 - Diagram 5: A circular loop with a wavy line entering from the left and exiting to the right. A red vertical line is drawn through the loop.
 - Diagram 6: Similar to Diagram 5, but with a vertical wavy line inside the loop.
 - Diagram 7: Similar to Diagram 5, but with a vertical wavy line inside the loop.
 - Diagram 8: Similar to Diagram 5, but with a blue shaded region on the left and a green shaded region on the right, and a diagonal wavy line inside the loop.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\begin{aligned}
 \sigma_{\gamma^* \rightarrow d\bar{d}} &= \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^* \\
 &+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^* \\
 \sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} &= \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \\
 &+ \text{Diagram 8} + \text{Diagram 9}
 \end{aligned}$$

The diagrams are Feynman-like graphs for the process $\gamma^* \rightarrow d\bar{d}$.
 - Diagram 1: A wavy line (photon) enters from the left and splits into a quark and an antiquark.
 - Diagram 2: Similar to Diagram 1, but with a vertical wavy line (gluon) loop between the quark and antiquark.
 - Diagram 3: Similar to Diagram 1, but with a diagonal wavy line (gluon) loop.
 - Diagram 4: Similar to Diagram 3, but with a vertical wavy line (gluon) loop.
 - Diagram 5: A circular loop with a wavy line entering from the left and exiting to the right. A red vertical line is drawn through the loop.
 - Diagram 6: Similar to Diagram 5, but with a vertical wavy line loop inside the circle.
 - Diagram 7: Similar to Diagram 6, but with a diagonal wavy line loop inside the circle.
 - Diagram 8: Similar to Diagram 5, but with a diagonal wavy line loop inside the circle.
 - Diagram 9: Similar to Diagram 8, but with a vertical wavy line loop inside the circle.
 - Diagrams 2, 4, 8, and 9 are highlighted with blue and green squares.
 - Diagrams 3 and 4 are highlighted with a blue square.
 - Diagram 9 is highlighted with a green square.
 - Diagrams 5, 6, 7, and 8 have a red vertical line drawn through them.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The first row shows two diagrams in parentheses, multiplied together. The first diagram in each pair is a wavy line entering a semi-circle from the left, with a vertical wavy line inside the semi-circle. The second diagram in each pair is a wavy line entering a semi-circle from the left, with a vertical wavy line inside the semi-circle. The second row shows two diagrams in parentheses, multiplied together. The first diagram in each pair is a wavy line entering a semi-circle from the left, with a diagonal wavy line inside the semi-circle. The second diagram in each pair is a wavy line entering a semi-circle from the left, with a diagonal wavy line inside the semi-circle. The first diagram in the second row is highlighted with a blue square, and the second diagram is highlighted with a green square.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(LU)} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7}$$

$$+ \text{Diagram 8} + \text{Diagram 9}$$

$$+ \text{Diagram 10}$$

The first row shows three diagrams: a circle with a vertical red line, a circle with a vertical wavy line and a vertical red line, and a circle with a vertical wavy line and a vertical red line. The second row shows two diagrams: a circle with a diagonal wavy line and a vertical red line, and a circle with a diagonal wavy line and a vertical red line. The third row shows one diagram: a circle with a diagonal wavy line and a vertical red line, highlighted with a blue square on the left and a green square on the right.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The first row shows two terms in large parentheses, each containing a sum of two diagrams. The first diagram in each sum is a wavy line entering a semi-circle from the left, with a vertical wavy line inside. The second diagram is similar but with a vertical wavy line on the right. The second row shows two terms in large parentheses, each containing a sum of two diagrams. The first diagram in each sum is a wavy line entering a semi-circle from the left, with a diagonal wavy line inside. The second diagram is similar but with a diagonal wavy line on the right. The second diagrams in both rows are highlighted with colored squares: blue for the first row and green for the second row.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3}$$

$$+ \text{Diagram 4} + \text{Diagram 5}$$

$$+ \text{Diagram 6} + \text{Diagram 7}$$

The equation shows a sum of seven diagrams. Each diagram consists of a circle with a wavy line entering from the left and exiting to the right. A red vertical line is drawn through each circle. The first three diagrams have a vertical wavy line inside the circle. The next two diagrams have a diagonal wavy line inside. The last diagram has a diagonal wavy line and is highlighted with a blue and green square. The red lines represent forward-scattering graphs.

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

The diagrams in the first row show a wavy line entering a semi-circle from the left. The first semi-circle is empty, and the second contains a vertical chain of small circles. The diagrams in the second row show a wavy line entering a semi-circle from the left, with a diagonal chain of small circles crossing the semi-circle.

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

The diagrams in this equation are circles with a wavy line entering from the left and another exiting to the right. A red vertical line is drawn through each circle. Diagram 5 is enclosed in a green box. Diagrams 6 and 7 have a vertical chain of small circles inside. Diagrams 8 and 9 have a diagonal chain of small circles. Diagram 10 has a diagonal chain of small circles and a dashed diagonal line.

 LO

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

█ LO
█ NLO, Double-Triangle (DT)

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

- █ LO
- █ NLO, Double-Triangle (DT)
- █ NLO, Self-Energy (SE)

REMEDY: FORWARD-SCATTERING GRAPHS

$$\sigma_{\gamma^* \rightarrow d\bar{d}} = \left(\text{Diagram 1} + \text{Diagram 2} \right) \times \left(\text{Diagram 1} + \text{Diagram 2} \right)^*$$

$$+ \left(\text{Diagram 3} + \text{Diagram 4} \right) \times \left(\text{Diagram 3} + \text{Diagram 4} \right)^*$$

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10}$$

Diagram 5 LO
Diagram 6, 7, 8, 9 NLO, Double-Triangle (DT)
Diagram 10, 11 NLO, Self-Energy (SE)

$\text{Diagram 12} \equiv \frac{p^2}{2p^0} \delta(p^2) \Theta(p^0)$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi(\text{phase-space}) \left| \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right|^2$$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi(\text{phase-space}) \left| \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right|^2$$

↓

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right] + \text{LU} \left[\begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right] + \text{LU} \left[2 \times \begin{array}{c} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right]$$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi^{(\text{phase-space})} \left| \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right|^2$$

↓

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right] + \text{LU} \left[\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right] + \text{LU} \left[2 \times \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right]$$

$$\sum_{c \in \{RRR, RRV, RVV, \dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|_{\text{truncated}}^2$$

IR-subtraction numerical $d = 4$

analytic $d = 4 - 2\epsilon$

LOCAL UNITARITY: A CONCEPTUAL SHIFT

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{normal})} = \int \Pi^{(\text{phase-space})} \left| \text{---} \left(\text{---} + \text{---} + \text{---} + \text{---} \right) \right|^2$$

↓

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{---} \right] + \text{LU} \left[\text{---} \right] + \text{LU} \left[2 \times \text{---} \right]$$

$$\sum_{c \in \{RRR, RRV, RVV, \dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|_{\text{truncated}}^2$$

IR-subtraction numerical $d = 4$

analytic $d = 4 - 2\epsilon$

$$\sum_{j=1}^{n_{\text{supergraphs}}} \int \Pi g_j^{(\text{LU})}$$

numerical $d = 4$
NO IR-subtraction

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Diagram 1} \right] + \text{LU} \left[\text{Diagram 2} \right] + \text{LU} \left[2 \times \text{Diagram 3} \right]$$

The equation shows the Local Unitarity (LU) contribution to the cross-section $\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})}$. It is expressed as the sum of three terms, each enclosed in red brackets and preceded by "LU".

- Diagram 1:** A circular loop with two external wavy lines (representing photons) and two internal arrows indicating a clockwise flow.
- Diagram 2:** A circular loop with two external wavy lines and two internal arrows. A vertical dashed line with small circles (representing a ghost loop) is drawn across the center of the loop.
- Diagram 3:** A circular loop with two external wavy lines and two internal arrows. A smaller circular loop with a dashed line and small circles is attached to the top of the main loop.

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Diagram 1} \right] + \text{LU} \left[\text{Diagram 2} \right] + \text{LU} \left[2 \times \text{Diagram 3} \right]$$

- At the **integral** level, the **Optical Theorem** simply gives: $\text{LU}[\cdot] \propto \text{Im}[\cdot]$

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

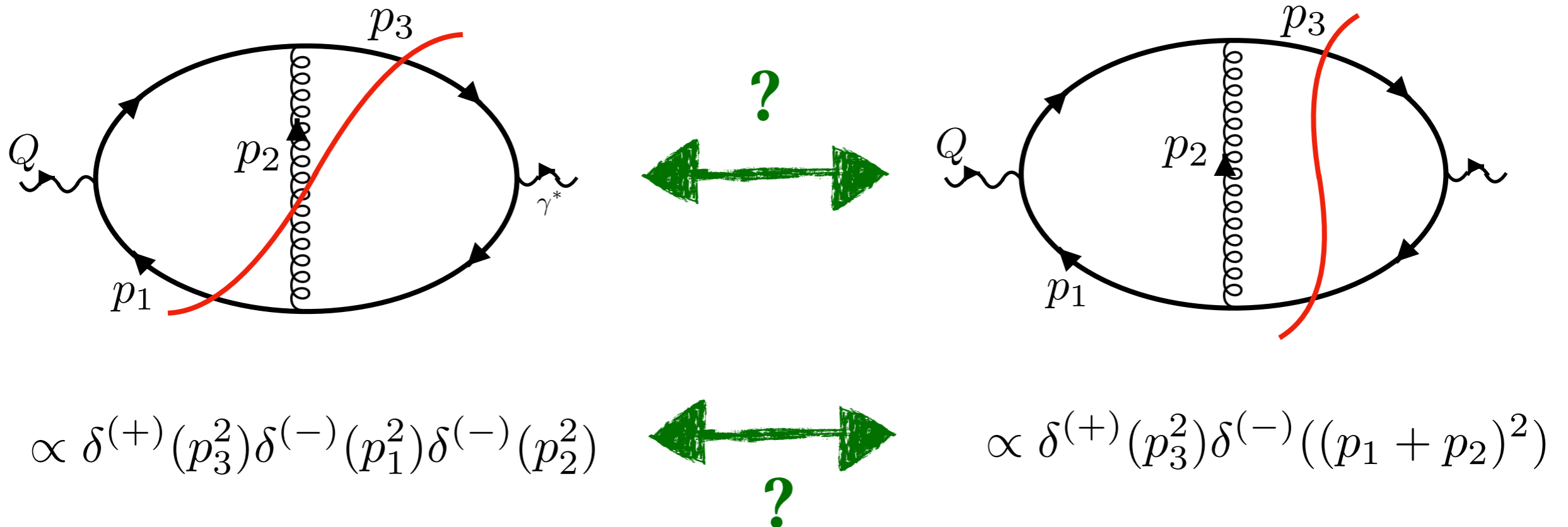
$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Diagram 1} \right] + \text{LU} \left[\text{Diagram 2} \right] + \text{LU} \left[2 \times \text{Diagram 3} \right]$$

- At the **integral** level, the **Optical Theorem** simply gives: $\text{LU}[\cdot] \propto \text{Im}[\cdot]$
- The **differential** version of **LU** is our new paradigm: **Local Unitarity**

SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Diagram 1} \right] + \text{LU} \left[\text{Diagram 2} \right] + \text{LU} \left[2 \times \text{Diagram 3} \right]$$

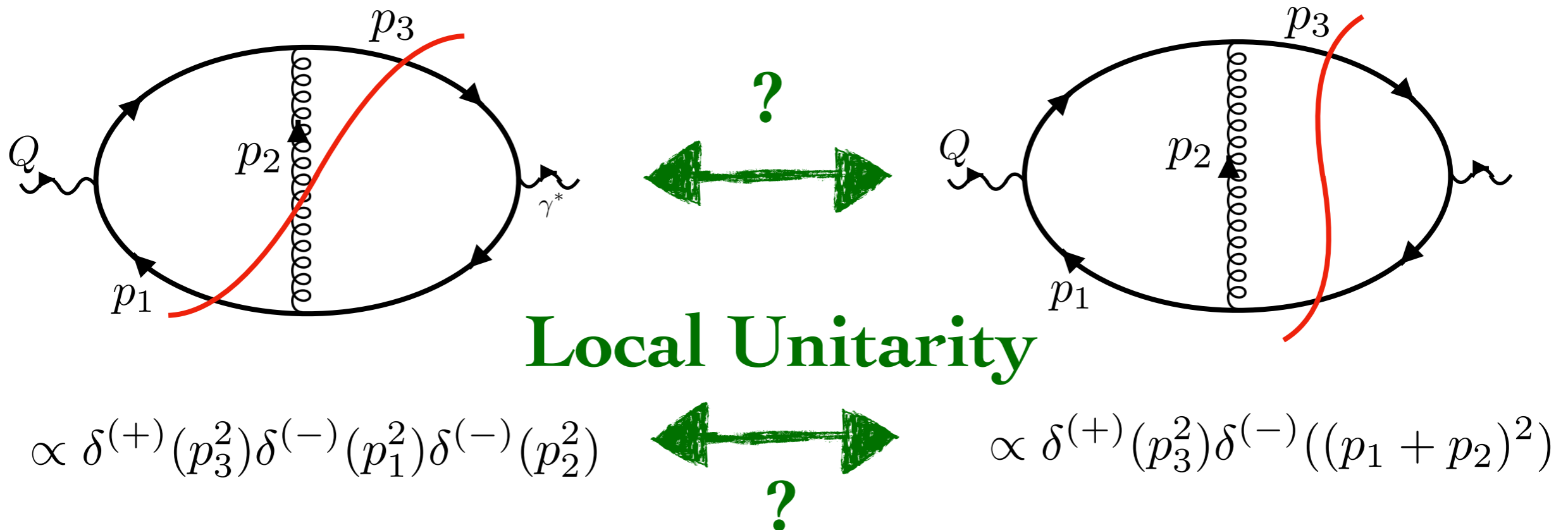
- At the **integral** level, the **Optical Theorem** simply gives: $\text{LU}[\cdot] \propto \text{Im}[\cdot]$
- The **differential** version of **LU** is our new paradigm: **Local Unitarity**
- Each supergraph is **individually locally finite**, how you ask?



SO WHAT? IT'S JUST THE OPTICAL THEOREM NO?

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Diagram 1} \right] + \text{LU} \left[\text{Diagram 2} \right] + \text{LU} \left[2 \times \text{Diagram 3} \right]$$

- At the **integral** level, the **Optical Theorem** simply gives: $\text{LU}[\cdot] \propto \text{Im}[\cdot]$
- The **differential** version of **LU** is our new paradigm: **Local Unitarity**
- Each supergraph is **individually locally finite**, how you ask?



MAIN INGREDIENT: LOOP TREE DUALITY

$$\int d^4 k \text{ (triangle diagram) } = \int d^3 \vec{k} \left[\text{(triangle diagram with red slash and +)} + \text{(triangle diagram with red slash and -)} + \text{(triangle diagram with red slash and +)} \right]$$

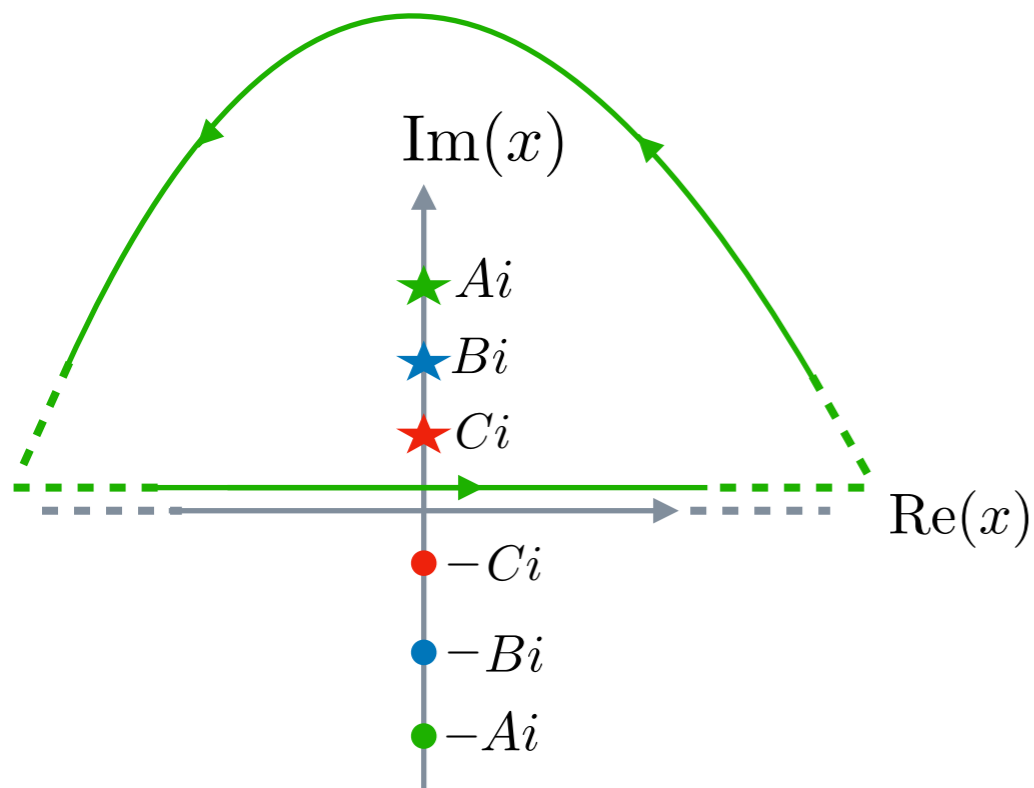
(CLOSELY RELATED TO THE FEYNMAN-TREE THEOREM
AND TIME-ORDERED PERTURBATION THEORY)

(ONE)-LOOP TREE DUALITY MOCK-UP

$$I = \int_{-\infty}^{+\infty} dx F(x) \quad F(x) = \frac{1}{x^2 + A^2} \frac{1}{x^2 + B^2} \frac{1}{x^2 + C^2}$$

$$F(x) = \frac{1}{(x - Ai)(x + Ai)} \frac{1}{(x - Bi)(x + Bi)} \frac{1}{(x - Ci)(x + Ci)}$$

(Assumptions $\rightarrow \{A > 0, B > 0, C > 0\}$)



Cauchy: $(R(x^*) \equiv \text{Res}(F, x = x^*))$

$$I = (-2\pi i) [R(Ai) + R(Bi) + R(Ci)]$$

What does it correspond to for a one-loop integral?

(ONE-)LOOP TREE DUALITY

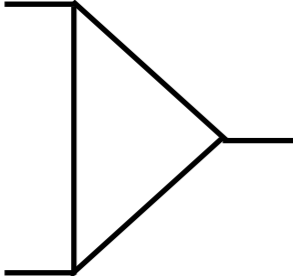
$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

Pole selected for each propagator

Then integrate the energy component using residue theorem


$$= \int d^3\vec{k} \left[\text{Res}_1 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_2 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_3 \left[\frac{N}{D_1 D_2 D_3} \right] \right]$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

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$$\text{Diagram} = \int d^3 \vec{k} \left[\text{Res}_1 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_2 \left[\frac{N}{D_1 D_2 D_3} \right] + \text{Res}_3 \left[\frac{N}{D_1 D_2 D_3} \right] \right]$$

Residues can be represented as cuts:

$$\text{Diagram} = \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3$$

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right) \left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

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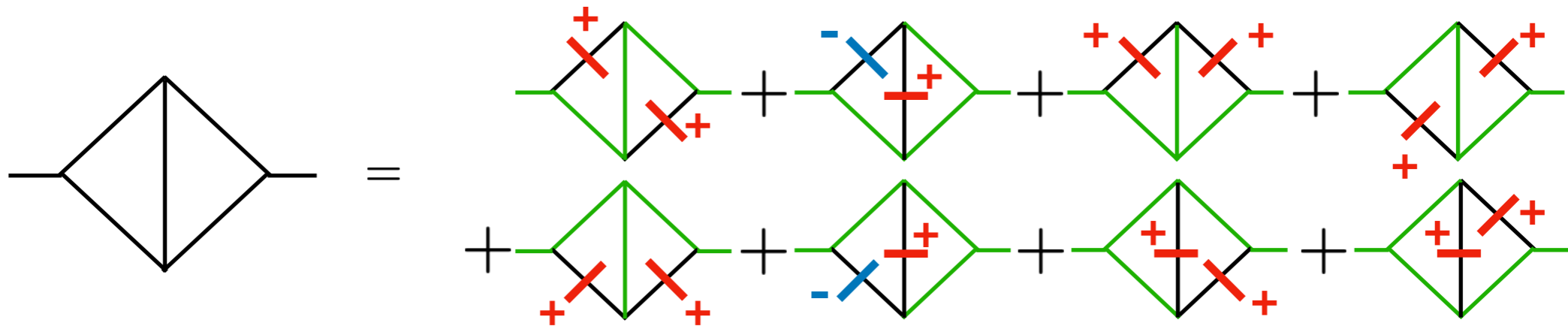
$$\text{Diagram} = \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3$$

Energy flow

$$= \int d^4 k \frac{N}{D_1 D_2 D_3} (D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3))$$

(MULTI-)LOOP TREE DUALITY

Applying LTD to a two-loop double-triangle: one residue per spanning tree



Interplay of momentum conservation and causal prescription is key to obtain the energy flow

- **Distributional identities:** [Bierenbaum, Catani, Draggiotis, Rodrigo,
- **Averaging procedure:** [Runkel, Scór, Vesga, Weinzierl,
- **Iterative procedure:** [Capatti, VH, Kermanschah, Ruijl,
- **Manifestly causal:** [Capatti, VH, Kermanschah, Pelloni, Ruijl,
- **Cross-Free Family** [Capatti, arxiv: 2211.09653]
(the best 3D repr!)

Codes : [<https://github.com/apelloni/cLTD>]
[<https://bitbucket.org/wjtorresb/lotty>]

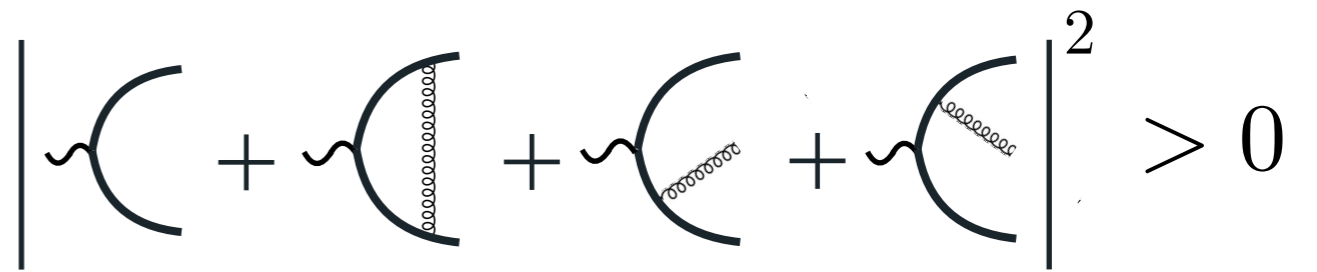
{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

POSITIVITY

$$\left| \text{Candy 1} + \text{Candy 2} + \text{Candy 3} + \text{Candy 4} \right|^2 > 0$$

{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

~~POSITIVITY~~

$$\left| \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \right|^2 > 0$$


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~~POSITIVITY~~

LORENTZ INVARIANCE

$$\left| \text{~} \left(\text{~} + \text{~} + \text{~} + \text{~} \right) \right|^2 > 0$$

$$\mathcal{M}^\mu (\{p_i^\mu\}) = \Lambda_\nu^\mu \mathcal{M}^\nu (\{\Lambda_\nu^\mu p_i^\nu\})$$

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$$k^\mu \mathcal{M}_\mu = 0$$

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UNITARITY

$$i(T^\dagger - T) = T^\dagger T$$

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~~POSITIVITY~~

~~LORENTZ INVARIANCE~~

~~GAUGE INVARIANCE~~

UNITARITY

$$\left| \text{tree} + \text{ghost} + \text{ghost} + \text{ghost} \right|^2 > 0$$

$$\mathcal{M}^\mu(\{p_i^\mu\}) = \Lambda_\nu^\mu \mathcal{M}^\nu(\{\Lambda_\nu^\mu p_i^\nu\})$$

$$k^\mu \mathcal{M}_\mu = 0$$

$$i(T^\dagger - T) = T^\dagger T$$

{ PICK YOUR CANDY: CANNOT ALL BE MANIFEST }

~~POSITIVITY~~

~~LORENTZ INVARIANCE~~

~~GAUGE INVARIANCE~~

UNITARITY

LOCALLY FINITE

$$\left| \text{tree} + \text{tree with ghost} + \text{tree with ghost} + \text{tree with ghost} \right|^2 > 0$$

$$\mathcal{M}^\mu(\{p_i^\mu\}) = \Lambda_\nu^\mu \mathcal{M}^\nu(\{\Lambda_\nu^\mu p_i^\nu\})$$

$$k^\mu \mathcal{M}_\mu = 0$$

$$i(T^\dagger - T) = T^\dagger T$$

$$\lim_{k \rightarrow \text{soft, colli, UV}} I(k) = \mathcal{O}(1)$$

FOUR TYPES OF SINGULARITIES :

THRESHOLDS

**CONTOUR-DEFORMATION
OR
THRES. SUBTRACTION**

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(TROPICAL) SAMPLING

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OR
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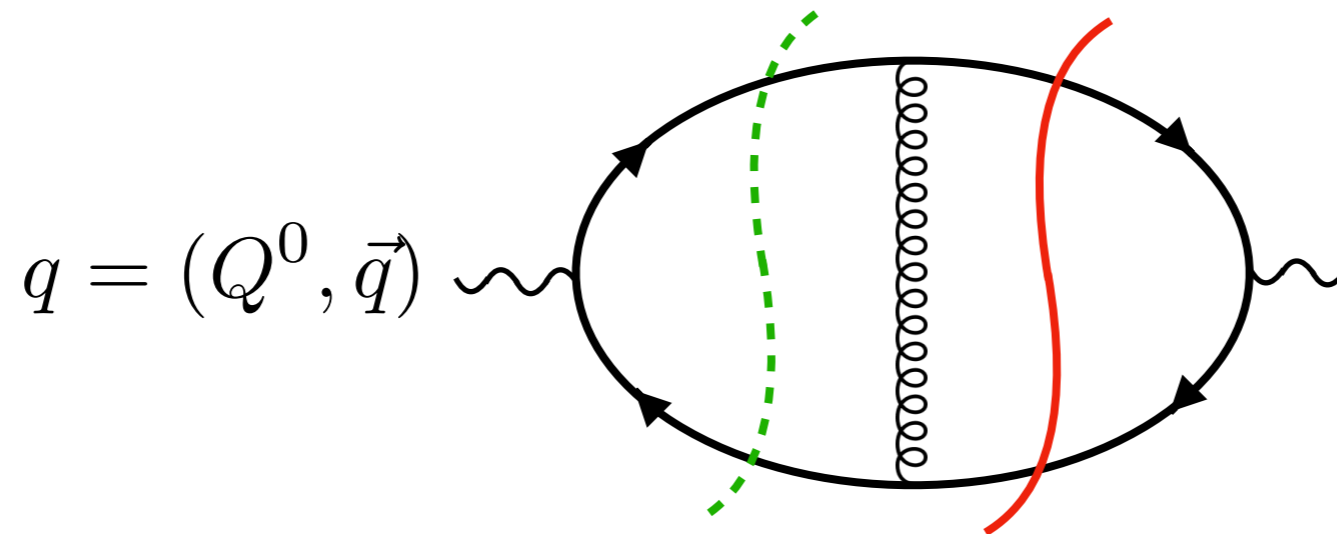
(TROPICAL) SAMPLING

ULTRAVIOLET

LOCAL BPHZ

THRESHOLDS

$$E_1 = \sqrt{|\vec{k}|^2 + m^2 - i\epsilon}$$



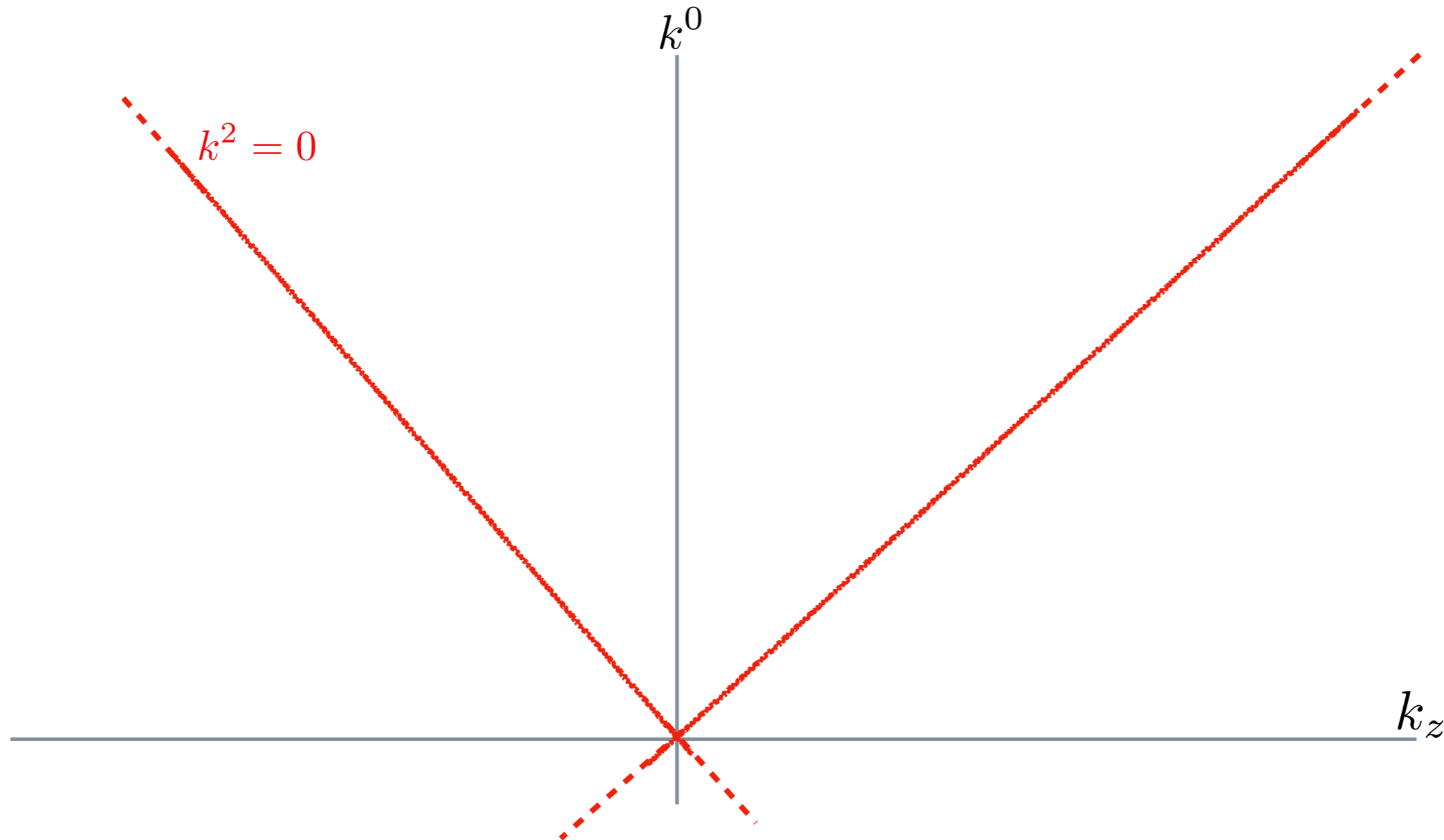
$$E_2 = \sqrt{|\vec{k} - \vec{q}|^2 + m^2 - i\epsilon}$$

$$\int d^3 \vec{k} I^{(\text{Local Unitarity})} \supset \int d^3 \vec{k} \frac{1}{E_1 E_2 E_3} \left(\frac{1}{(E_1 + E_2 - Q^0)(E_1 + E_2 + Q^0)} \right)$$

$$\eta(\vec{k}) = E_1 + E_2 - Q^0 \stackrel{\vec{Q}=0 \quad m=0}{=} 2|\vec{k}| - Q^0$$

SINGULAR SURFACES IN MINKOWSKI SPACE

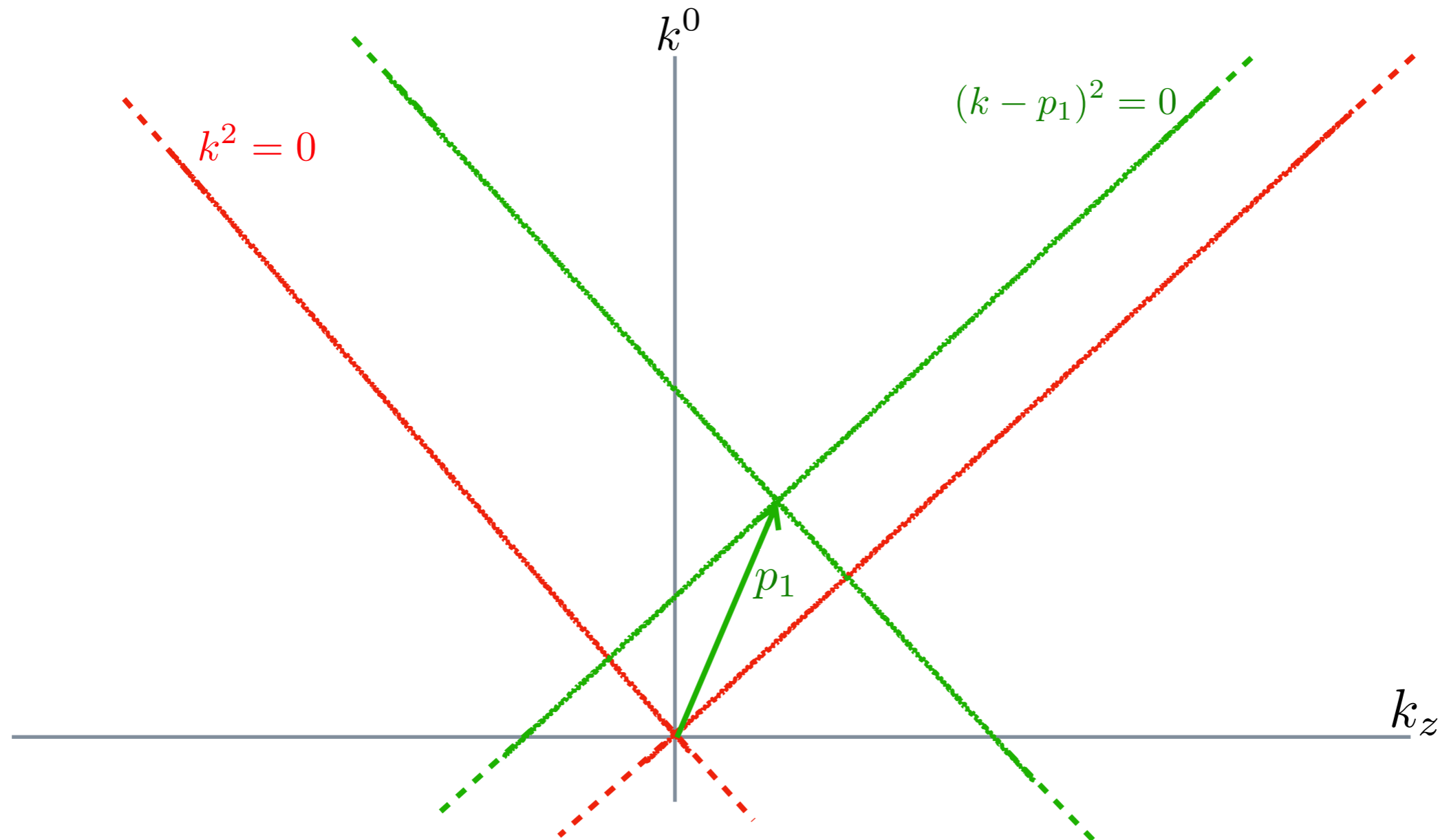
$$\int d^4 k \frac{1}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2}$$



The integrand is singular along each of the coloured surface

SINGULAR SURFACES IN MINKOWSKI SPACE

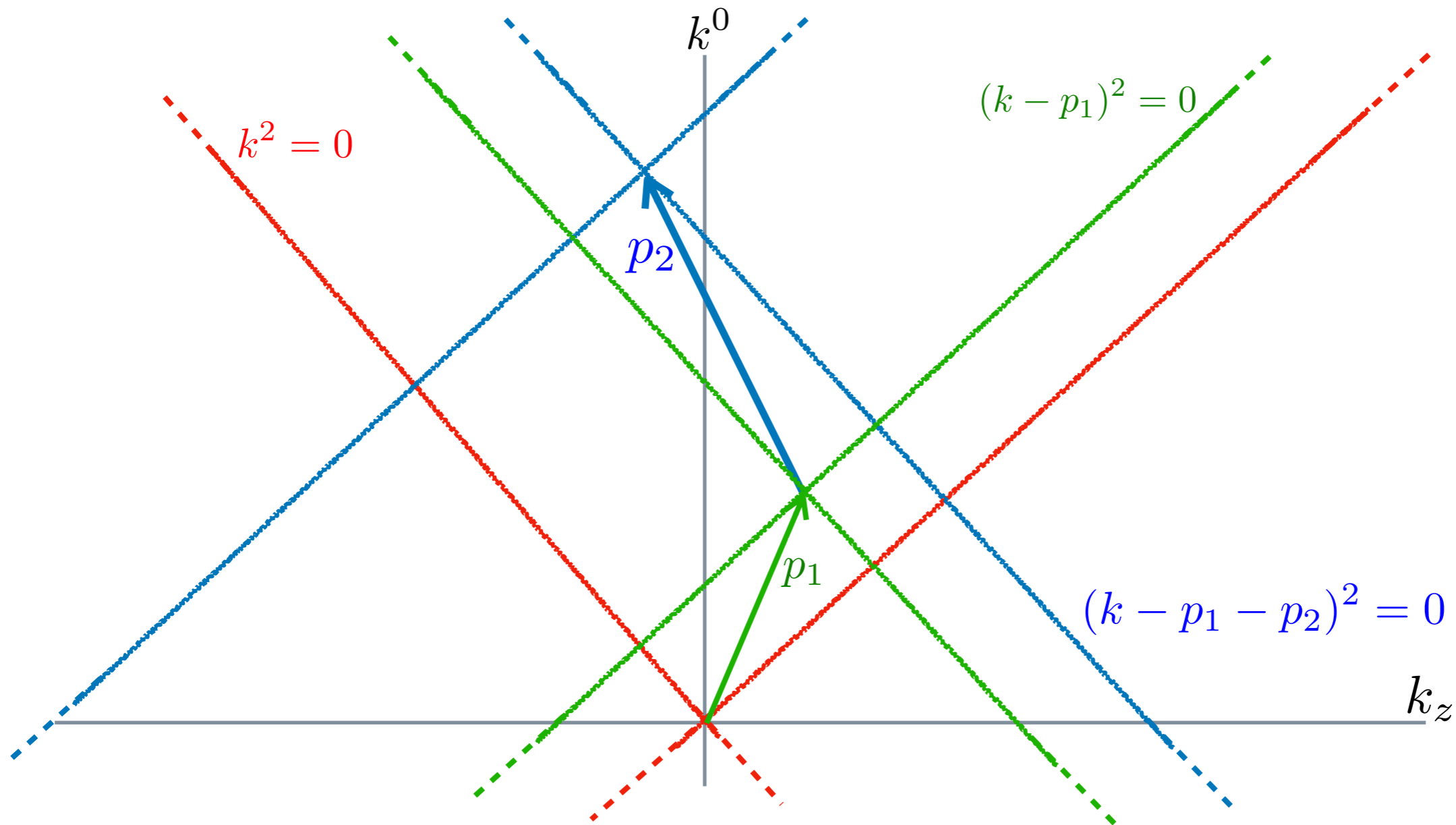
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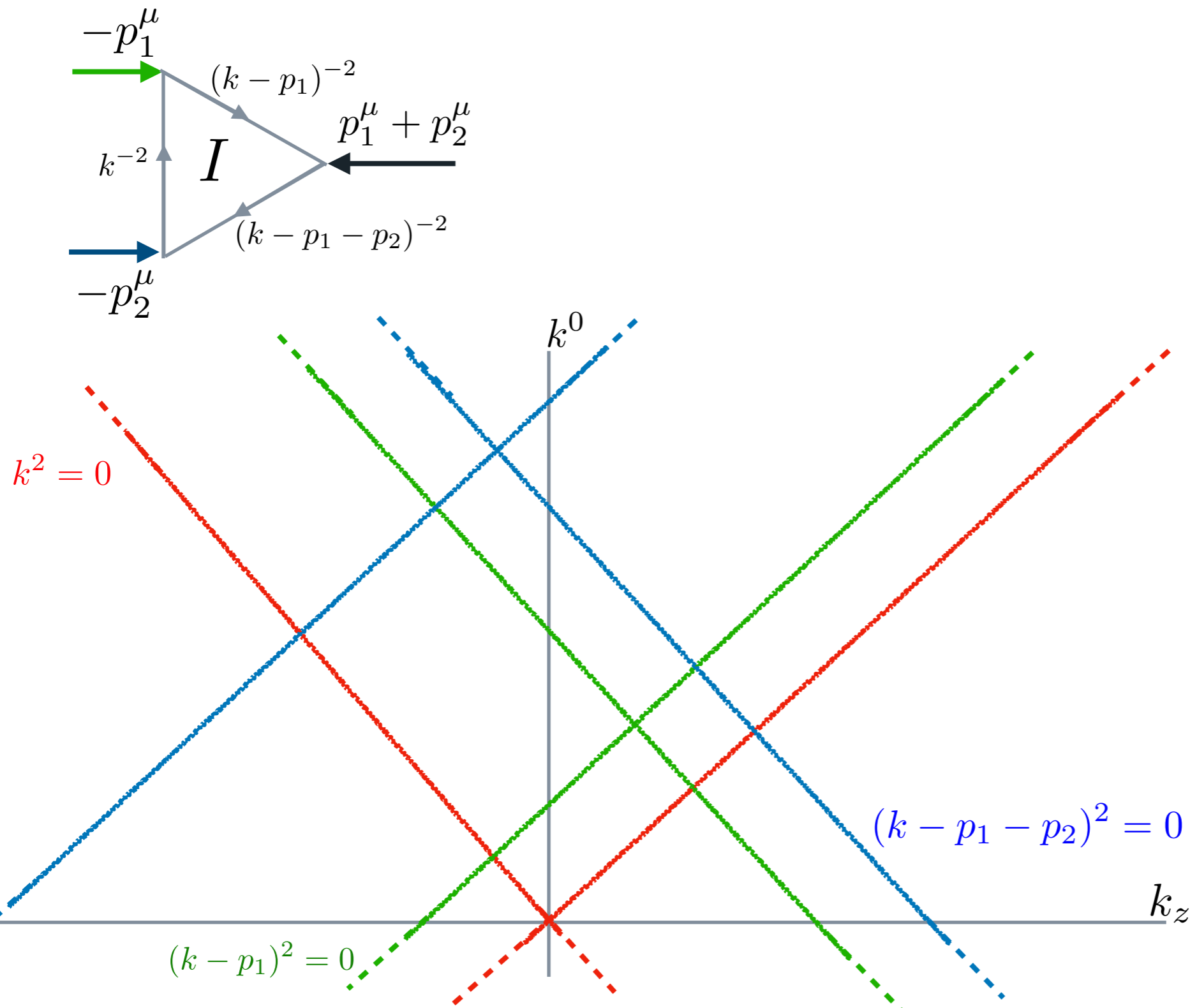
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SINGULAR SURFACES OF THE LTD REPRESENTATION

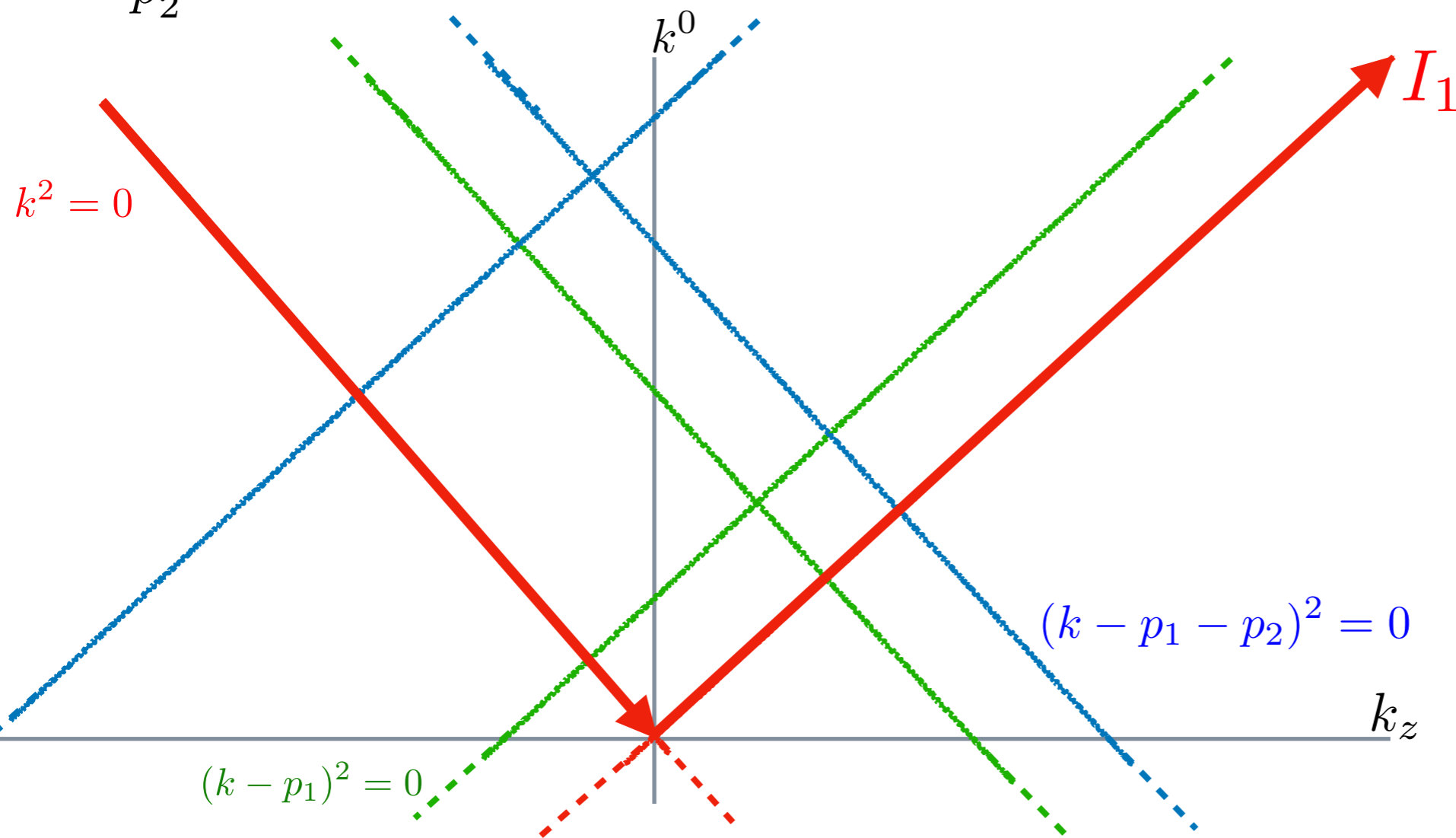
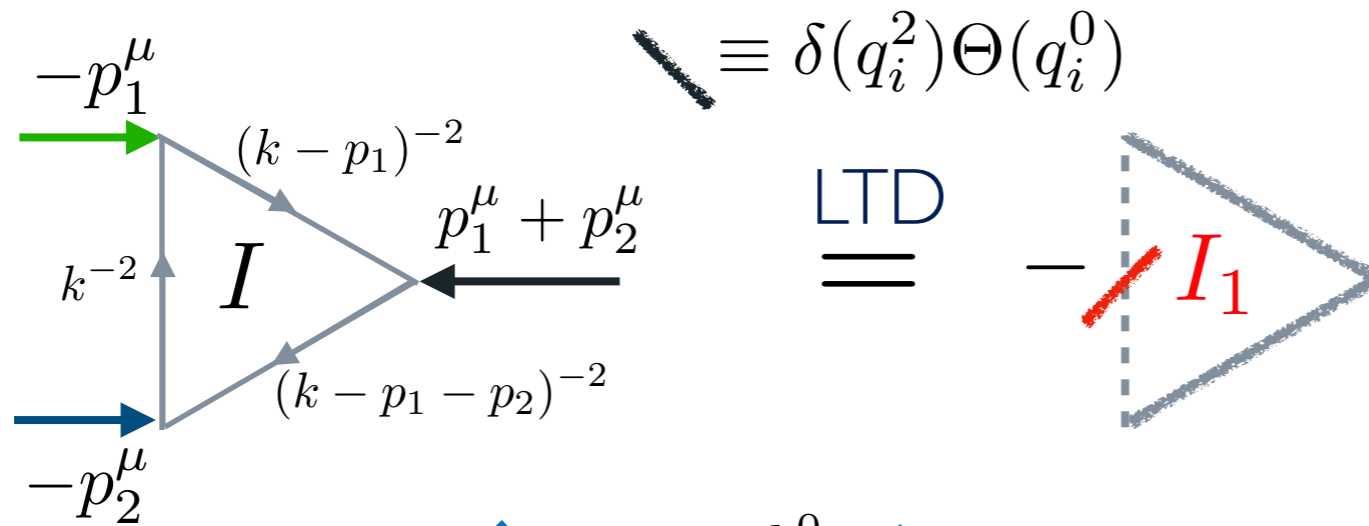
Analytically integrate over the loop energies using Cauchy's theorem (LTD):



$$p_i^2 > 0 \quad \forall i$$

SINGULAR SURFACES OF THE LTD REPRESENTATION

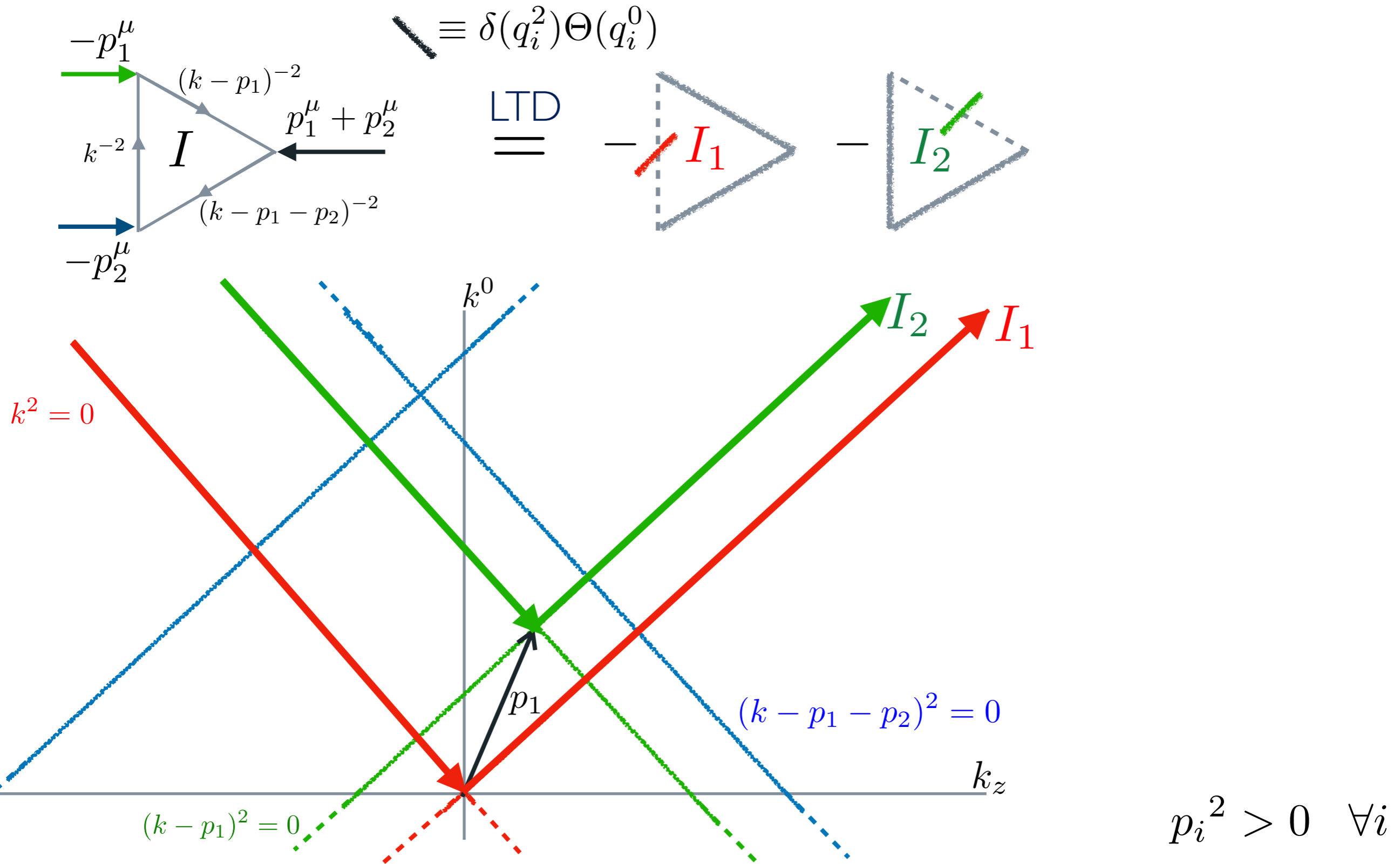
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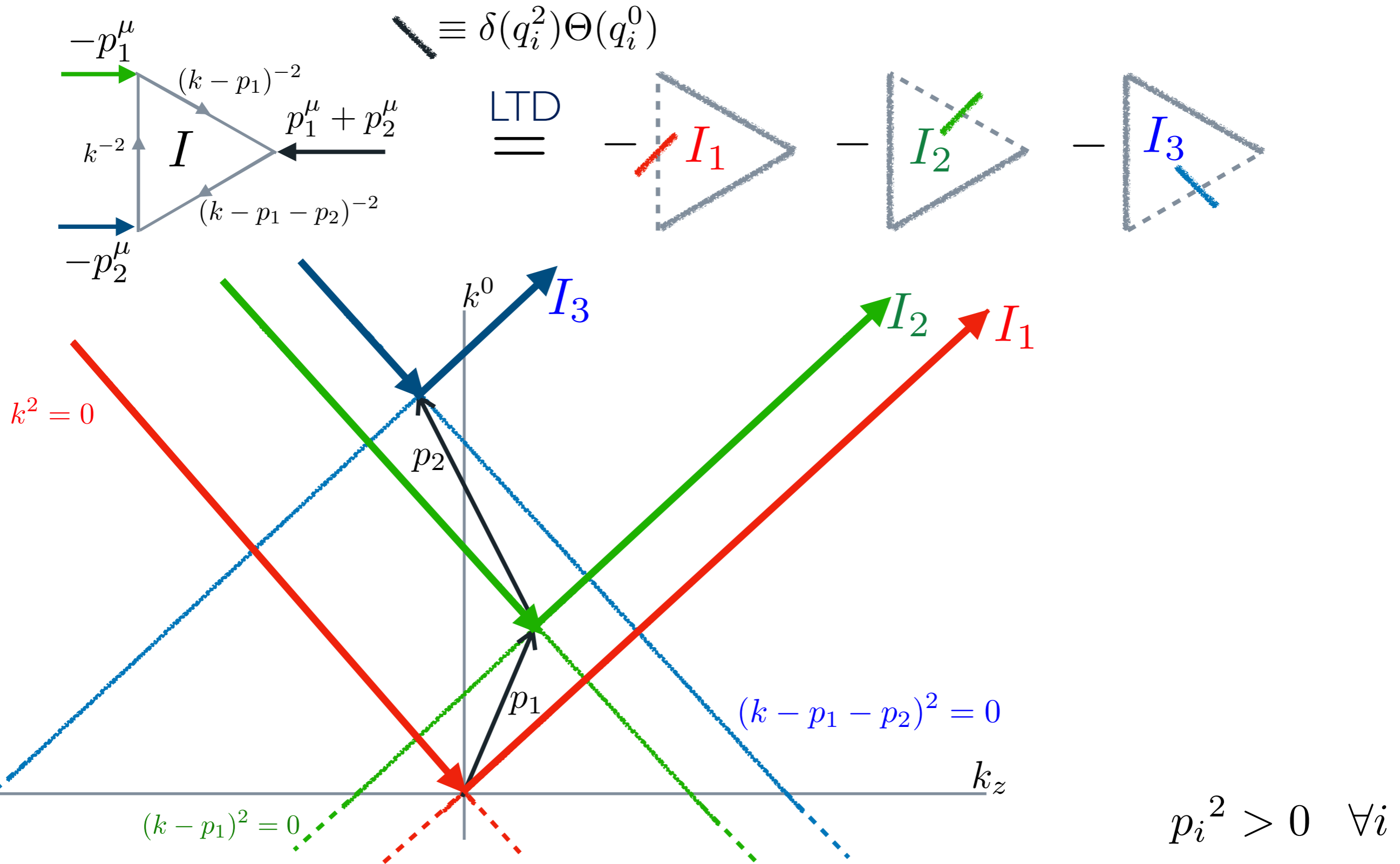
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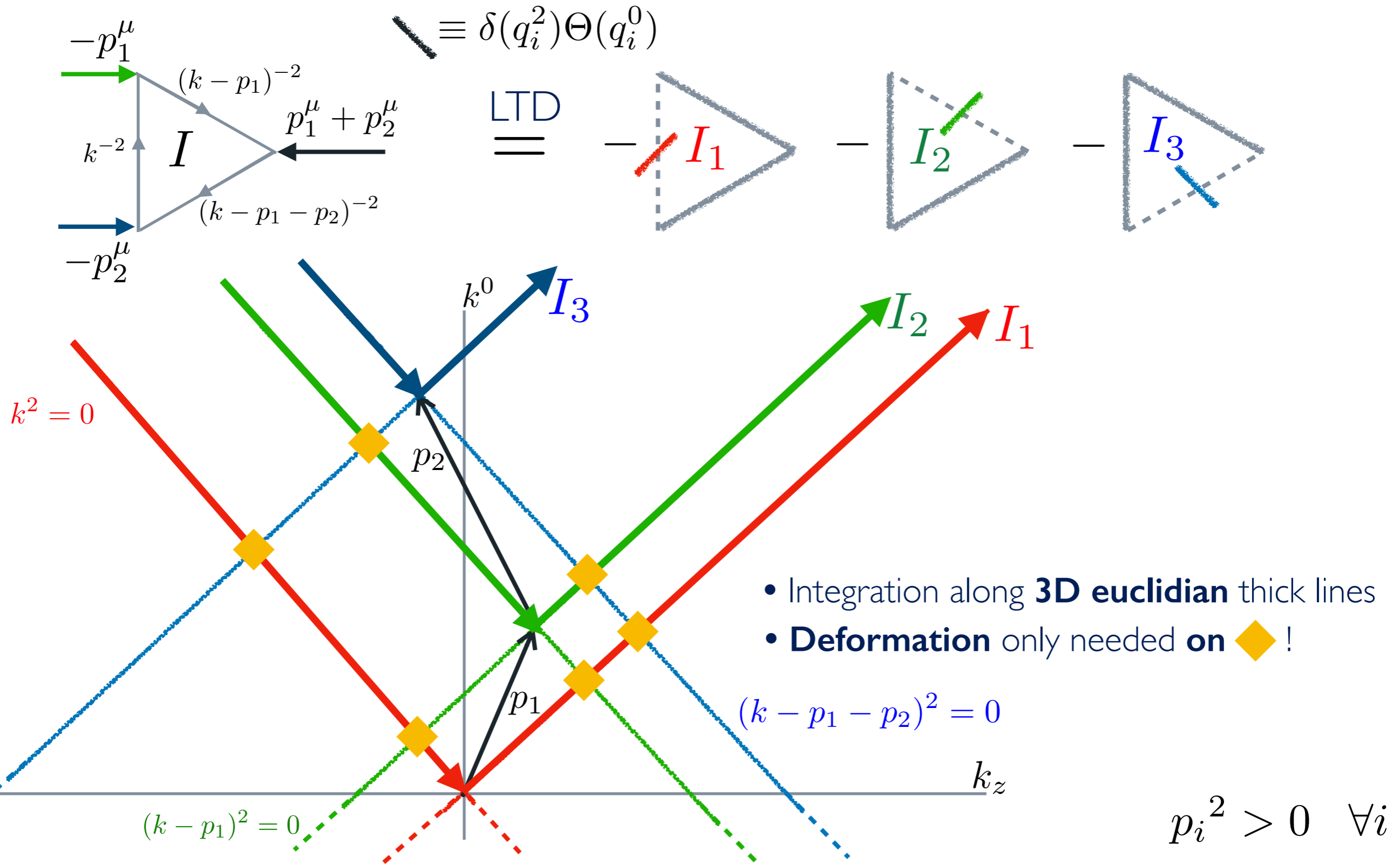
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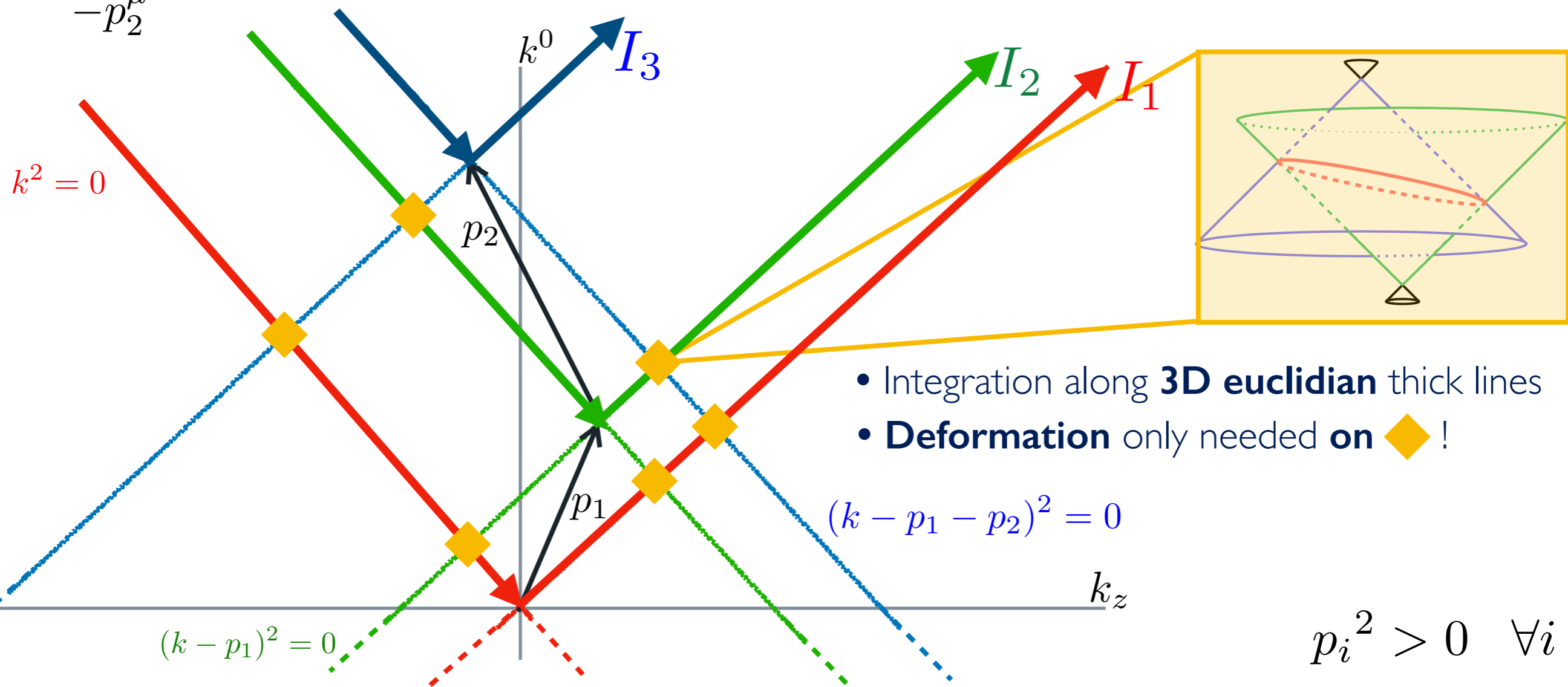
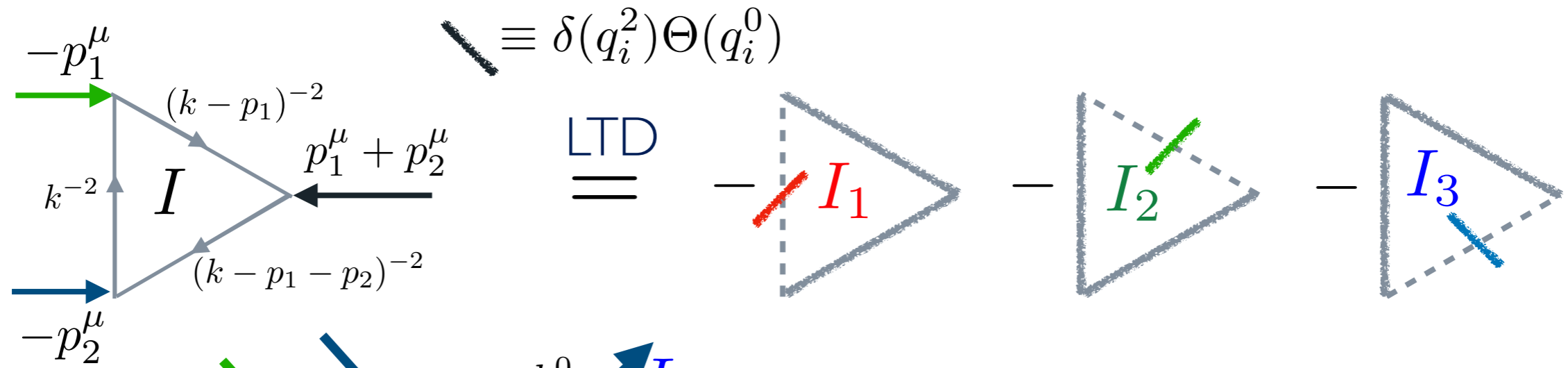
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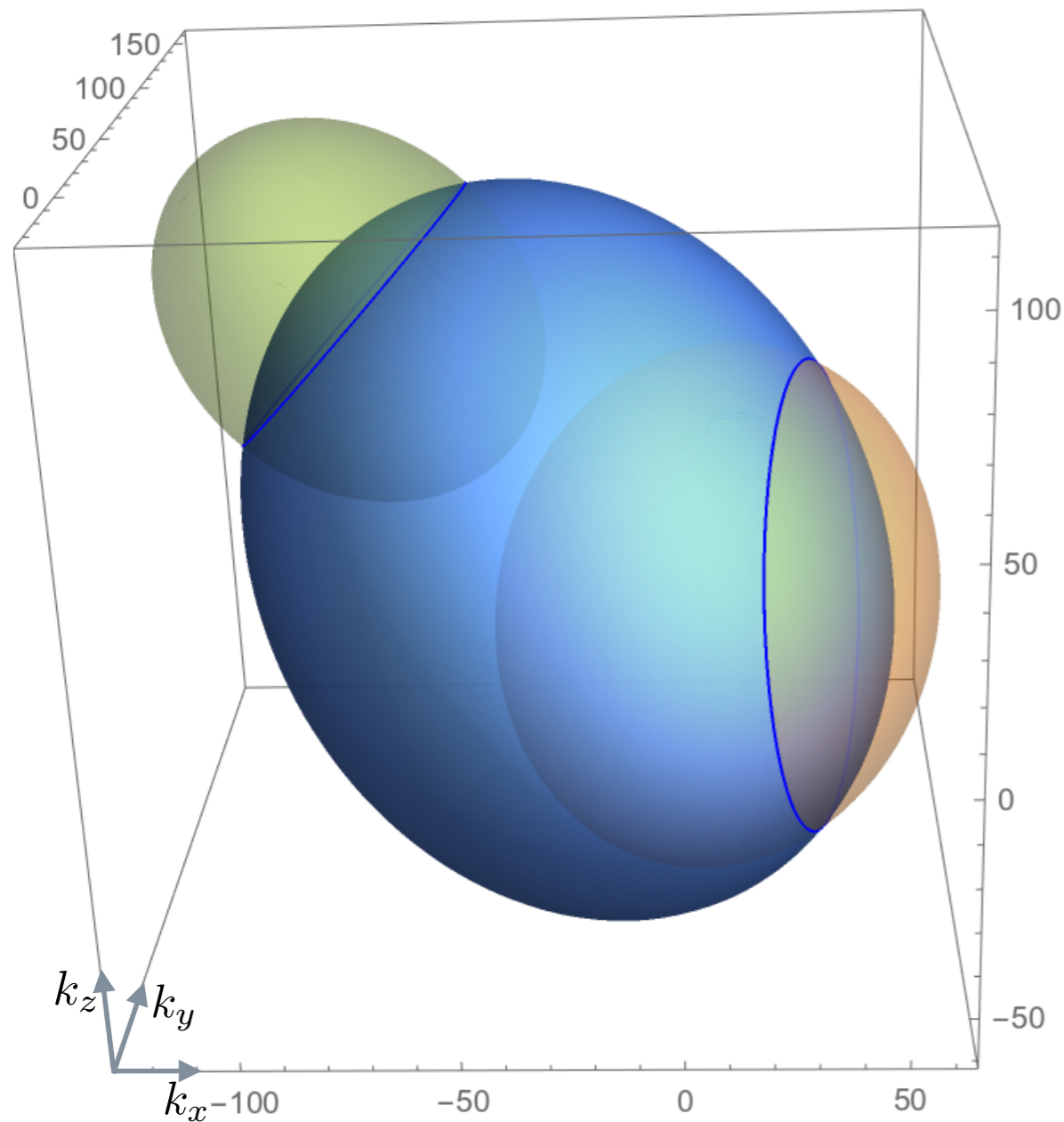


SINGULAR SURFACES OF THE LTD REPRESENTATION

Analytically integrate over the loop energies using Cauchy's theorem (LTD):



SINGULAR SURFACES - 2D ELLIPSOIDS



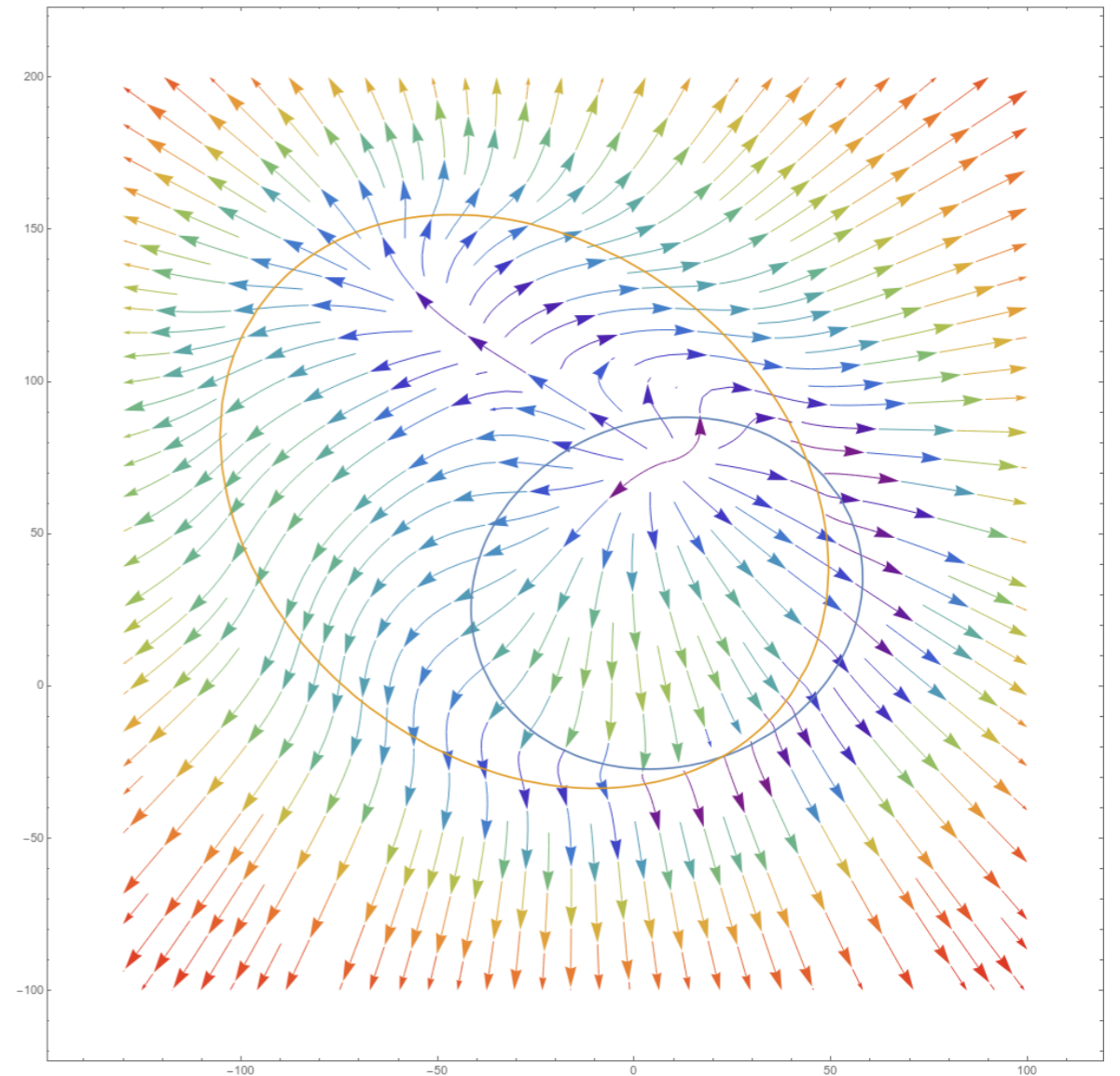
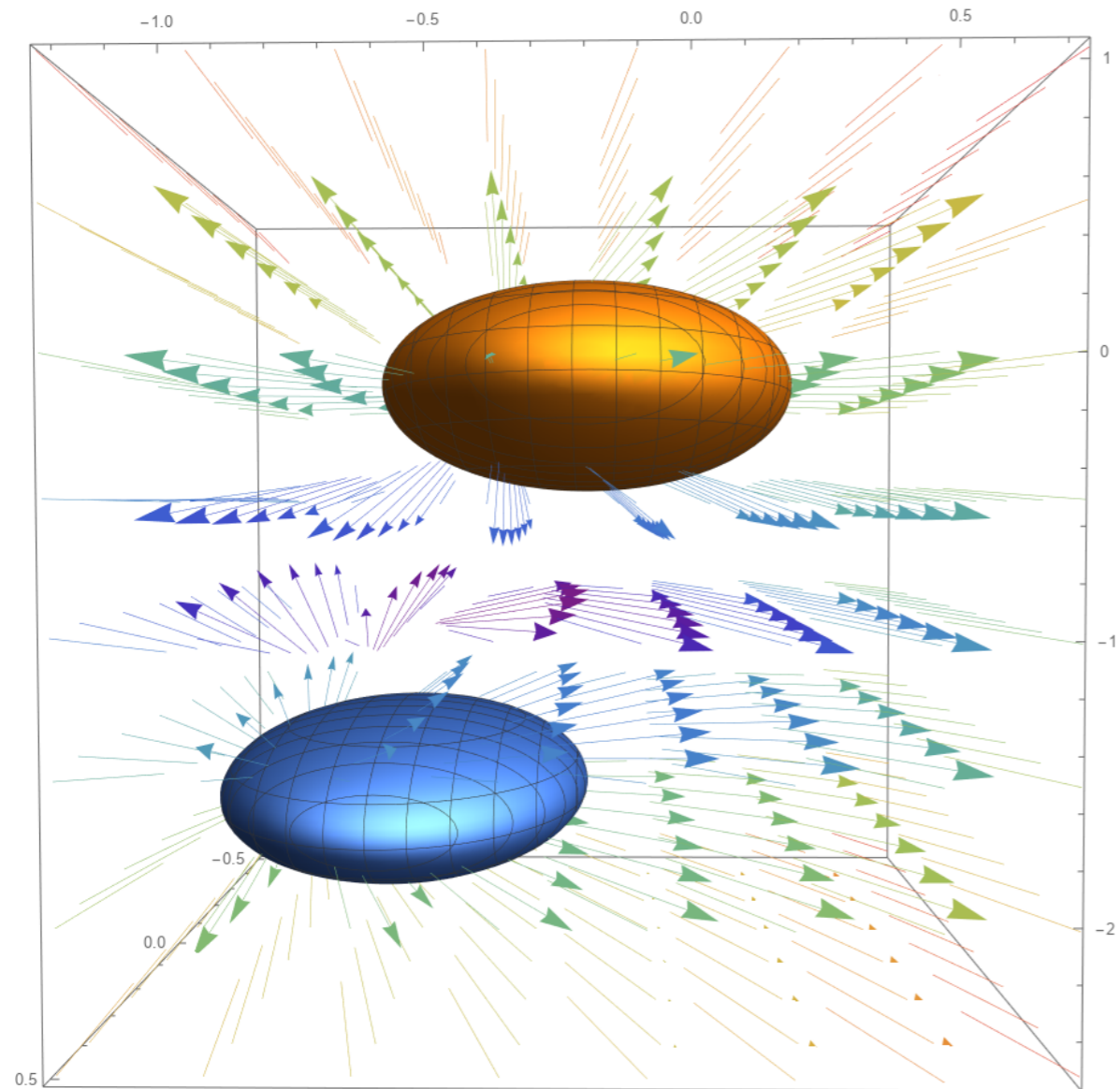
- General **one-loop** ellipsoid **expression**:

$$E_{ij}(\vec{k}) = \sqrt{(\vec{k} + \vec{p}_i)^2 + m_i^2 - i\delta} + \sqrt{(\vec{k} + \vec{p}_j)^2 + m_j^2 - i\delta} - p_i^0 + p_j^0$$

DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

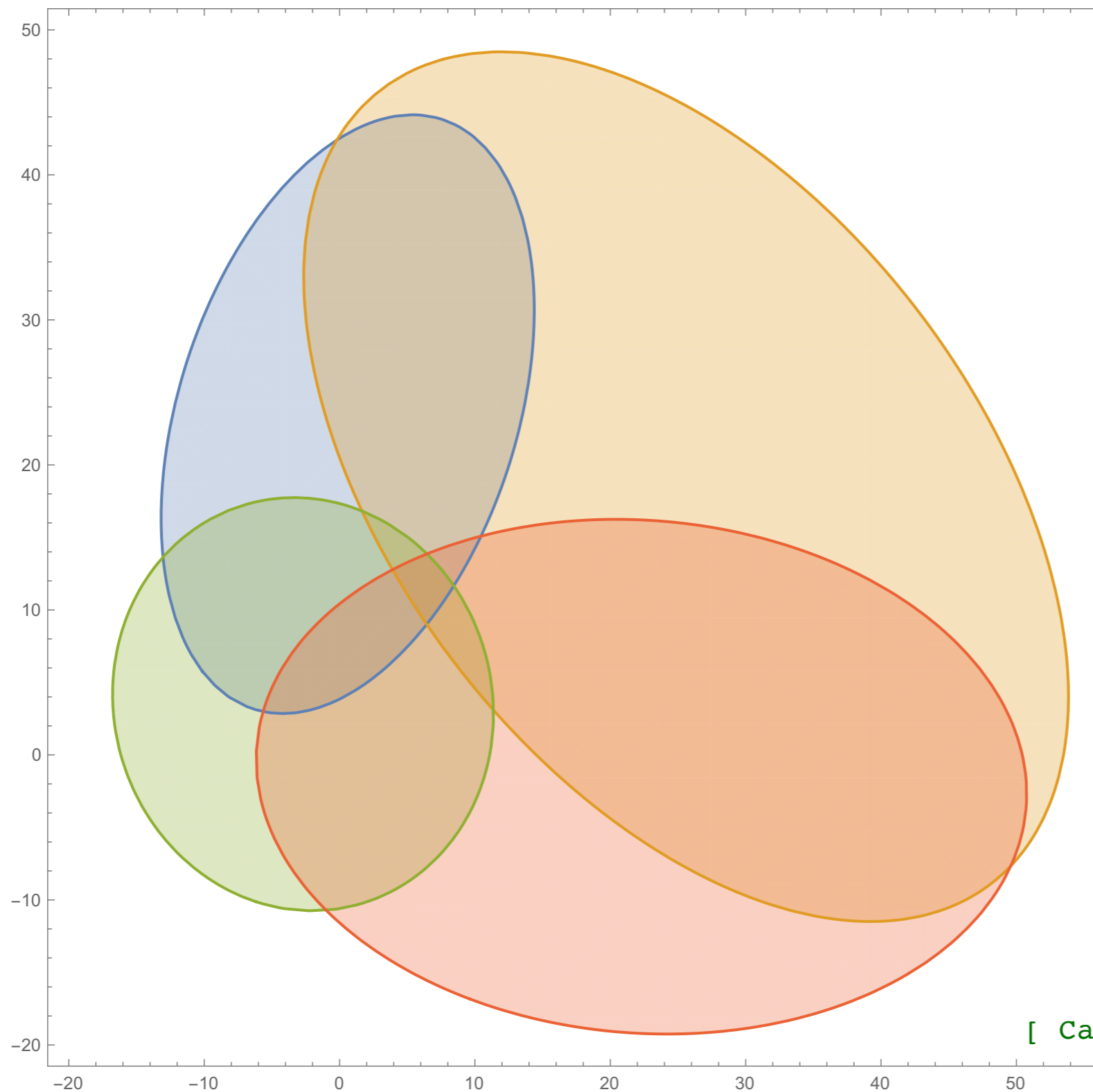
Deformation: $\vec{k} \rightarrow \vec{k} - i\vec{\kappa}$

Causal prescription imposes: $\vec{\kappa} \cdot \vec{n}_{E_{ij}} > 0$



DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

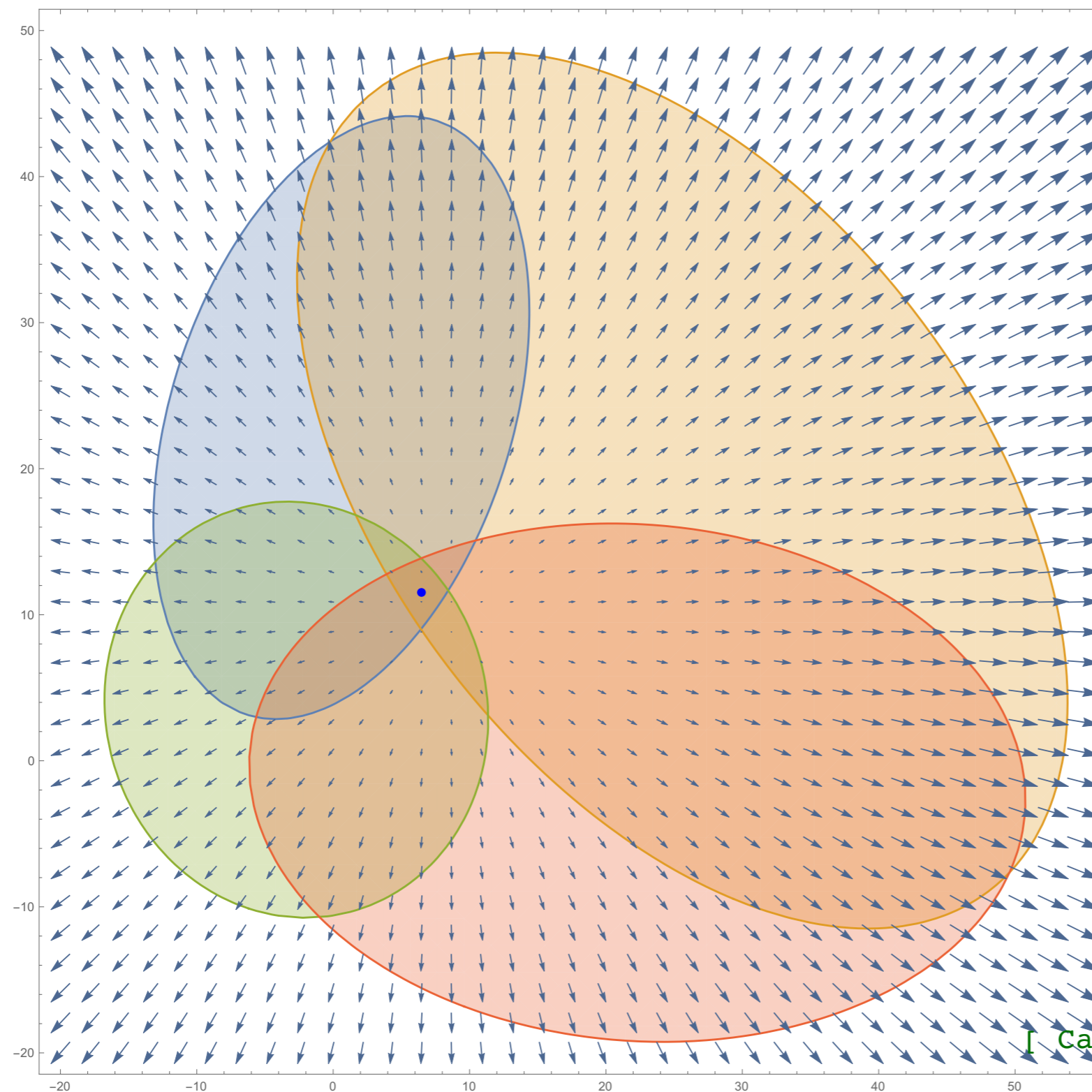
How to construct such a field? For example for this case:



[Capatti, VH, Kermanschah, Pelloni, Ru
[arxiv:1906.06138]

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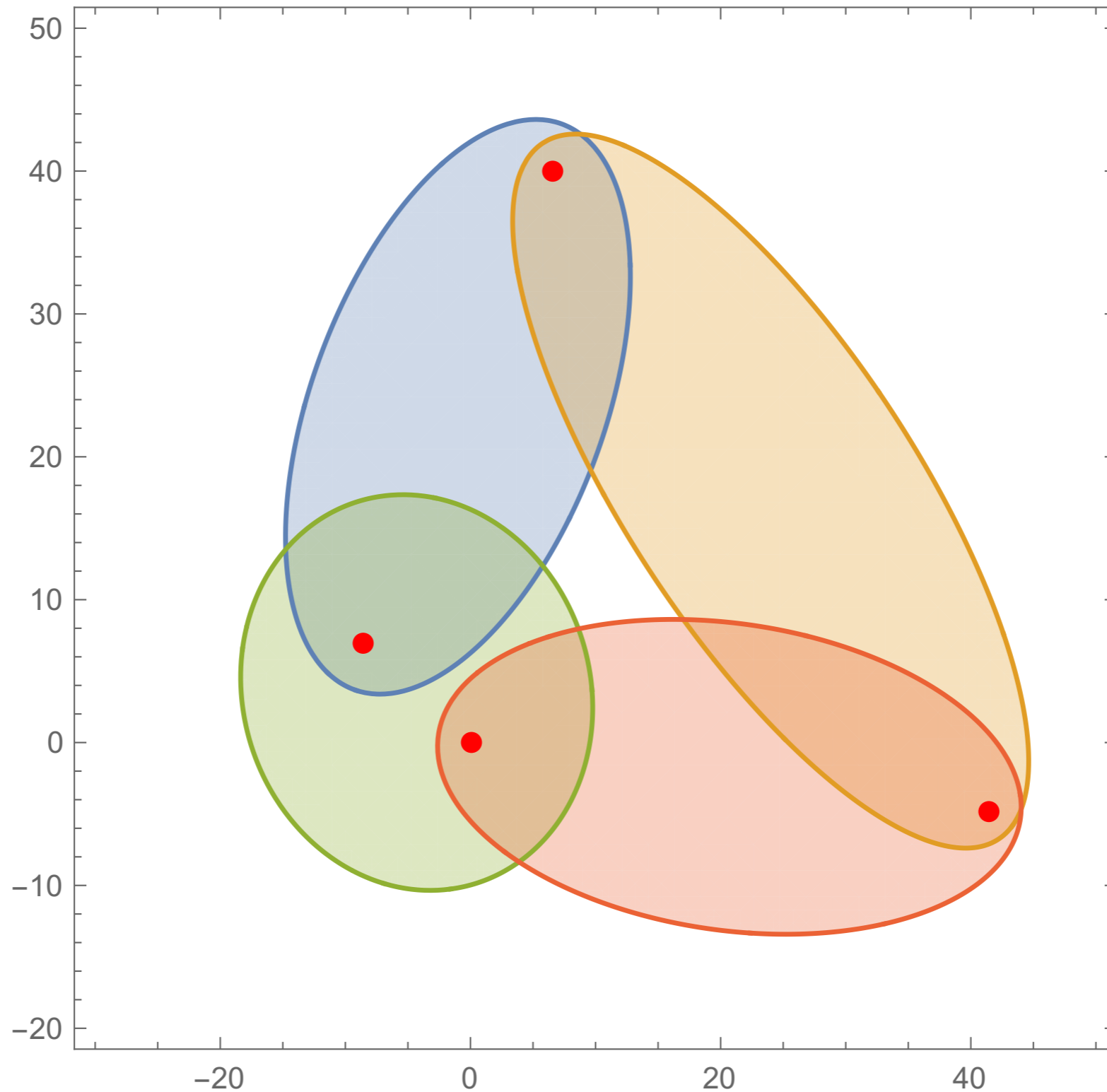


A **radial field** centered in the inside of all ellipsoids!

[Capatti, VH, Kermanschah, Pelloni, Ru
[arxiv:1906.06138]

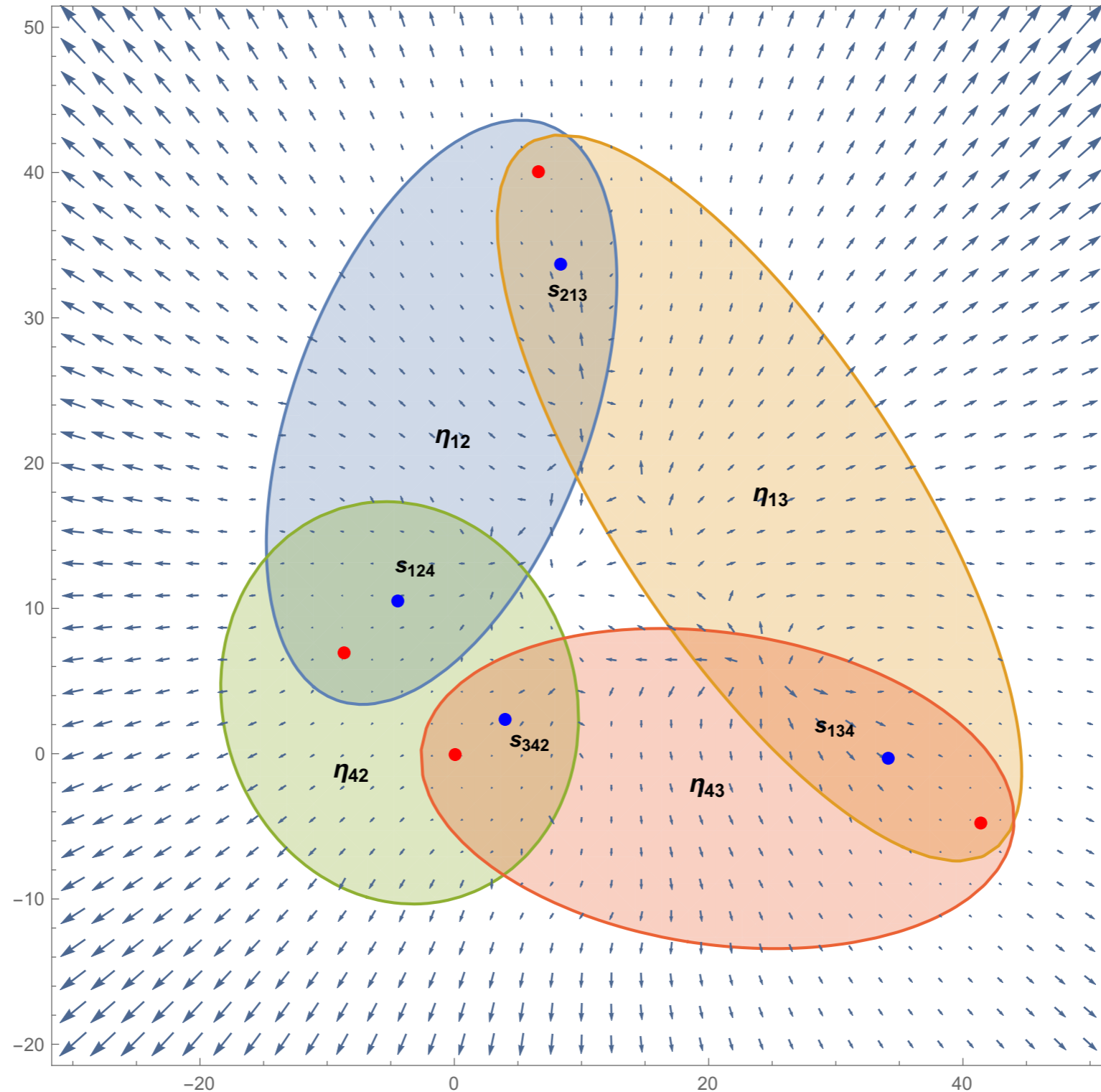
DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

But then what if there is no point in the inside of **all ellipsoids** (Box4E example) ?



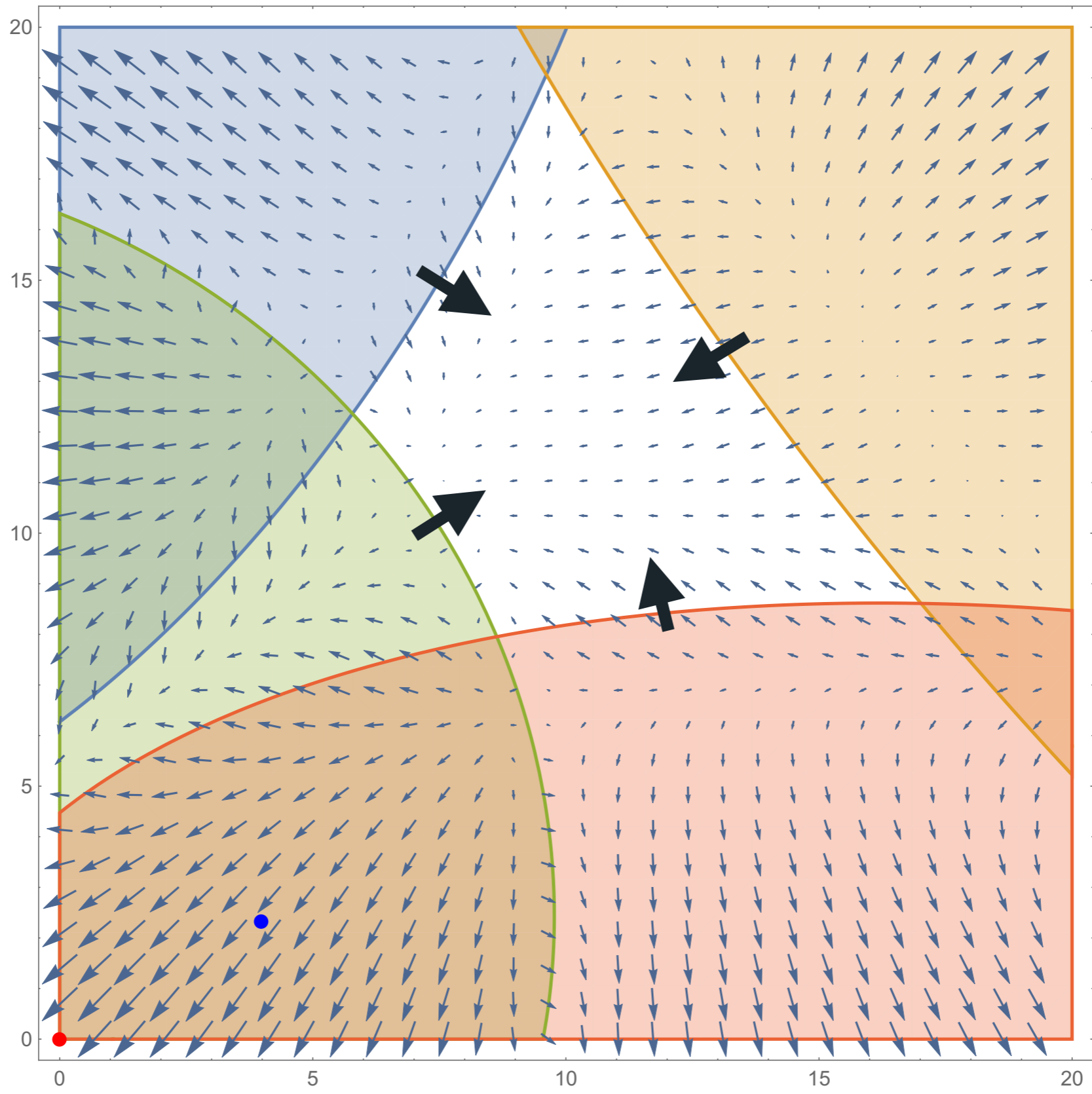
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DEFORMING AROUND SINGULAR 2D-ELLIPSOIDS

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THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : [2110.06869](https://arxiv.org/abs/2110.06869)]

$$\frac{1}{E_1 + E_2 - p_1^0} = \frac{1}{|\vec{k}| + |\vec{k} - \vec{p}_1| - p_1^0}$$
$$\underset{p_1^\mu = (2, \vec{0})}{=} \frac{1}{2|\vec{k}| - 2} \propto \frac{1}{\sqrt{k_x^2 + k_y^2} - 1}$$

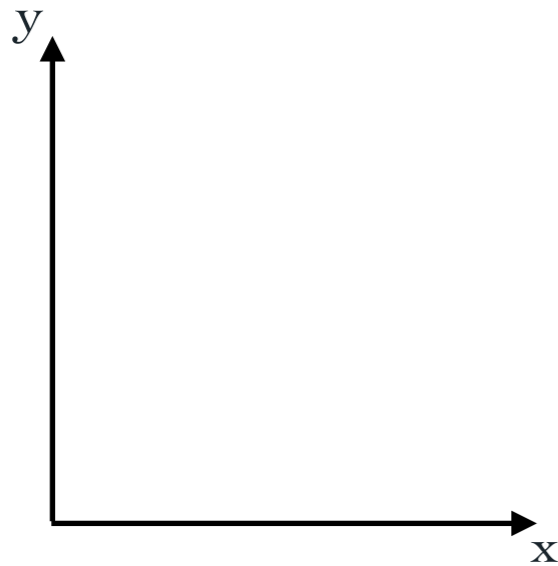
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$$\lim_{\delta \rightarrow 0^+} \int_{-\infty}^{\infty} dx dy \frac{2}{\pi^2} \frac{1}{x^2 + y^2 + 1} \frac{1}{\sqrt{x^2 + y^2} - 1 \pm i\delta} = 1 \mp 2i$$



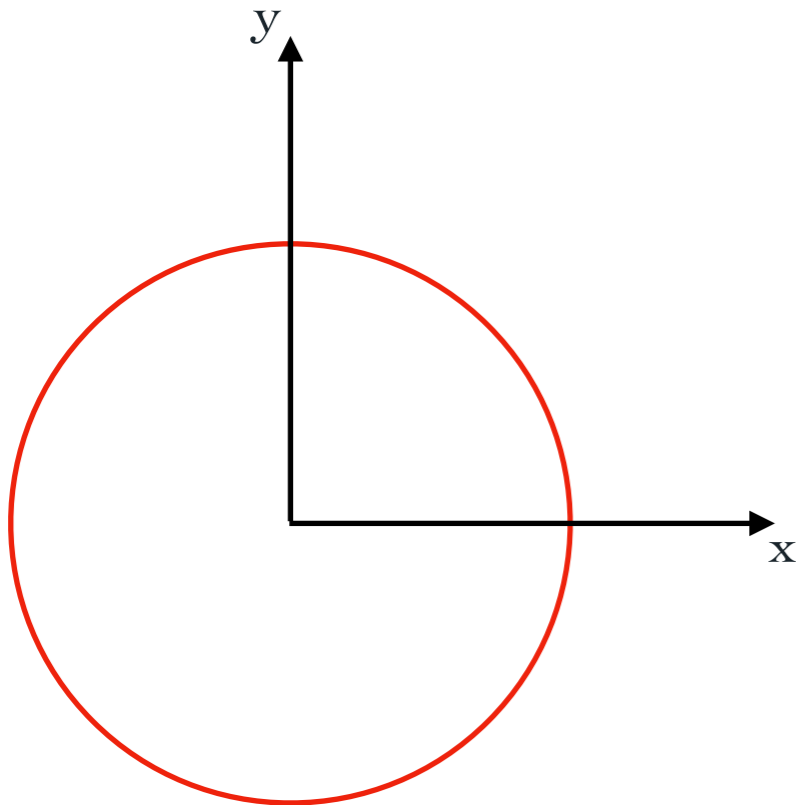
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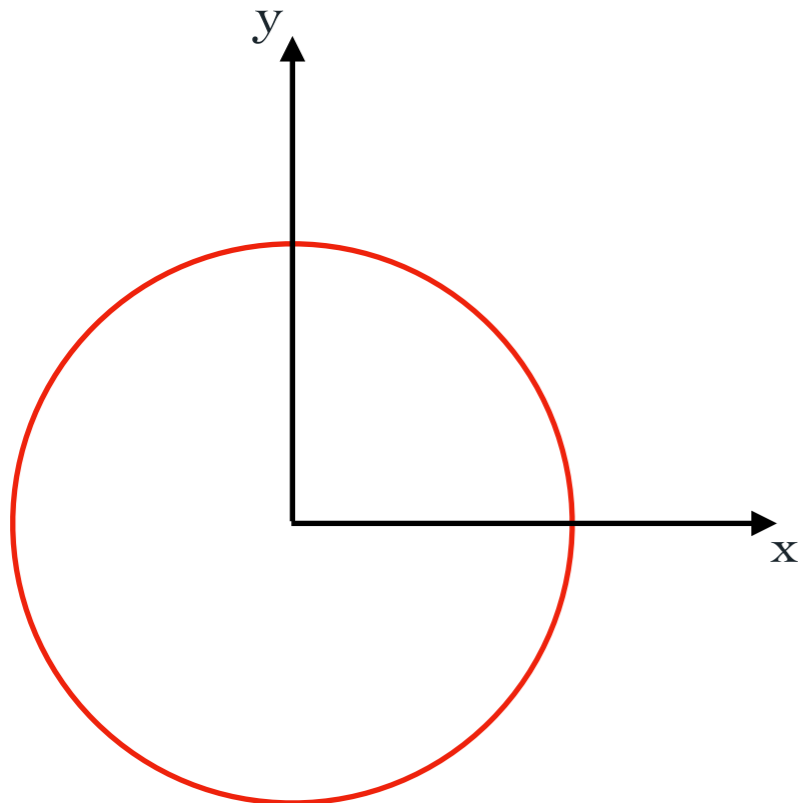
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$$= \lim_{\delta \rightarrow 0^+} \frac{4}{\pi} \left[\int_0^\infty dr \left(\frac{1}{r^2 + 1} \right) \frac{1}{1 - r \pm i\delta} \right]$$



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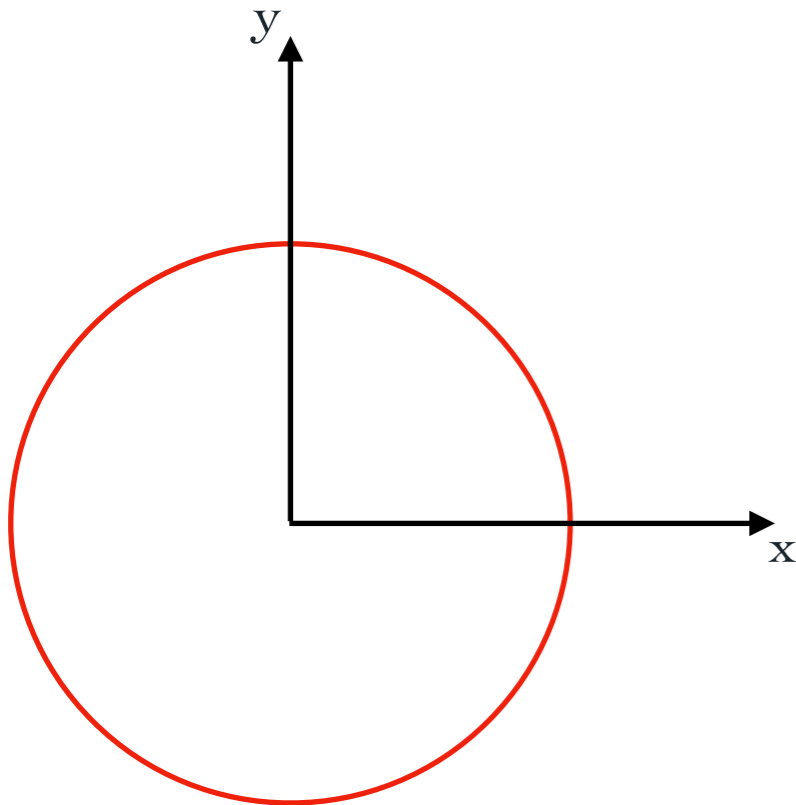
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Local threshold counterterm

$$= \lim_{\delta \rightarrow 0^+} \frac{4}{\pi} \left[\int_0^\infty dr \left(\frac{1}{r^2 + 1} \right) \frac{1}{1 - r \pm i\delta} - \int_{1-\Delta}^{1+\Delta} dr \left(\frac{1}{(1)^2 + 1} \right) \frac{1}{1 - r \pm i\delta} \right]$$



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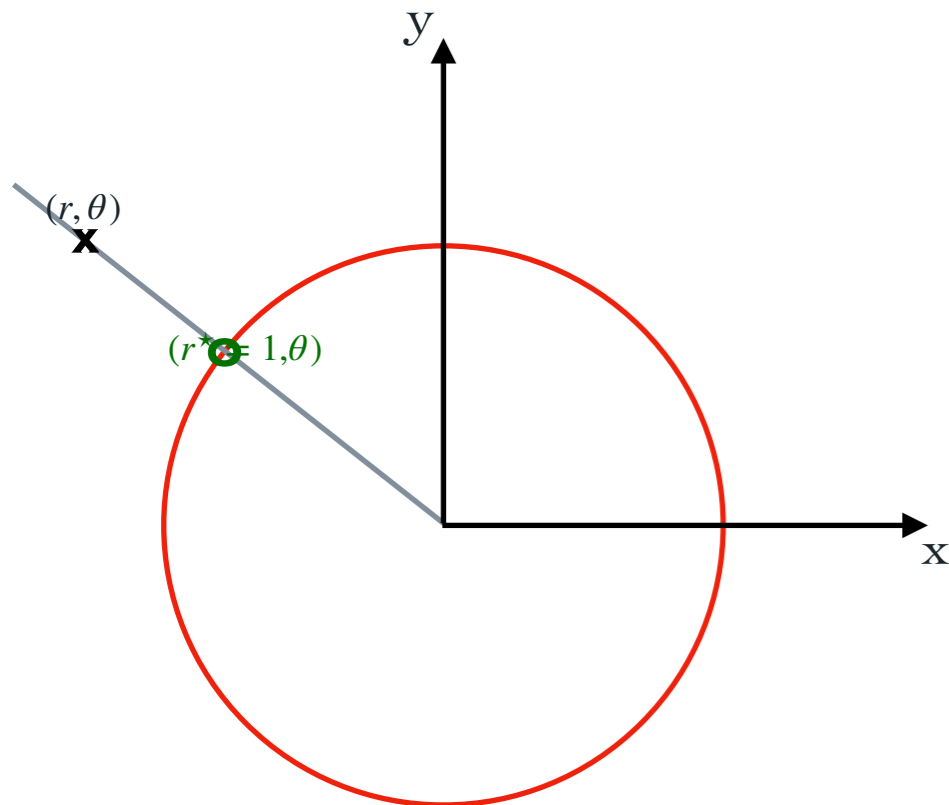
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THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : [2110.06869](https://arxiv.org/abs/2110.06869)]

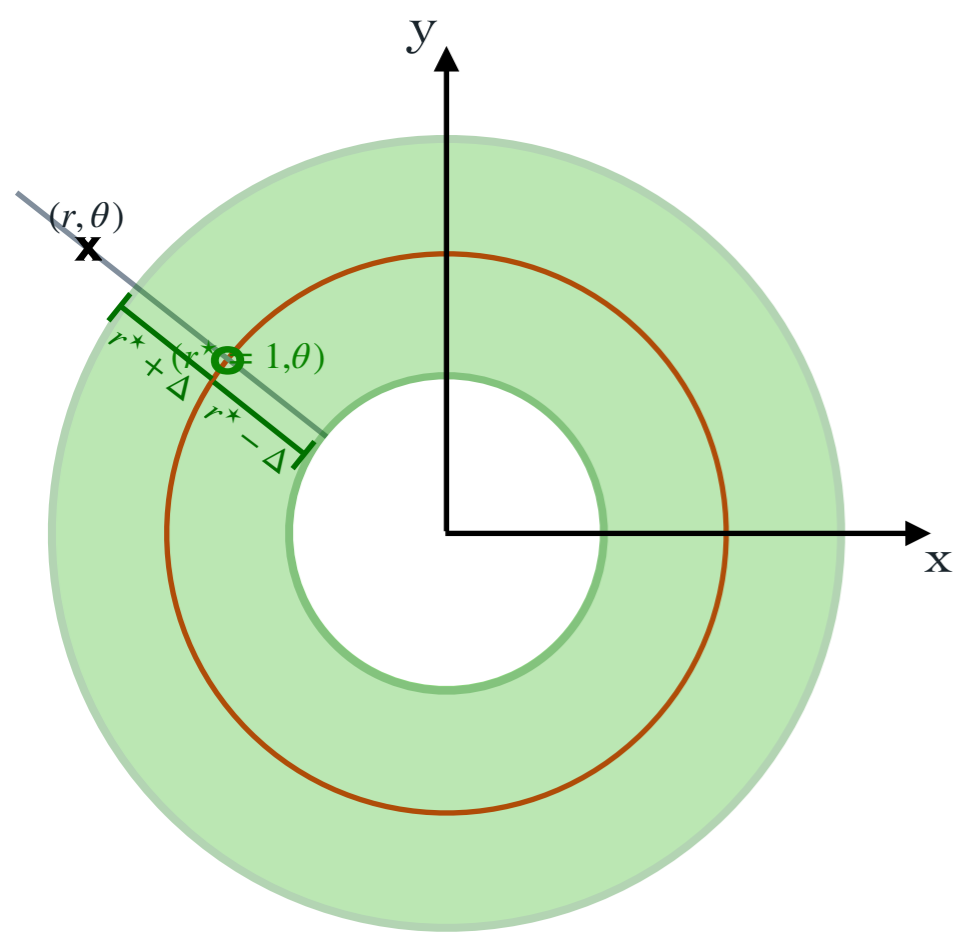
$$\frac{1}{E_1 + E_2 - p_1^0} = \frac{1}{|\vec{k}| + |\vec{k} - \vec{p}_1| - p_1^0}$$

$$\underset{p_1^\mu = (2, \vec{0})}{=} \frac{1}{2|\vec{k}| - 2} \propto \frac{1}{\sqrt{k_x^2 + k_y^2} - 1}$$

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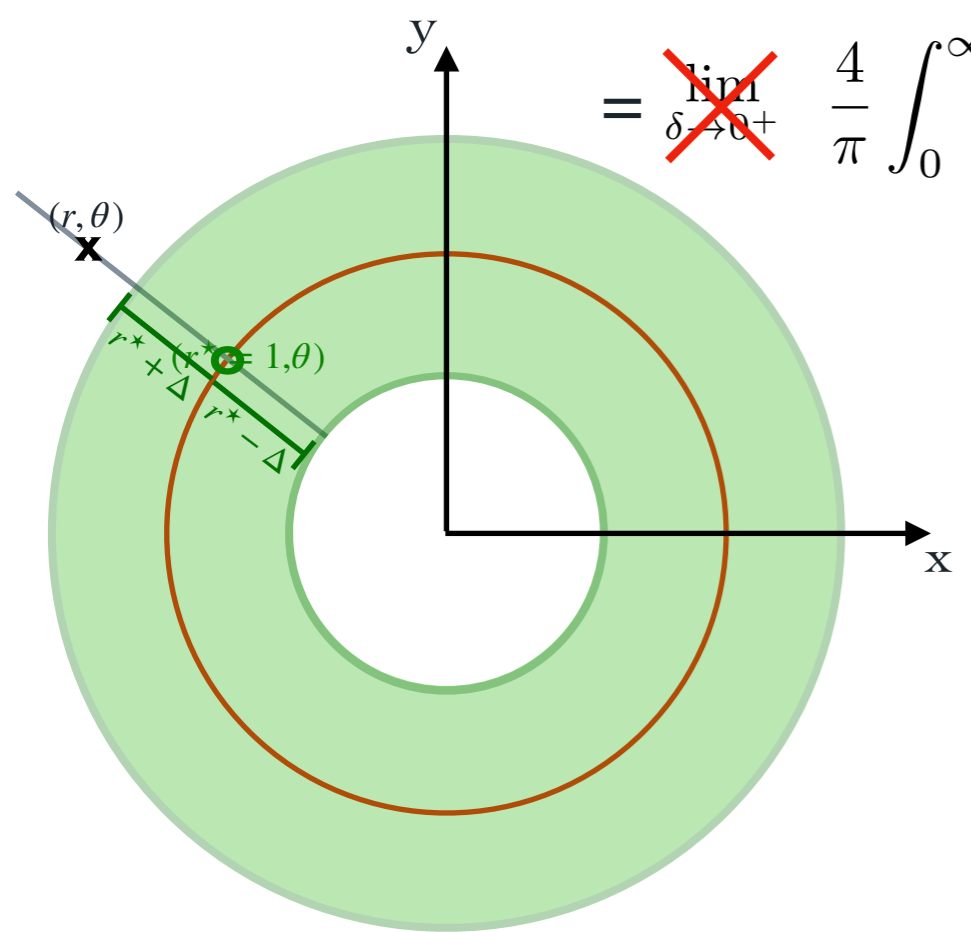
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$\delta = 0$ can be taken safely !



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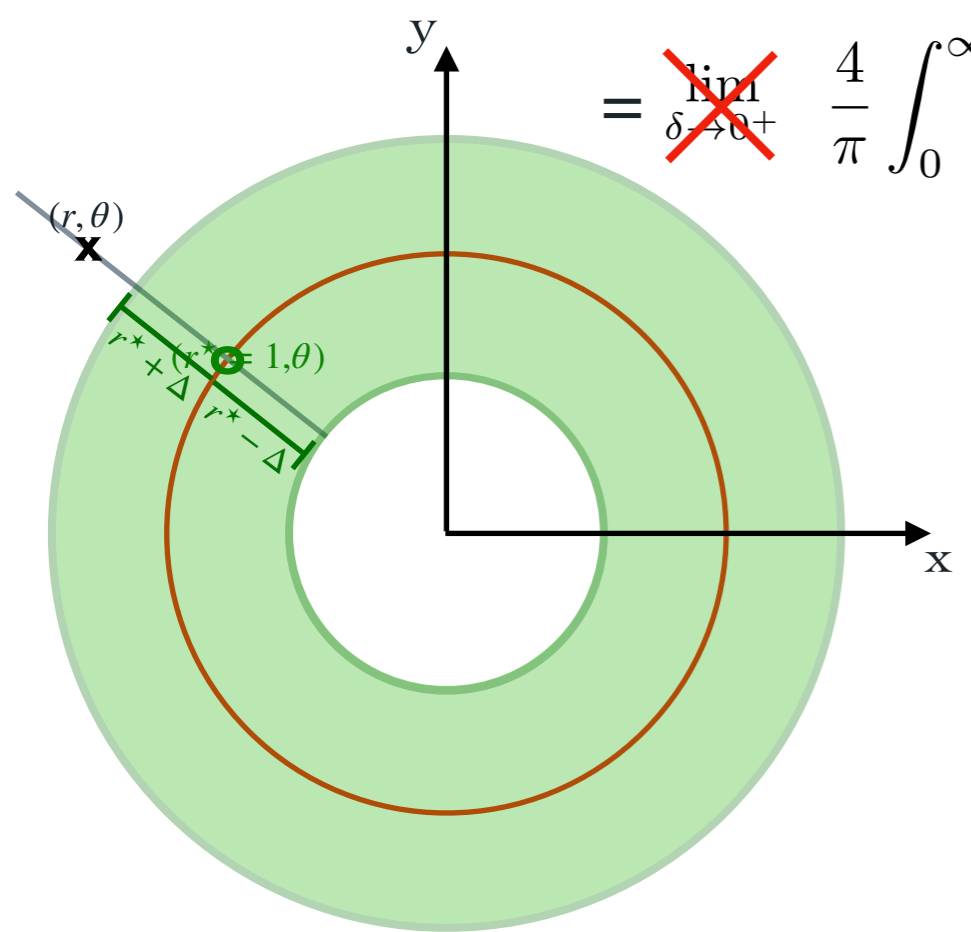
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Local threshold counterterm Integrated threshold counterterm

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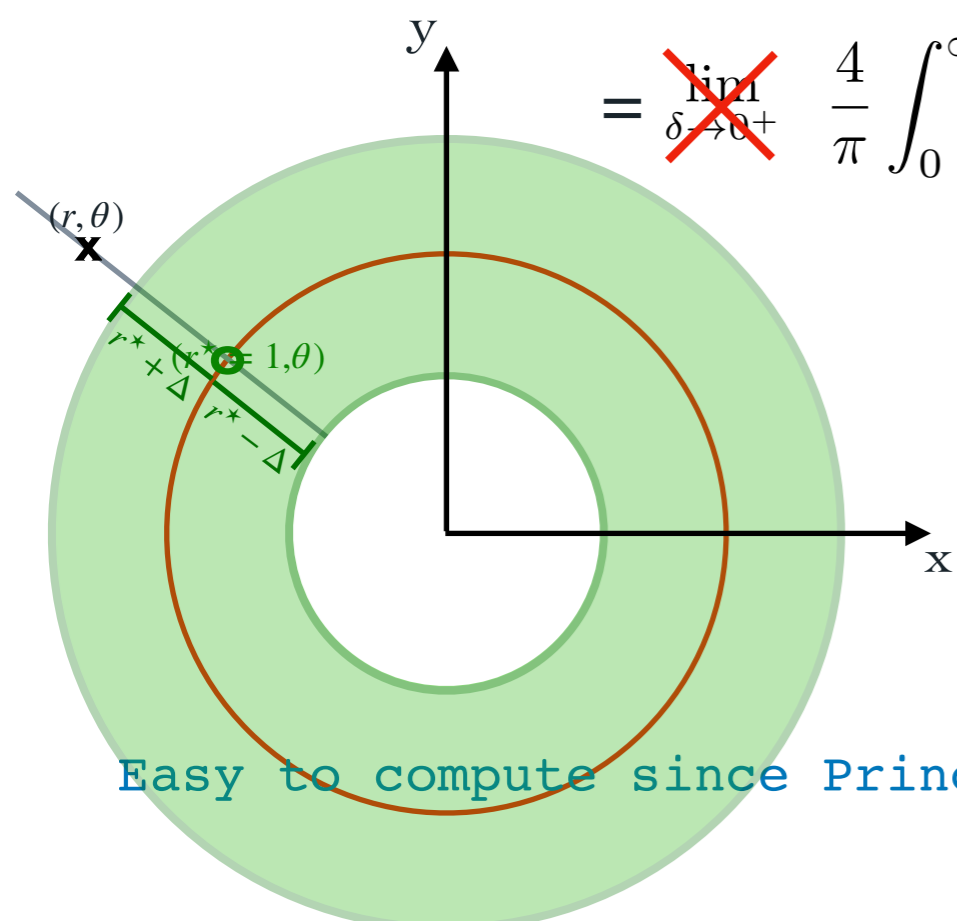
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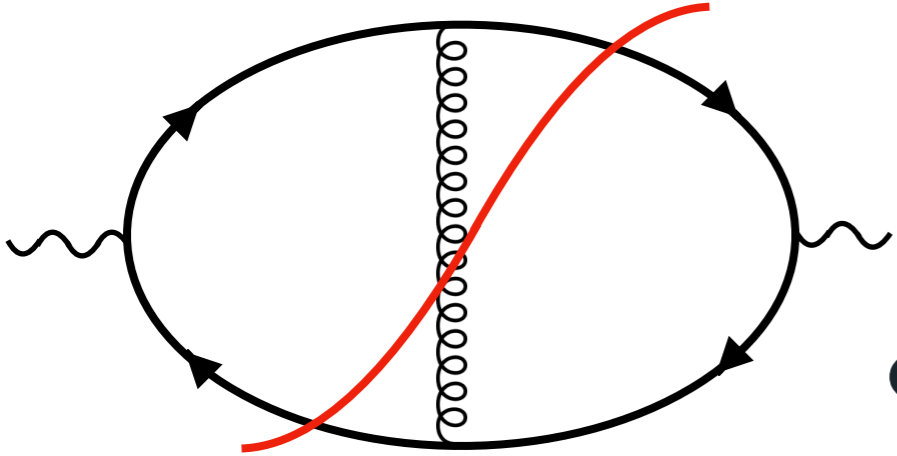
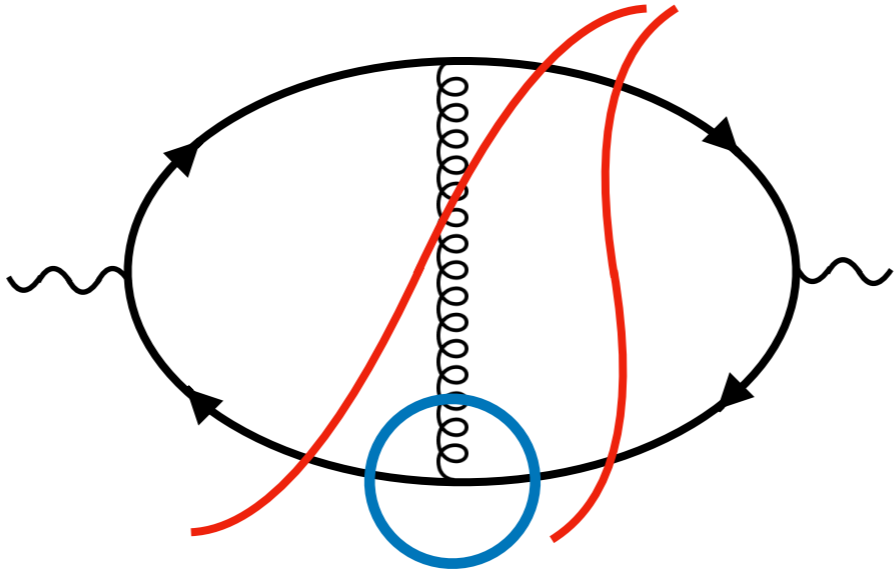
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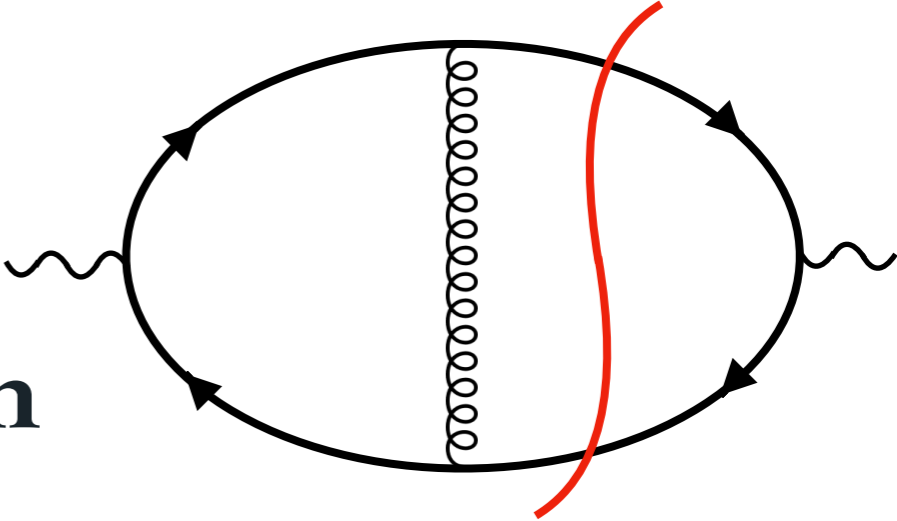
$$+ \lim_{\delta \rightarrow 0^+} \int_{1-\Delta}^{1+\Delta} dr \frac{4}{\pi} \frac{1}{2} \frac{1}{r - 1 \pm i\delta} \stackrel{\Delta \leq 1}{=} \underbrace{\frac{2}{\pi} \text{PV} \left[\frac{1}{r - 1 \pm i\delta} \right]}_0 \mp 2i$$

Easy to compute since Principal Value is zero by construction !

LOCAL UNITARITY



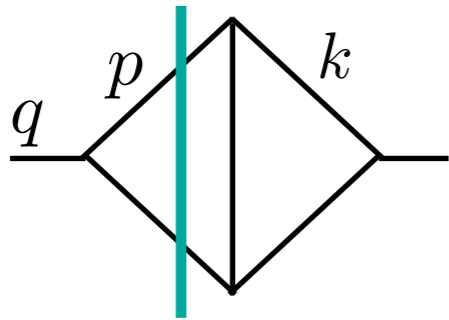
local
↔
cancellation



LOCALITY UNITARITY

We convert the **four-dimensional Minkowski loop integration measure** into a **three-dimensional Euclidean phase-space measure**:

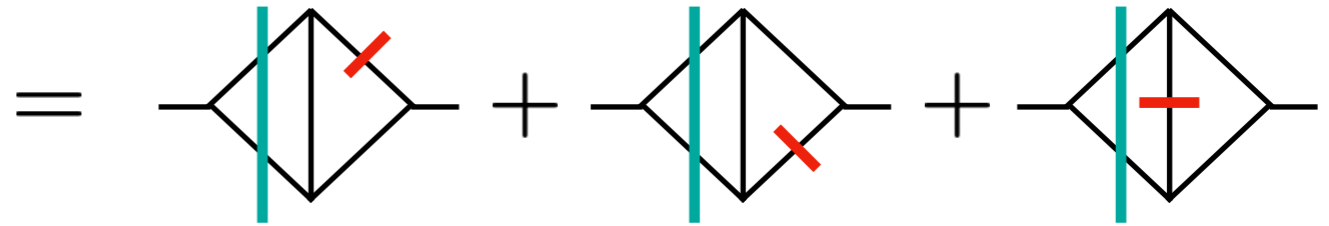
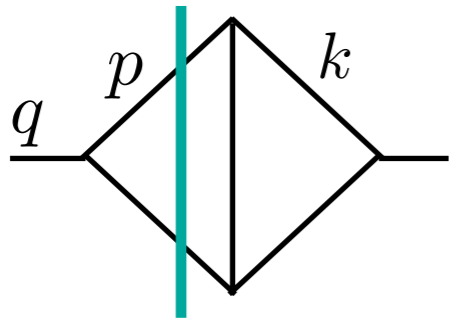
$$\frac{d^3 \vec{p}}{2|\vec{p}|} d^4 k \delta(|\vec{p}| + |\vec{p} - \vec{q}| - Q_0)$$



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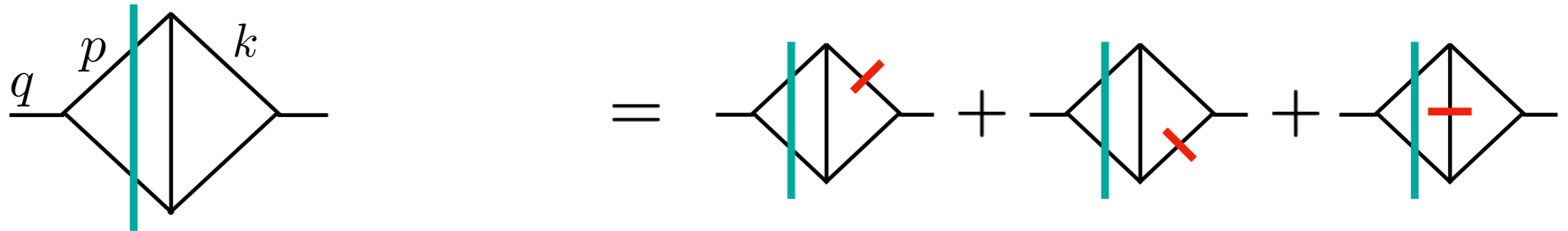
$$\frac{d^3\vec{p}}{2|\vec{p}|} d^4k \delta(|\vec{p}| + |\vec{p} - \vec{q}| - Q_0) \rightarrow \frac{d^3\vec{p}}{2|\vec{p}|} \frac{d^3\vec{k}}{2|\vec{k}|} \delta(|\vec{p}| + |\vec{p} - \vec{q}| - Q_0)$$



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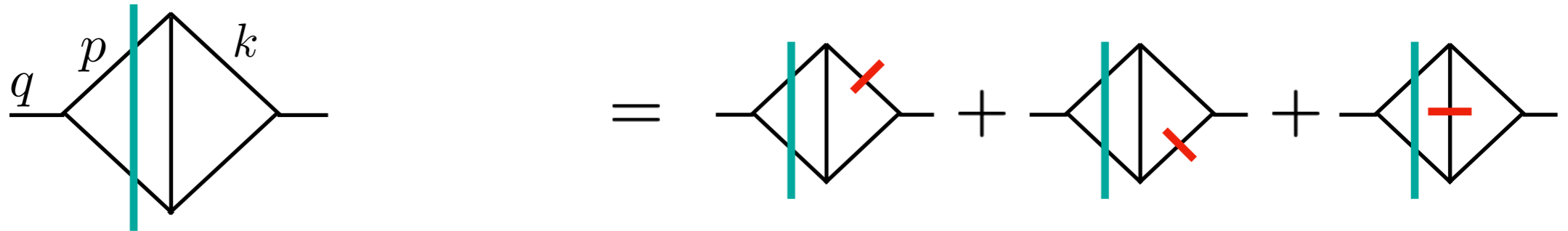
But the measure is **not yet fully aligned**:

$$\left. \begin{array}{c} E_2 \\ E_5 \\ E_3 \\ E_1 \\ E_4 \end{array} \right\} \left| \begin{array}{c} \text{Teal line} \\ \text{Red slash} \end{array} \right. = \int d^3\vec{k} d^3\vec{p} (\delta(E_1 + E_2 - Q_0) f_{\text{virt}} + \delta(E_1 + E_3 + E_5 - Q_0) f_{\text{real}})$$

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$\eta_v(\vec{k}, \vec{p})$

$\eta_r(\vec{k}, \vec{p})$

(on-shell energies: $E_i(\vec{k}_i) = \sqrt{\vec{k}_i^2 + m_i^2 - i\delta}$)

CAUSAL FLOW

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The measure now differs only in the **delta enforcing on shell energy conservation**

$$\text{Diagram 1} \sim \delta(E_1 + E_2 - Q_0)$$

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Objective: find a common variable to solve both deltas.

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A different perspective on the usual phase space mapping problem

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$$\delta(|\vec{k}| - Q_0) \xrightarrow{\vec{k} \rightarrow t\vec{k}} \delta(t|\vec{k}| - Q_0) \rightarrow t = \frac{Q_0}{|\vec{k}|}$$

Soper,
arXiv: [9804454](#) (1998)

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arXiv: [0102031](#) (2001 @ RADCOR)

ZC, Hirschi, Pelloni, Ruijl
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**General FSR cancellations
For N to M N^kLO processes**

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**General FSR cancellations
For N to M N^kLO processes**

A toy example:

$$\int d^3\vec{k} \delta(|\vec{k}| - Q_0) f(\vec{k})$$

CAUSAL FLOW : TOY INTEGRAL

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$$= \int d^3 \vec{k} \int dt h(t) \delta(|\vec{k}| - Q_0) f(\vec{k}) \quad \text{using} \quad 1 = \int dt h(t)$$

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$$= \int d^3 \vec{k} \frac{Q_0^3}{|\vec{k}|^4} h(Q_0/|\vec{k}|) f(Q_0\vec{k}/|\vec{k}|) \quad \text{with} \quad t^* = Q_0/|\vec{k}|$$

Solve all deltas in the common scaling variable. This completes the alignment of the measure!

CAUSAL FLOW : TOY INTEGRAL

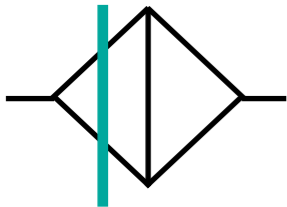
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When applying this construction to LU we get:



$$= \int d^3 \vec{k} d^3 \vec{p} \delta(E_1 + E_2 - Q_0) f_{\text{virt}} = \int d^3 \vec{k} d^3 \vec{p} g_v(t_v^*)$$

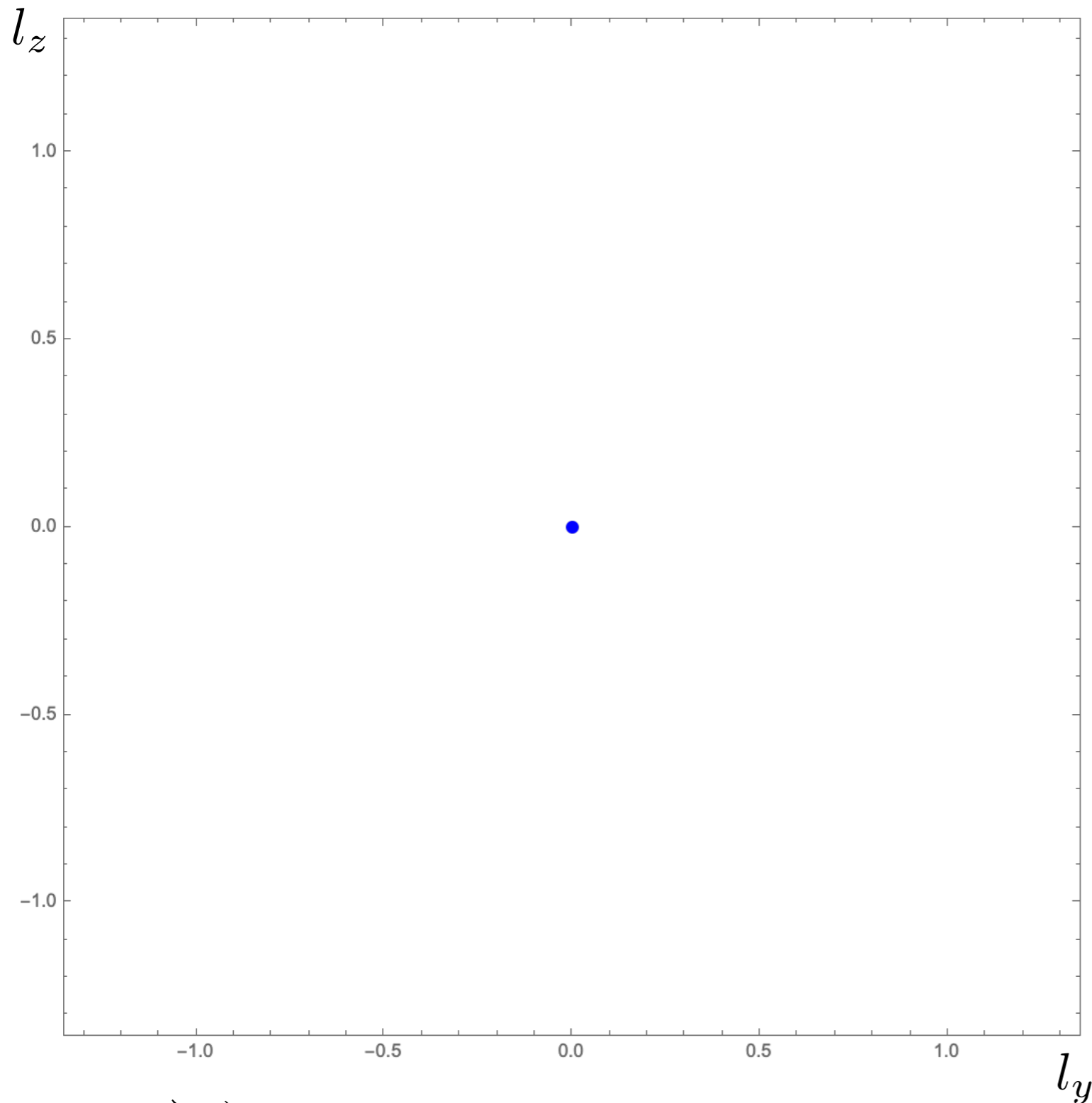
where $t_v^* = t_v^*(\vec{k}, \vec{p}) = \frac{Q_0}{E_1 + E_2}$

$(\vec{p}, \vec{k}) \rightarrow \vec{\phi}(t, (\vec{p}, \vec{k}))$

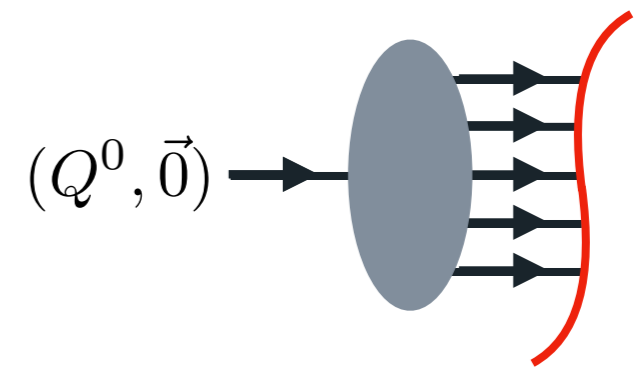
“Causal flow” is called like this because it is the generalisation of the Soper, derive from a contour deformation field satisfying the causal constraints.

$$\begin{cases} \partial_t \vec{\phi} = \vec{k} \circ \vec{\phi} \\ \vec{\phi}(0, (\vec{k}, \vec{l})) = (\vec{k}, \vec{l}) \end{cases}$$

LOCALITY UNITARITY: VISUALISATION

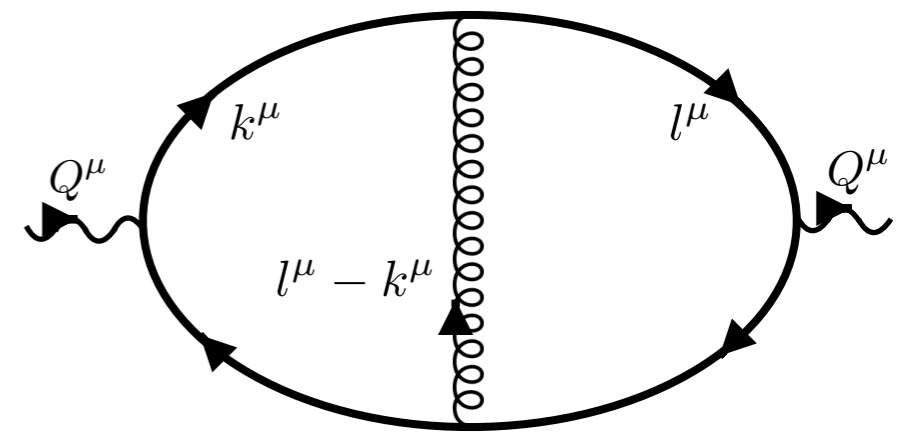


$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$



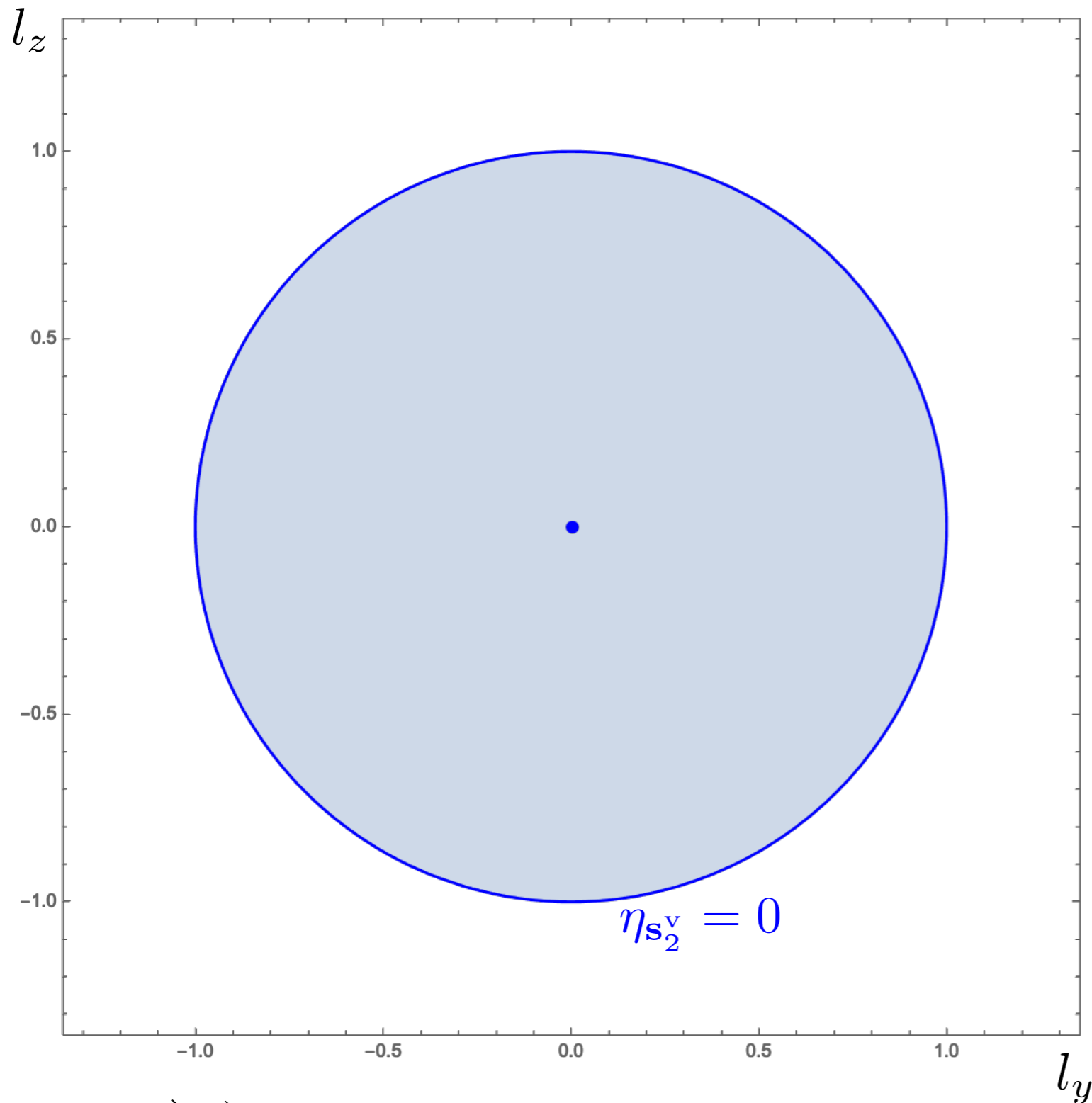
$$Q^\mu = (2, 0, 0, 0)$$

$$(\vec{k}, \vec{l}) = ((0, 0.5, 0.5), (0, l_y, l_z))$$

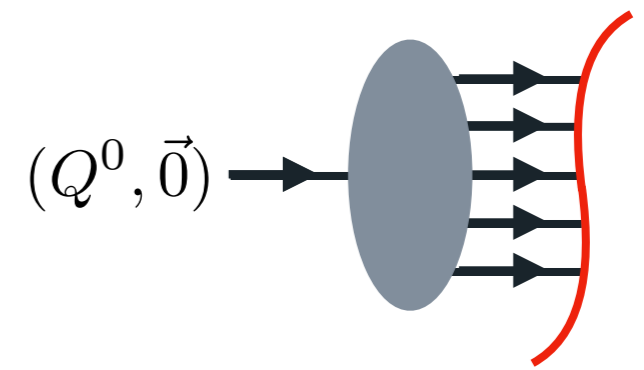


— = Cutkosky cut \equiv threshold

LOCALITY UNITARITY: VISUALISATION



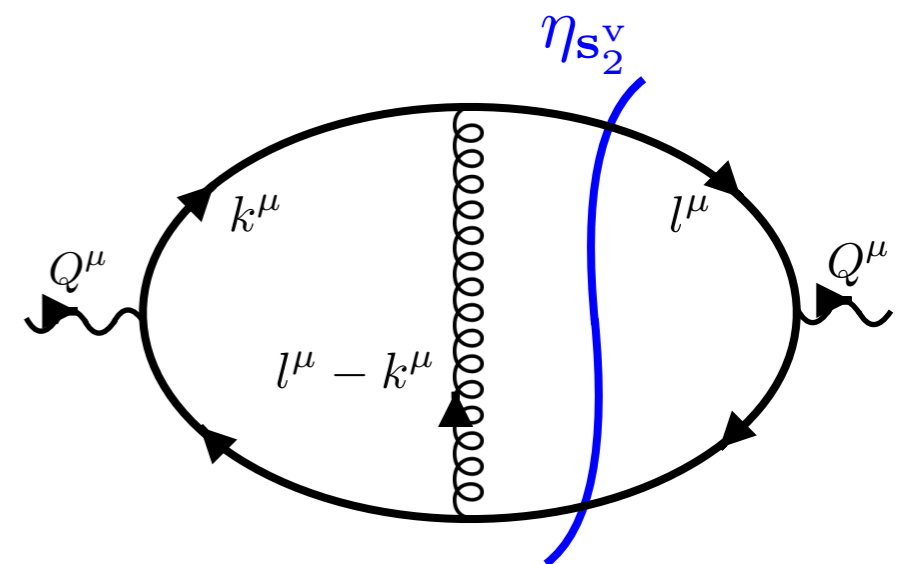
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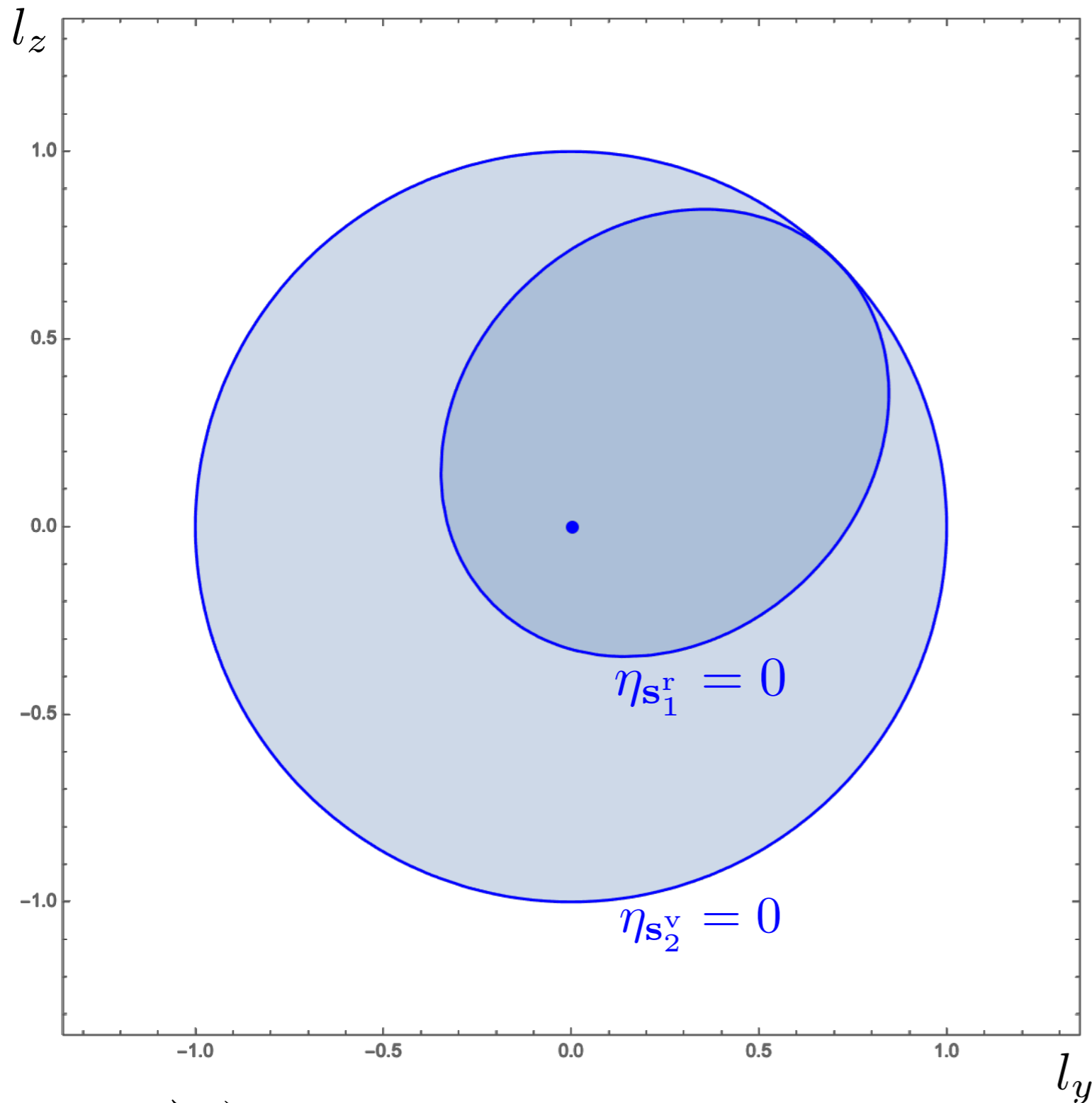
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$$\eta_{s_2^v} \rightarrow 2|\vec{l}'| = Q^0$$

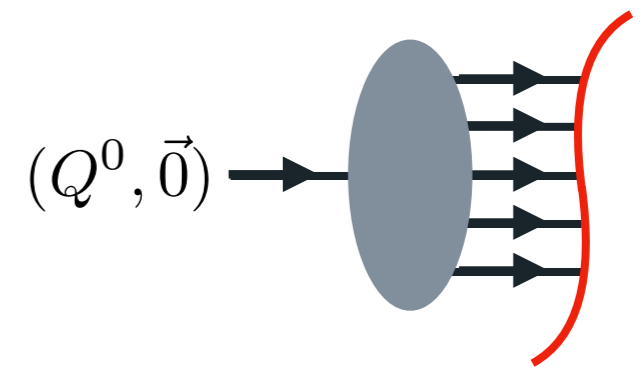


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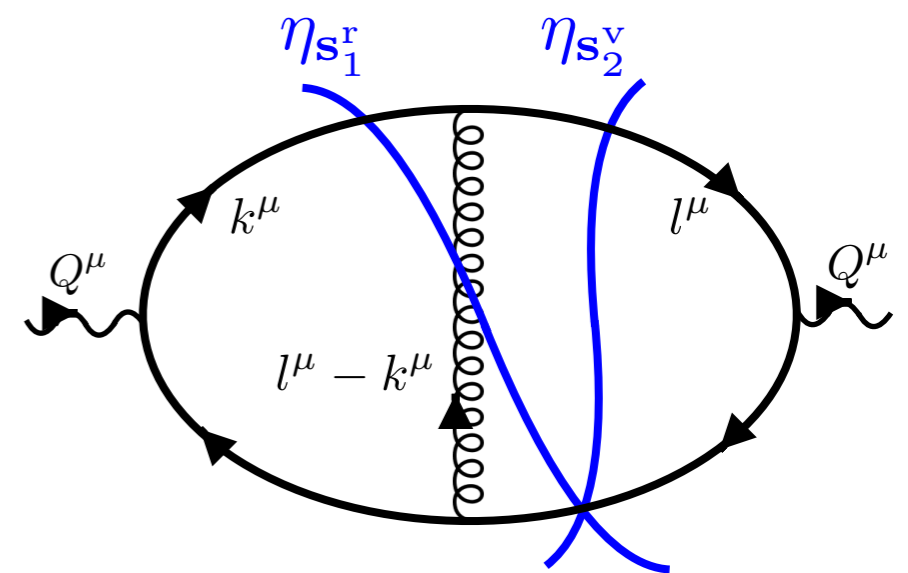


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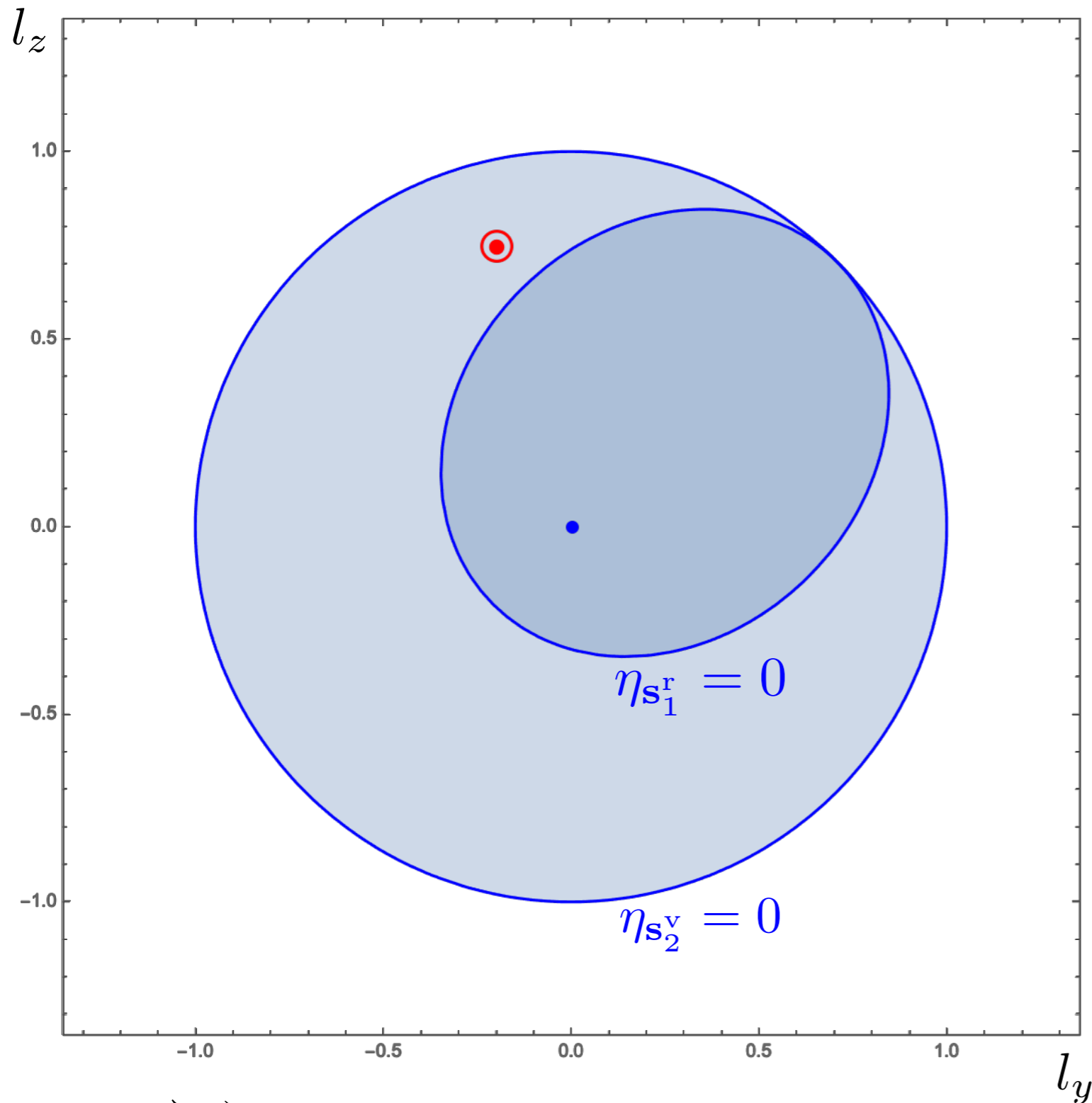
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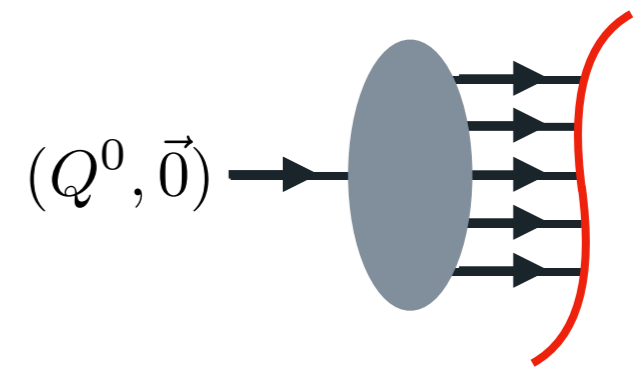


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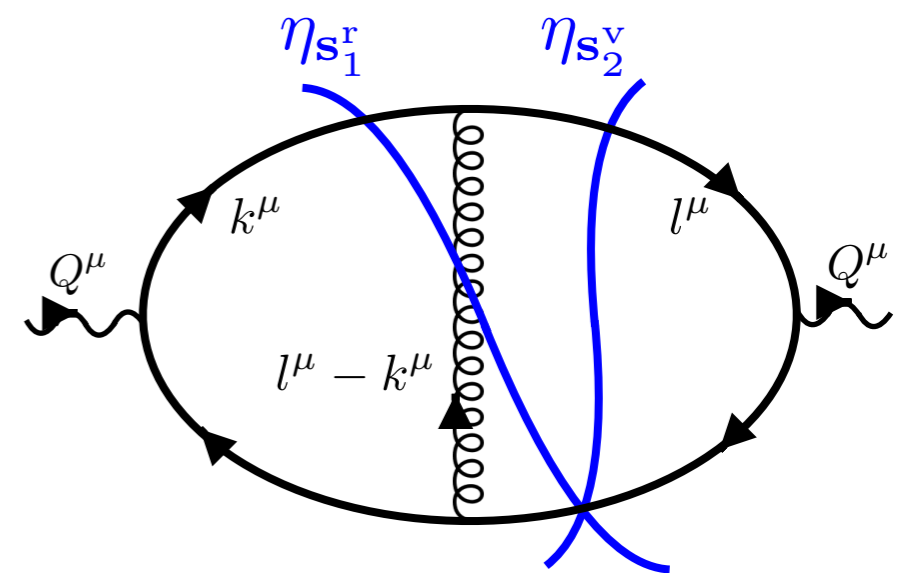


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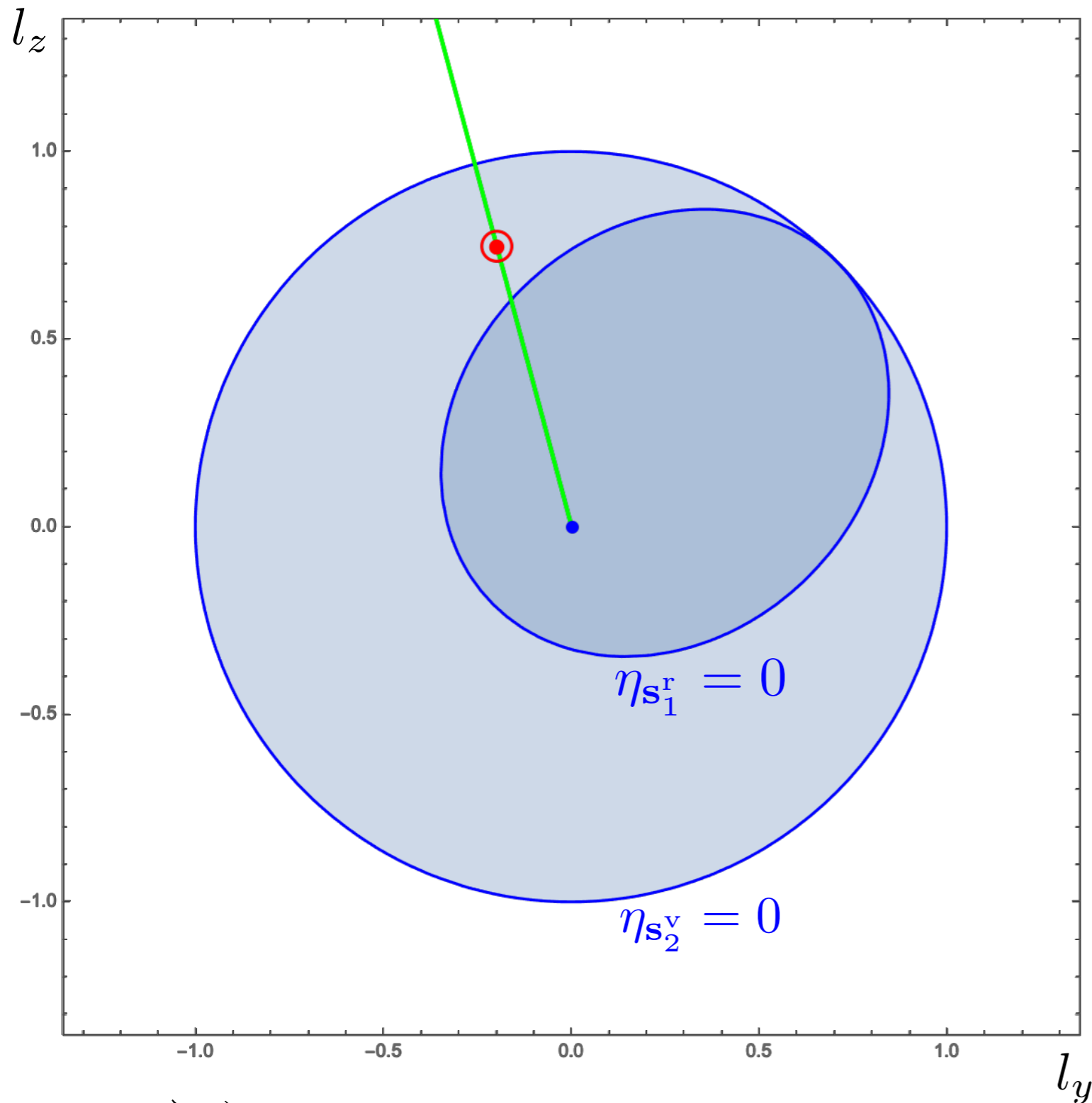
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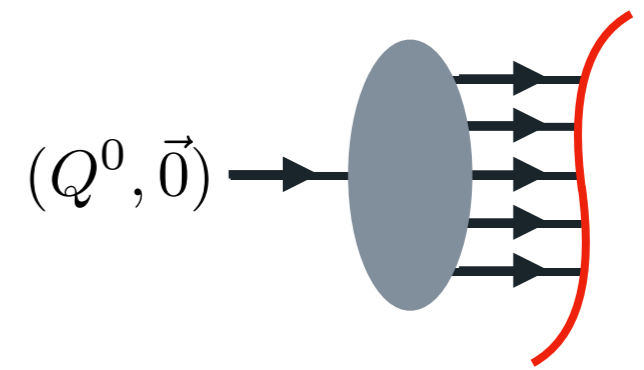


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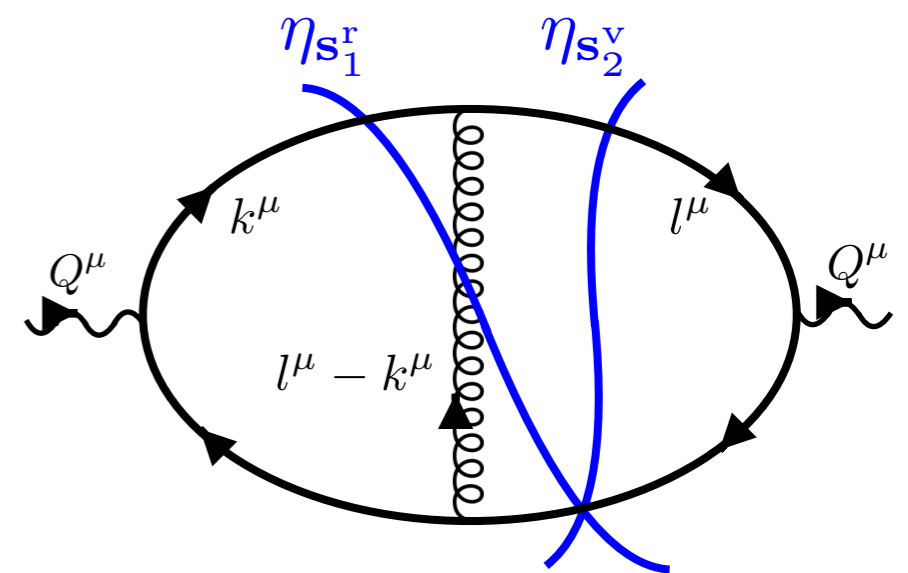


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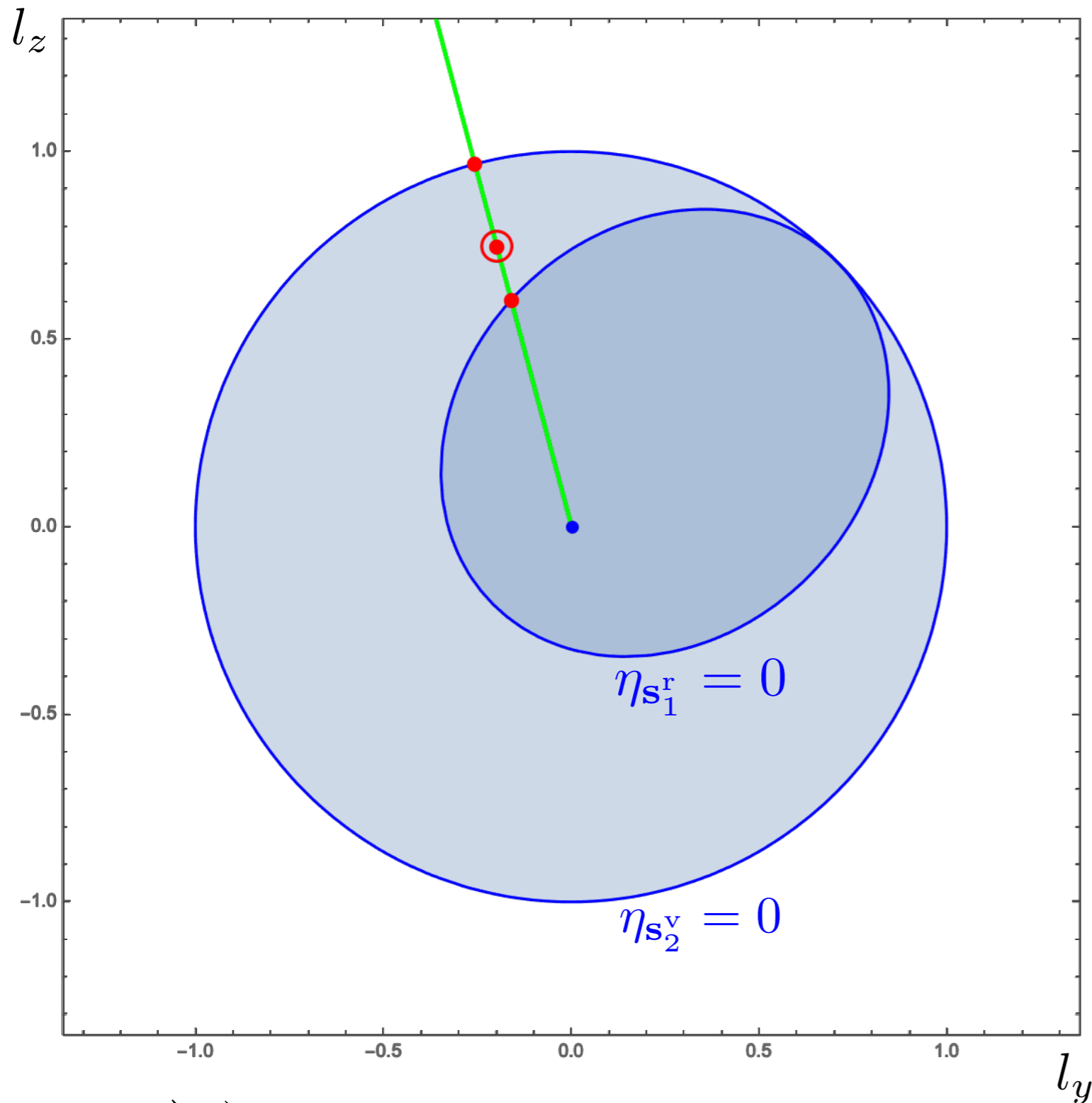
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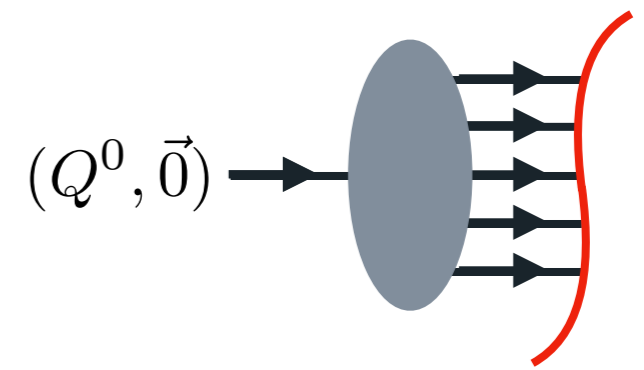


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LOCALITY UNITARITY: VISUALISATION



$(\vec{k}, \vec{l}) \in \mathbb{R}^3 \times \mathbb{R}^3$ projected to $(l_y, l_z) \in \mathbb{R}^2$

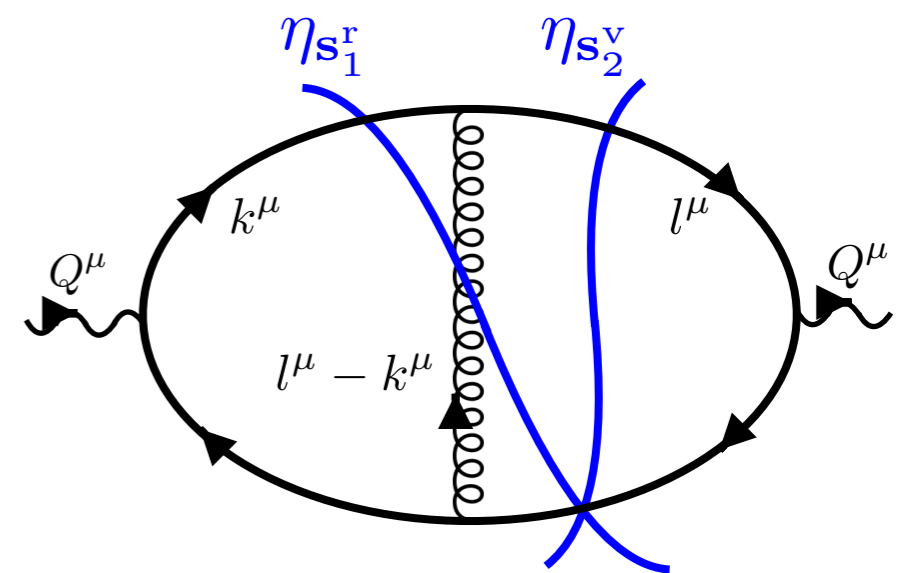


$$Q^\mu = (2, 0, 0, 0)$$

$$(\vec{k}, \vec{l}) = ((0, 0.5, 0.5), (0, l_y, l_z))$$

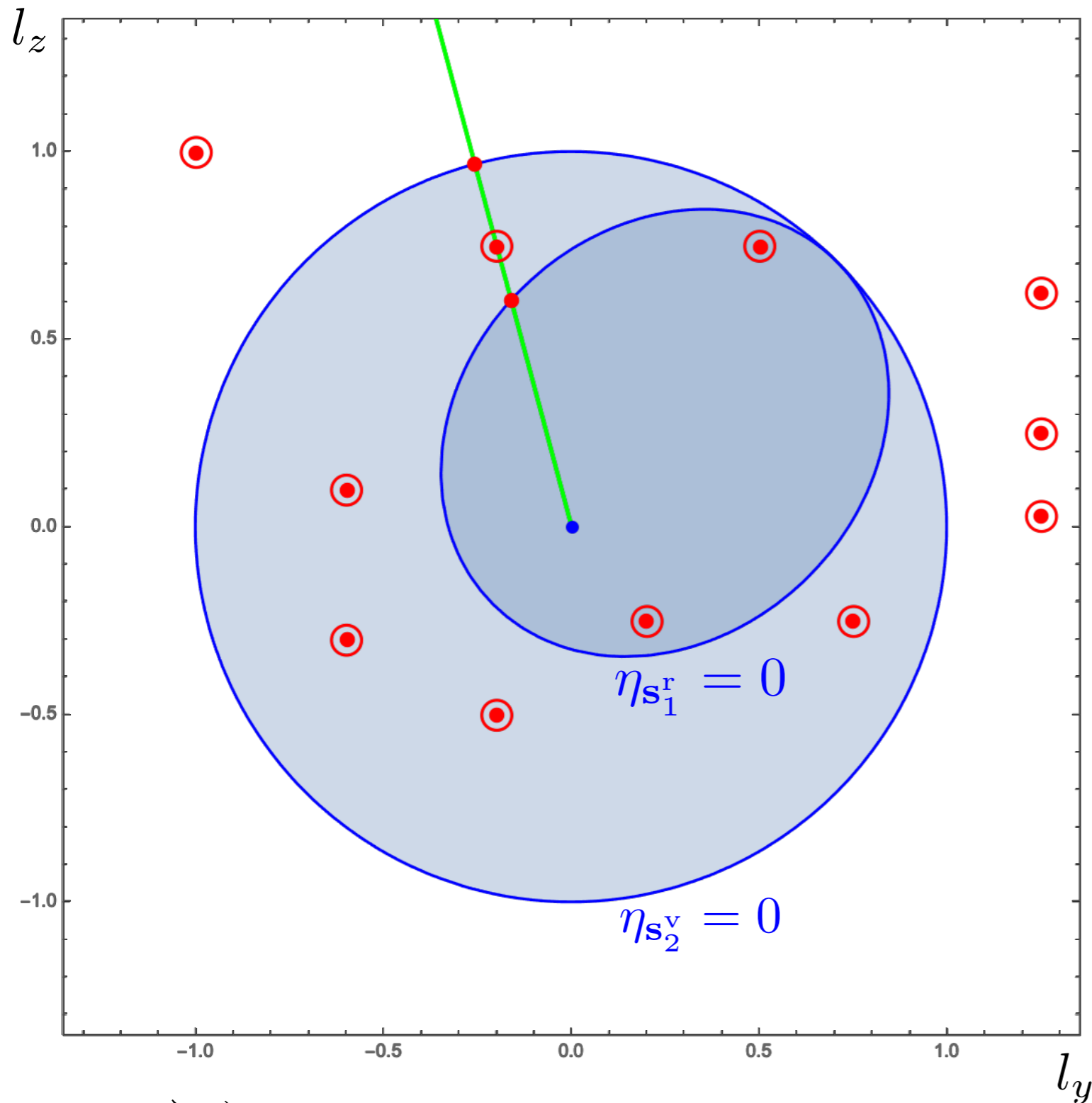
$$\eta_{s_2^v} \rightarrow 2|\vec{l}'| = Q^0$$

$$\eta_{s_1^r} \rightarrow |\vec{l}'| + |\vec{l}' - \vec{k}'| = Q^0 - |\vec{k}'|$$

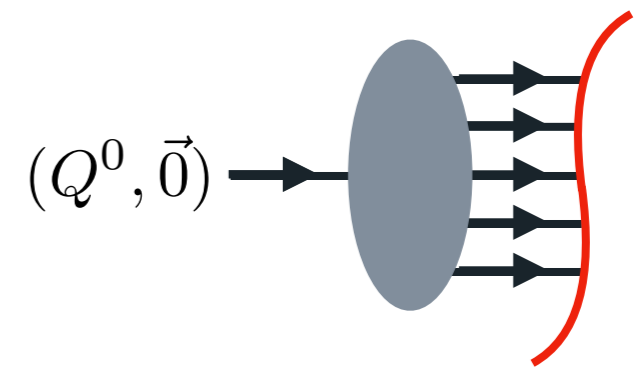


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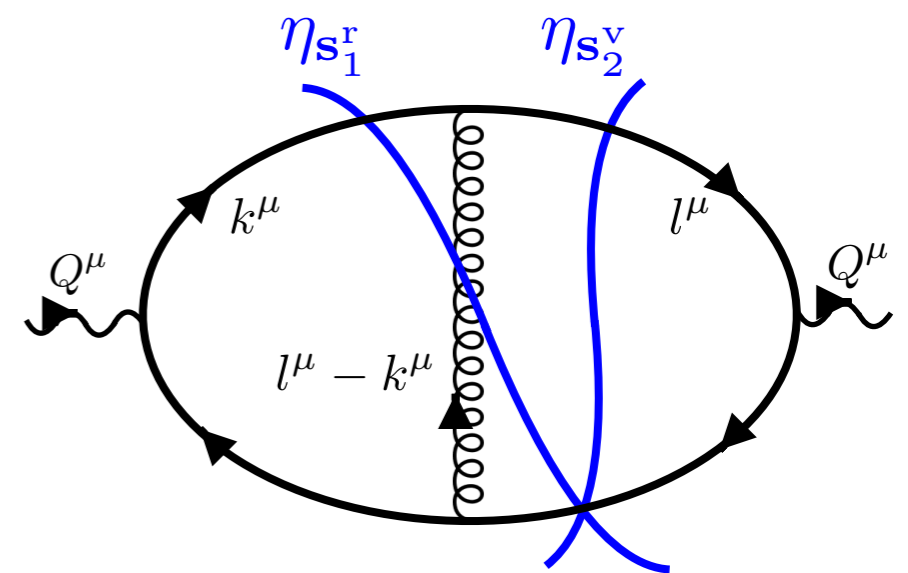


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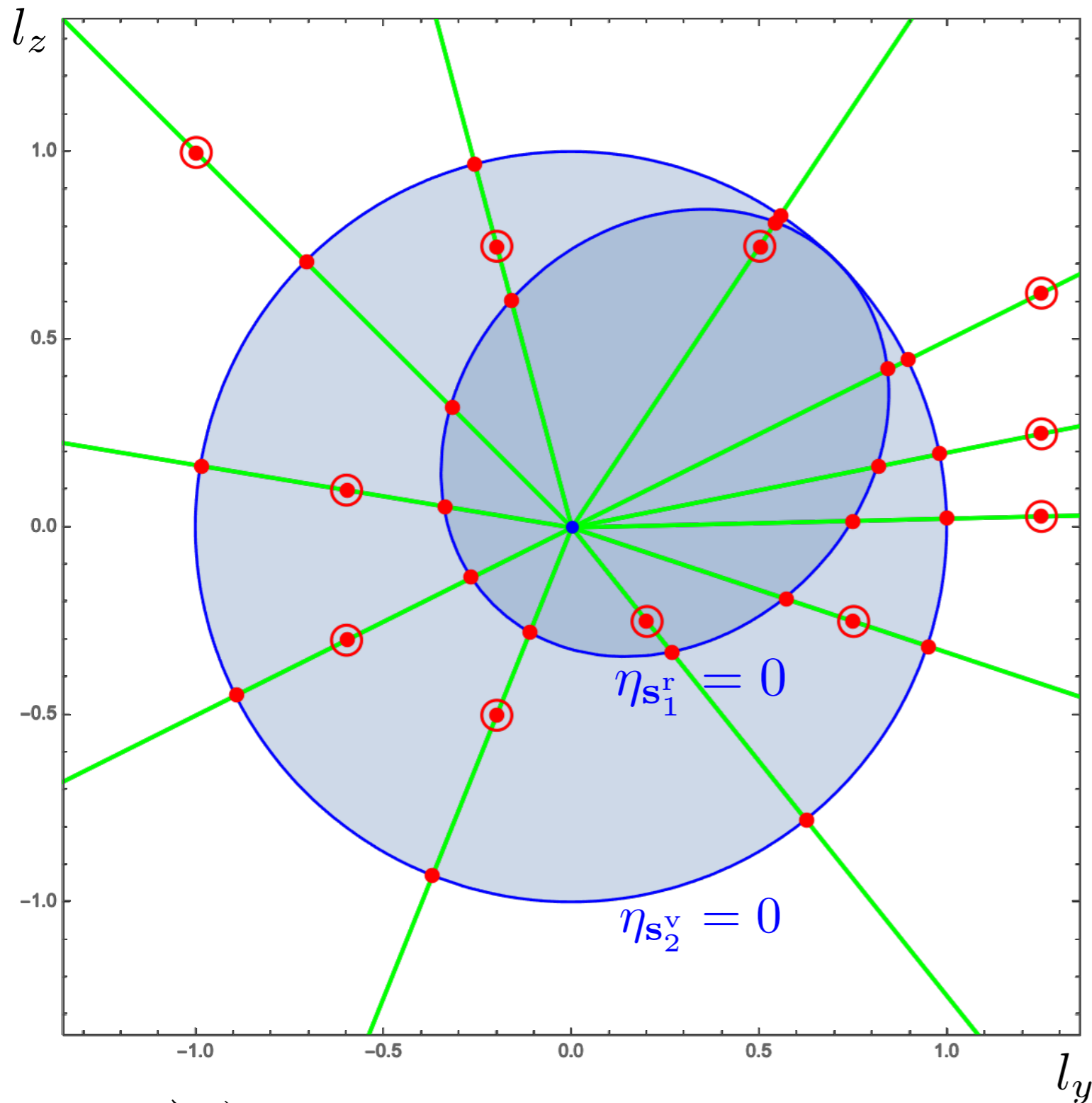
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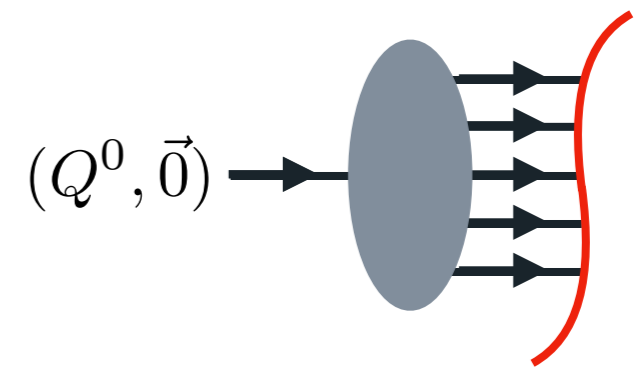


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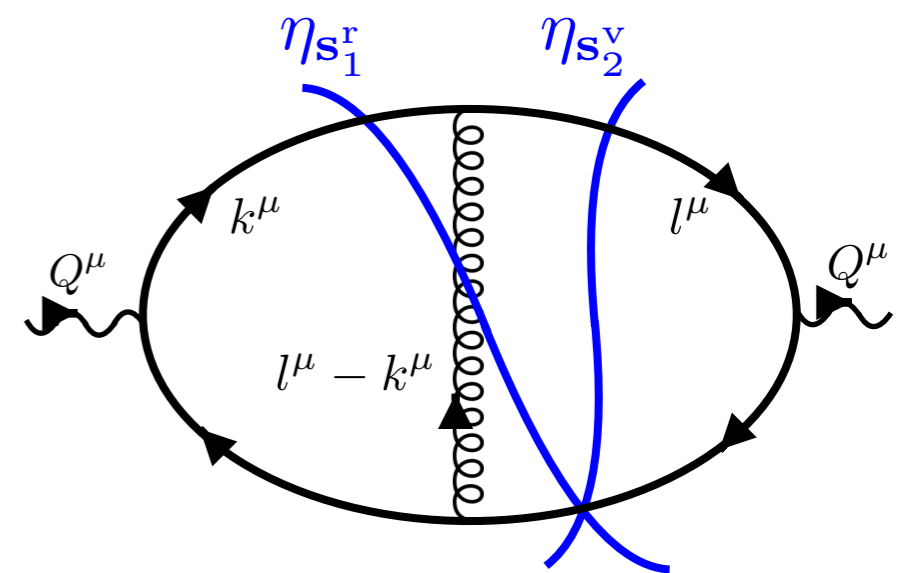


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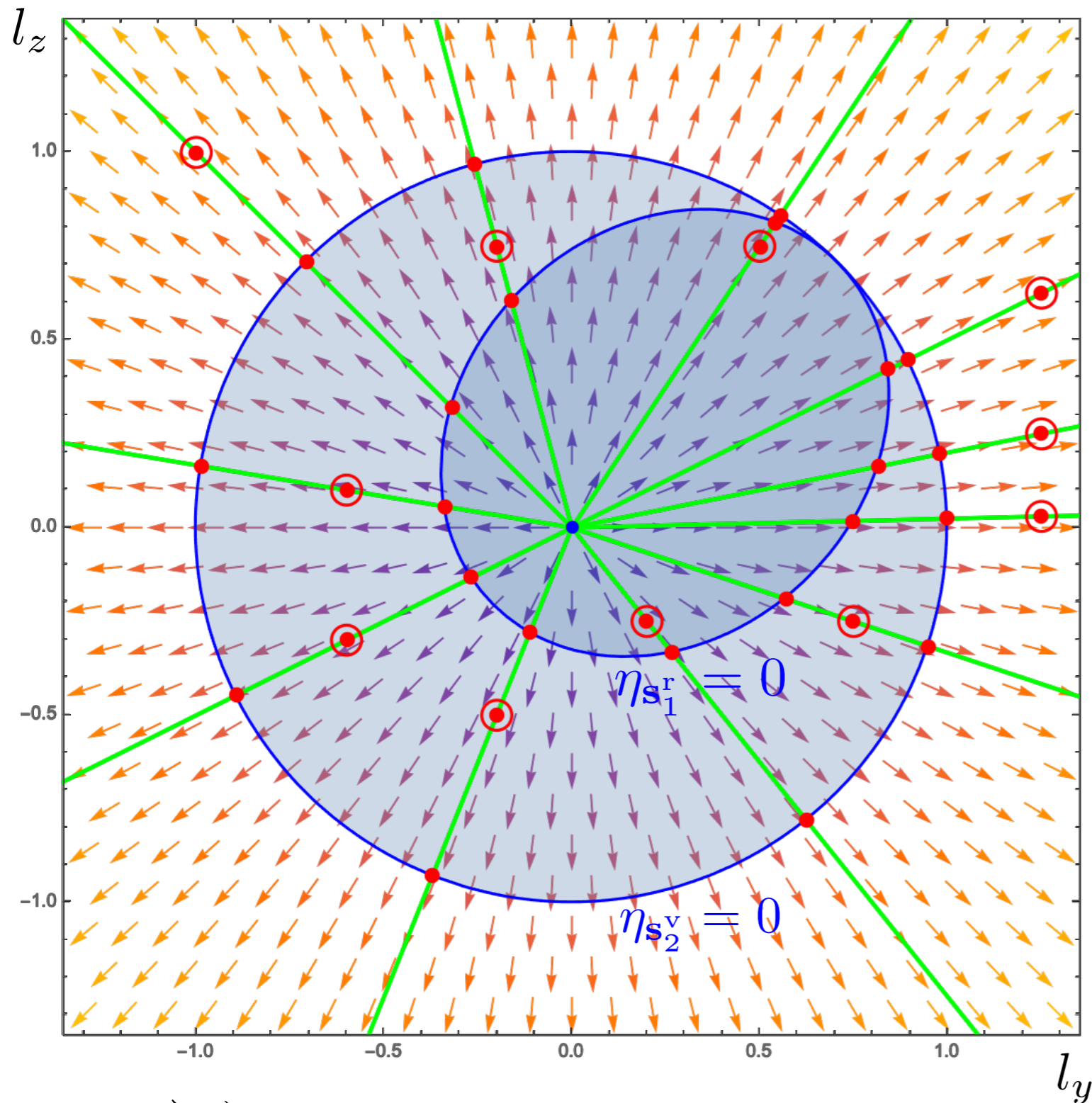
$$\eta_{s_2^v} \rightarrow 2|\vec{l}'| = Q^0$$

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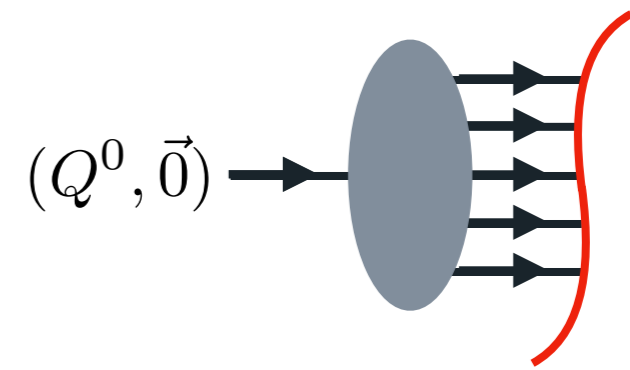


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LOCALITY UNITARITY: VISUALISATION



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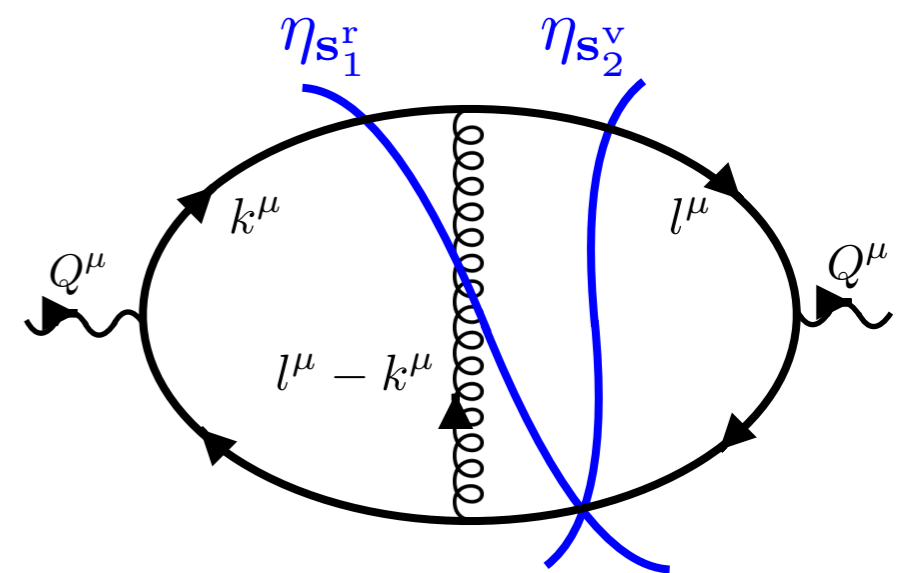


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LOCALITY UNITARITY: ALL-ORDERS PROOF

ni, B. Ruijl, arXiv : [2010.01068](https://arxiv.org/abs/2010.01068)] in proceedings, arXiv : [2110.15662](https://arxiv.org/abs/2110.15662)]

LOCALITY UNITARITY: ALL-ORDERS PROOF

Valentin Hirschi, B. Ruijl, arXiv : [2010.01068](https://arxiv.org/abs/2010.01068) [in proceedings, arXiv : [2110.15662](https://arxiv.org/abs/2110.15662)]

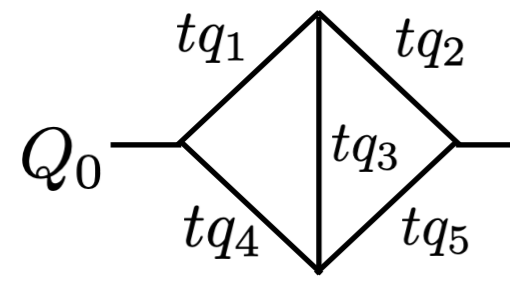
The **LTD representation** of the double triangle with rescaled momenta is

$$f_{\text{ltd}} \left(\text{double triangle} \right) \Big|_{tq_i} = \left[\begin{array}{cccc} \text{diag 1} & + & \text{diag 2} & + & \text{diag 3} & + & \text{diag 4} \\ \text{diag 5} & + & \text{diag 6} & + & \text{diag 7} & + & \text{diag 8} \end{array} \right] q_i \rightarrow tq_i$$

LOCALITY UNITARITY: ALL-ORDERS PROOF

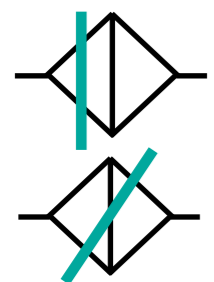
Hirschi, B. Ruijl, arXiv : [2010.01068](https://arxiv.org/abs/2010.01068) [in proceedings, arXiv : [2110.15662](https://arxiv.org/abs/2110.15662)]

The **LTD representation** of the double triangle with rescaled momenta is



$$f_{\text{ltd}} \left(\text{triangle} \right) \Big|_{tq_i} = \left[\begin{array}{cccc} \text{triangle with red crosses} & + & \text{triangle with red crosses} & + & \text{triangle with red crosses} & + & \text{triangle with red crosses} \\ + & \text{triangle with red crosses} & + & \text{triangle with red crosses} & + & \text{triangle with red crosses} & + & \text{triangle with red crosses} \end{array} \right] q_i \rightarrow tq_i$$

Then one can capture the **thresholds** of this forward-scattering graphs with



$$= \int d^3 \vec{p} d^3 \vec{k} \left[\lim_{t \rightarrow t_v^*} (t - t_v^*) f_{\text{ltd}} \left(\text{triangle} \right) \Big|_{tq_i} + \lim_{t \rightarrow t_r^*} (t - t_r^*) f_{\text{ltd}} \left(\text{triangle} \right) \Big|_{tq_i} \right]$$

g_v , g_r can be written as different limits of the same function!

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Hirschi, B. Ruijl, arXiv : [2010.01068](https://arxiv.org/abs/2010.01068) [in proceedings, arXiv : [2110.15662](https://arxiv.org/abs/2110.15662)]

The **LTD representation** of the double triangle with rescaled momenta is

$$Q_0 \text{ (double triangle)} \quad f_{\text{ltd}} \left(\text{diamond} \right) \Big|_{tq_i} = \left[\text{sum of 8 diamond graphs with red crosses} \right]_{q_i \rightarrow tq_i}$$

Then one can capture the **thresholds** of this forward-scattering graphs with

$$\text{(diamond with blue line)} \quad = \int d^3 \vec{p} d^3 \vec{k} \left[\lim_{t \rightarrow t_v^*} (t - t_v^*) f_{\text{ltd}} \left(\text{diamond} \right) \Big|_{tq_i} + \lim_{t \rightarrow t_r^*} (t - t_r^*) f_{\text{ltd}} \left(\text{diamond} \right) \Big|_{tq_i} \right]$$

g_v , g_r can be written as different limits of the same function!

Solving delta in the scaling variable \Rightarrow **1d residue theorem along the line** $\gamma(t) = (t\vec{k}, t\vec{p})$

$$\text{(diamonds with blue and red lines)} \quad = \int d^3 \vec{p} d^3 \vec{k} \left[\sum_{i=1}^4 \lim_{t \rightarrow t_i^*} (t - t_i^*) f_{\text{ltd}} \left(\text{diamond} \right) \Big|_{tq_i} \right] \quad \text{LU representation}$$

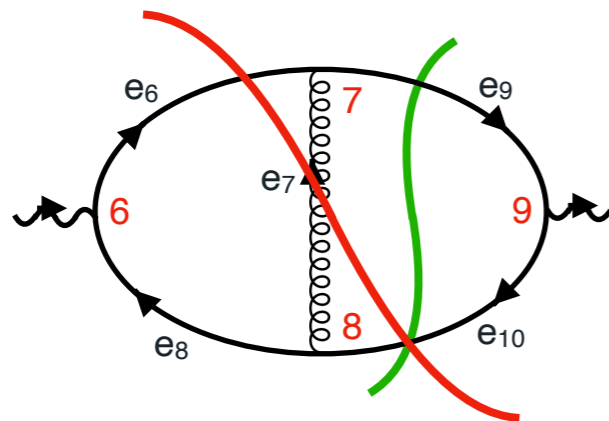
Cutkosky, but at the local level! We prove cancellations by studying the limit $t_r^* \rightarrow t_v^*$

LOCALITY UNITARITY

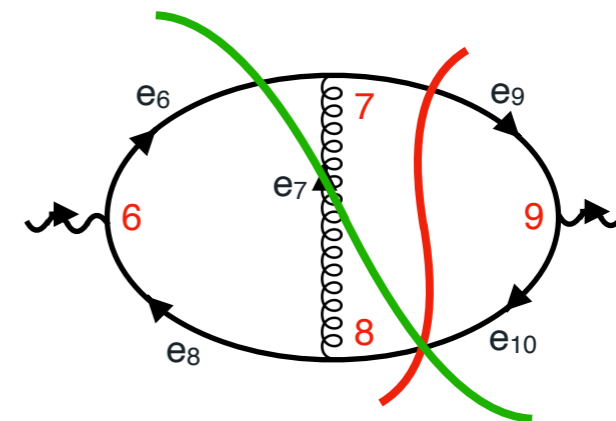
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

— = Cutkosky cut — = threshold singularity



cancel

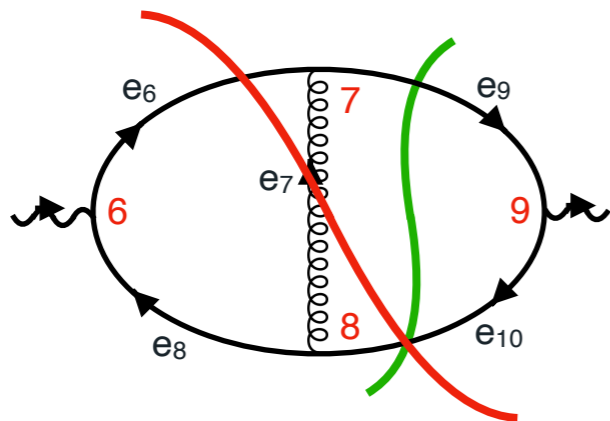


LOCALITY UNITARITY

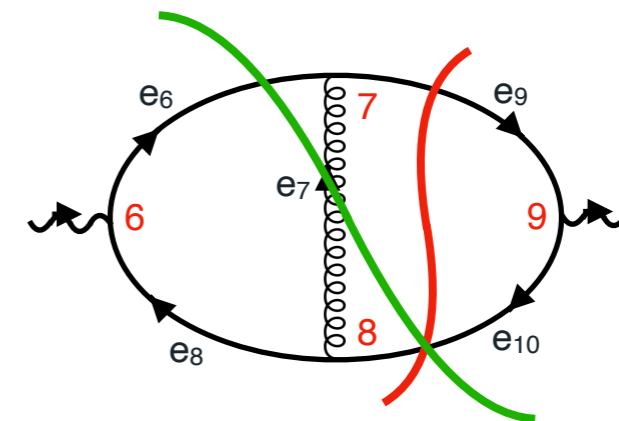
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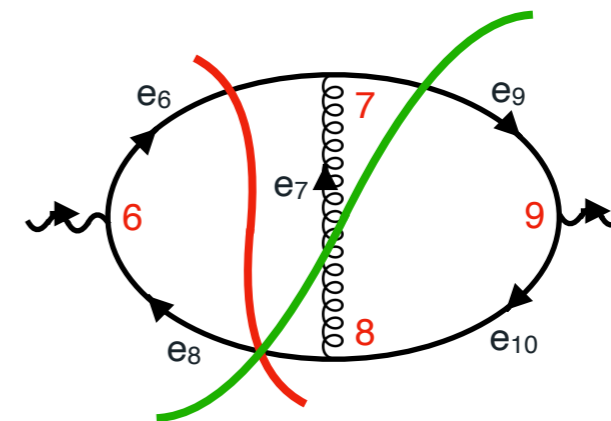
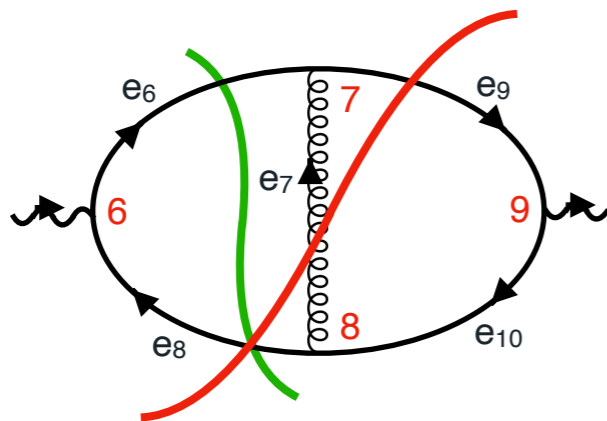
— = Cutkosky cut — = threshold singularity



— cancels —



— cancels —

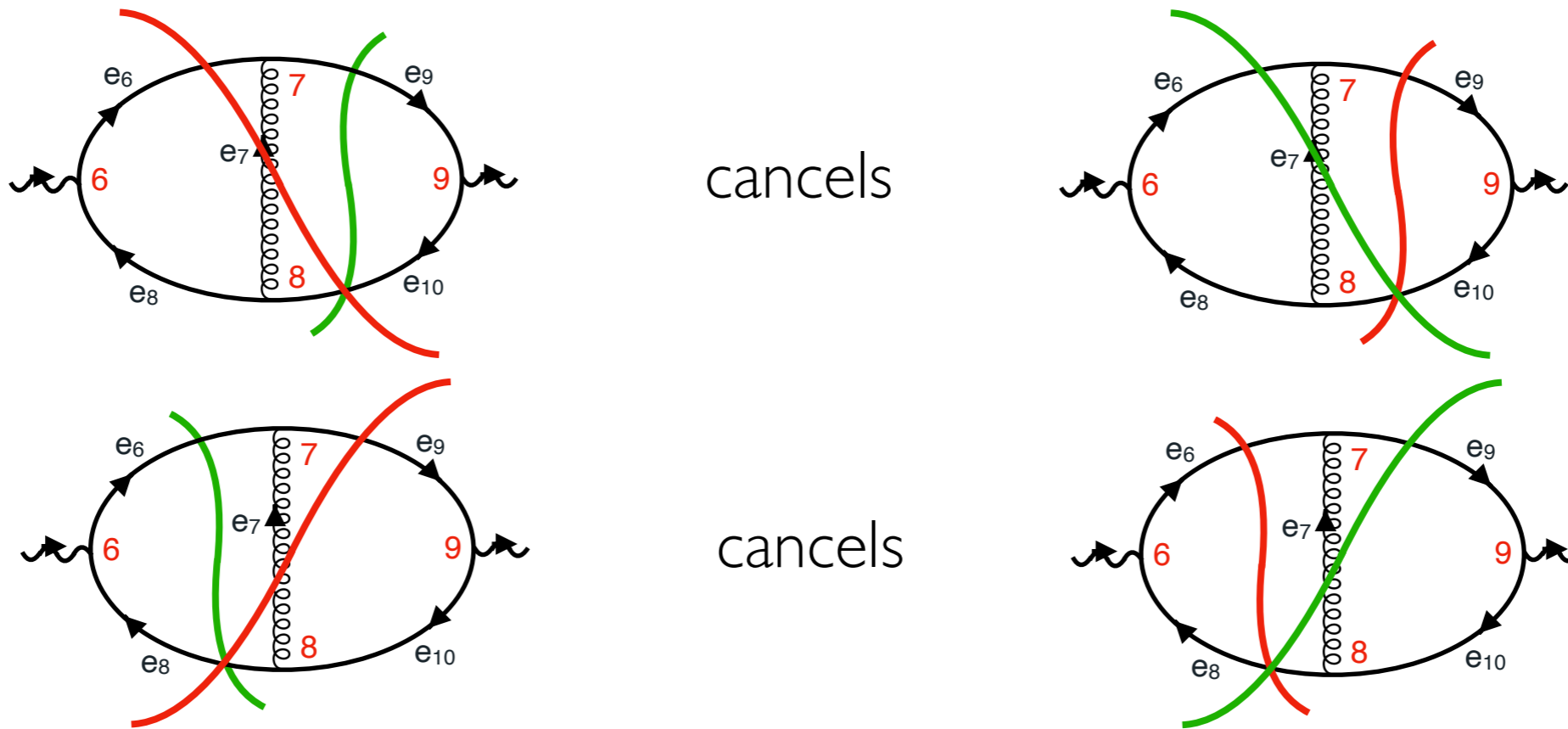


LOCALITY UNITARITY

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

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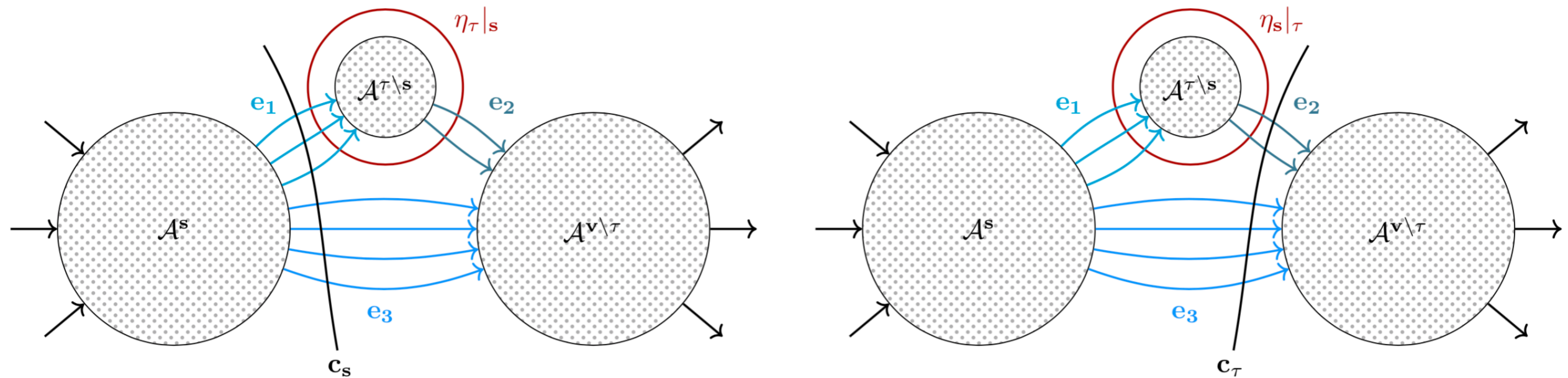
Even for **non-pinched singular threshold** ! (when $\mathcal{O}_s \equiv 1$) :



LOCALITY UNITARITY

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :



TROPICAL SAMPLING IN MOMENTUM SPACE

FOR TAMING INTEGRABLE SINGULARITIES

$$I^{(\text{eucl.})}[f] = \int d^{DL} \mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}}$$

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$$= \int_{[0,1]^{(DL+2|E|+1)}} dz \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

TROPICAL SAMPLING OF EUCLIDEAN FEYNMAN INTEGRALS

$$I^{\text{eucl.}} [f = 1]$$

E	$\ell(G)$	σ_I/I	samples per second	preprocessing time	RAM
6	3	0.9	$1.1 \cdot 10^6 / s$	$3.0 \cdot 10^{-5} s$	1 KB
8	4	1.1	$7.5 \cdot 10^5 / s$	$1.3 \cdot 10^{-4} s$	4 KB
10	5	1.3	$5.1 \cdot 10^5 / s$	$6.0 \cdot 10^{-4} s$	16 KB
12	6	1.6	$4.1 \cdot 10^5 / s$	$2.7 \cdot 10^{-3} s$	64 KB
14	7	1.8	$3.2 \cdot 10^5 / s$	$1.2 \cdot 10^{-2} s$	256 KB
16	8	2.1	$2.6 \cdot 10^5 / s$	$5.3 \cdot 10^{-2} s$	1 MB
18	9	2.5	$2.1 \cdot 10^5 / s$	$2.3 \cdot 10^{-1} s$	4 MB
20	10	2.8	$1.4 \cdot 10^5 / s$	$1.1 \cdot 10^0 s$	16 MB
22	11	3.2	$1.0 \cdot 10^5 / s$	$4.7 \cdot 10^0 s$	64 MB
24	12	3.7	$8.6 \cdot 10^4 / s$	$2.1 \cdot 10^1 s$	256 MB
26	13	4.2	$6.9 \cdot 10^4 / s$	$9.5 \cdot 10^1 s$	1 GB
28	14	4.8	$5.9 \cdot 10^4 / s$	$4.4 \cdot 10^2 s$	4 GB
30	15	5.3	$5.1 \cdot 10^4 / s$	$1.9 \cdot 10^3 s$	16 GB
32	16	6.3	$4.3 \cdot 10^4 / s$	$8.7 \cdot 10^3 s$	64 GB
34	17	7.2	$3.6 \cdot 10^4 / s$	$3.9 \cdot 10^4 s$	256 GB

Table 1: Benchmark of Feynman integral evaluations with different numbers of edges.

[M. Borinsky, arXiv : [2008.12310](https://arxiv.org/abs/2008.12310)]

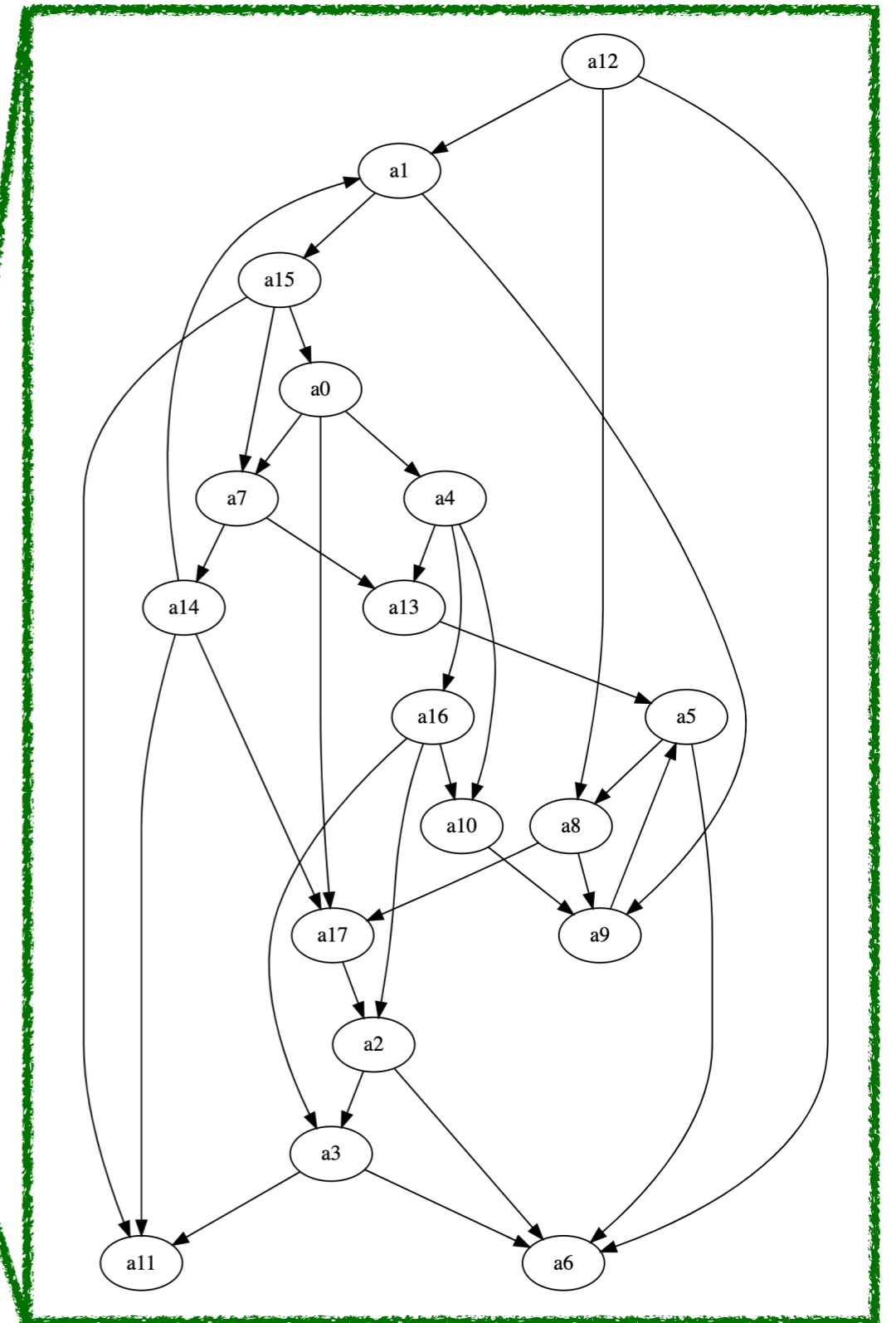
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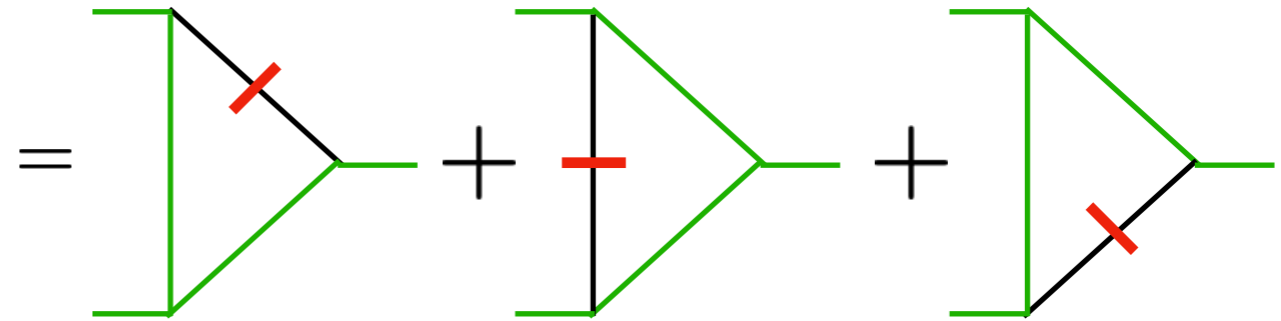
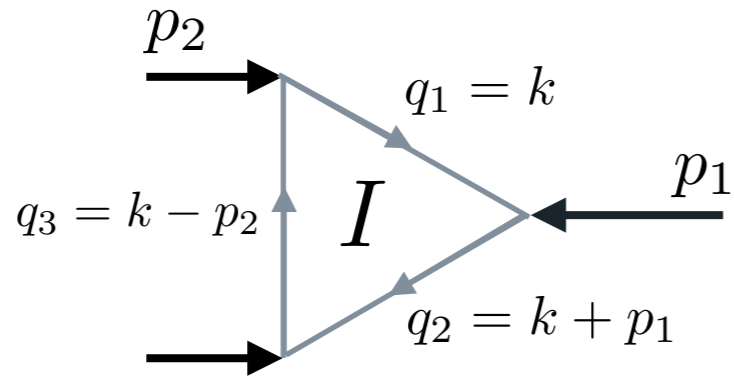


TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

[Inspired from and in collaboration with M. Borinsky, arXiv : [2008.12310](https://arxiv.org/abs/2008.12310)]

[Investigation from **Mathijs Fraije**]

Recall:

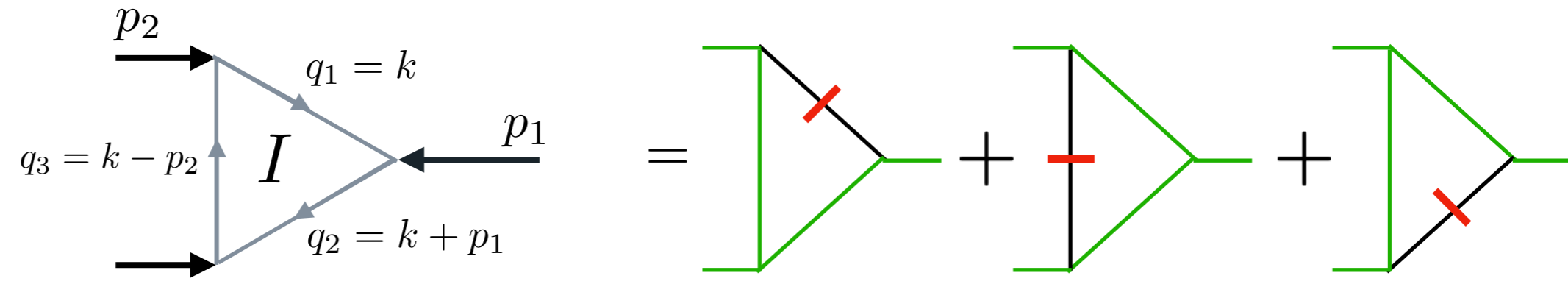


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[Inspired from and in collaboration with M. Borinsky, arXiv : [2008.12310](https://arxiv.org/abs/2008.12310)]

[Investigation from Mathijs Fraije]

Recall:



$$= \int d^4k \frac{1}{D_1 D_2 D_3} (D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3)) \supset \int d^3\vec{k} \frac{1}{E_1 E_2 E_3} \frac{1}{E_1 + E_2 - p_1^0} \frac{1}{E_1 + E_2 + p_1^0}$$

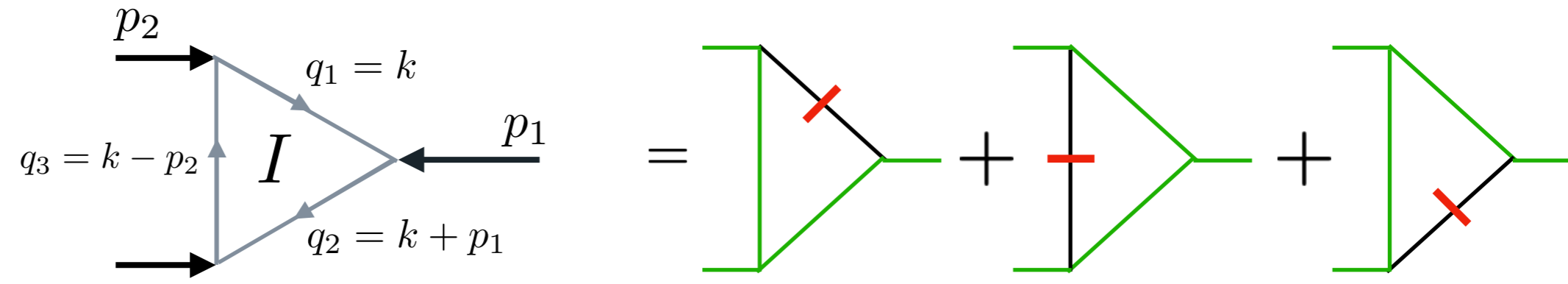
With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \stackrel{m=0}{=} |\vec{q}_i|$.

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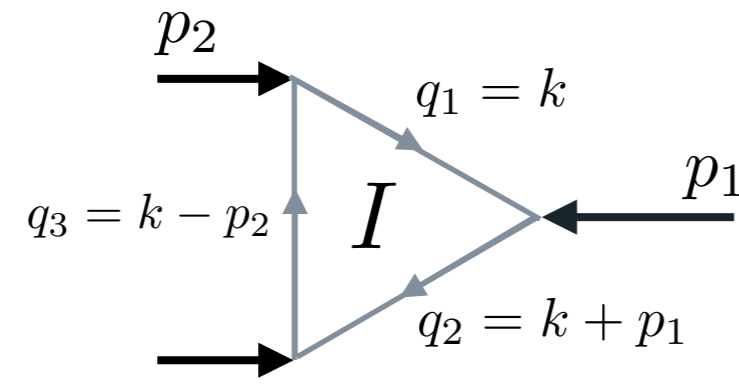
With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \stackrel{m=0}{=} |\vec{q}_i|$. The prefactor $\frac{1}{E_1 E_2 E_3}$ contains point-like integrable singularities :

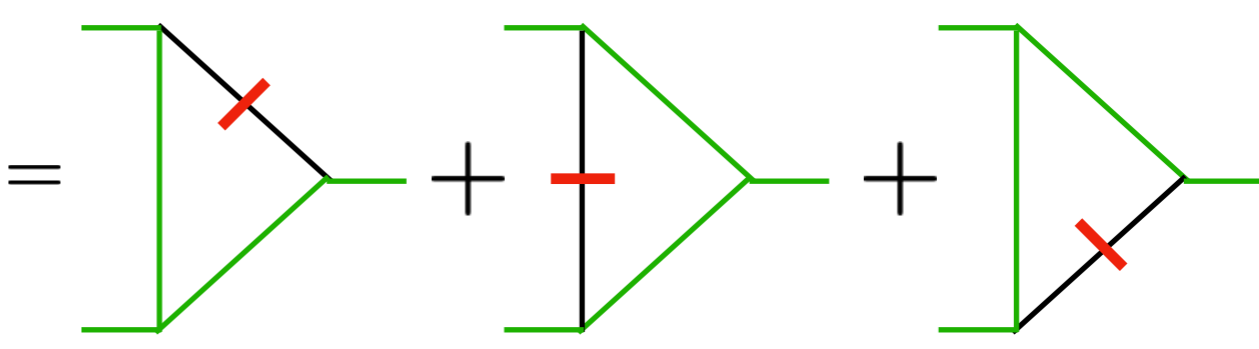
TROPICAL SAMPLING OF LOOP-TREE DUALITY INTEGRANDS

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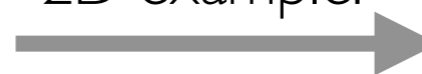


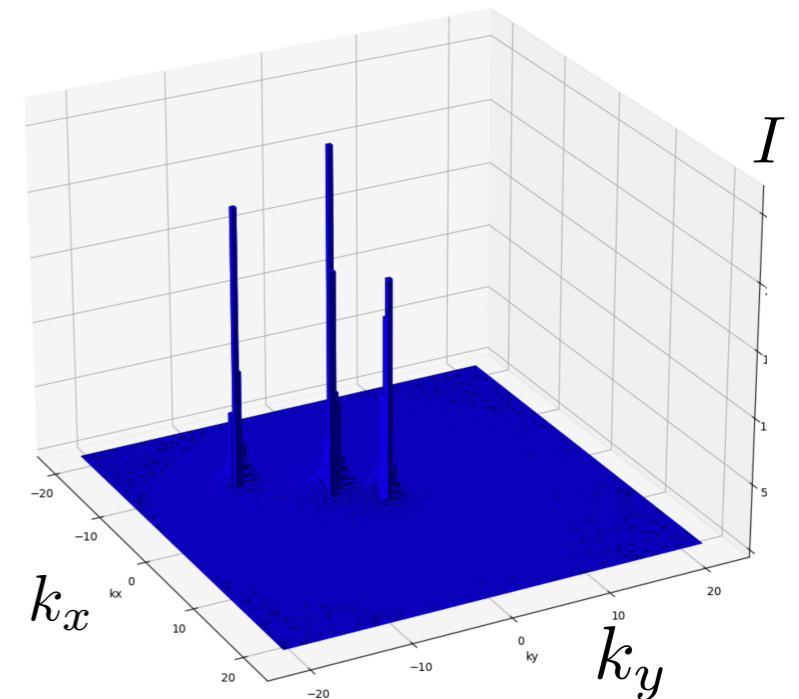
$$=$$


$$= \int d^4k \frac{1}{D_1 D_2 D_3} (D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3)) \supset \int d^3\vec{k} \frac{1}{E_1 E_2 E_3} \frac{1}{E_1 + E_2 - p_1^0} \frac{1}{E_1 + E_2 + p_1^0}$$

With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \stackrel{m=0}{=} |\vec{q}_i|$. The prefactor $\frac{1}{E_1 E_2 E_3}$ contains point-like integrable singularities :

$$\frac{1}{E_1 E_2 E_3} = \frac{1}{|\vec{k}| |\vec{k} + \vec{p}_1| |\vec{k} - \vec{p}_2|}$$

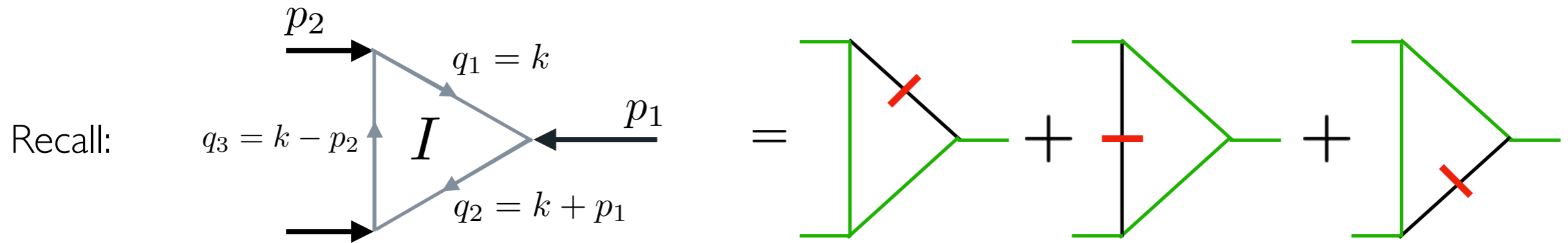
2D example: 



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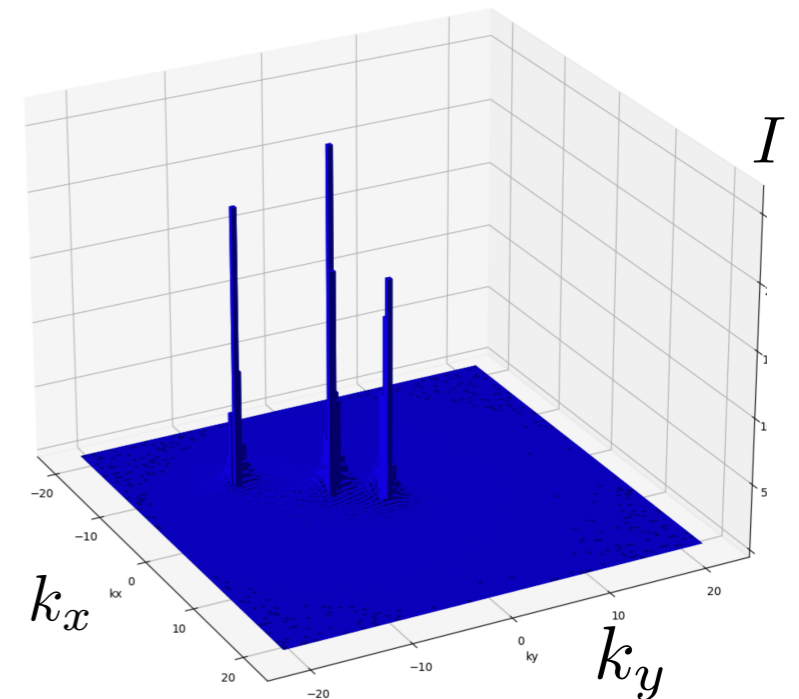


$$= \int d^4k \frac{1}{D_1 D_2 D_3} (D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3)) \supset \int d^3\vec{k} \frac{1}{E_1 E_2 E_3} \frac{1}{E_1 + E_2 - p_1^0} \frac{1}{E_1 + E_2 + p_1^0}$$

With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \stackrel{m=0}{=} |\vec{q}_i|$. The prefactor $\frac{1}{E_1 E_2 E_3}$ contains point-like integrable singularities :

$$\frac{1}{E_1 E_2 E_3} = \frac{1}{|\vec{k}| |\vec{k} + \vec{p}_1| |\vec{k} - \vec{p}_2|}$$

2D example:



[Originates from $\frac{1}{(k^0 - E(\vec{k})) (k^0 + E(\vec{k}))} = \frac{1}{2E(\vec{k})} \left(\frac{1}{E(\vec{k}) - k^0} + \frac{1}{E(\vec{k}) + k^0} \right)$]

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simultaneously at all three points $\vec{k} = \{ \vec{0}, -\vec{p}_1, \vec{p}_2 \}$:

$$= \int d^3 \vec{k} \frac{1}{D_1^{\frac{1}{2}} D_2^{\frac{1}{2}} D_3^{\frac{1}{2}}} f(\vec{k})$$

with **euclidean** propagators :

$$D_1 = \vec{k} \cdot \vec{k}$$

$$D_2 = (\vec{k} + \vec{p}_1) \cdot (\vec{k} + \vec{p}_1)$$

$$D_3 = (\vec{k} - \vec{p}_2) \cdot (\vec{k} - \vec{p}_2)$$

Ready for being tropical-sampled !

$$= \int_{[0,1]^{(DL+2|E|+1)}} dz \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

TROPICAL SAMPLING IN MOMENTUM SPACE

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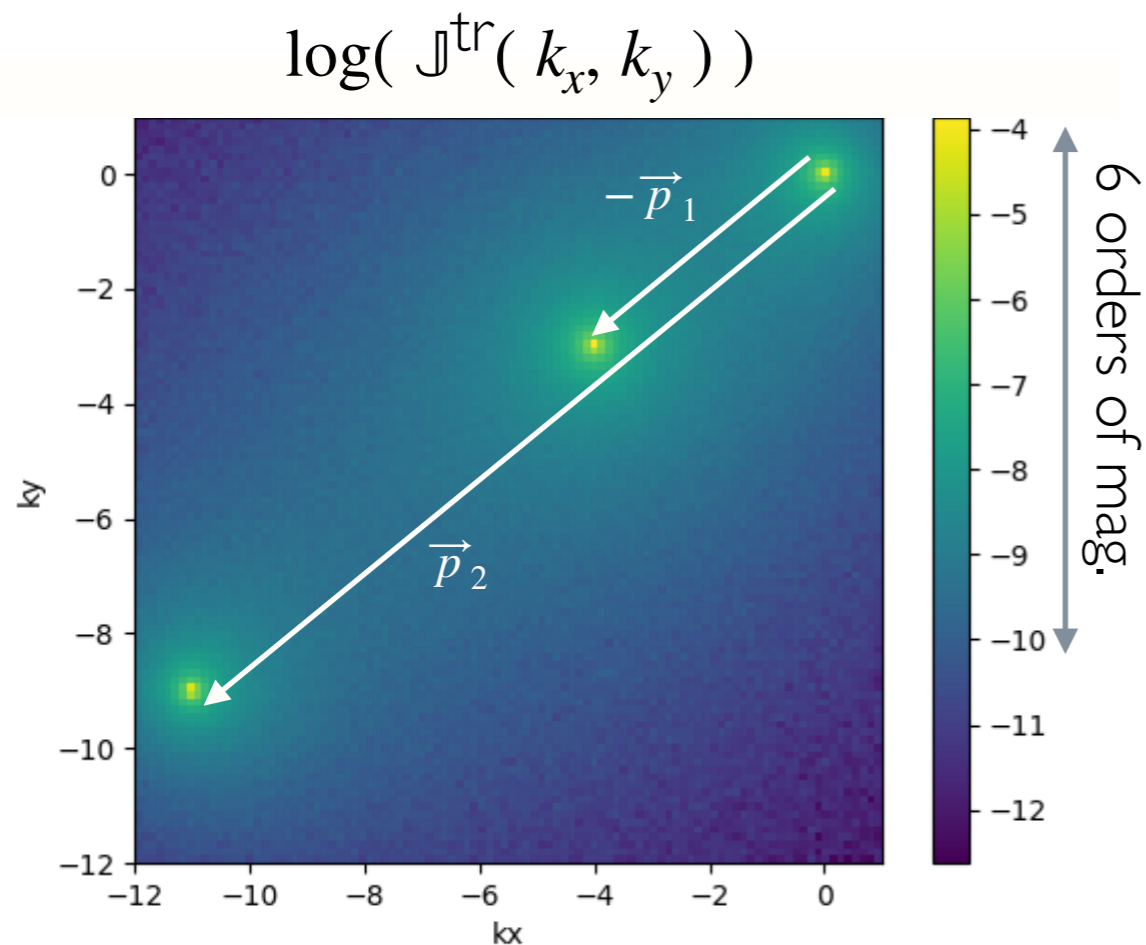
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- M. Fraije and M. Borinsky will soon publish a **Rust** implementation of the map: $\phi^{\text{tr}}[\Gamma](\mathbf{z}) \rightarrow (K(\mathbf{z}), \mathbf{k}(\mathbf{z}))$

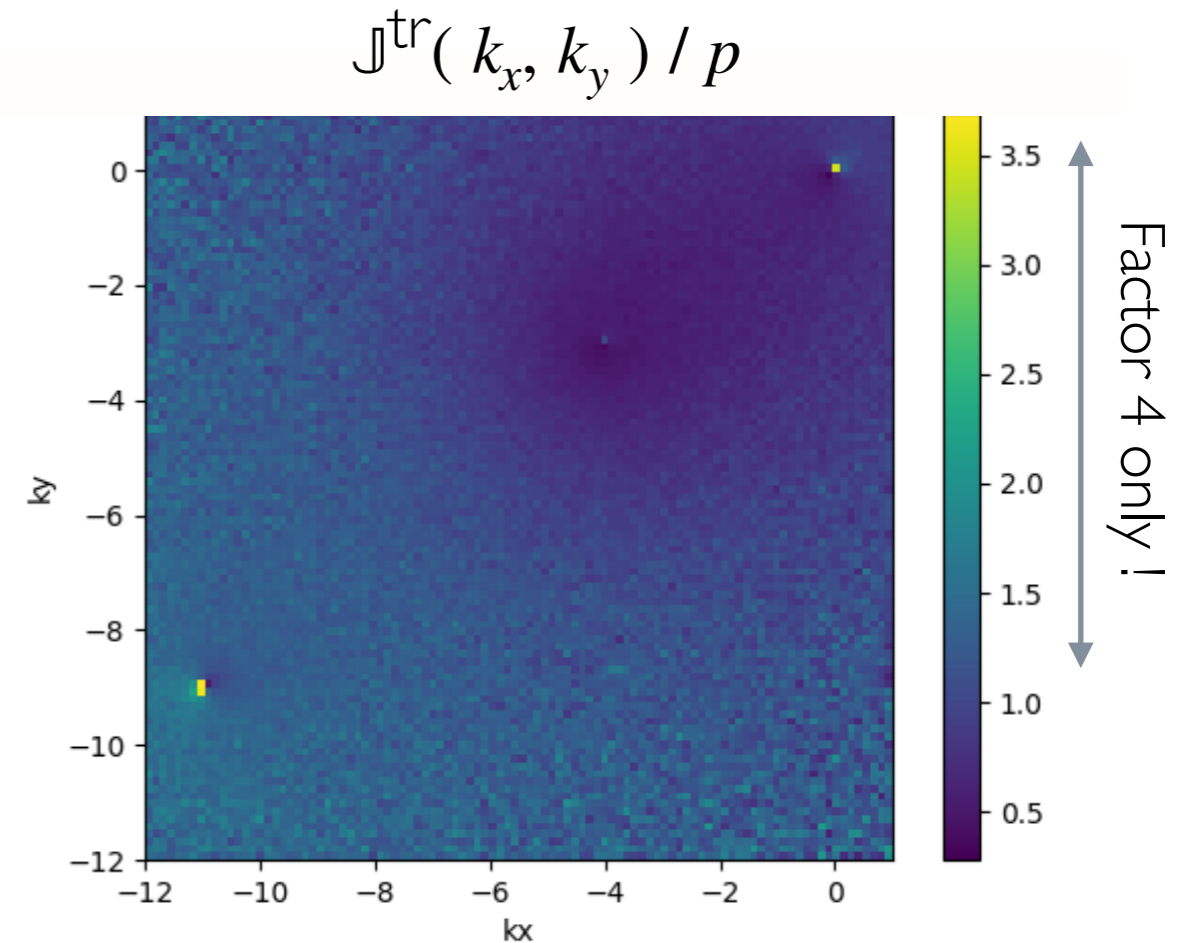
TROPICAL SAMPLING OF THE LTD TRIANGLE INTEGRAL

Let's apply this approach to our example.

In 2D with tropical sampling for removing blow ups from $p := \frac{1}{|\vec{k}||\vec{k} + \vec{p}_1||\vec{k} - \vec{p}_2|}$:



Log of tropical \sim sampling density

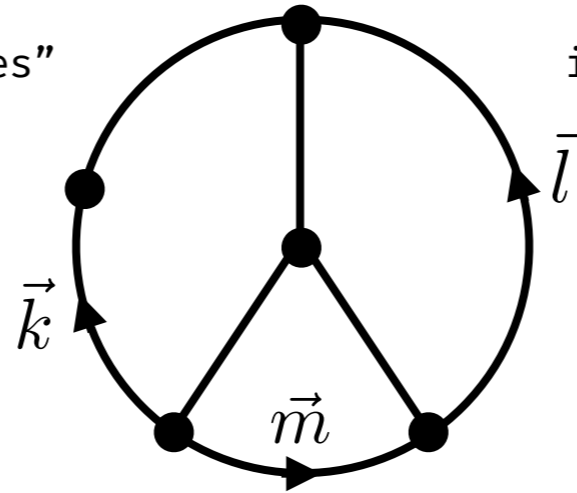


Integrable singularities conquered !

- Many details of this approach omitted here. In our implementation: **arbitrary numerators** supported!

TROPICAL SAMPLING RESULTS

“scalar vacuum mercedes”



internal mass: $m = 1$

SAMPLING STRATEGY	# DIM.	IM. CENTRAL VALUE	PULL
Analytic result	-	$-3.0525714828514e-7$	ref.
Prod. of spherical params [†]	9	$-2.990(50)e-7$	-1.2σ

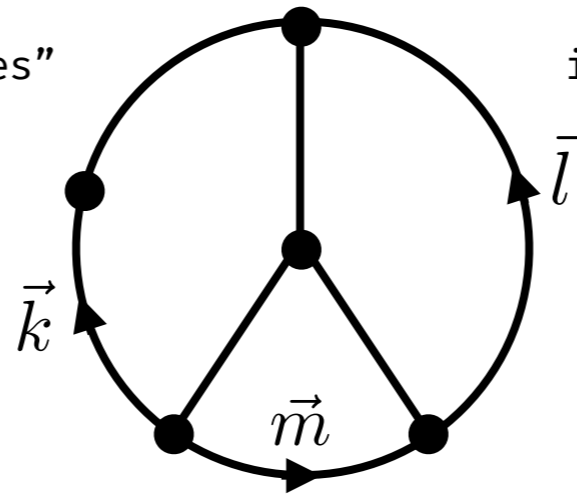
$$\int [d\Omega_k |\vec{k}|^2 d|\vec{k}|] \times [d\Omega_l |\vec{l}|^2 d|\vec{l}|] \times [d\Omega_m |\vec{m}|^2 d|\vec{m}|]$$

[†]: 5 x 2M samples, 10 μs/sample

Credits: **Mathijs Fraaije**

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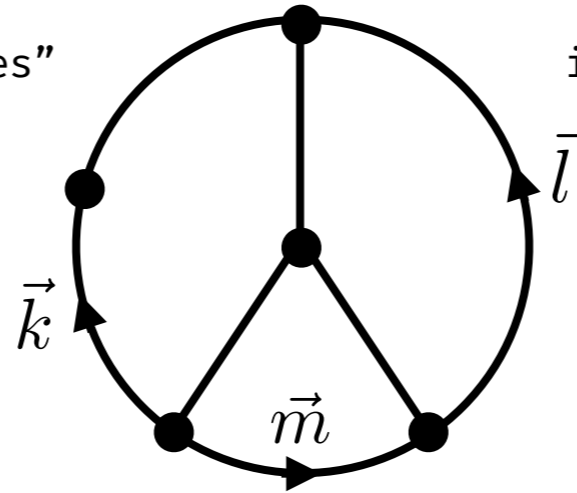
$$1 = \sum_{\text{lmb}} \left[\frac{\prod_{i \in \text{lmb}} E_i^{-\alpha}}{\left(\sum_{\text{lmb}'} \prod_{j \in \text{lmb}'} E_j^{-\alpha} \right)} \right]$$

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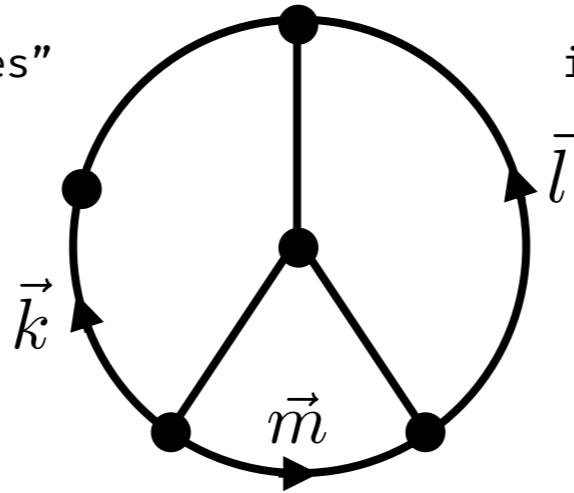
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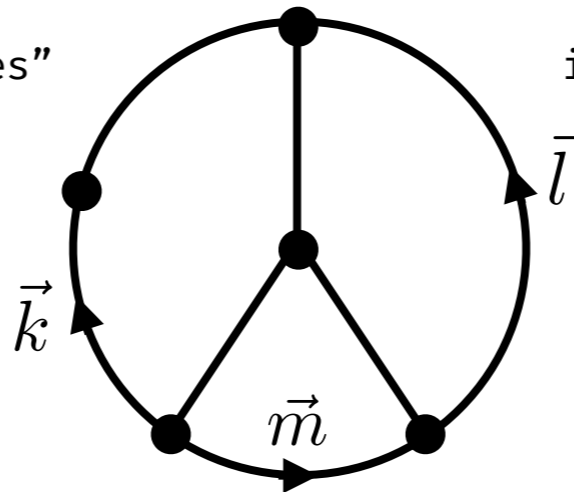
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Neural Importance Sampling	dep. on sampling	Theo Heime1 prelim: <u>2x+ var. reduction !</u>	\o/

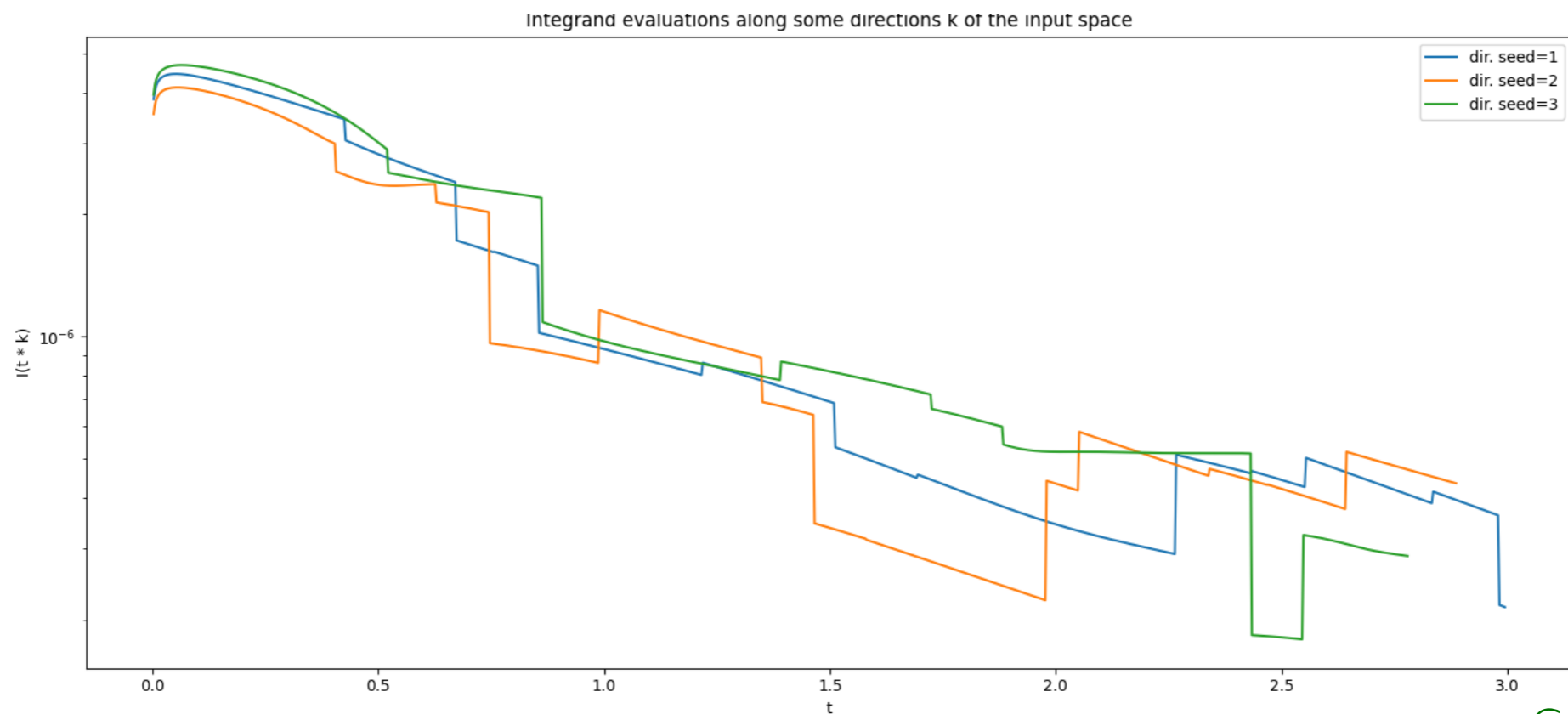
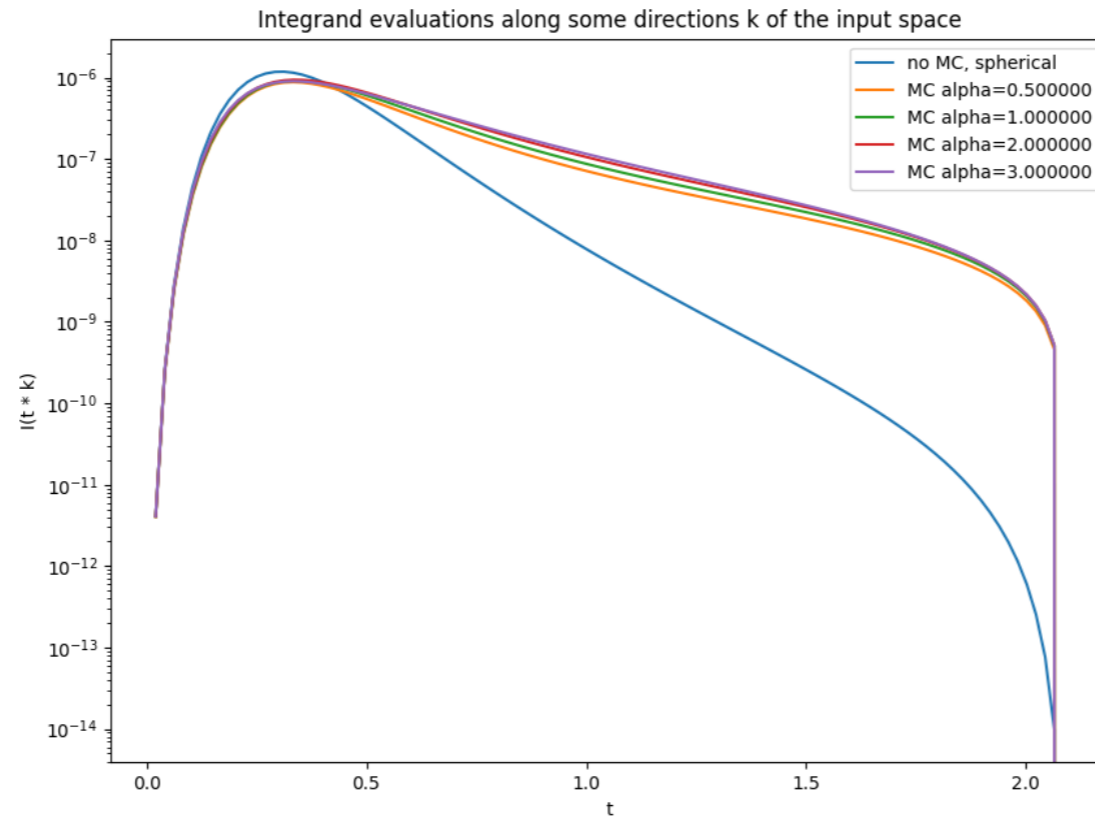
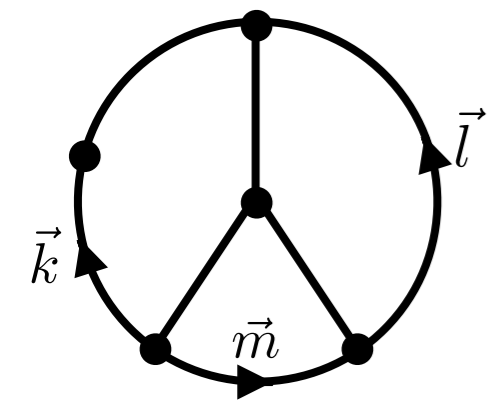
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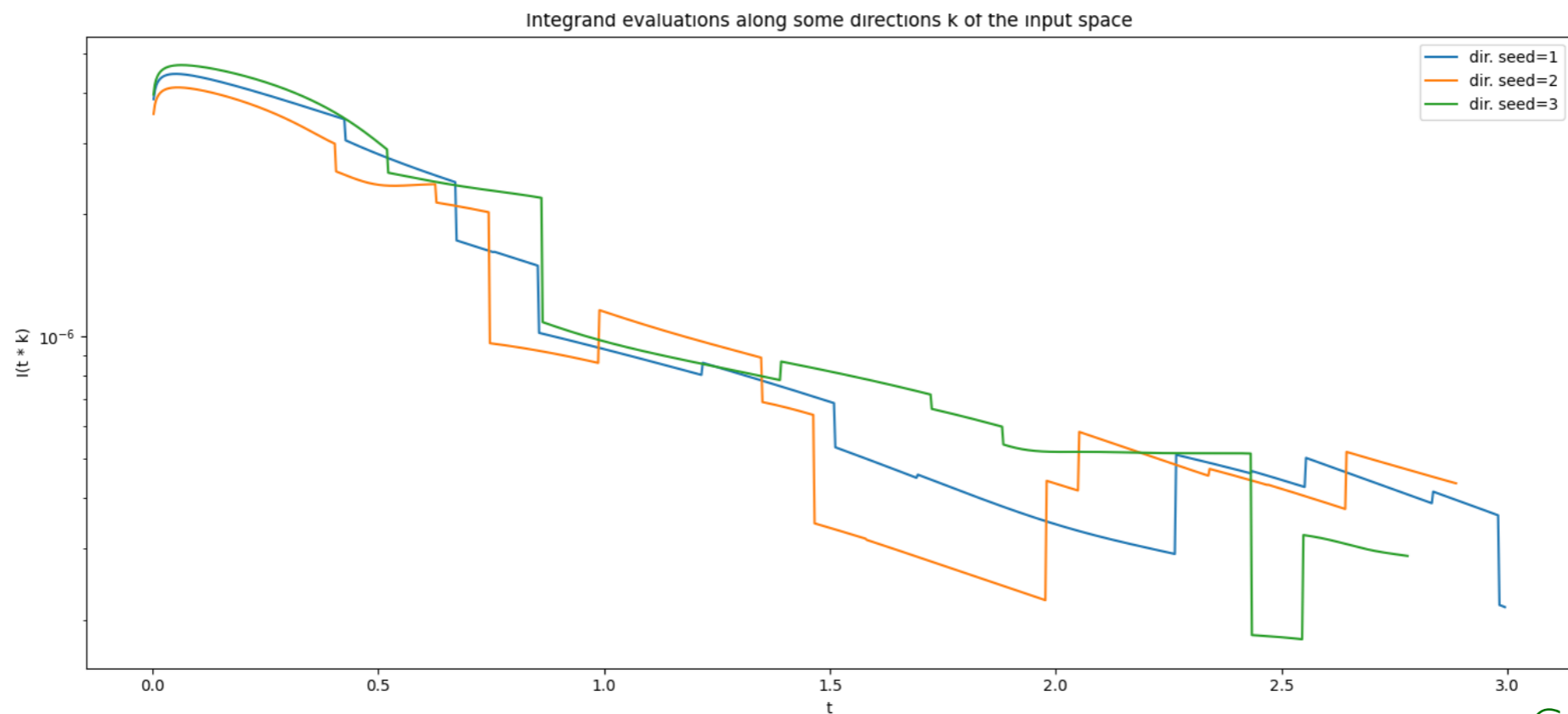
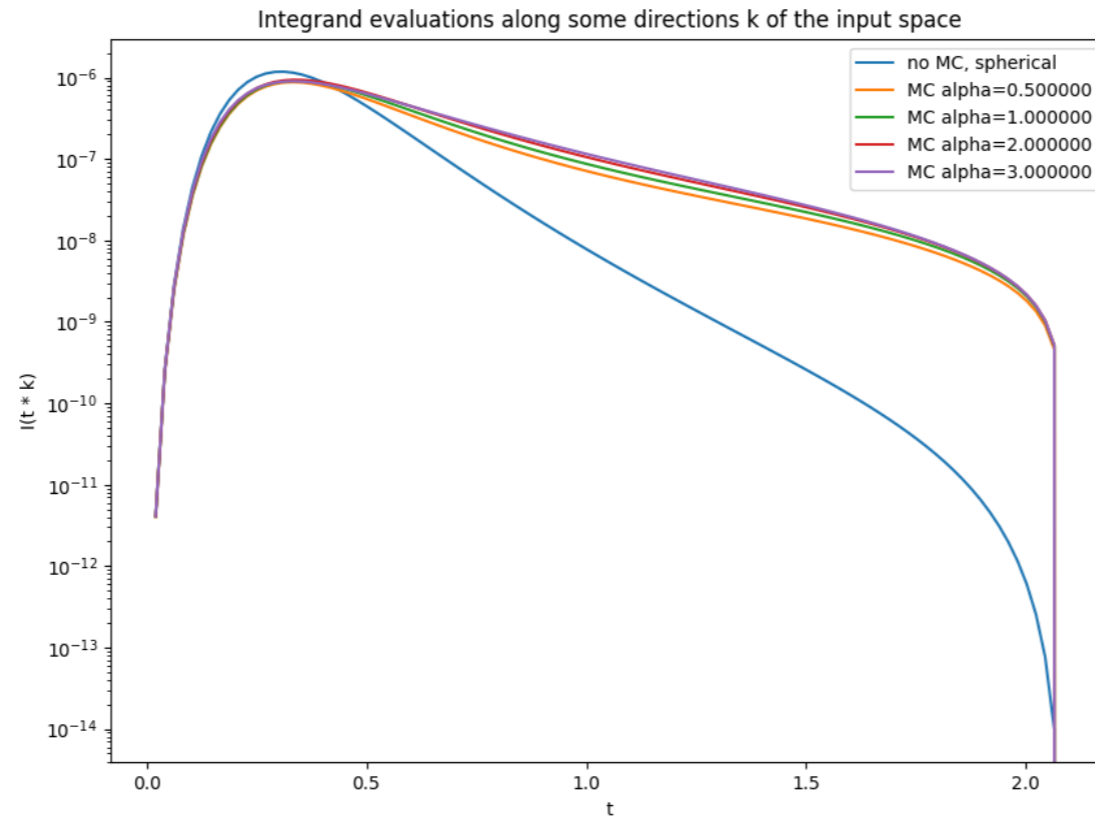
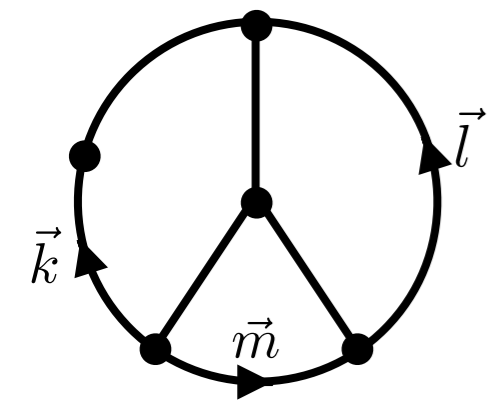
Credits: **Mathijs Fraaije**

TROPICAL SAMPLING PDF



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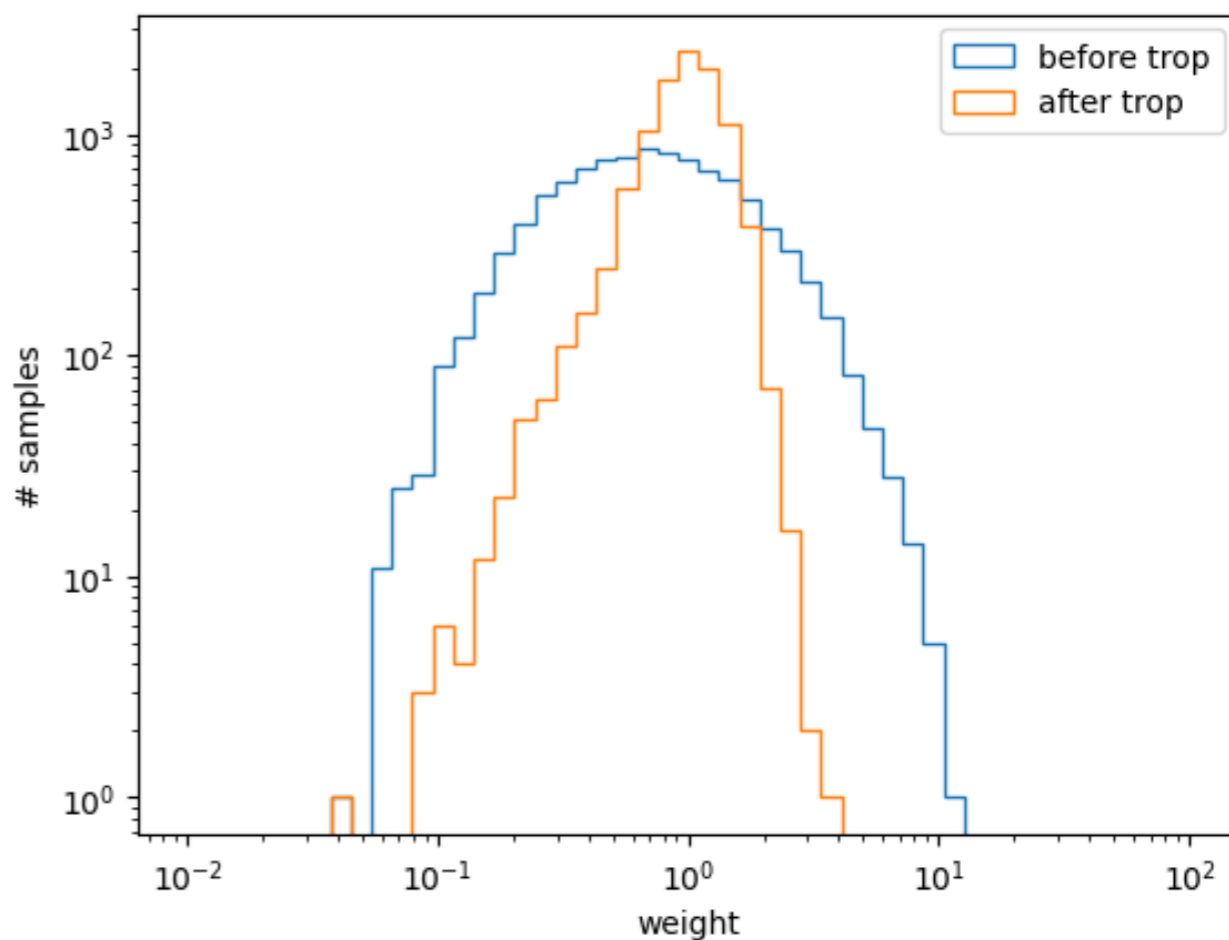
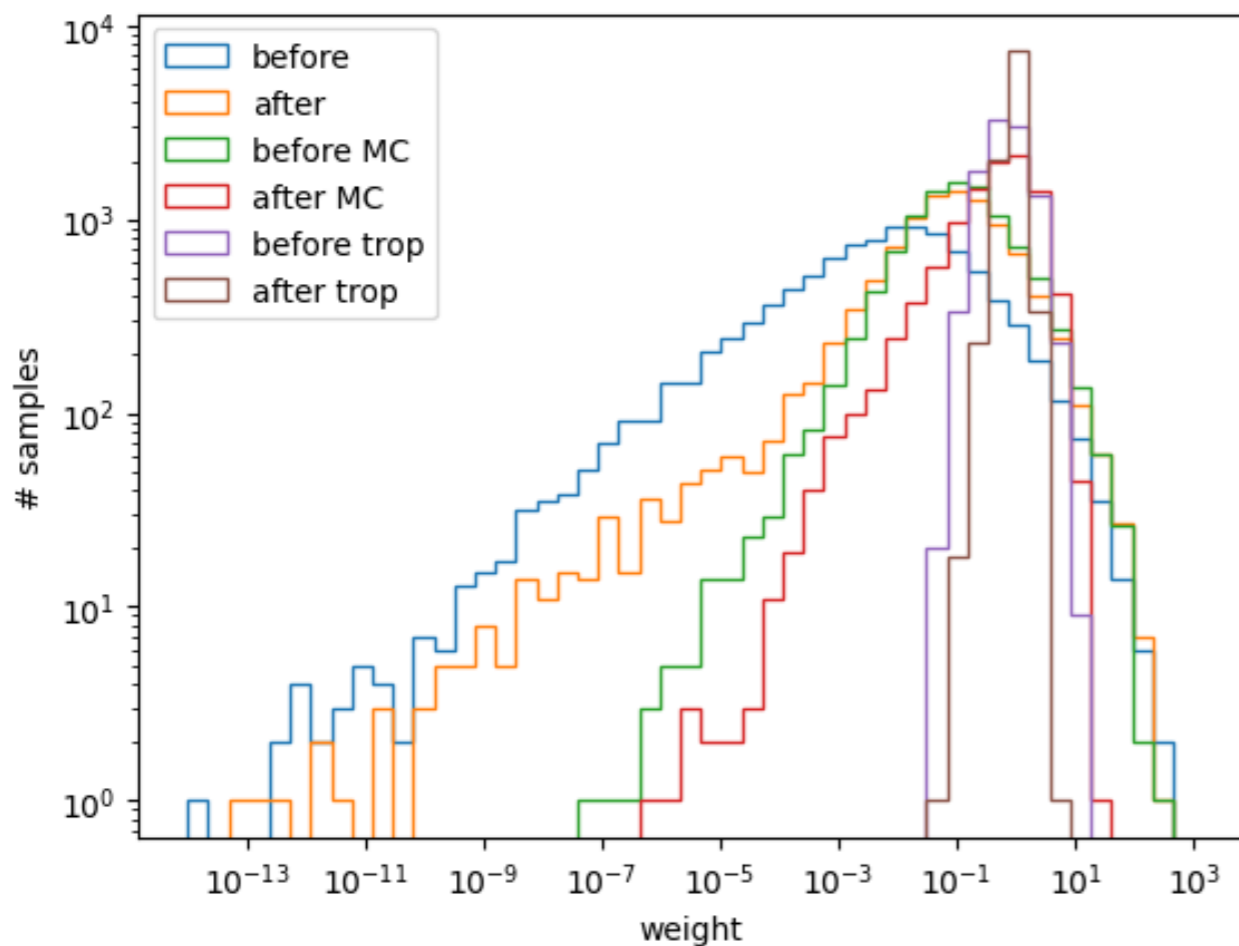
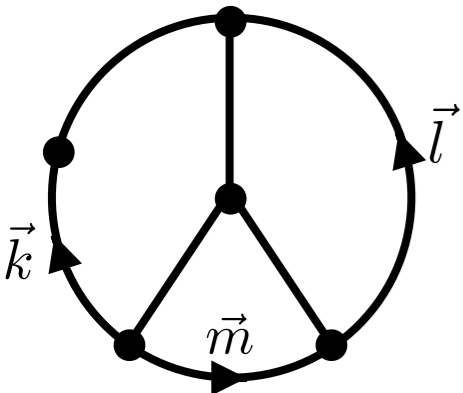
TROPICAL SAMPLING PDF



Credits: **Mathijs Fraaije**

NEURAL IMPORTANCE SAMPLING

PRELIMINARY

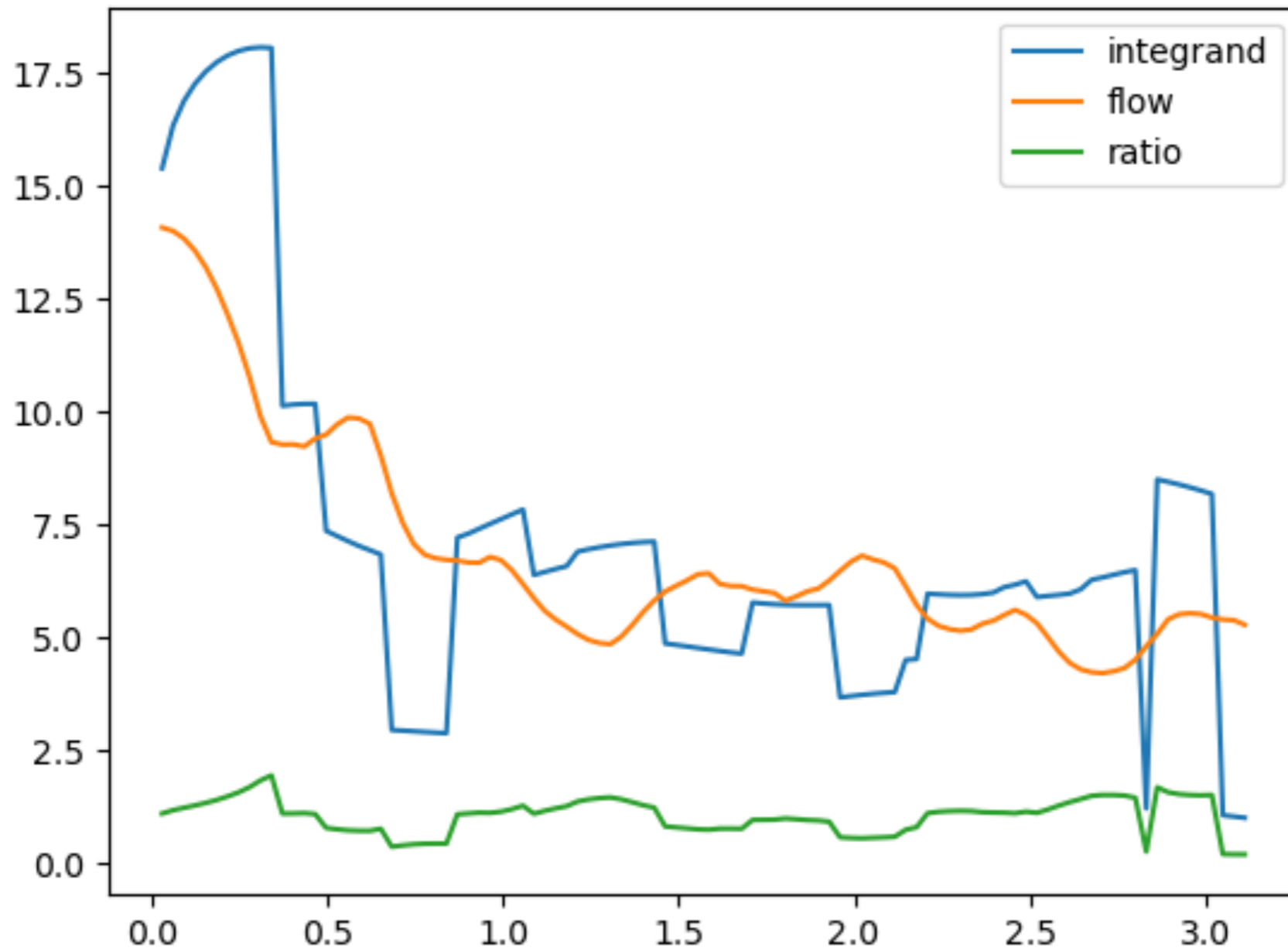
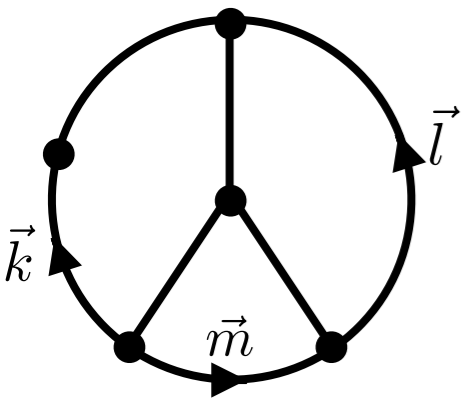


EASY ~2X VARIANCE REDUCTION !

Credits: **Theo Heime**

NEURAL IMPORTANCE SAMPLING

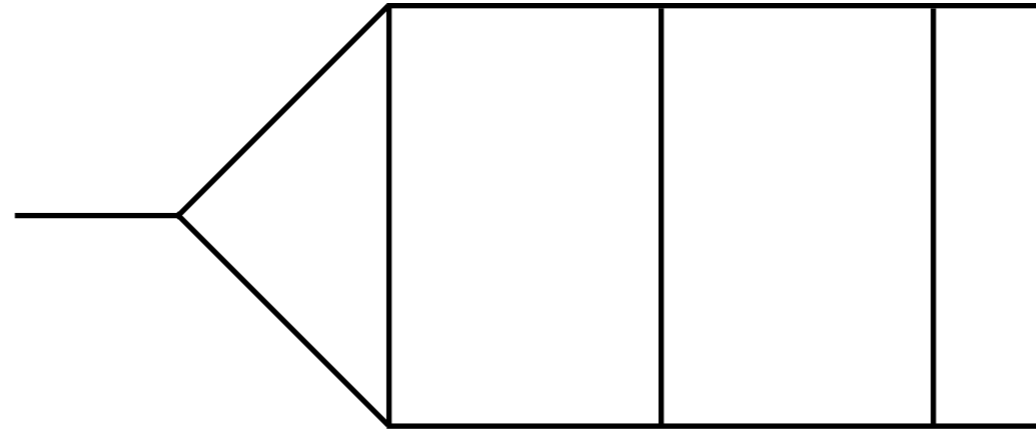
PRELIMINARY



NIS SEEMINGLY FINE WITH IRREGULAR INTEGRAND!

Credits: **Theo Heimel**

TROPICAL SAMPLING RESULTS

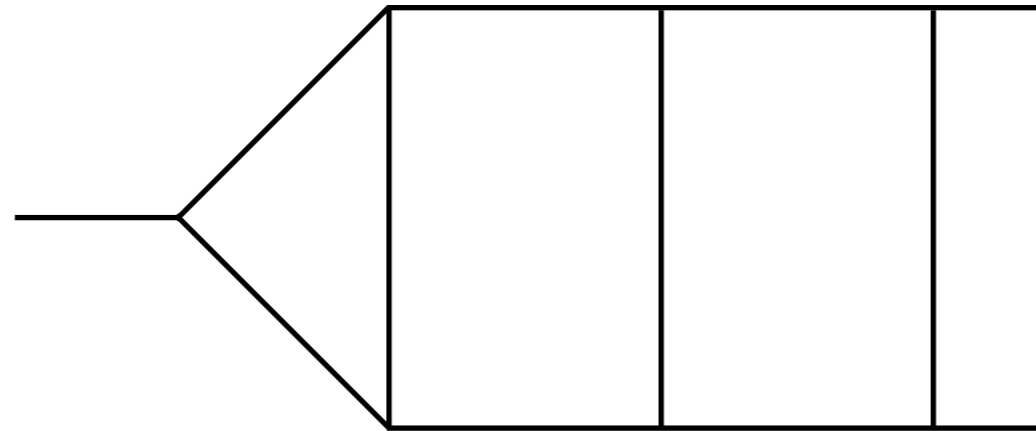


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/18$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	3.51(63)e-8 18%	
0.1M 0.3 s	3.78(24)e-8 6%	
1M 3 s	3.99(11)e-8 2.7%	
10M 30 s	4.045(36)e-8 0.9%	

Credits: Mathijs Fraaije

TROPICAL SAMPLING RESULTS

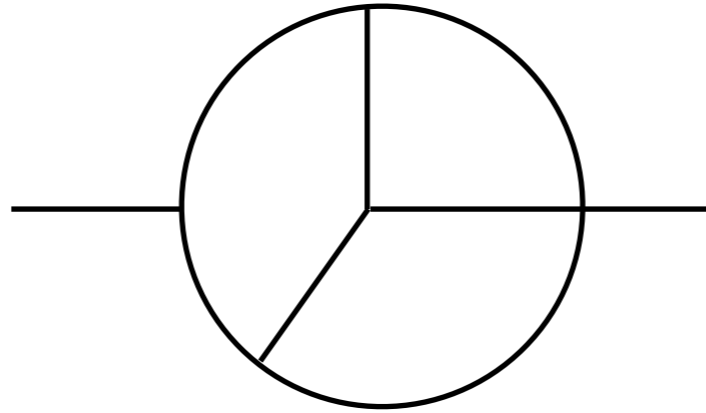


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/18$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	3.51(63)e-8 18%	4.050(35)e-8 0.9%
0.1M 0.3 s	3.78(24)e-8 6%	4.030(11)e-8 0.3%
1M 3 s	3.99(11)e-8 2.7%	4.0379(35)e-8 0.09%
10M 30 s	4.045(36)e-8 0.9%	4.0358(11)e-8 0.03%

Credits: Mathijs Fraaije

TROPICAL SAMPLING RESULTS

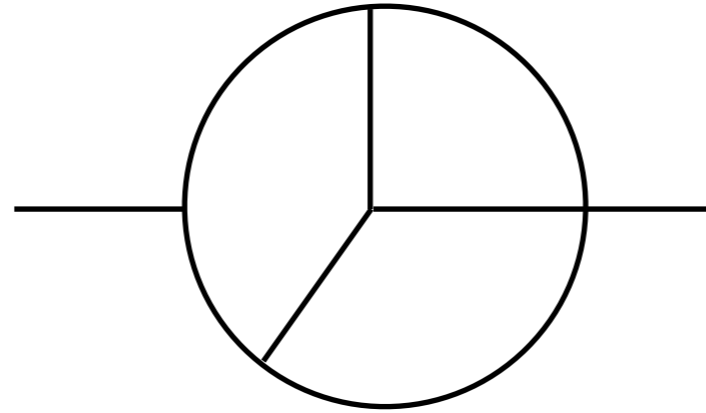


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/14$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	8.9(2.9)e-7 33%	
0.1M 0.3 s	3.5(1.0)e-6 29%	
1M 3 s	5.6(1.2)e-6 22%	
10M 30 s	1.23(41)e-5 34%	

Credits: Mathijs Fraaije

TROPICAL SAMPLING RESULTS

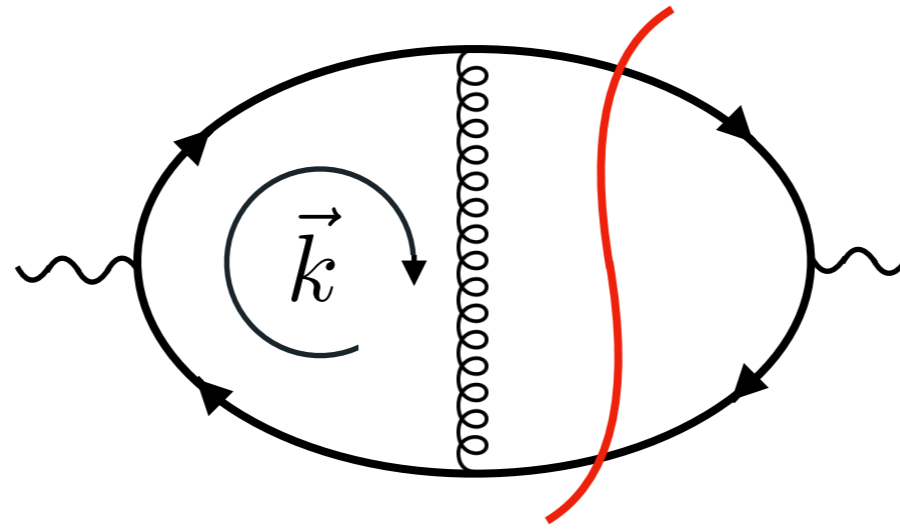


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/14$)

N SAMPLES	NAIVE	TROPICAL
0.01M 30 ms	8.9(2.9)e-7 33%	9.403(60)e-6 0.6%
0.1M 0.3 s	3.5(1.0)e-6 29%	9.518(19)e-6 0.2%
1M 3 s	5.6(1.2)e-6 22%	9.4984(60)e-6 0.06%
10M 30 s	1.23(41)e-5 34%	9.4986(19)e-6 0.02%

Credits: Mathijs Fraaije

LOCALISED RENORMALISATION: BPHZ



$$\lim_{|\vec{k}| \rightarrow \infty} I(\text{Local Unitarity}) \rightarrow \infty$$

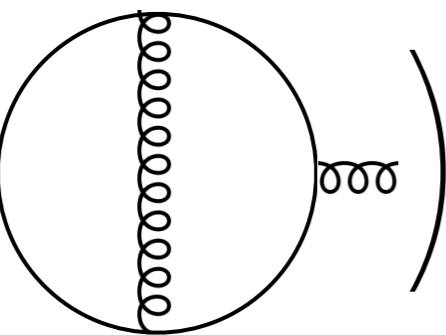
LOCALISED RENORMALISATION

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ [refs.](#)]

$$R(\Gamma) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$

LOCALISED RENORMALISATION

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ [refs.](#)]

$$R \left(\Gamma = \text{diagram} \right) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$


LOCALISED RENORMALISATION

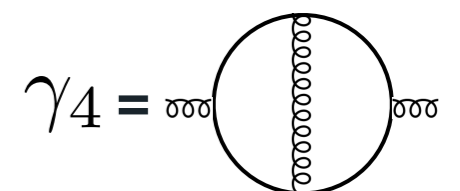
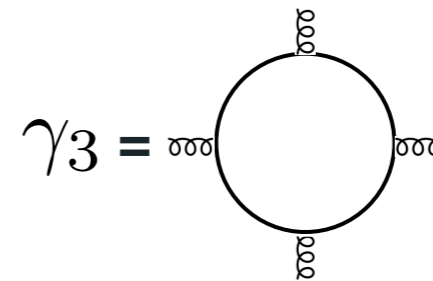
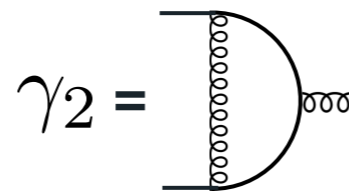
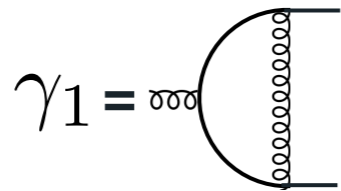
[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ [refs.](#)]

$$R \left(\Gamma = \text{circle with vertical wavy line} \right) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$

UV subgraphs :

$$\text{dod}(\gamma_{\{1,2,3\}}) = 0$$

$$\text{dod}(\gamma_4) = 2$$



LOCALISED RENORMALISATION

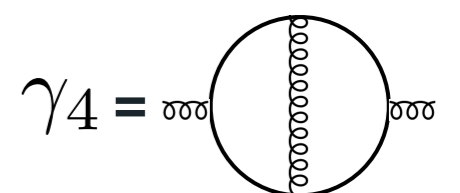
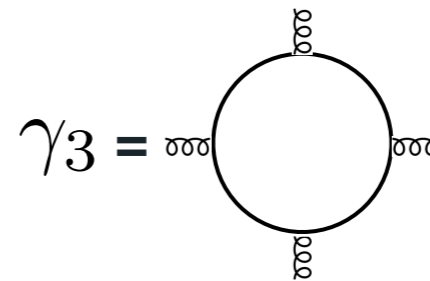
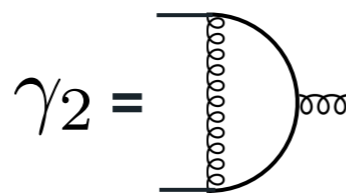
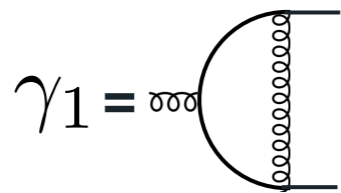
[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ [refs.](#)]

$$R \left(\Gamma = \text{circle with vertical wavy line} \right) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$

UV subgraphs :

$$\text{dod}(\gamma_{\{1,2,3\}}) = 0$$

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$$R(\Gamma) = \Gamma - K(\gamma_1) * \Gamma \setminus \gamma_1 - K(\gamma_2) * \Gamma \setminus \gamma_2 - K(\gamma_3) * \Gamma \setminus \gamma_3 - K(\gamma_4) * \Gamma \setminus \gamma_4 \\ + K(K(\gamma_1) * \Gamma \setminus \gamma_1) + K(K(\gamma_2) * \Gamma \setminus \gamma_2) + K(K(\gamma_3) * \Gamma \setminus \gamma_3)$$

LOCALISED RENORMALISATION

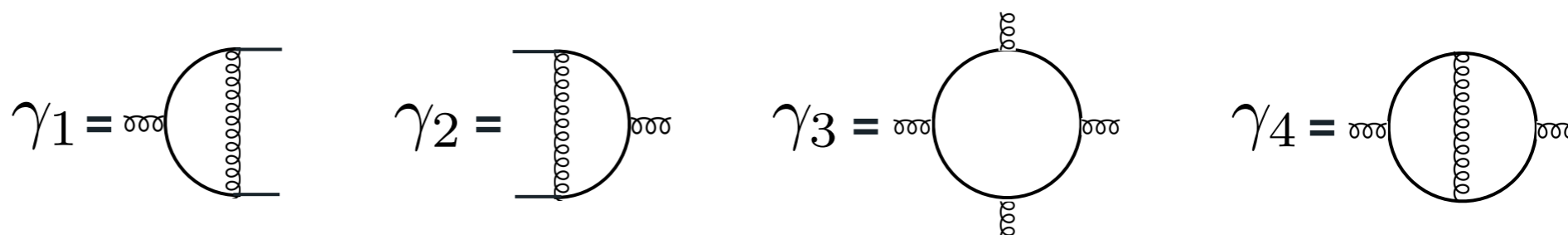
[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ [refs.](#)]

$$R \left(\Gamma = \text{diagram} \right) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$

UV subgraphs :

$$\text{dod}(\gamma_{\{1,2,3\}}) = 0$$

$$\text{dod}(\gamma_4) = 2$$



$$R(\Gamma) = \Gamma - K(\gamma_1) * \Gamma \setminus \gamma_1 - K(\gamma_2) * \Gamma \setminus \gamma_2 - K(\gamma_3) * \Gamma \setminus \gamma_3 - K(\gamma_4) * \Gamma \setminus \gamma_4$$

$$+ K(K(\gamma_1) * \Gamma \setminus \gamma_1) + K(K(\gamma_2) * \Gamma \setminus \gamma_2) + K(K(\gamma_3) * \Gamma \setminus \gamma_3)$$

What is the operator $K(\gamma)$? Anything we want ! so long as it:

- Locally cancels UV divergences of γ , even in the presence of nestings
- Yields results immediately renormalised in the chosen scheme ($\overline{\text{MS}} + \text{OS}$)
- Minimal analytics: at most single-scale all-massive vacuum integrals

LOCAL RENORMALISATION OPERATOR K

Our solution: $K(\gamma) := T(\gamma)$

$T(\gamma) :=$ **Local CT** : Taylor expansion around the “UV point” up to $\text{dod}(\gamma)$

LOCAL RENORMALISATION OPERATOR K

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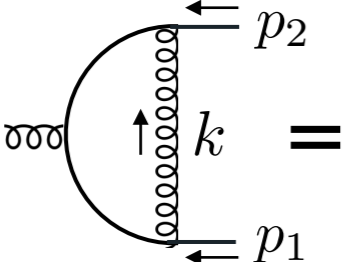
$$\gamma_1 = \text{diagram} = \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{((k - p_1)^2 - m^2)(k^2)((k + p_2)^2 - m^2)}$$

The diagram shows a loop with a wavy internal line. The loop is bounded by a vertical line on the right and a curved line on the left. The vertical line has an upward arrow labeled k . The curved line has an upward arrow labeled k . The top horizontal line has a leftward arrow labeled p_2 . The bottom horizontal line has a leftward arrow labeled p_1 . The left side of the loop is labeled with three small circles.

LOCAL RENORMALISATION OPERATOR K

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$$\gamma_1^\lambda := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k - \lambda p_1)^2 - m_{UV}^2 - \lambda^2(m^2 - m_{UV}^2)}(k^2 - m_{UV}^2)((k + \lambda p_2)^2 - m_{UV}^2 - \lambda^2(m^2 - m_{UV}^2))$$

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$$T(\gamma) = T_{\text{dod}(\gamma)}(\gamma^\lambda) = \sum_{j=0}^{\text{dod}(\gamma)} \frac{1}{j!} \frac{d^j}{d\lambda^j} \gamma^\lambda \Big|_{\lambda=0}, \quad T_0(\gamma_1) = \frac{\mathcal{N}(k, 0, 0, 0)}{(k^2 - m_{UV}^2)^3} \sim \text{Diagram}$$

LOCAL RENORMALISATION OPERATOR K

Our solution: $K(\gamma) := T(\gamma) - [T](\gamma)$

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$$\gamma_1^\lambda := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k - \lambda p_1)^2 - m_{UV}^2 - \lambda^2(m^2 - m_{UV}^2)}(k^2 - m_{UV}^2)((k + \lambda p_2)^2 - m_{UV}^2 - \lambda^2(m^2 - m_{UV}^2))$$

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$$[T](\gamma) := \text{Integrated CT}, \quad [T](\gamma_1) = \left(\frac{\mu_r^2}{4\pi e^{-\gamma_E}} \right)^\epsilon \int d^{4-2\epsilon} k \text{Diagram} = \sum_{k=-\infty}^{+\infty} \alpha_k \epsilon^k$$

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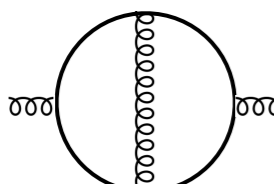
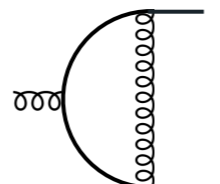
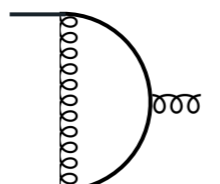
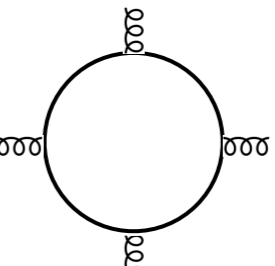
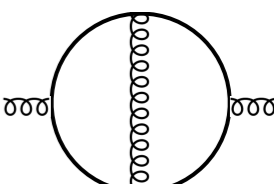
$$\gamma_1^\lambda := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k - \lambda p_1)^2 - m_{\text{UV}}^2 - \lambda^2(m^2 - m_{\text{UV}}^2)}(k^2 - m_{\text{UV}}^2)((k + \lambda p_2)^2 - m_{\text{UV}}^2 - \lambda^2(m^2 - m_{\text{UV}}^2))$$

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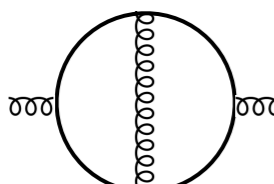
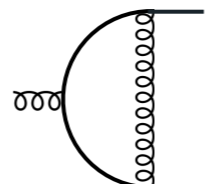
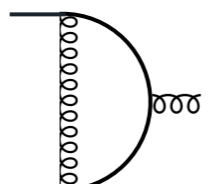
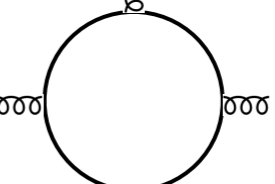
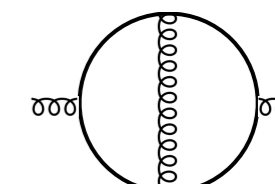
$$\delta^X(\gamma) := \text{Renormalisation CT in scheme X}, \quad (-[T] + \delta^{\overline{\text{MS}}}) := \bar{K}, \quad \bar{K}(\gamma_1) = \sum_{k=0}^{+\infty} \alpha_k \epsilon^k$$

R-OPERATOR UNFOLDING

$\Gamma =$  with UV subgraphs
 $\gamma_1 =$ 
 $\gamma_2 =$ 
 $\gamma_3 =$ 
 $\gamma_4 =$ 

$$\begin{aligned}
 R(\Gamma) = \Gamma & - T_0(\gamma_1) * \Gamma \setminus \gamma_1 & - T_0(\gamma_2) * \Gamma \setminus \gamma_2 & - T_0(\gamma_3) * \Gamma \setminus \gamma_3 & - T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\
 & + T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) & + T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) & + T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) & + \bar{K} \text{ terms}
 \end{aligned}$$

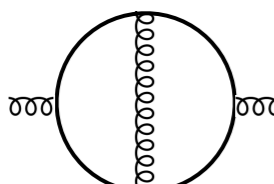
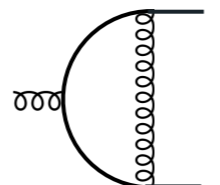
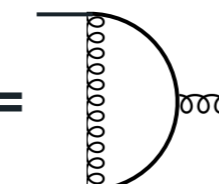
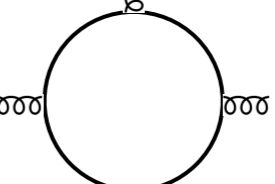
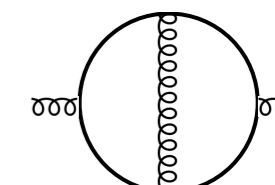
R-OPERATOR UNFOLDING

$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$\begin{aligned}
 R(\Gamma) = & \Gamma - T_0(\gamma_1) * \Gamma \setminus \gamma_1 - T_0(\gamma_2) * \Gamma \setminus \gamma_2 - T_0(\gamma_3) * \Gamma \setminus \gamma_3 - T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\
 & + T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}
 \end{aligned}$$

$$\begin{aligned}
 = & \text{Diagram } \Gamma - \text{Diagram } T_0(\gamma_1) * \Gamma \setminus \gamma_1 - \text{Diagram } T_0(\gamma_2) * \Gamma \setminus \gamma_2 - \text{Diagram } T_0(\gamma_3) * \Gamma \setminus \gamma_3 - \text{Diagram } T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\
 & + \text{Diagram } T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + \text{Diagram } T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + \text{Diagram } T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}
 \end{aligned}$$

R-OPERATOR UNFOLDING

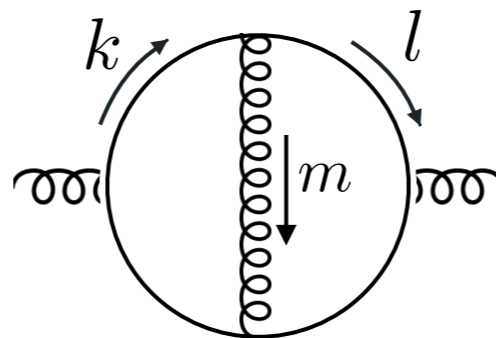
$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$\begin{aligned}
 R(\Gamma) = & \Gamma - T_0(\gamma_1) * \Gamma \setminus \gamma_1 - T_0(\gamma_2) * \Gamma \setminus \gamma_2 - T_0(\gamma_3) * \Gamma \setminus \gamma_3 - T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\
 & + T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}
 \end{aligned}$$

$$\begin{aligned}
 = & \text{Diagram 1} - \text{Diagram 2} - \text{Diagram 3} - \text{Diagram 4} - \text{Diagram 5} \\
 & + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \bar{K} \text{ terms}
 \end{aligned}$$

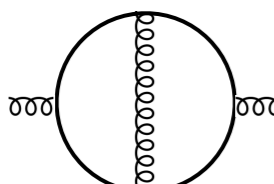
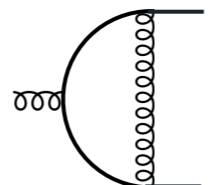
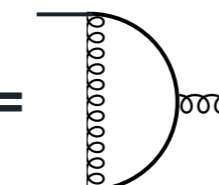
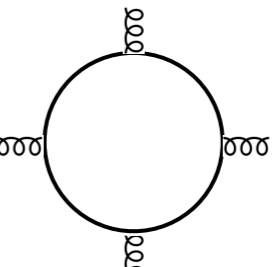
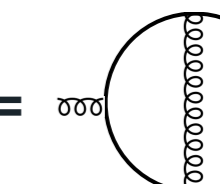
(Note: The diagrams in the above equation represent the graphical expansion of the R-operator formula, showing the subtraction of UV subgraphs and the addition of their second-order counterterms.)

The four different types of UV limits are now **finite** !



- $k, m \rightarrow \infty, l \text{ finite}$
- $l, m \rightarrow \infty, k \text{ finite}$
- $k, l \rightarrow \infty, m \text{ finite}$
- $k, l, m \rightarrow \infty$

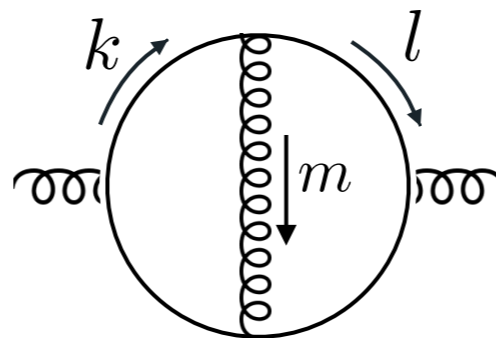
R-OPERATOR UNFOLDING

$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$\begin{aligned}
 R(\Gamma) = & \Gamma - T_0(\gamma_1) * \Gamma \setminus \gamma_1 - T_0(\gamma_2) * \Gamma \setminus \gamma_2 - T_0(\gamma_3) * \Gamma \setminus \gamma_3 - T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\
 & + T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}
 \end{aligned}$$

$$\begin{aligned}
 = & \text{~~Diagram of } \Gamma \text{ with a red slash}~~ - \text{Diagram of } \Gamma \setminus \gamma_1 \text{ with } T_0(\gamma_1) \text{ attached} - \text{Diagram of } \Gamma \setminus \gamma_2 \text{ with } T_0(\gamma_2) \text{ attached} - \text{Diagram of } \Gamma \setminus \gamma_3 \text{ with } T_0(\gamma_3) \text{ attached} - \text{Diagram of } \Gamma \setminus \gamma_4 \text{ with } T_2(\gamma_4) \text{ attached} \\
 & + \text{Diagram of } T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + \text{Diagram of } T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + \text{Diagram of } T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}
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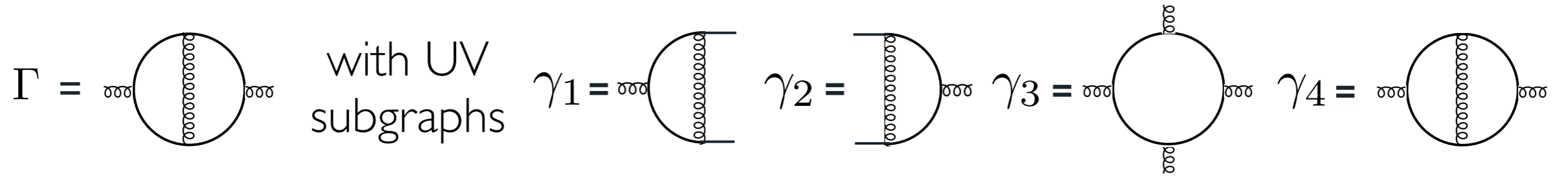
$$k, m \rightarrow \infty, l \text{ finite}$$

$$l, m \rightarrow \infty, k \text{ finite}$$

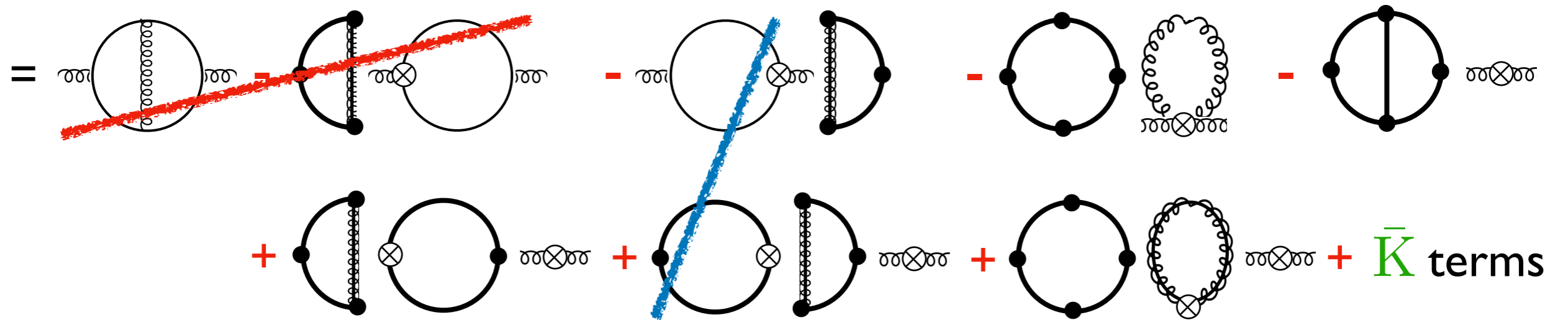
$$k, l \rightarrow \infty, m \text{ finite}$$

$$k, l, m \rightarrow \infty$$

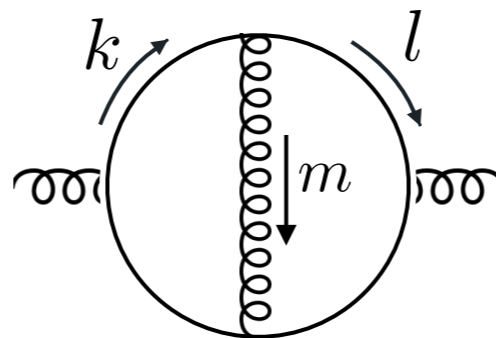
R-OPERATOR UNFOLDING



$$\begin{aligned}
 R(\Gamma) = & \Gamma - T_0(\gamma_1) * \Gamma \setminus \gamma_1 - T_0(\gamma_2) * \Gamma \setminus \gamma_2 - T_0(\gamma_3) * \Gamma \setminus \gamma_3 - T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\
 & + T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}
 \end{aligned}$$



The four different types of UV limits are now **finite** !



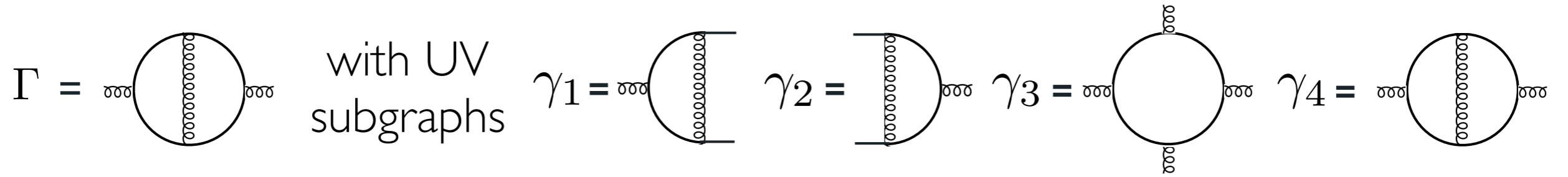
$$k, m \rightarrow \infty, l \text{ finite}$$

$$l, m \rightarrow \infty, k \text{ finite}$$

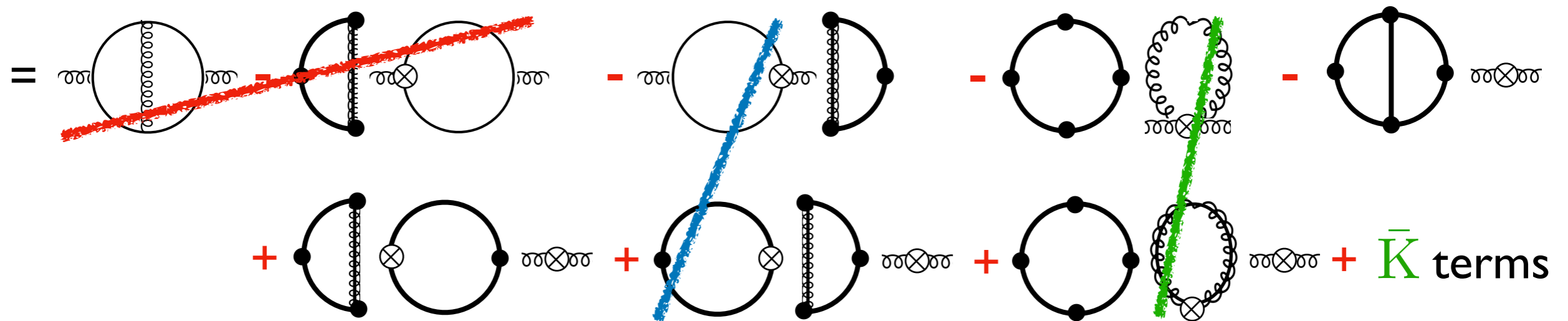
$$k, l \rightarrow \infty, m \text{ finite}$$

$$k, l, m \rightarrow \infty$$

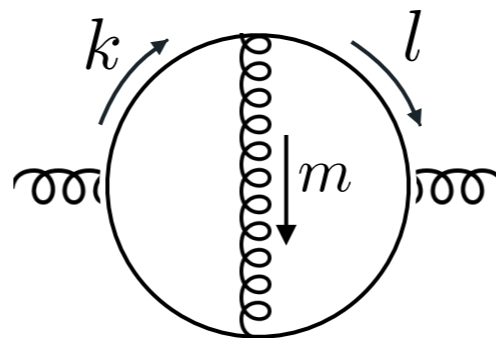
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 R(\Gamma) = & \Gamma - T_0(\gamma_1) * \Gamma \setminus \gamma_1 - T_0(\gamma_2) * \Gamma \setminus \gamma_2 - T_0(\gamma_3) * \Gamma \setminus \gamma_3 - T_2(\gamma_4) * \Gamma \setminus \gamma_4 \\
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 \end{aligned}$$



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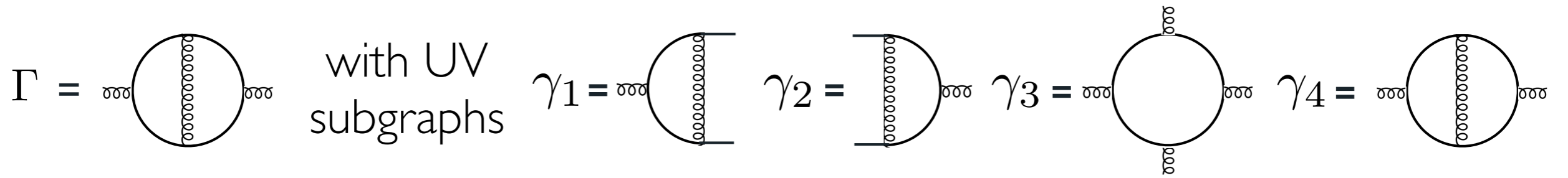
$$k, m \rightarrow \infty, l \text{ finite}$$

$$l, m \rightarrow \infty, k \text{ finite}$$

$$k, l \rightarrow \infty, m \text{ finite}$$

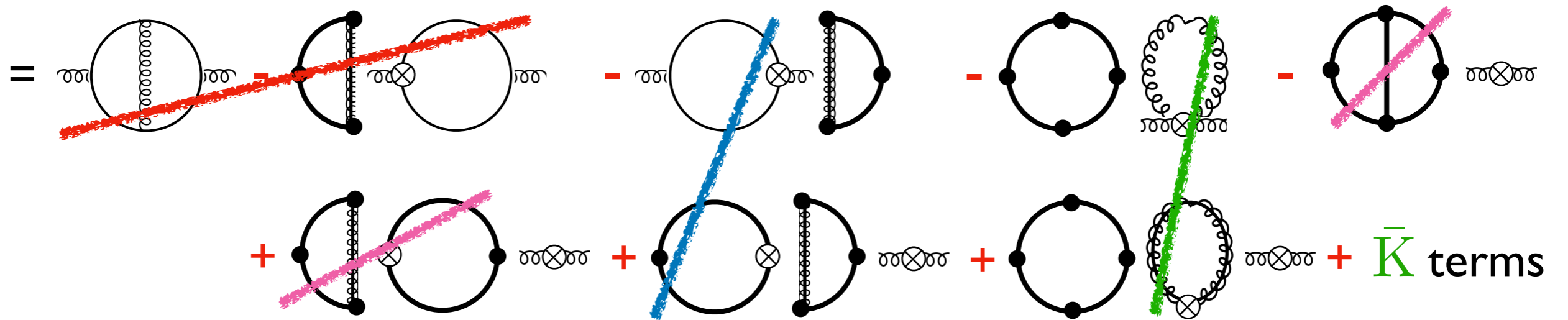
$$k, l, m \rightarrow \infty$$

R-OPERATOR UNFOLDING

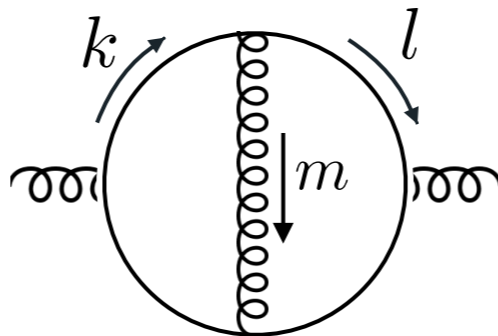


$$R(\Gamma) = \Gamma - T_0(\gamma_1) * \Gamma \setminus \gamma_1 - T_0(\gamma_2) * \Gamma \setminus \gamma_2 - T_0(\gamma_3) * \Gamma \setminus \gamma_3 - T_2(\gamma_4) * \Gamma \setminus \gamma_4$$

$$+ T_2(T_0(\gamma_1) * \Gamma \setminus \gamma_1) + T_2(T_0(\gamma_2) * \Gamma \setminus \gamma_2) + T_2(T_0(\gamma_3) * \Gamma \setminus \gamma_3) + \bar{K} \text{ terms}$$



The four different types of UV limits are now **finite**!



$$k, m \rightarrow \infty, l \text{ finite}$$

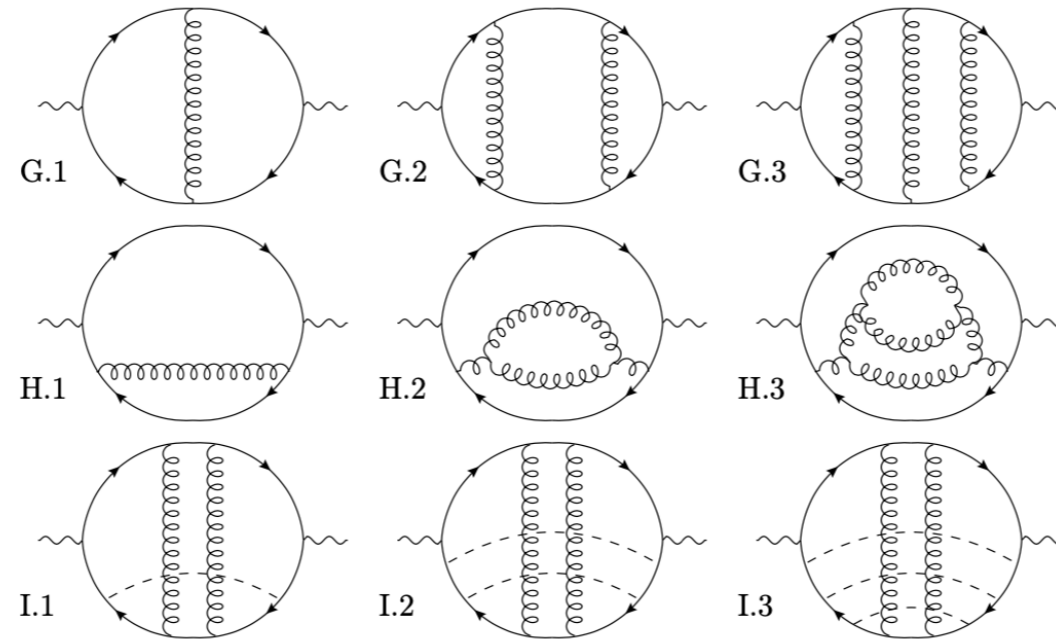
$$l, m \rightarrow \infty, k \text{ finite}$$

$$k, l \rightarrow \infty, m \text{ finite}$$

$$k, l, m \rightarrow \infty$$

EFFICIENCY IN LU IMPLEMENTATION

α Loop \longrightarrow γ Loop



SG	proc.	order	t_{gen} [s]	M_{disk} [MB]	N_{sg} [-]	N_{cuts} [-]	t_{eval} [ms]	$t_{\text{eval}}^{(\text{f128})}$ [ms]
G.1	1 \rightarrow 2	NLO	0.1	0.13	2	4	0.004	0.13
G.2	1 \rightarrow 2	NNLO	4.7	3.0	17	9	0.04	2.1
G.3	1 \rightarrow 2	N3LO	36K	509	220	16	17.6	281
H.1	1 \rightarrow 2	NLO	0.07	0.12	2	2	0.006	0.14
H.2	1 \rightarrow 2	NNLO	1.5	1.3	17	3	0.056	1.9
H.3	1 \rightarrow 2	N3LO	255	43	220	4	2.35	56
I.1	1 \rightarrow 3	NNLO	126	22	266	9	0.32	12.4
I.2	1 \rightarrow 4	NNLO	1.9K	120	4492	9	4.4	67
I.3	1 \rightarrow 5	NNLO	36K	20K	$\mathcal{O}(100\text{K})$	9	3.6K	17.3K

NB: most recent version of α Loop does better, but scaling is similar

NUMERICAL GAMMA CHAINS WITH SPENSO.RS

spenso.rs

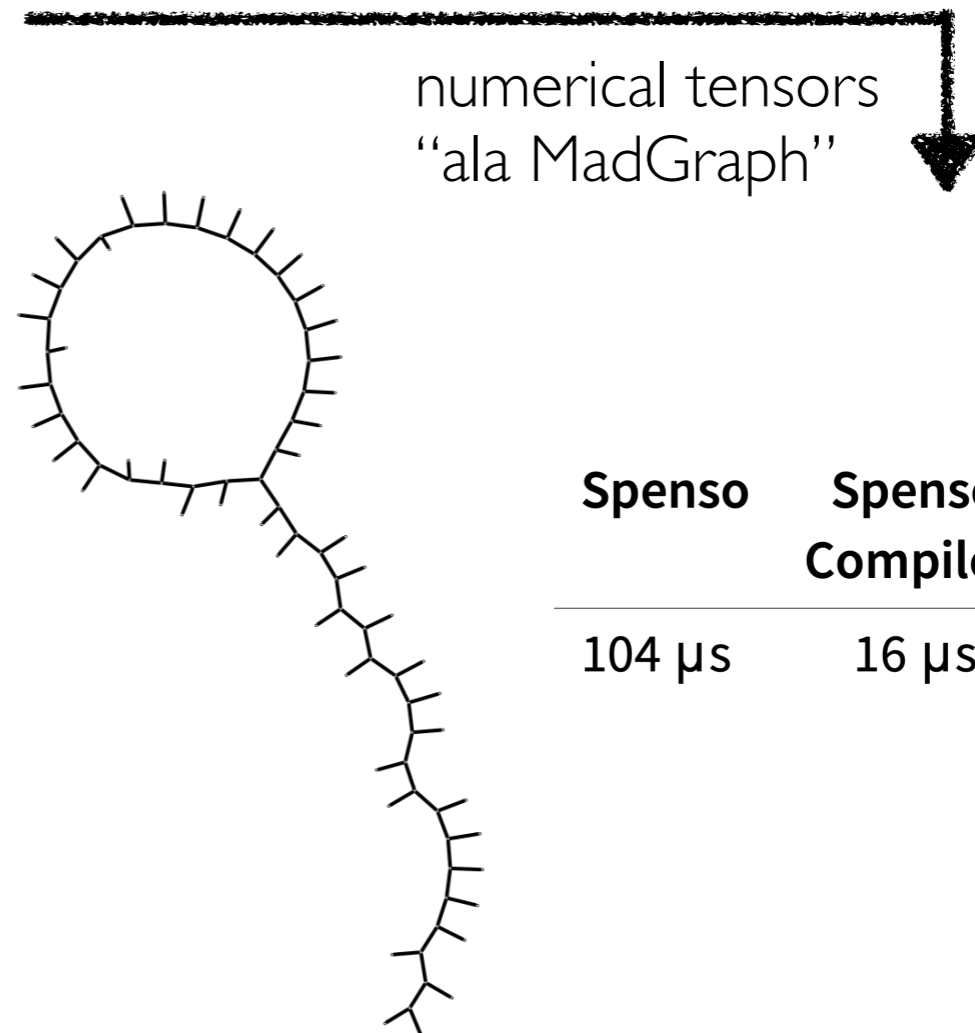
[Lucien Huber : crates.io/crates/spenso]

$$p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} \text{Tr}(\gamma_\mu \gamma_{\mu_1} \gamma_\nu \gamma_{\mu_2} \gamma_\rho \gamma_{\mu_3})$$

↓ γ -algebra

$$\begin{aligned} & -4(p_1 \cdot p_2) p_{3,\mu} \eta_{\nu\rho} \\ & + 4(p_1 \cdot p_2) p_{3,\nu} \eta_{\mu\rho} \\ & - 4(p_1 \cdot p_2) p_{3,\rho} \eta_{\mu\nu} \\ & + 4(p_1 \cdot p_3) p_{2,\mu} \eta_{\nu\rho} \\ & - 4(p_1 \cdot p_3) p_{2,\nu} \eta_{\mu\rho} \\ & - 4(p_1 \cdot p_3) p_{2,\rho} \eta_{\mu\nu} \\ & - 4p_{1,\mu} (p_2 \cdot p_3) \eta_{\nu\rho} \\ & + 4p_{1,\mu} p_{2,\nu} p_{3,\rho} \\ & + 4p_{1,\mu} p_{2,\rho} p_{3,\nu} \\ & - 4p_{1,\nu} (p_2 \cdot p_3) \eta_{\mu\rho} \end{aligned}$$

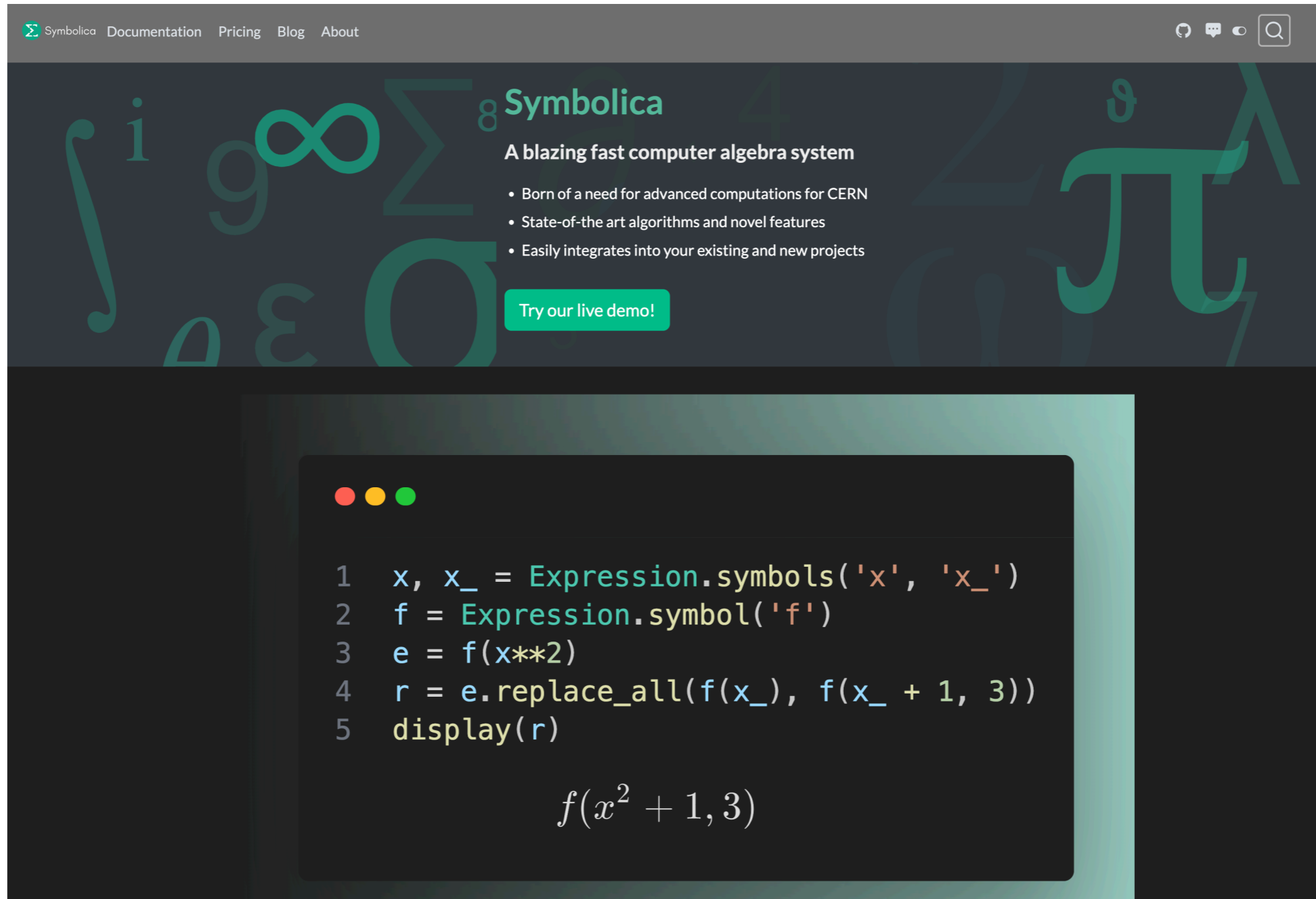
necessary for d-dimensions
but scales badly for $d = 4$



Spenso	Spenso Compiled	Hardcode Fortran
104 μ s	16 μ s	31 μ s

SYMBOLICA

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```
1 x, x_ = Expression.symbols('x', 'x_')
2 f = Expression.symbol('f')
3 e = f(x**2)
4 r = e.replace_all(f(x_), f(x_ + 1, 3))
5 display(r)
```

$$f(x^2 + 1, 3)$$

Try it for yourself in this [colab notebook](#) !

SUMMARY - POSSIBLE OVERLAP WITH LSS

Formalism :

- **Local** cancellation of all **final-state IR singularities**
- Completely **generic** (masses, kinematics, topologies, observables...)
- New **theoretical perspectives** on perturbative expansions
- **Automated renormalisation** with minimal analytical computations
- **Falsehood** : “Analytical = solved && Numerical = black box”

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Outlook :

- Rewrite **LU** impl. **cleaner and more efficient**: $\alpha\text{Loop} \rightarrow \gamma\text{Loop}$
- Apply **LU** to **new unknown corrections** : e.g. $b\bar{b}$ N³LO AFB or decays
- Applications in finite **T** and finite μ pQFT : see [Navarrete & al.: [2403.02180](#)]
- Generalise **LU** so as to apply of **initial-state singularities** as well
- Match **LU** to some form of **numerical resummation** (PSMC).
- Method likely well-suited for deployment on GPU.



BACK-UP SLIDES

LOCAL UNITARITY: X-SEC RESULTS

NUMERICAL RESULTS FOR $e^+e^- \rightarrow \gamma^* \rightarrow d\bar{d}$ @ NLO

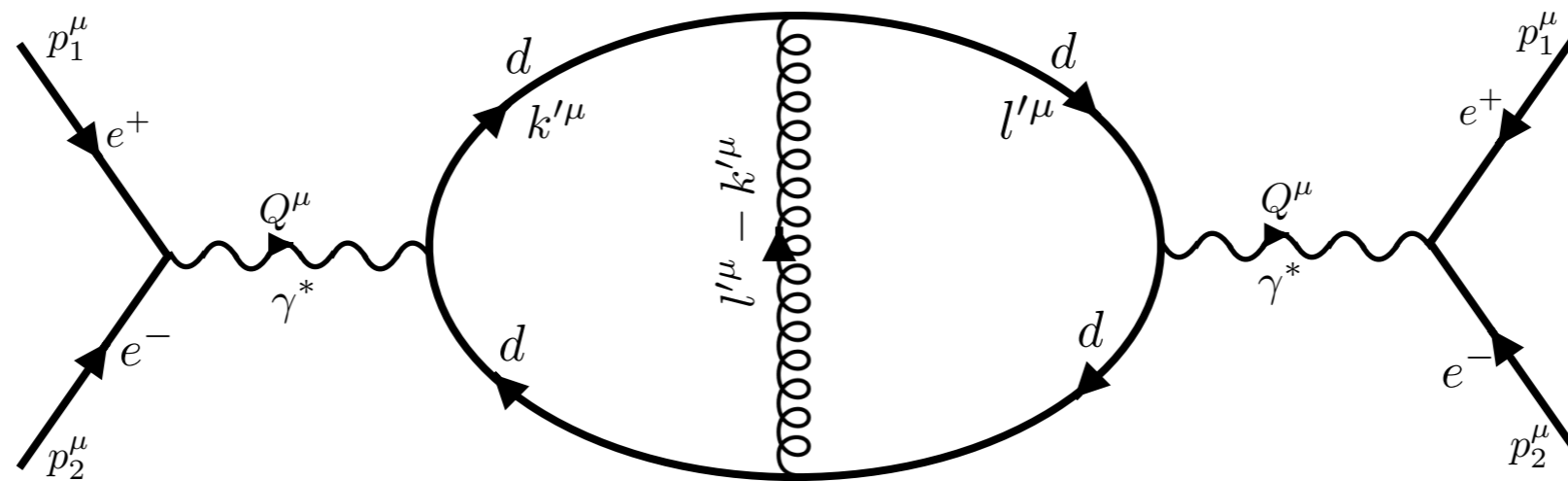
$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Diagram 1} + \text{Diagram 2} + 2 \times \text{Diagram 3} \right]$$

The equation shows the Local Unitarity (LU) correction to the cross-section for the process $\gamma^* \rightarrow d\bar{d}$. The correction is expressed as a sum of three diagrams enclosed in large red square brackets. The first diagram is a simple loop with two external wavy lines. The second diagram is a loop with a vertical dashed line representing a ghost loop. The third diagram is a loop with a smaller loop on top, representing a two-loop correction.

NUMERICAL RESULTS FOR $e^+e^- \rightarrow \gamma^* \rightarrow d\bar{d}$ @ NLO

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Loop 1} + \text{Loop 2} + 2 \times \text{Loop 3} \right]$$

Visualisation of the LU integrand for the Double-Triangle supergraph and :



$$p_1^\mu = (1, 0, 0, 1)$$

$$0.4 < p_{t,j_1} < 0.8$$

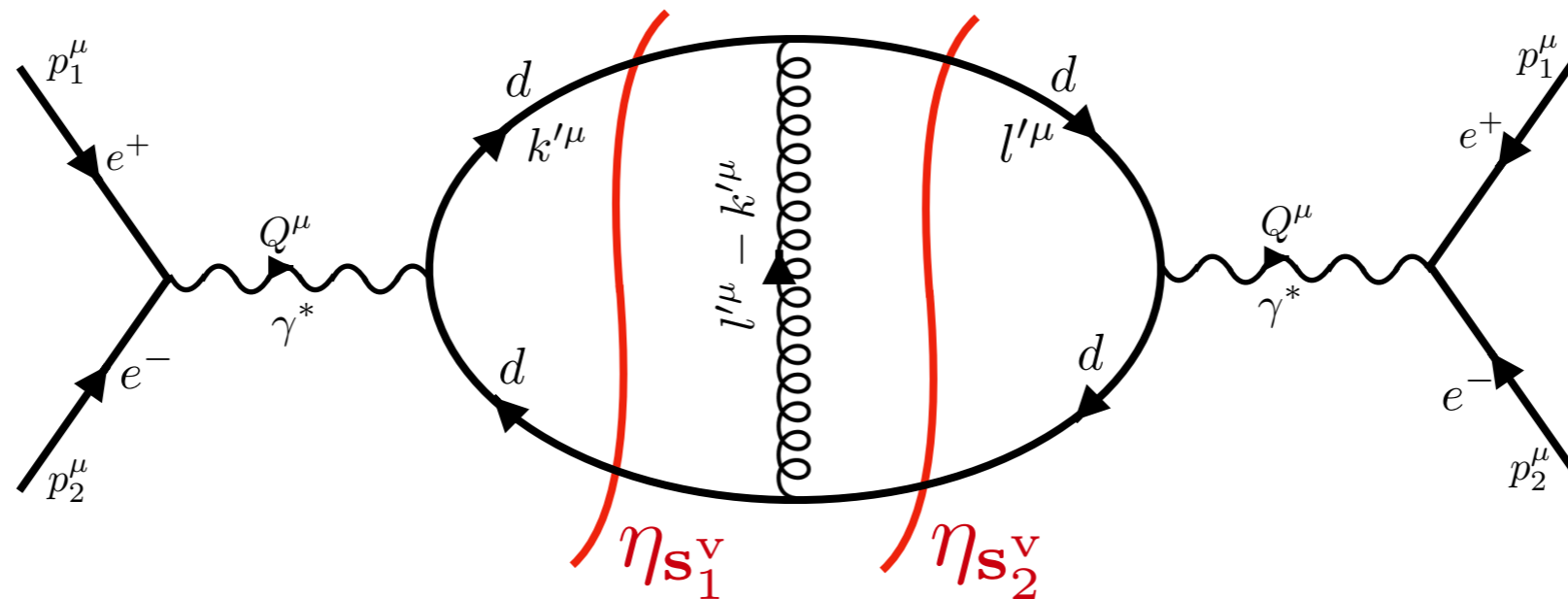
$$p_2^\mu = (1, 0, 0, -1)$$

$$(\vec{k}, \vec{l}) = \left(\left(0, k_y, \frac{1}{\sqrt{2}} \right), \left(0, \frac{1}{\sqrt{2}}, l_z \right) \right)$$

NUMERICAL RESULTS FOR $e^+e^- \rightarrow \gamma^* \rightarrow d\bar{d}$ @ NLO

$$\sigma_{\gamma^* \rightarrow d\bar{d}}^{(\text{LU})} = \text{LU} \left[\text{Loop} + \text{Double-Triangle} + 2 \times \text{Triangle} \right]$$

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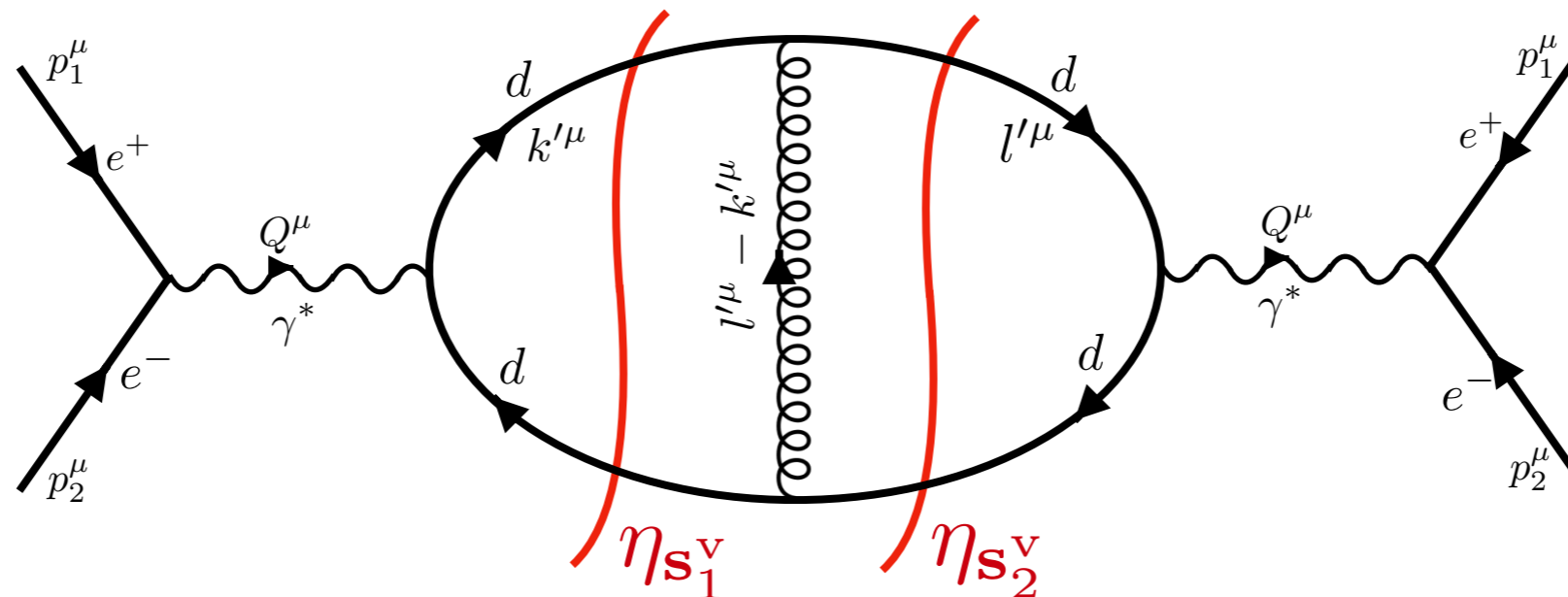
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Cancellation of non-pinched E-surfaces for : $\eta_{s_1^v} = \eta_{s_2^v} \rightarrow k'_y = l'_z$

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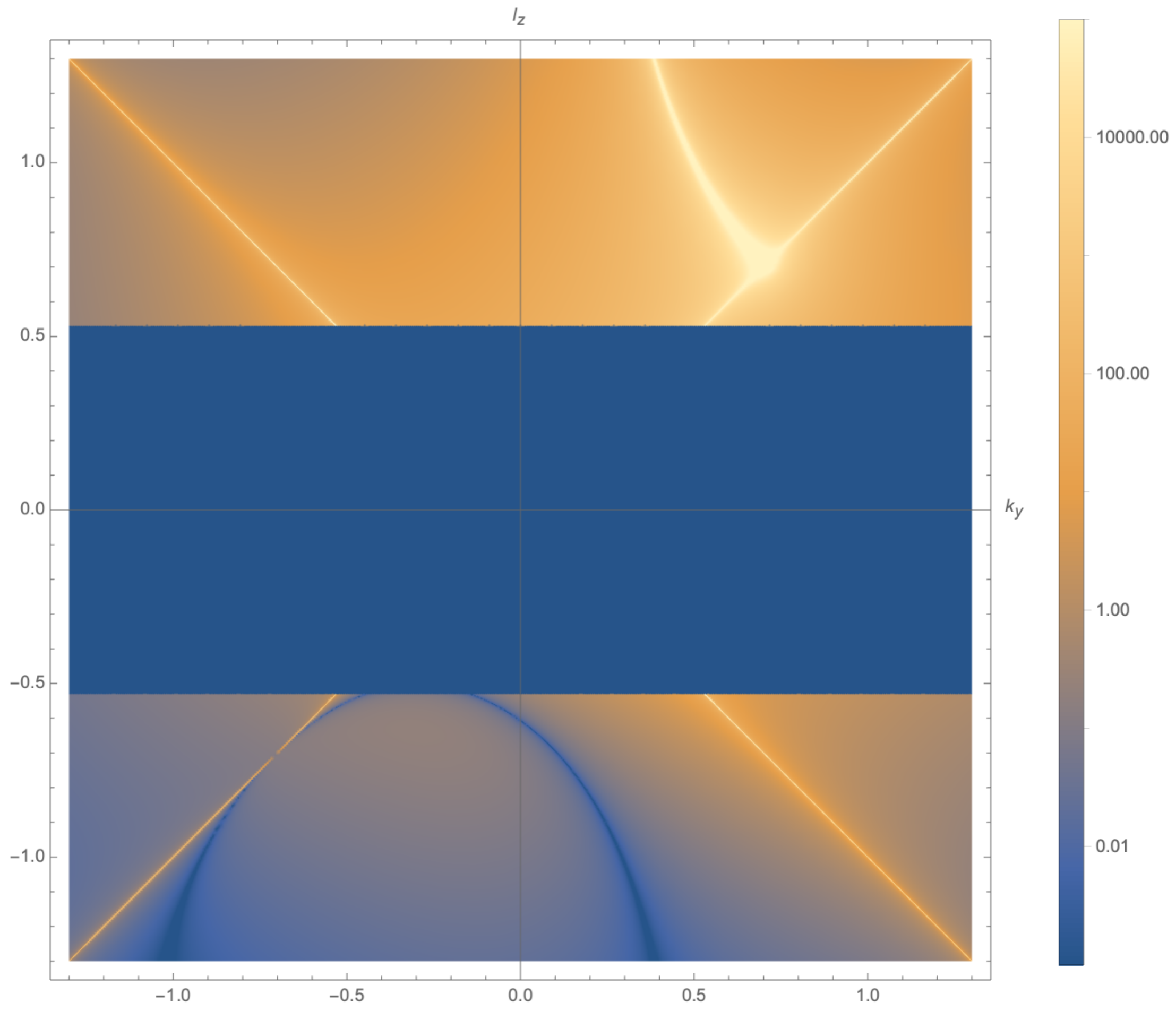
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Cancellation of non-pinched E-surfaces for : $\eta_{S_1^V} = \eta_{S_2^V} \rightarrow k'_y = l'_z$

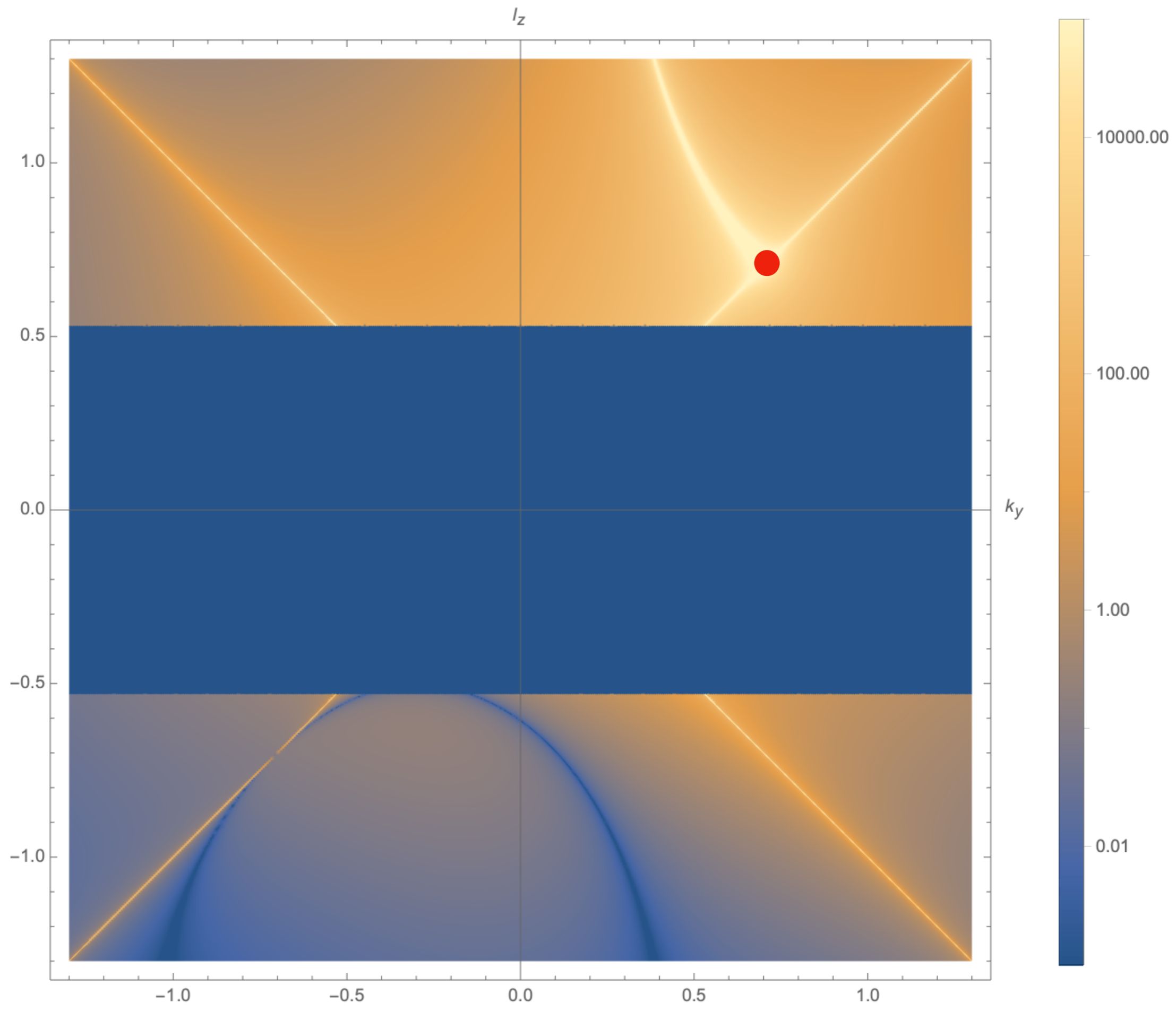
Soft configuration for : $|\vec{l}' - \vec{k}| = 0 \rightarrow k'_y = l'_z = \frac{1}{\sqrt{2}}$

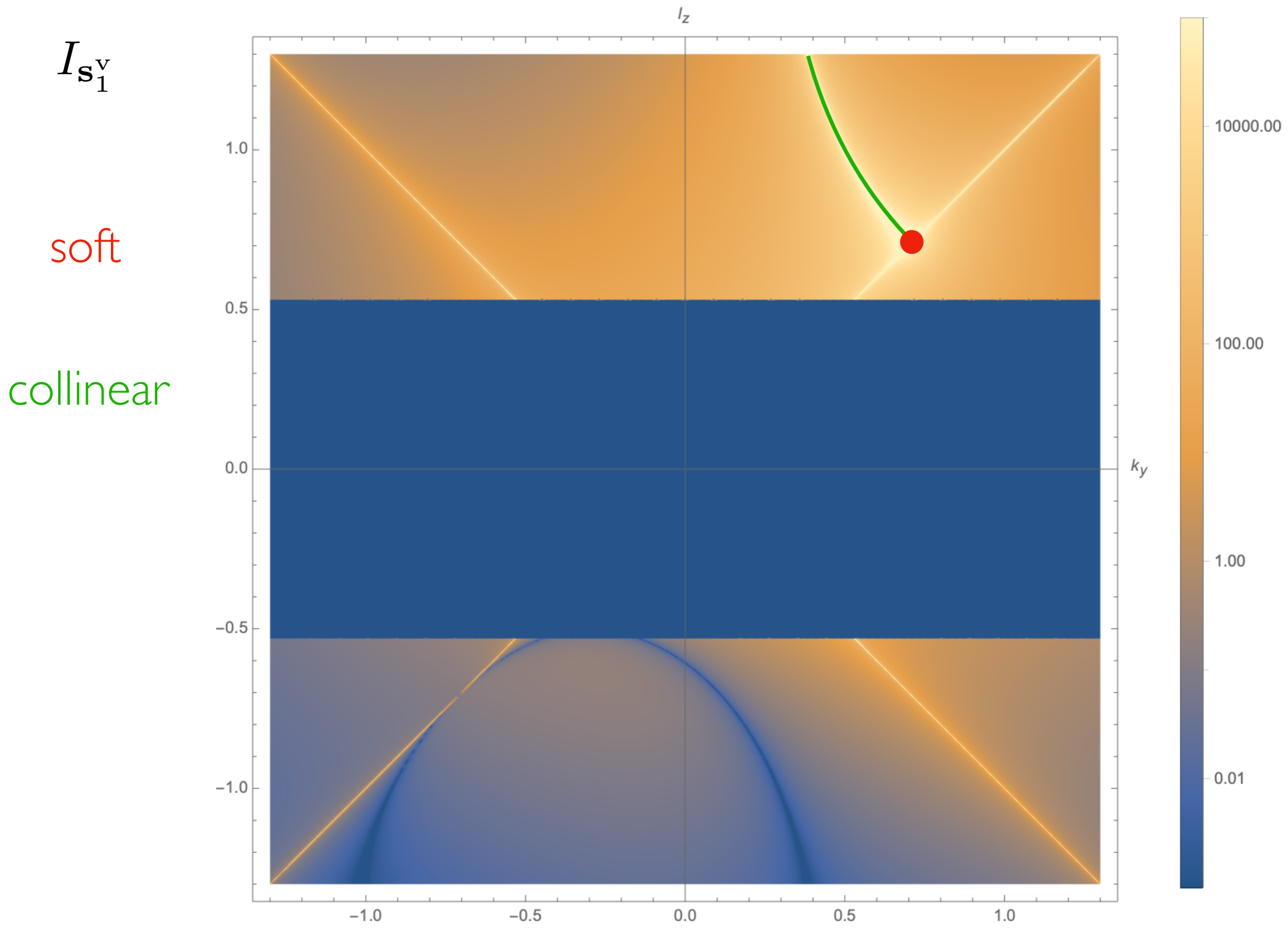
$I_{s_1^v}$

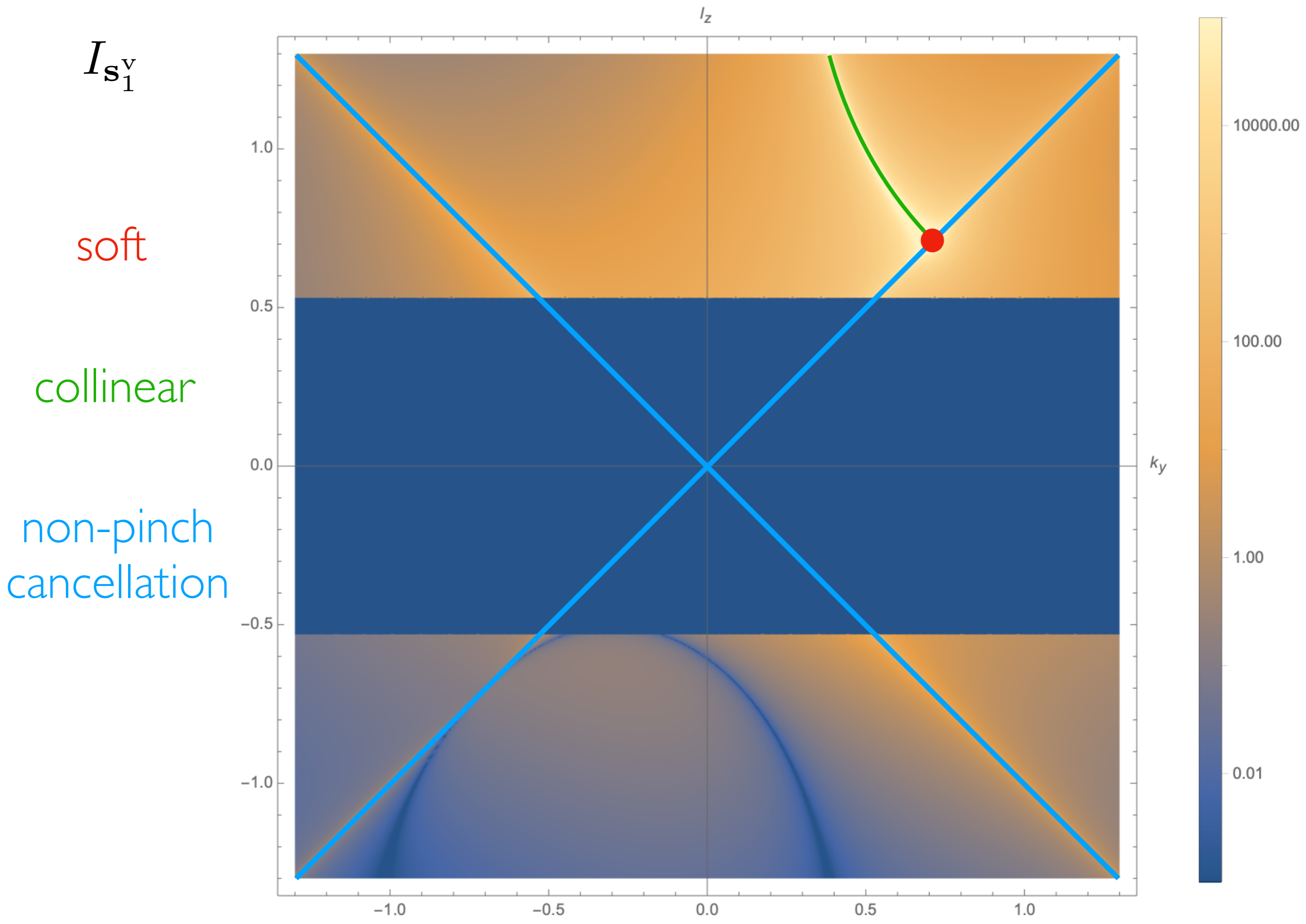


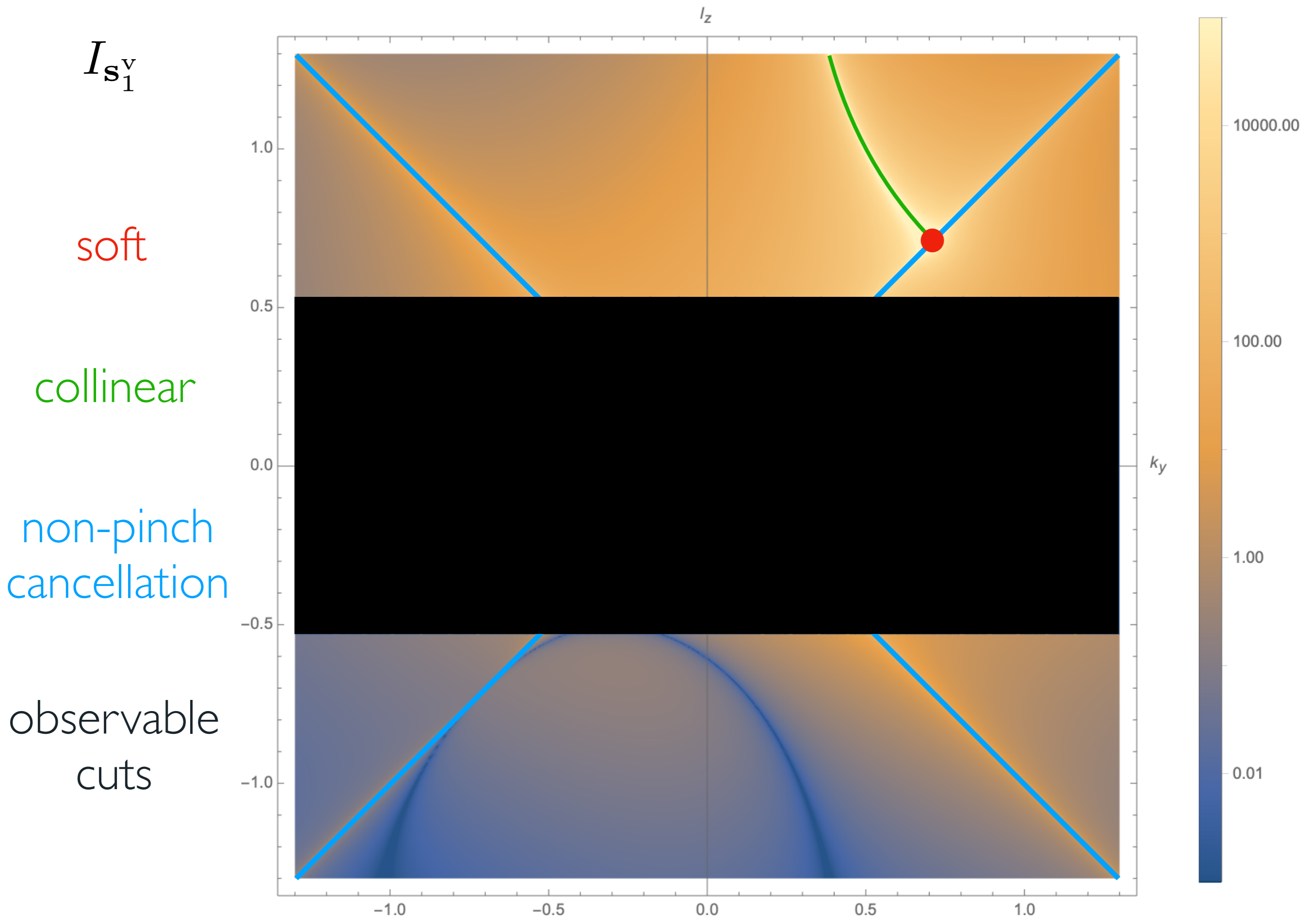
$I_{s_1^v}$

soft

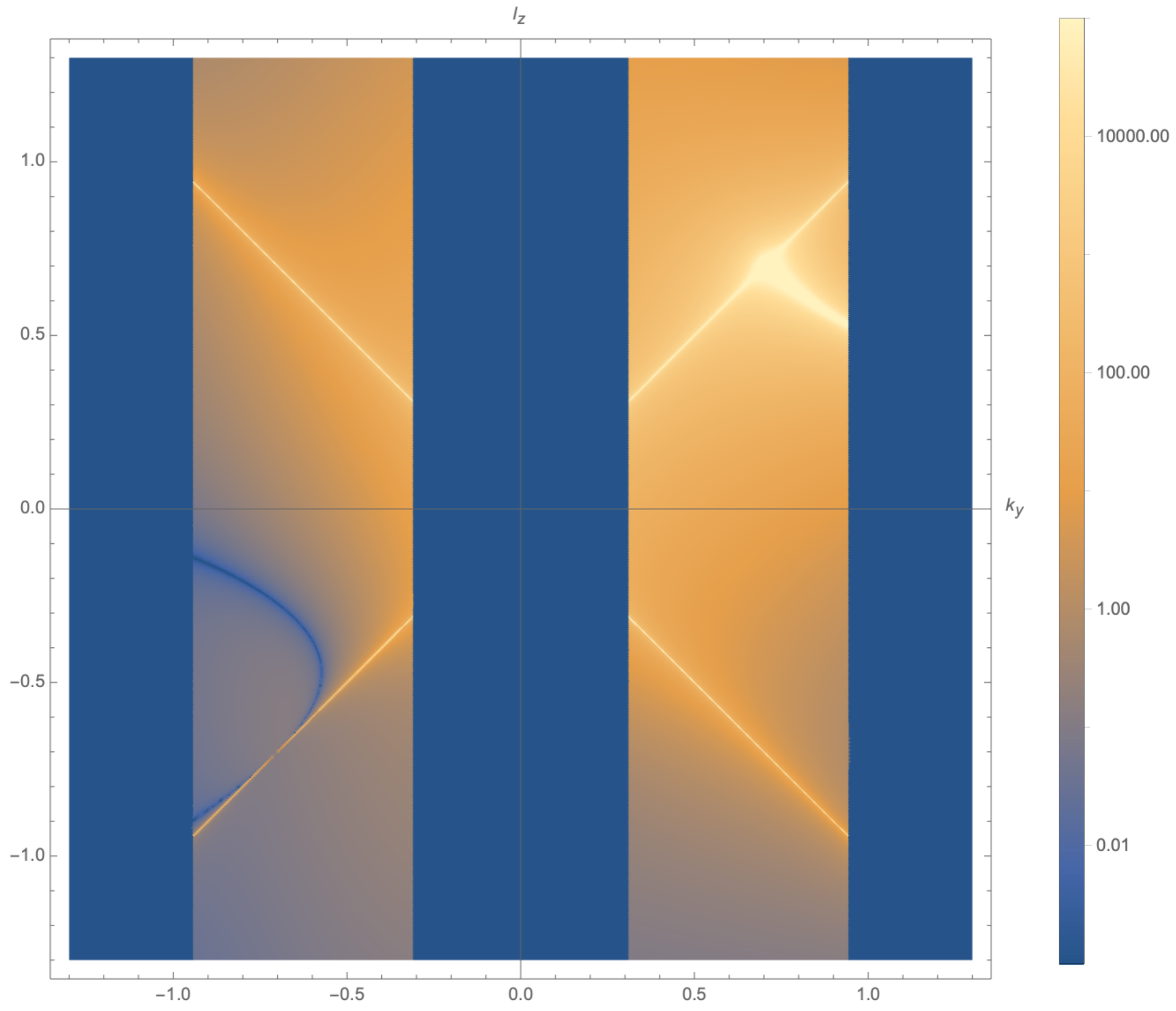




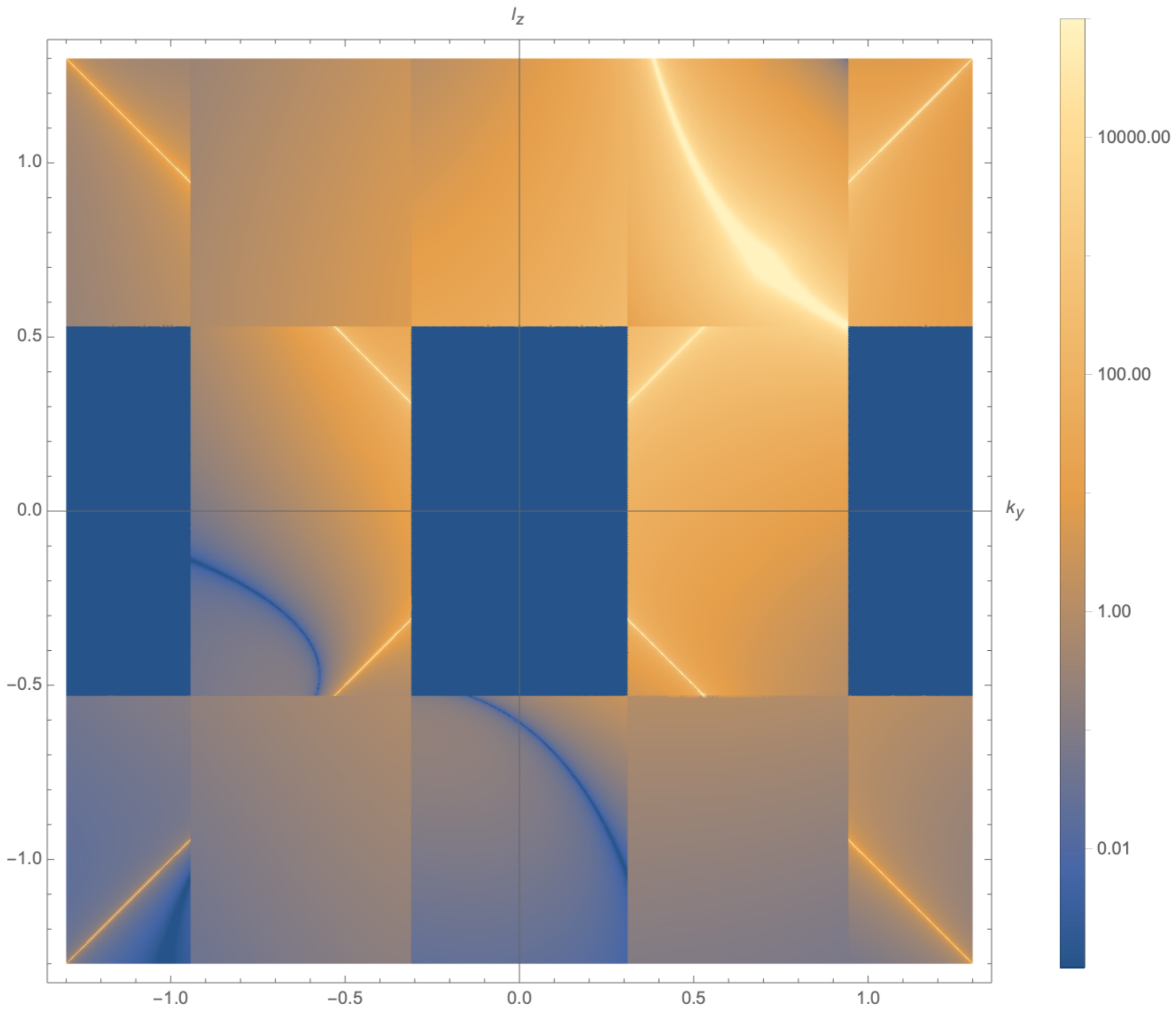




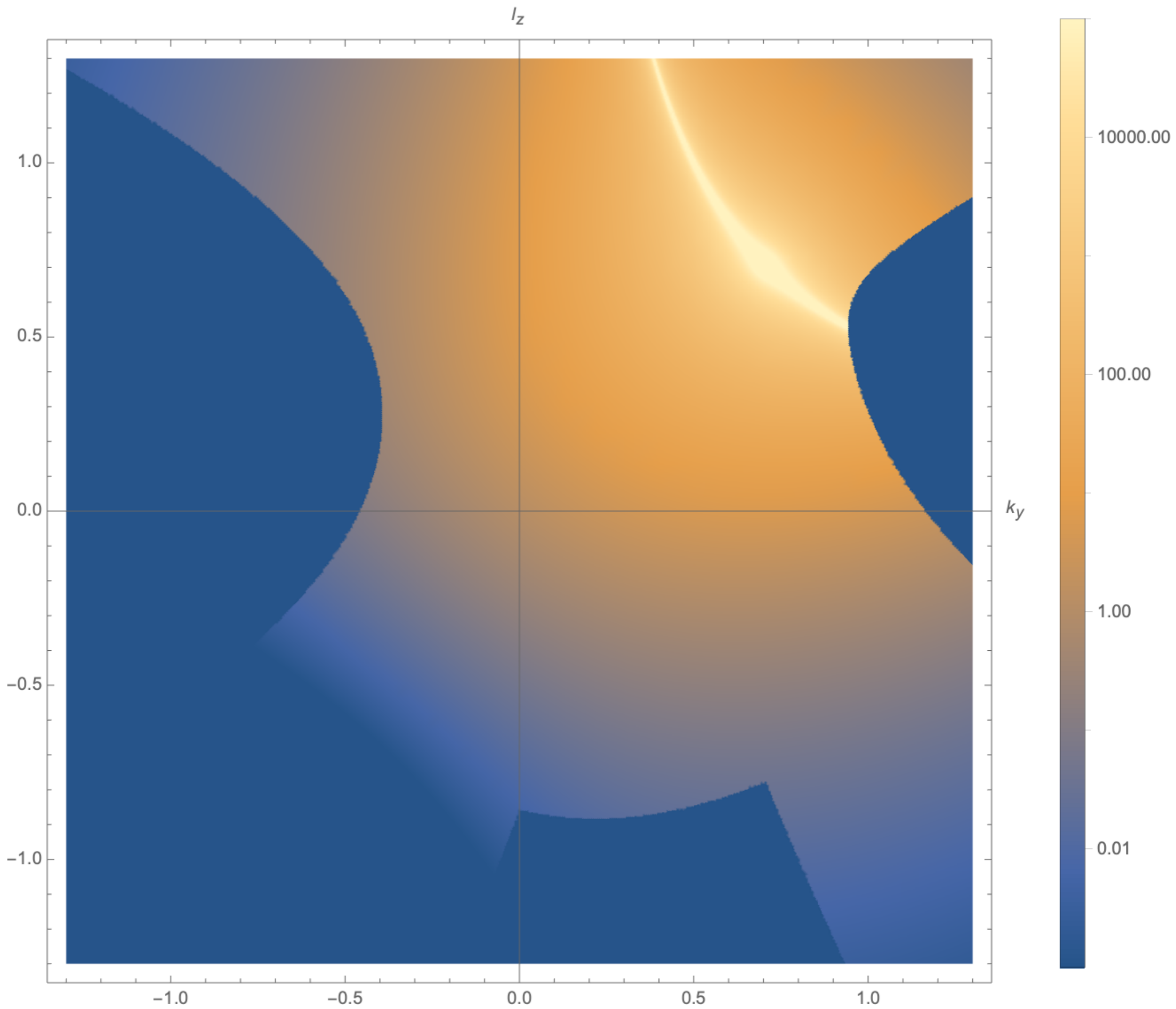
$I_{S_2^v}$

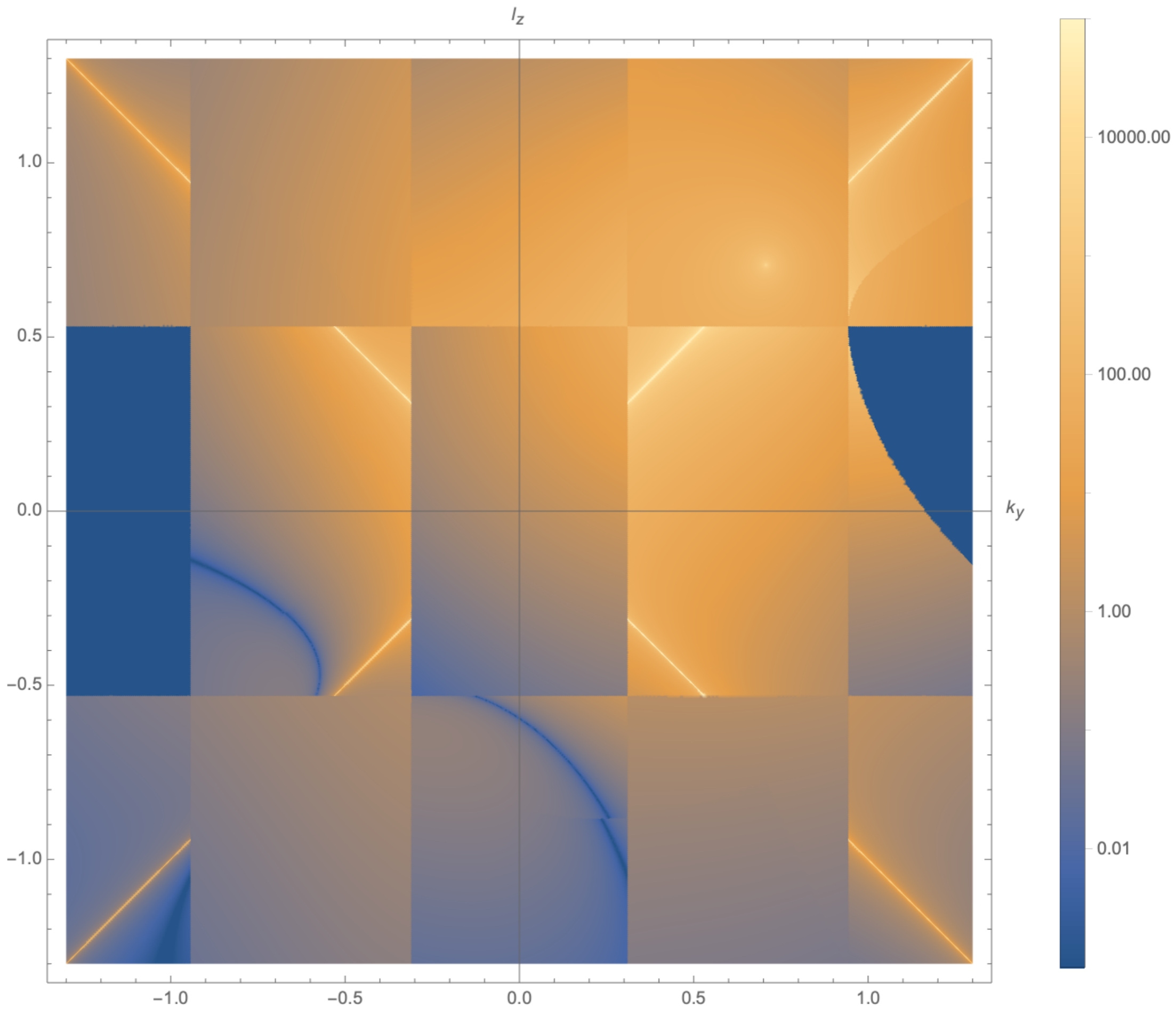


$$I_{\mathbf{s}_1^v} + I_{\mathbf{s}_2^v}$$

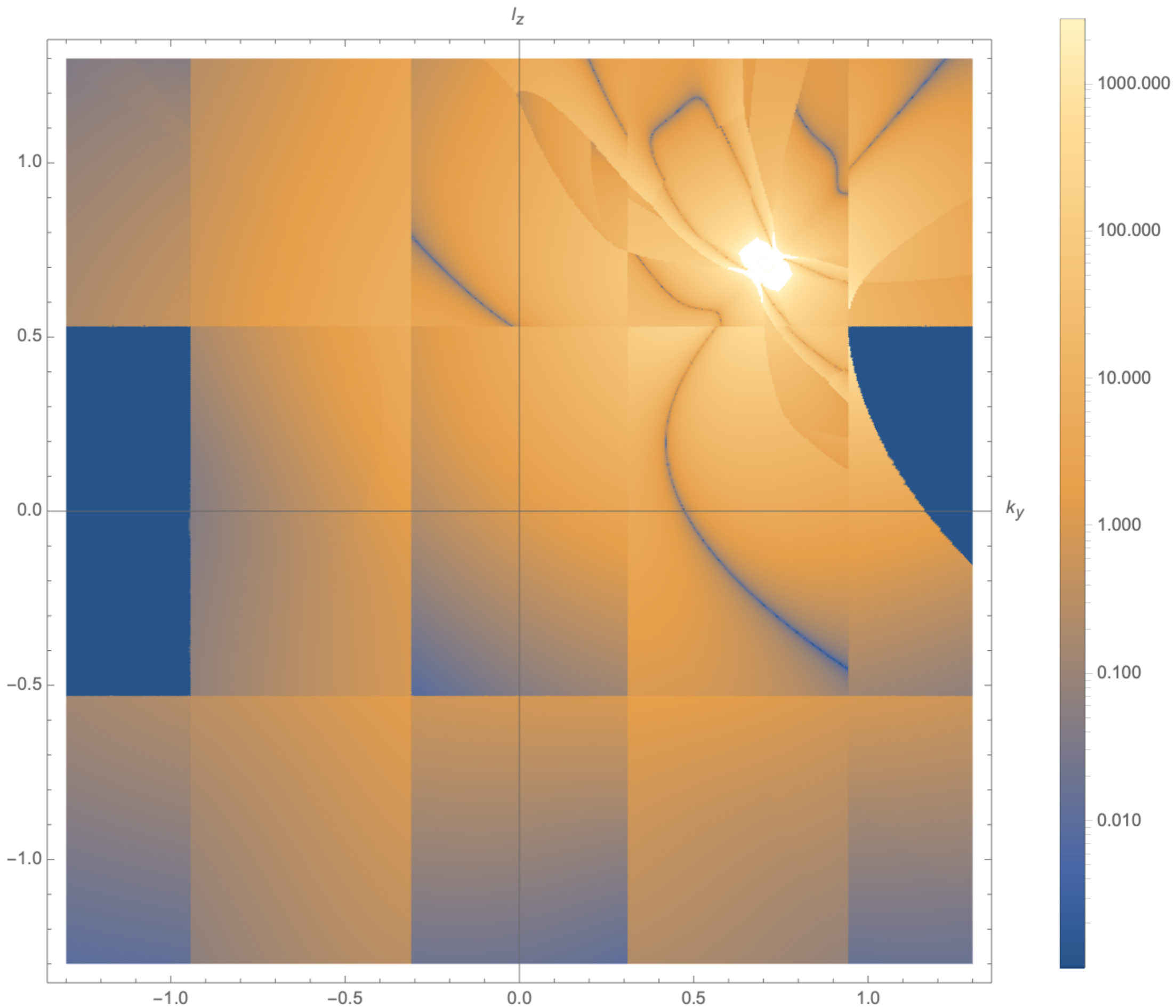


$$I_{\mathbf{s}_1^r} + I_{\mathbf{s}_2^r}$$

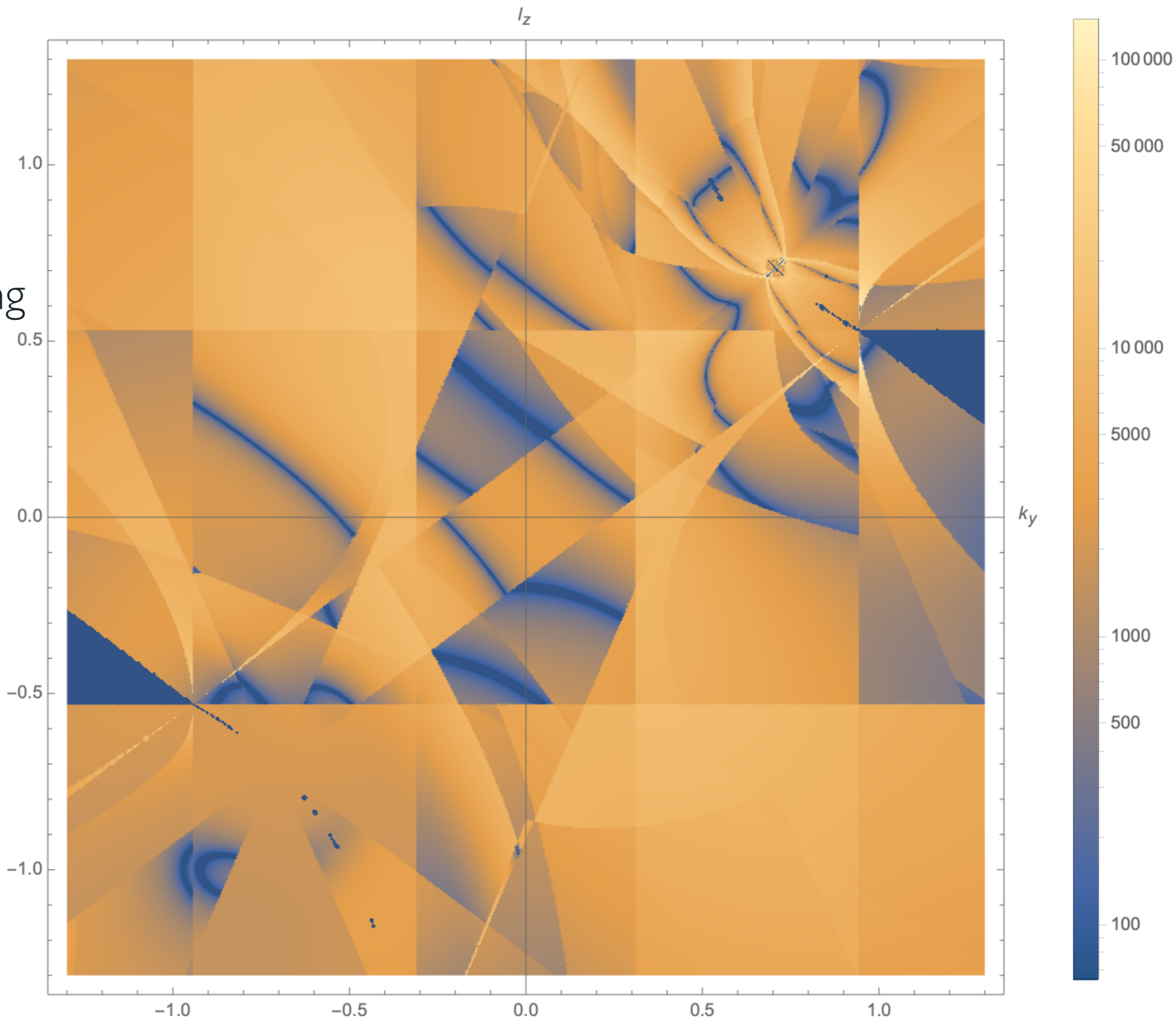


I_Σ 

$\text{Re} [I_\Sigma]$
with
deformation



$\text{Re} [I_\Sigma]$
with
deformation
and
multichanneling

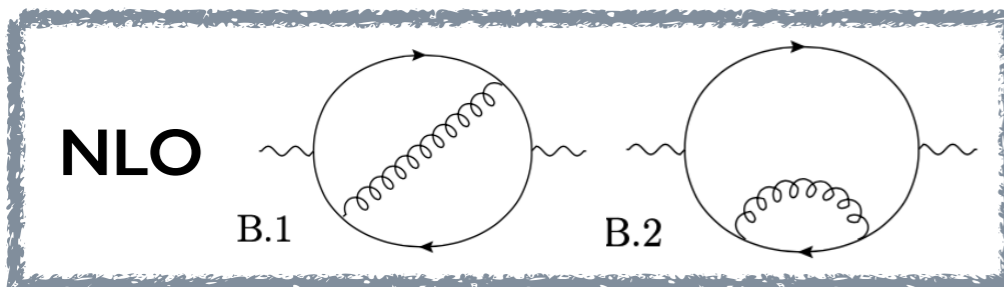
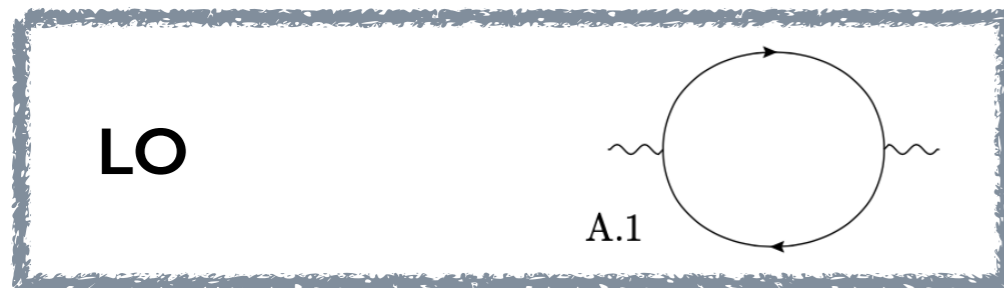


NUMERICAL RESULTS

We have computed the following fixed-order processes with **Local Unitarity**:

NLO	$e^+e^- \rightarrow \gamma \rightarrow jj$	$p_t(j_1)$ distribution	NNLO	$\gamma^* \rightarrow jj$	inclusive
	$e^+e^- \rightarrow \gamma \rightarrow jjj$	semi-inclusive		$\gamma^* \rightarrow t\bar{t}$	inclusive
	$e^+e^- \rightarrow \gamma \rightarrow t\bar{t}h$	(semi-)inclusive			

First **NNLO** cross-sections computed **fully numerically** in momentum space.

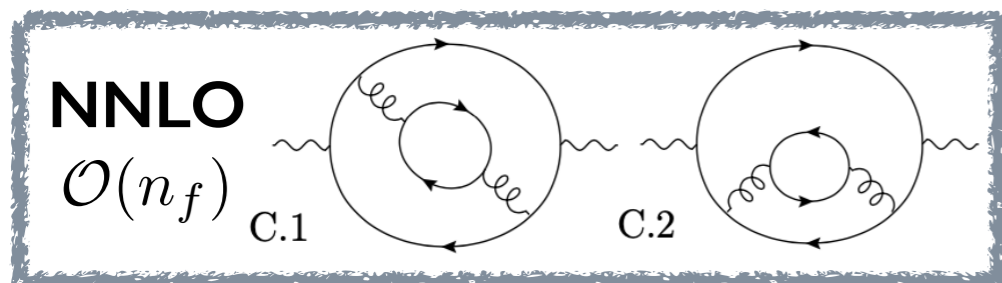
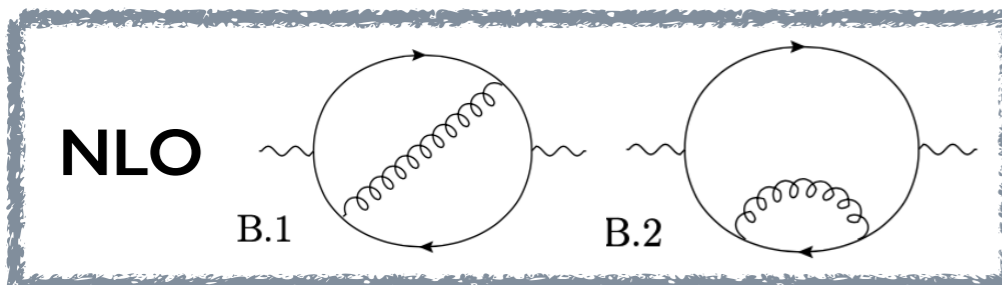
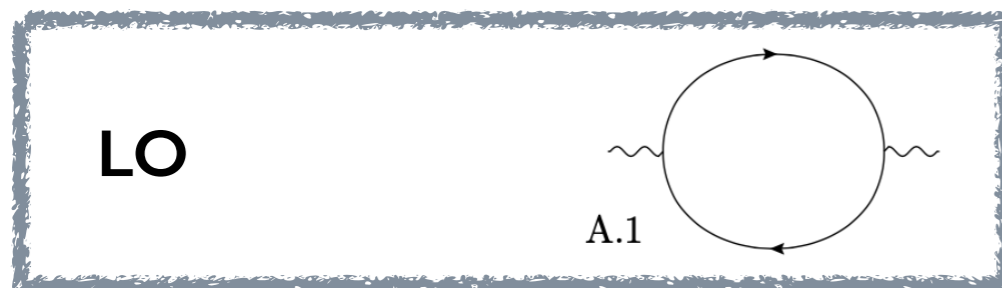


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		$e^+e^- \rightarrow \gamma \rightarrow jjj$ semi-inclusive			$\gamma^* \rightarrow t\bar{t}$ inclusive
		$e^+e^- \rightarrow \gamma \rightarrow t\bar{t}h$ (semi-)inclusive			

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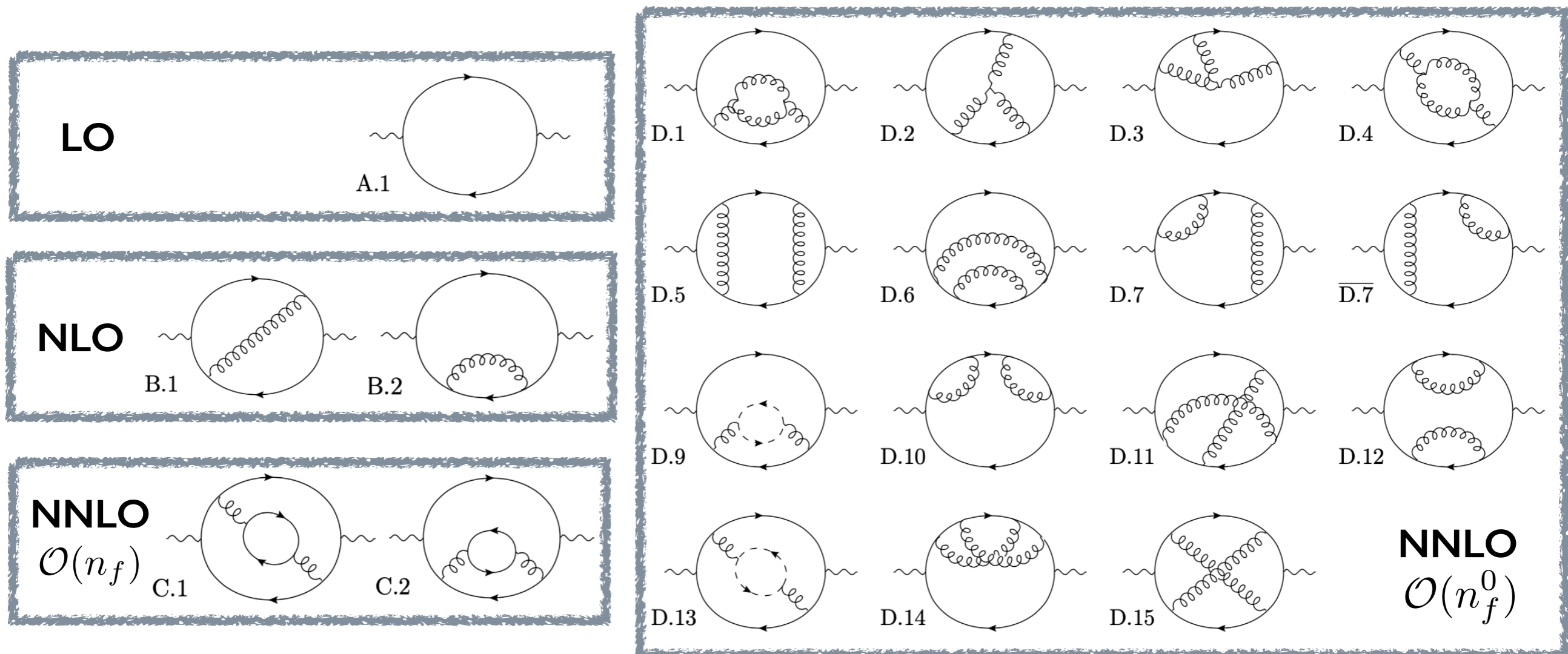


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First **NNLO** cross-sections computed **fully numerically** in momentum space.



NUMERICAL RESULTS

SG id	Ξ	$\sigma_{\gamma^* \rightarrow jj}^{(\overline{\text{MS}})}$ [GeV ⁻²] $p_{\gamma^*}^2 = \mu_r^2 = (400 \text{ GeV})^2$	Δ [%]	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]}$ [GeV ⁻²] $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (400 \text{ GeV})^2$	Δ [%]
LO $\mathcal{O}(\alpha_s^0)$					
A.1	1	$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
Total		$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
NLO $\mathcal{O}(\alpha_s^1)$					
B.1	1	$5.03926 \cdot 10^{-02}$	0.0075	$2.52705 \cdot 10^{-01}$	0.034
B.2	2	$-3.14956 \cdot 10^{-02}$	0.018	$1.80050 \cdot 10^{-01}$	0.049
Total		$1.88970 \cdot 10^{-02}$	0.036	$4.3276 \cdot 10^{-01}$	0.028
Benchmark		$1.889690 \cdot 10^{-02}$	0.00053	$4.32831 \cdot 10^{-01}$	-0.018
NNLO $\mathcal{O}(\alpha_s^2 n_f)$					
C.1	1	$-4.66342 \cdot 10^{-04}$	0.019	$-1.0022 \cdot 10^{-03}$	0.17
C.2	2	$3.8448 \cdot 10^{-04}$	0.036	$-4.6982 \cdot 10^{-03}$	0.081
Total		$-8.186 \cdot 10^{-05}$	0.20	$-5.7004 \cdot 10^{-03}$	0.073
Benchmark		$-8.1834 \cdot 10^{-05}$	0.036	$-5.6982 \cdot 10^{-03}$	0.038
NNLO $\mathcal{O}(\alpha_s^2)$					
D.1	2	$-2.30886 \cdot 10^{-03}$	0.017	$3.8886 \cdot 10^{-02}$	0.031
D.2	2	$6.42018 \cdot 10^{-03}$	0.0055	$5.6351 \cdot 10^{-03}$	0.14
D.3	2	$-6.91254 \cdot 10^{-03}$	0.0046	$1.76075 \cdot 10^{-02}$	0.055
D.4	1	$3.20278 \cdot 10^{-03}$	0.0084	$8.8163 \cdot 10^{-03}$	0.078
D.5	1	$1.68148 \cdot 10^{-03}$	0.013	$9.200 \cdot 10^{-04}$	0.79
D.6	2	$6.6698 \cdot 10^{-04}$	0.027	$5.1058 \cdot 10^{-03}$	0.15
D.7	2	$-1.30381 \cdot 10^{-03}$	0.013	$6.7284 \cdot 10^{-03}$	0.10
$\overline{\text{D.7}}$	2	$-1.30395 \cdot 10^{-03}$	0.013	$6.7300 \cdot 10^{-03}$	0.10
D.9	2	$-1.6661 \cdot 10^{-04}$	0.064	$2.3361 \cdot 10^{-03}$	0.12
D.10	2	$6.64155 \cdot 10^{-04}$	0.012	$3.7418 \cdot 10^{-03}$	0.14
D.11	2	$2.34300 \cdot 10^{-04}$	0.031	$2.0845 \cdot 10^{-03}$	0.083
D.12	1	$4.11063 \cdot 10^{-04}$	0.017	$3.5114 \cdot 10^{-03}$	0.12
D.13	1	$2.41514 \cdot 10^{-04}$	0.026	$8.222 \cdot 10^{-04}$	0.19
D.14	2	$5.8386 \cdot 10^{-05}$	0.088	$1.76075 \cdot 10^{-02}$	0.055
D.15	1	$-1.75957 \cdot 10^{-04}$	0.022	$-7.242 \cdot 10^{-04}$	0.14
Total		$1.40910 \cdot 10^{-03}$	0.056	$1.04214 \cdot 10^{-01}$	0.024
Benchmark		$1.40941 \cdot 10^{-03}$	-0.022	$1.0386 \cdot 10^{-01}$	0.34

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$$\sigma_{\gamma^* \rightarrow X}^{\overline{\text{MS}}}(p^2, \mu_r^2 = p^2) \supset K_X n_f \left(\frac{\alpha_s}{4\pi}\right)^2 \sigma_{\gamma^* \rightarrow X}^{(\text{LO})}(p^2)$$

NUMERICAL RESULTS

SG id	Ξ	$\sigma_{\gamma^* \rightarrow jj}^{(\overline{\text{MS}})}$ [GeV ⁻²] $p_{\gamma^*}^2 = \mu_r^2 = (400 \text{ GeV})^2$	Δ [%]	$\sigma_{\gamma^* \rightarrow t\bar{t}}^{[\alpha_s^{(\overline{\text{MS}})}, m_t^{(\text{OS})}]}$ [GeV ⁻²] $\mu_r^2 = m_t^2, p_{\gamma^*}^2 = (400 \text{ GeV})^2$	Δ [%]
LO $\mathcal{O}(\alpha_s^0)$					
A.1	1	$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
Total		$5.031049 \cdot 10^{-01}$	0.0018	$1.387586 \cdot 10^{+00}$	0.0011
NLO $\mathcal{O}(\alpha_s^1)$					
B.1	1	$5.03926 \cdot 10^{-02}$	0.0075	$2.52705 \cdot 10^{-01}$	0.034
B.2	2	$-3.14956 \cdot 10^{-02}$	0.018	$1.80050 \cdot 10^{-01}$	0.049
Total		$1.88970 \cdot 10^{-02}$	0.036	$4.3276 \cdot 10^{-01}$	0.028
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[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

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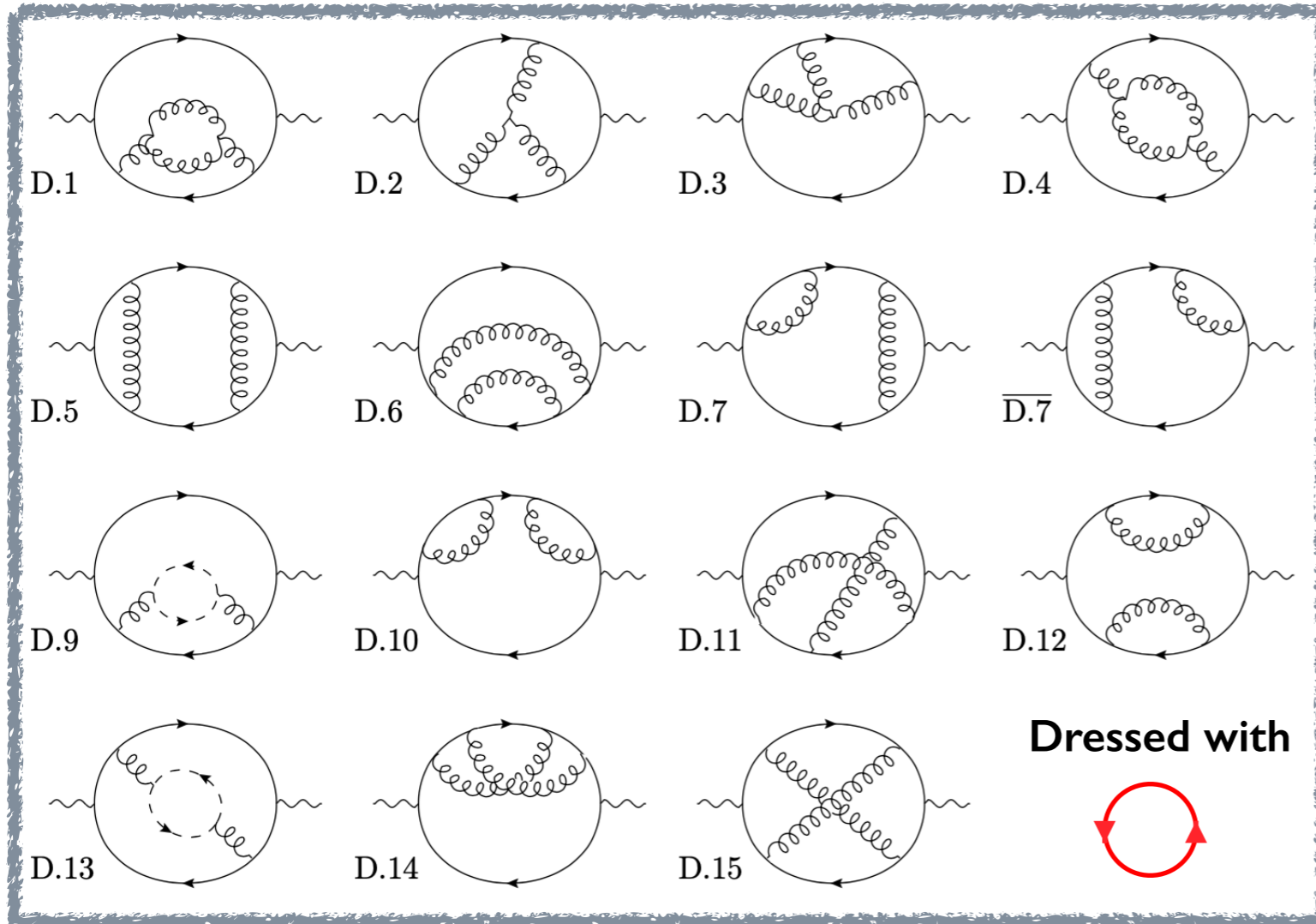
$$\gamma^* \rightarrow t\bar{t}$$

$$K_{t\bar{t}} = \delta^{(2)} = -\frac{(3-v^2)(1+v^2)}{6} \times \left\{ \text{Li}_3(p) - 2\text{Li}_3(1-p) - 3\text{Li}_3(p^2) - 4\text{Li}_3\left(\frac{p}{1+p}\right) - 5\text{Li}_3(1-p^2) + \frac{11}{2}\zeta(3) \right. \\ \left. + \text{Li}_2(p) \ln\left(\frac{4(1-v^2)}{v^4}\right) + 2\text{Li}_2(p^2) \ln\left(\frac{1-v^2}{2v^2}\right) + 2\zeta(2) \left[\ln p - \ln\left(\frac{1-v^2}{4v}\right) \right] \right. \\ \left. - \frac{1}{6} \ln\left(\frac{1+v}{2}\right) \left[36 \ln 2 \ln p - 44 \ln^2 p + 49 \ln p \ln\left(\frac{1-v^2}{4}\right) + \ln^2\left(\frac{1-v^2}{4}\right) \right] \right. \\ \left. - \frac{1}{2} \ln p \ln v \left[36 \ln 2 + 21 \ln p + 16 \ln v - 22 \ln(1-v^2) \right] \right\} \\ + \frac{1}{24} \left\{ (15 - 6v^2 - v^4) (\text{Li}_2(p) + \text{Li}_2(p^2)) + 3(7 - 22v^2 + 7v^4) \text{Li}_2(p) \right. \\ \left. - (1-v)(51 - 45v - 27v^2 + 5v^3) \zeta(2) \right. \\ \left. + \frac{(1+v)(-9 + 33v - 9v^2 - 15v^3 + 4v^4)}{v} \ln^2 p \right. \\ \left. + \left[(33 + 22v^2 - 7v^4) \ln 2 - 10(3-v^2)(1+v^2) \ln v \right. \right. \\ \left. \left. - (15 - 22v^2 + 3v^4) \ln\left(\frac{1-v^2}{4v^2}\right) \right] \ln p \right. \\ \left. + 2v(3-v^2) \ln\left(\frac{4(1-v^2)}{v^4}\right) \left[\ln v - 3 \ln\left(\frac{1-v^2}{4v}\right) \right] \right. \\ \left. + \frac{237 - 96v + 62v^2 + 32v^3 - 59v^4}{4} \ln p - 16v(3-v^2) \ln\left(\frac{1+v}{4}\right) \right. \\ \left. - 2v(39 - 17v^2) \ln\left(\frac{1-v^2}{2v^2}\right) - \frac{v(75 - 29v^2)}{2} \right\} \dots \quad (\text{B.3})$$

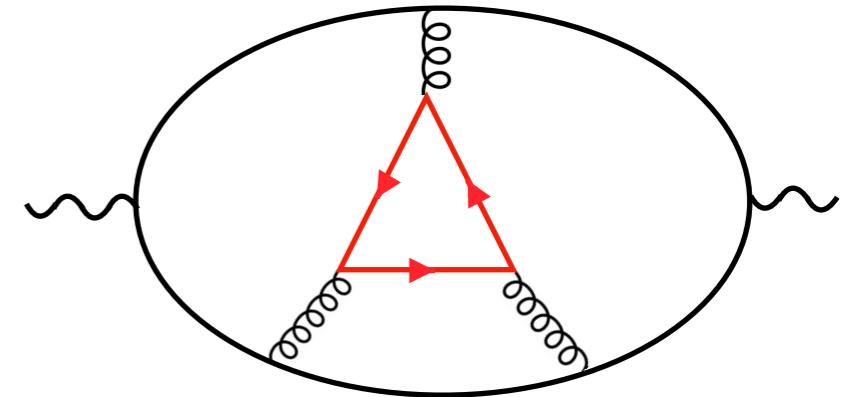
[Chetyrkin, Kuehn, Steinhauser, arxiv : 9606230]

PRELIMINARY N3LO RESULTS

n_f contributions :



+ new topologies, such as:



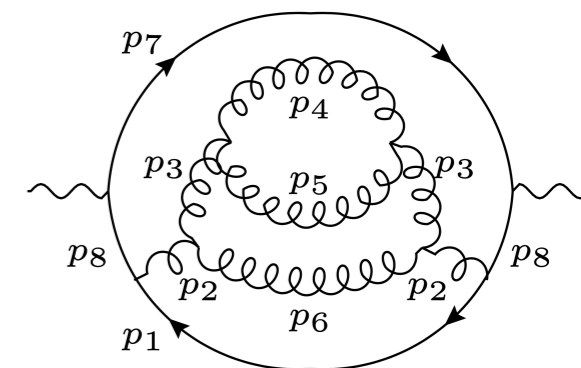
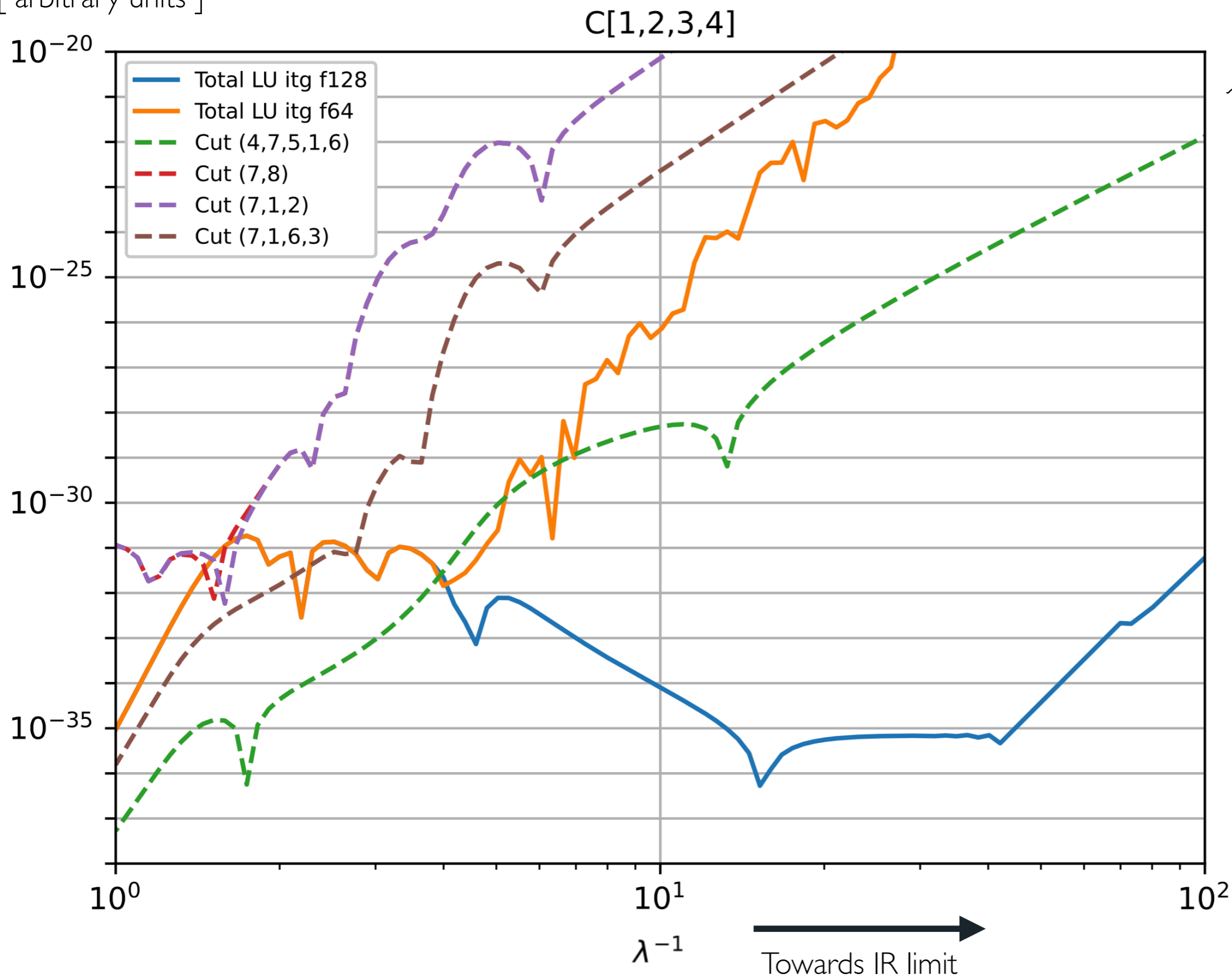
$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f), (\text{MC LU})} = -77.1(1.7)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f)} = -C_F^2 \left(\frac{29}{2} - 152\zeta_3 + 160\zeta_5 \right) - C_F C_A \left(\frac{15520}{27} - \frac{88}{3}\zeta_2 - \frac{3584}{9}\zeta_3 - \frac{80}{3}\zeta_5 \right) = -76.8086$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

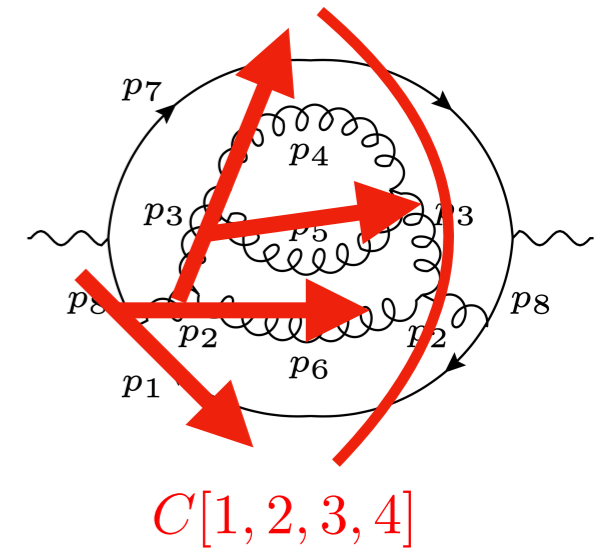
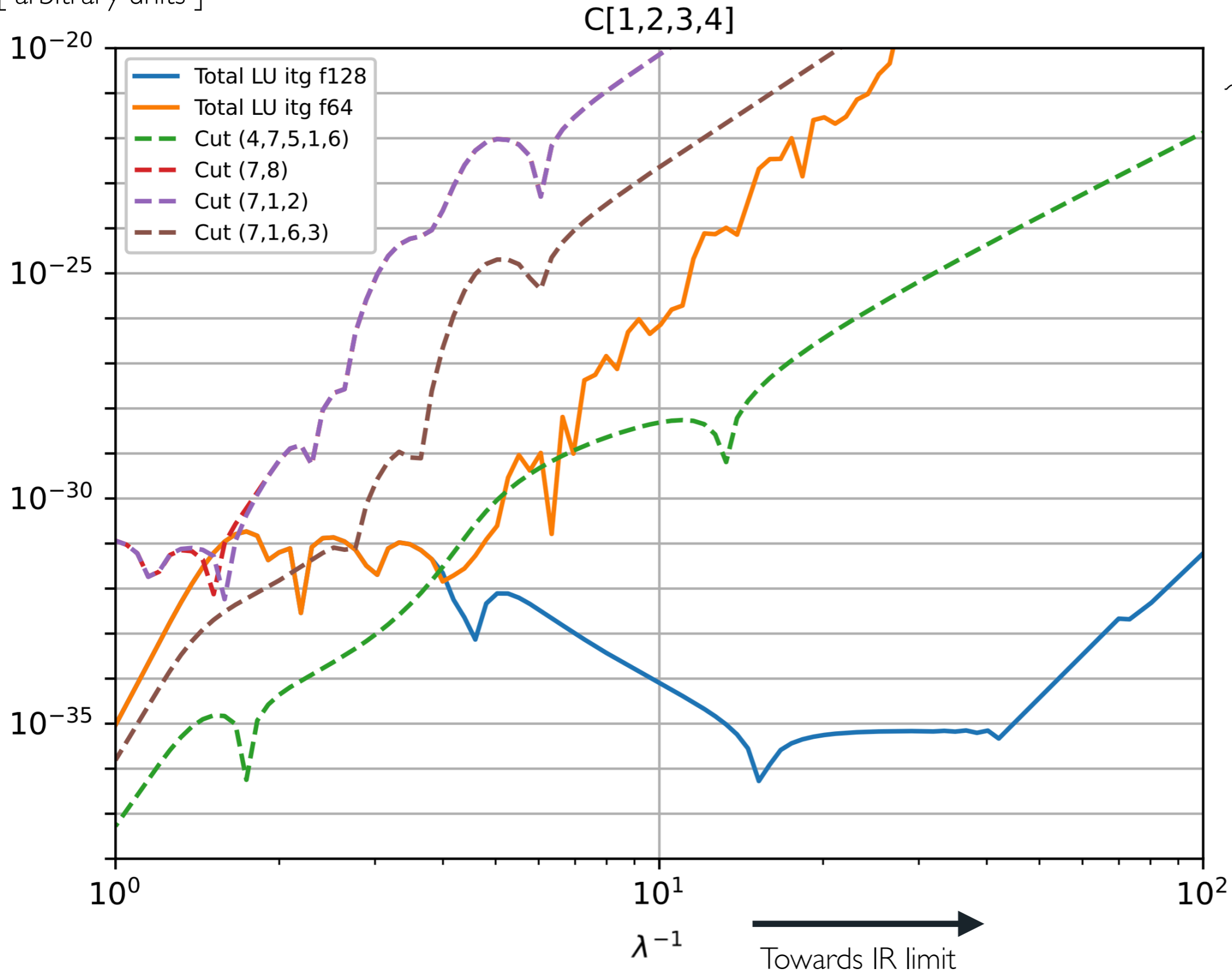
TESTING IR QUADRUPLE COLLINEAR LIMITS

[arbitrary units]



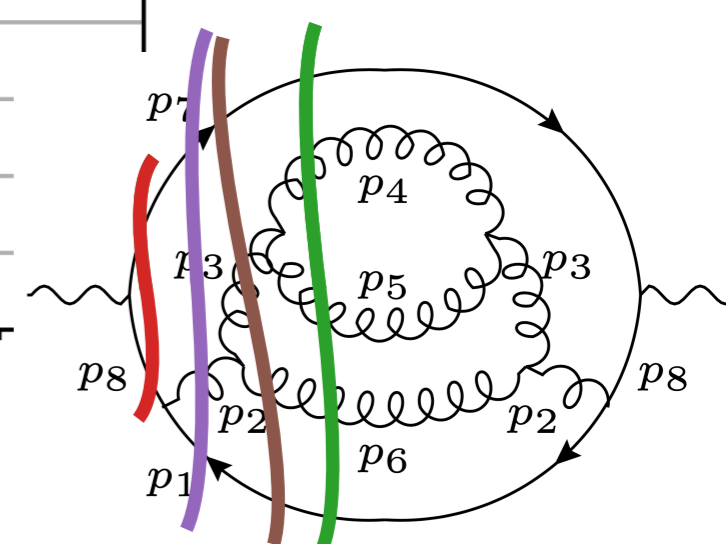
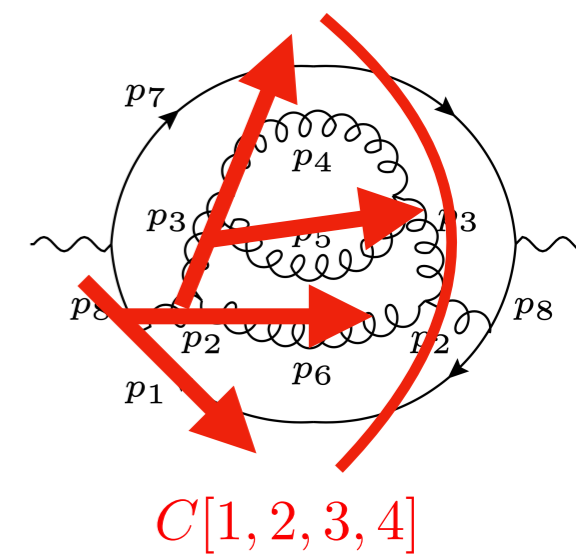
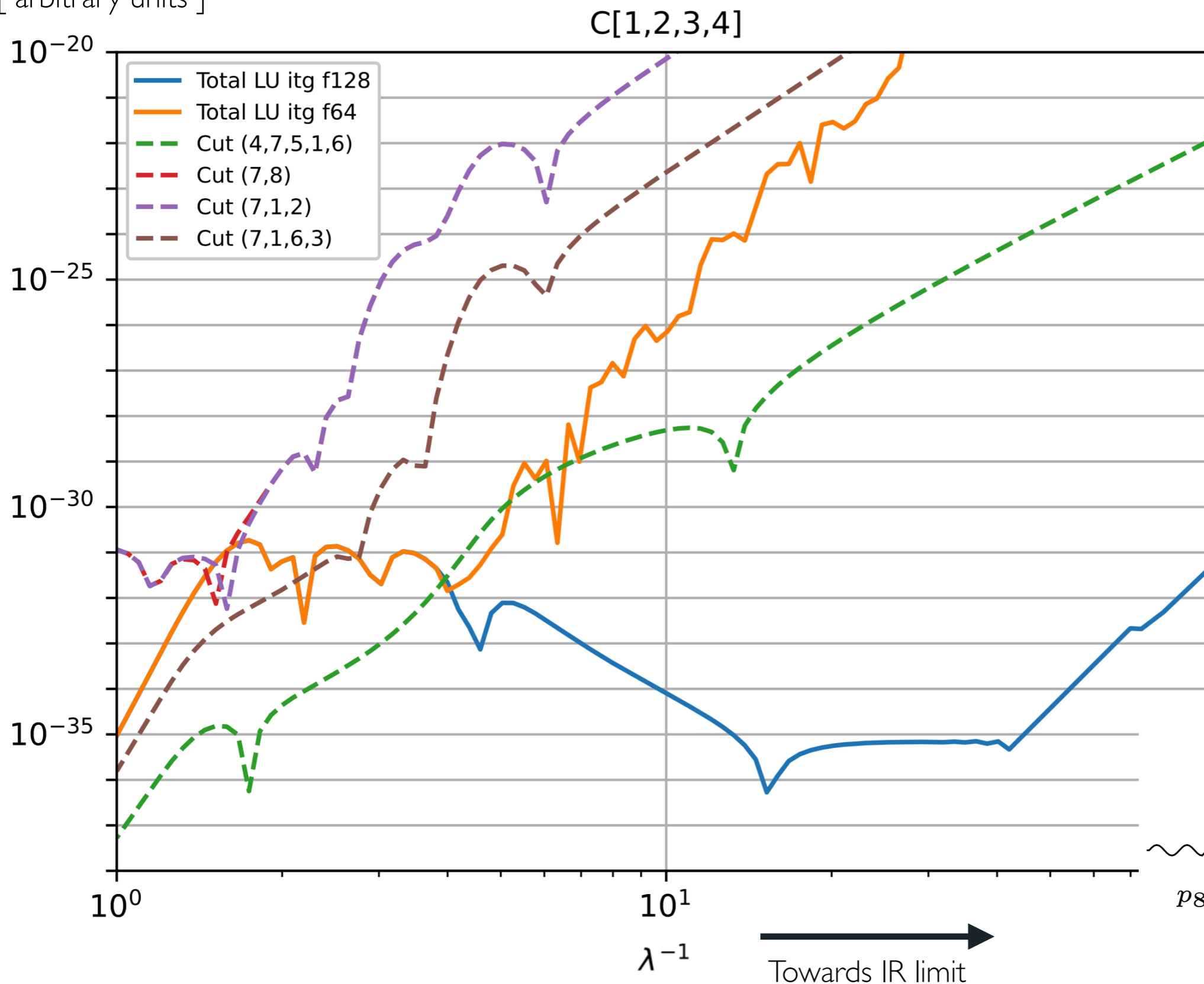
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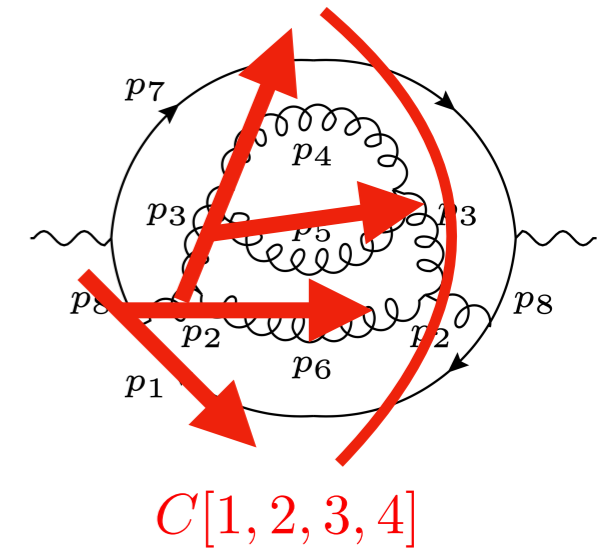
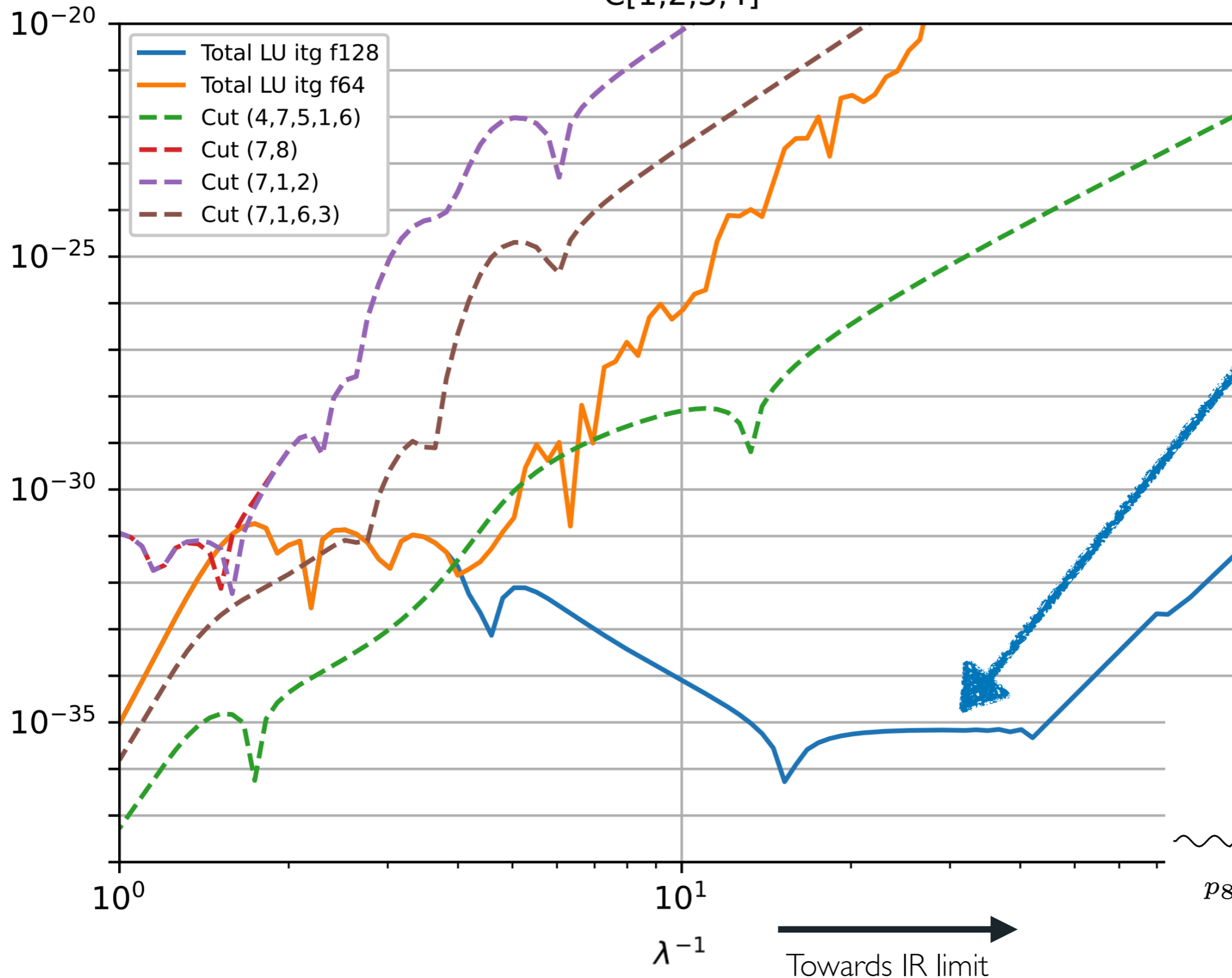
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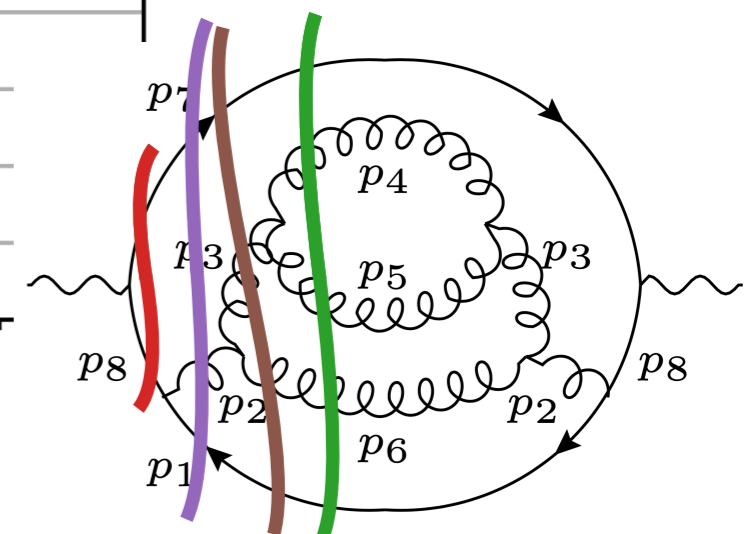


TESTING IR QUADRUPLE COLLINEAR LIMITS

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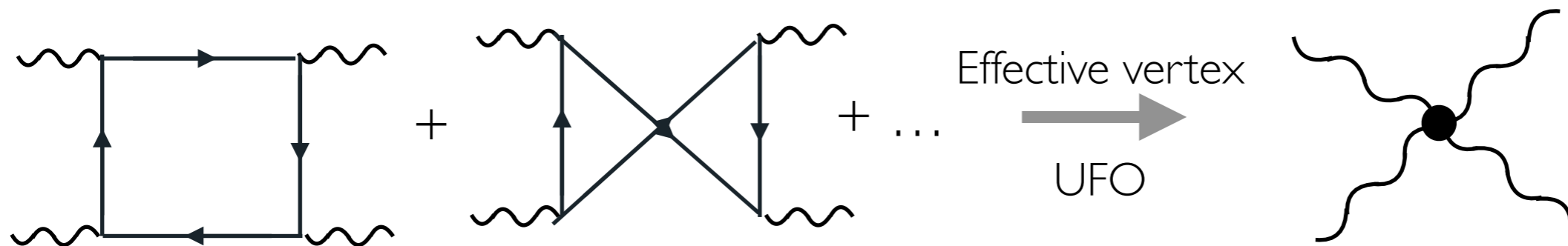
LU integrand always goes to a constant on **collinear limits** without incl. *any* scaling of the measure ! No residual integrable singularity.



PHOTON-PHOTON SCATTERING IN HI UPC

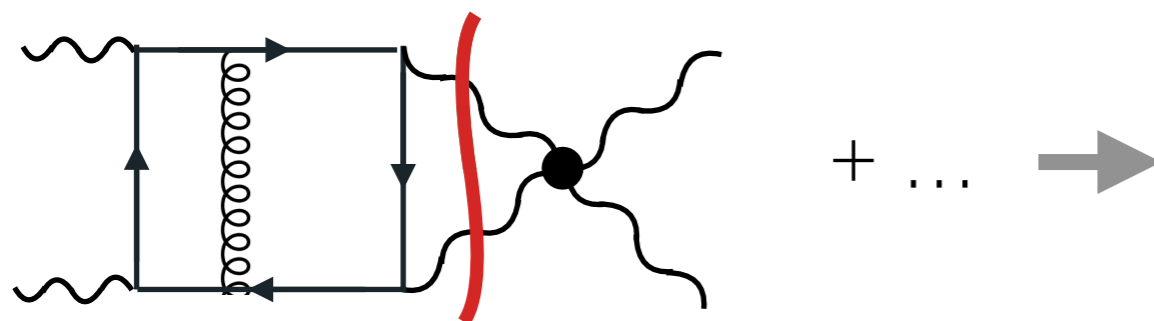
collaboration with H.S. [Shao, M. Fraaije, E. Chaubet]

- Too early to present / show much, but alphaLoop delivered a first NLO unknown x-sec!



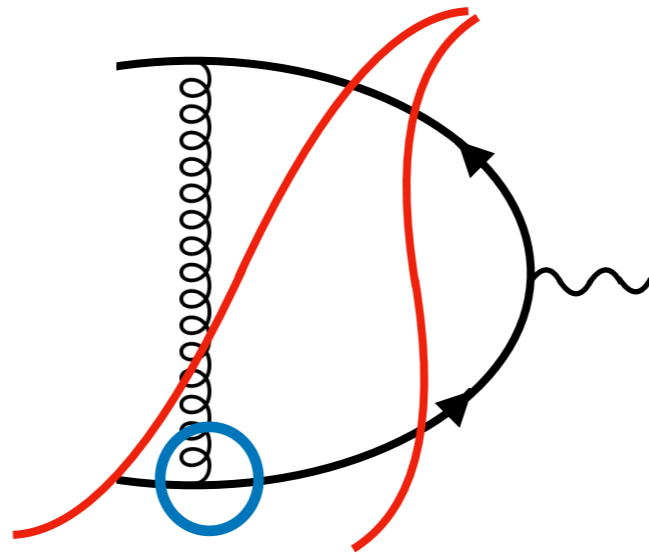
- In alphaloop we then do

$$\sim \sum_{i=1}^3 C_i(s, t, u, m_f^2) T_i^{\mu_1 \mu_2 \mu_3 \mu_4}(\{p_i\})$$



Yielding our first “prediction” for a piece of a yet unknown cross-section: NLO photon scattering.
Successful validation vs analytics!

INITIAL-STATE SINGULARITIES

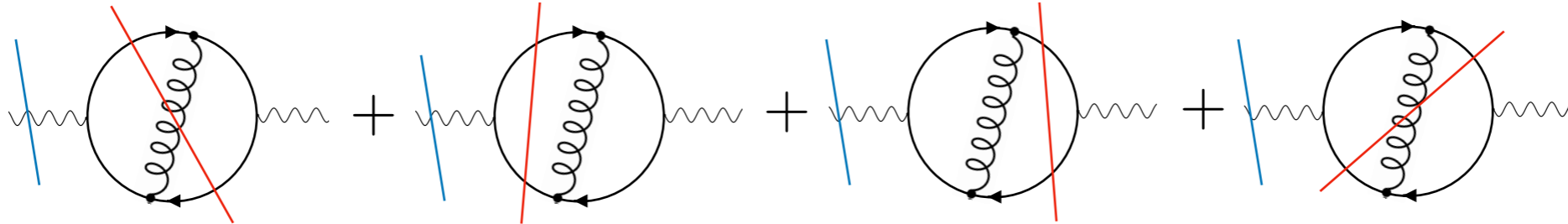


KLN CAN WORK FOR INITIAL-STATE !

INITIAL-STATE SINGULARITIES: IDEA

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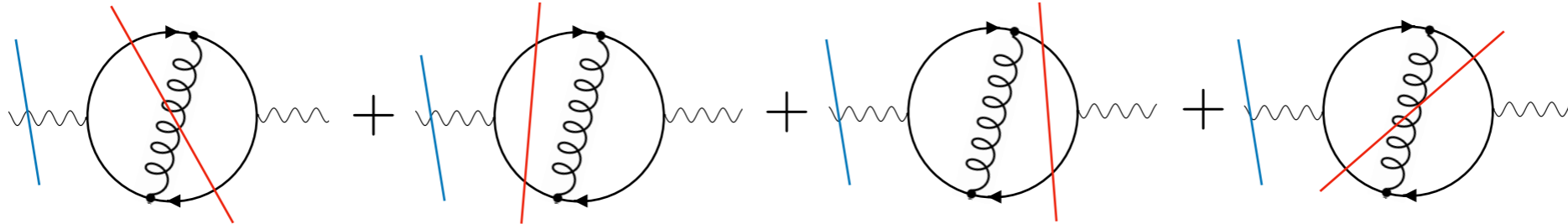
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(Including all degenerate configurations, higher final-state multiplicities)

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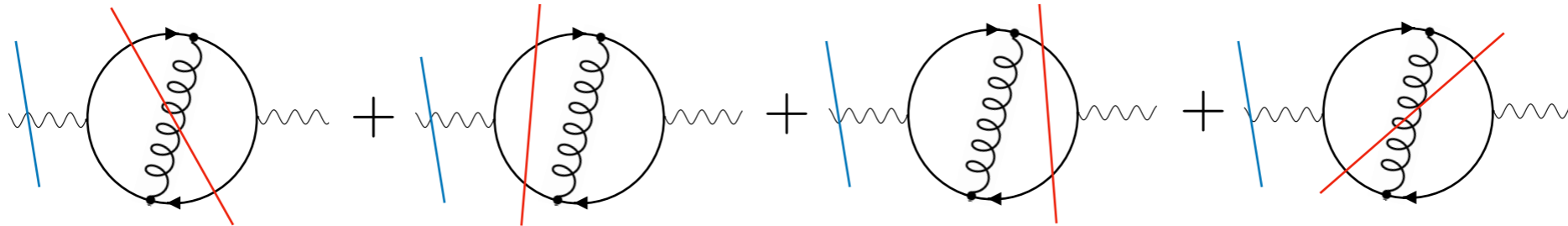


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Flip it, and obtain the answer for Drell-Yan, $2j \rightarrow e^+e^-$

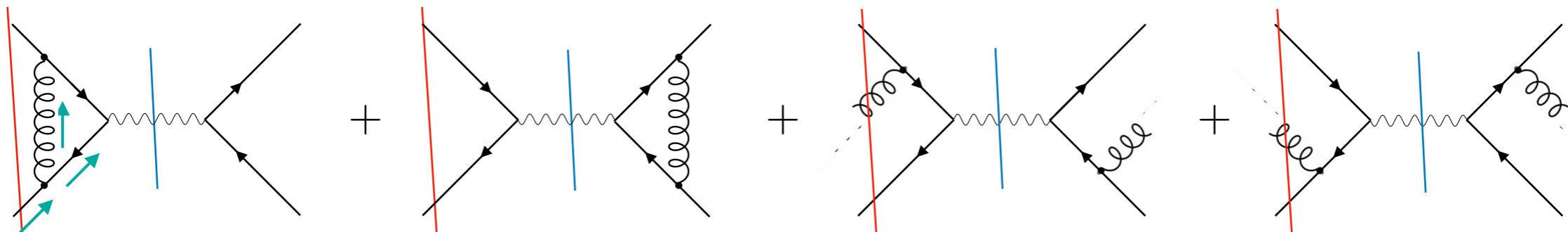
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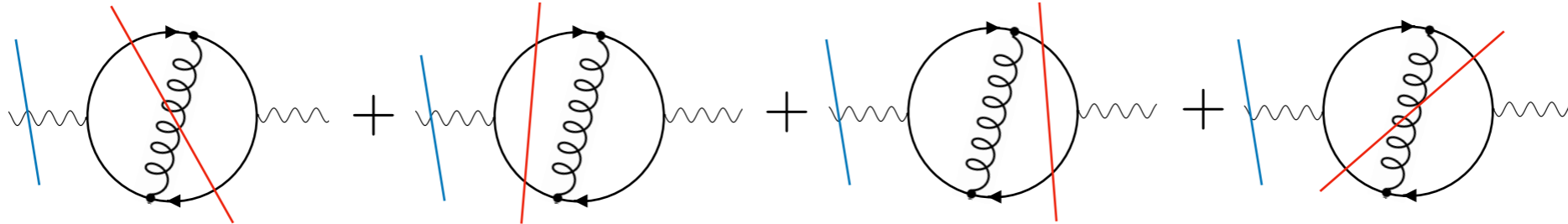
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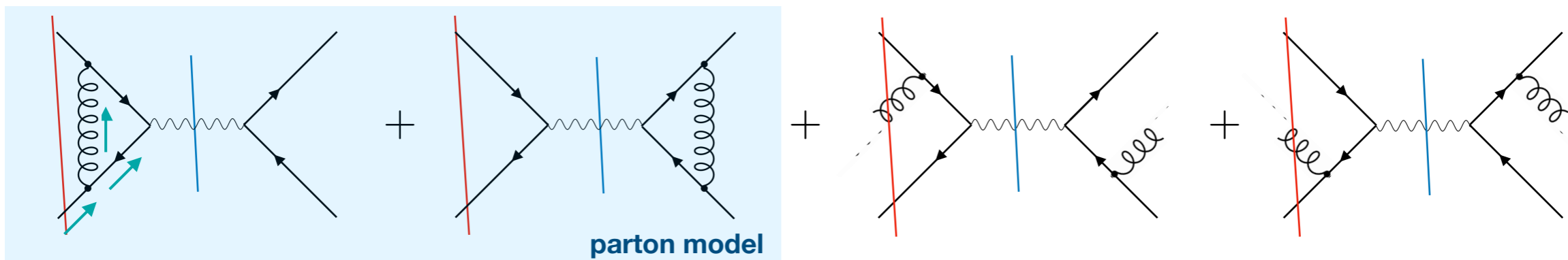
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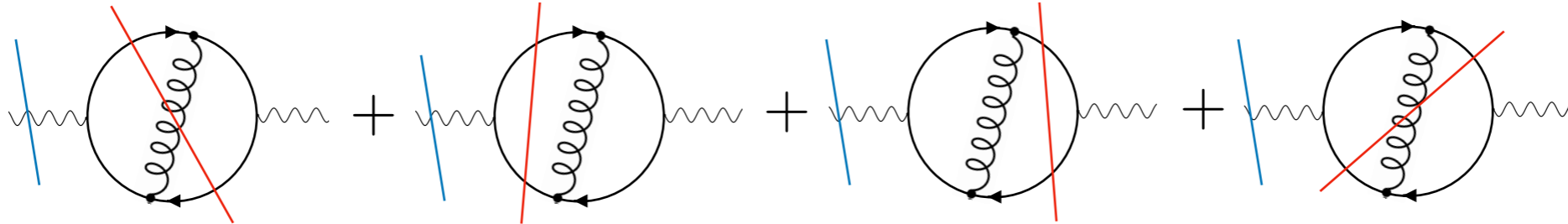
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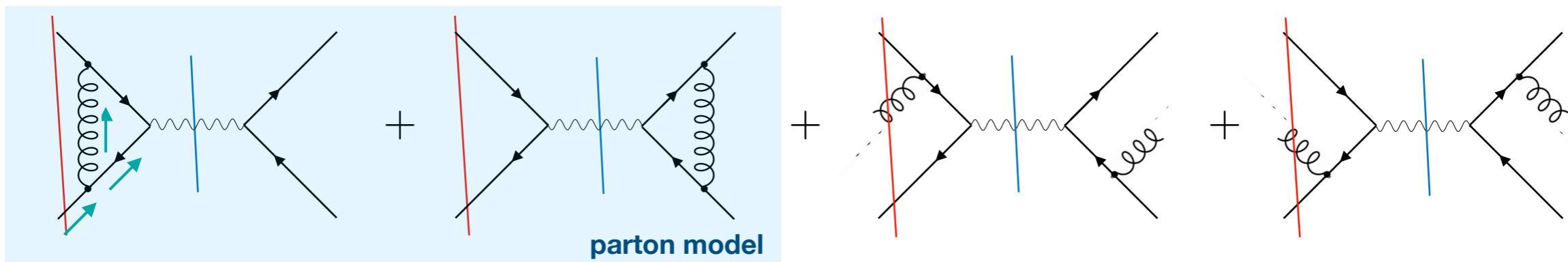
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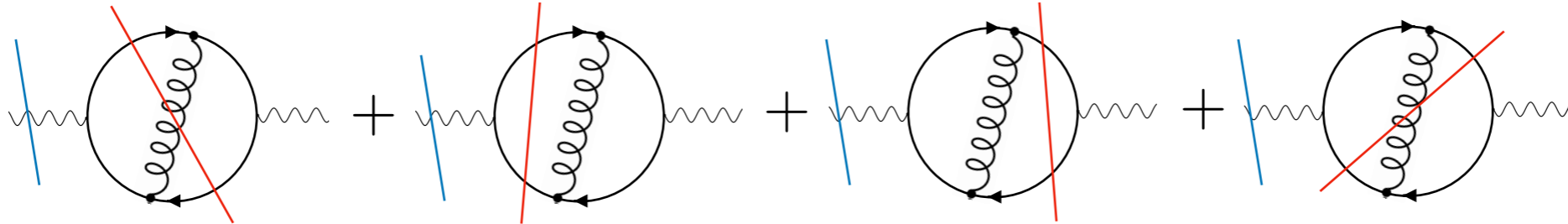
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The initial state singularity is now absent!

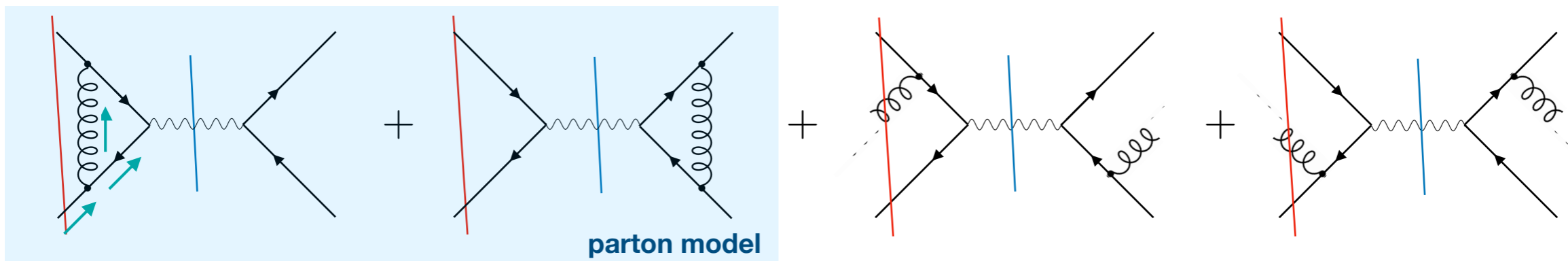
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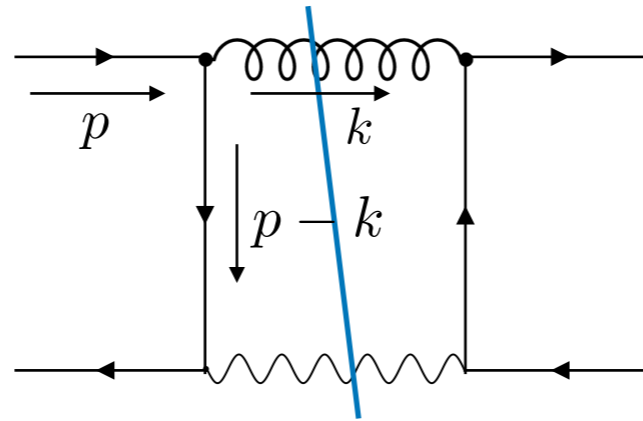
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Include degenerate initial states \rightarrow Higher multiplicity initial states

INITIAL-STATE SINGULARITIES: IDEA

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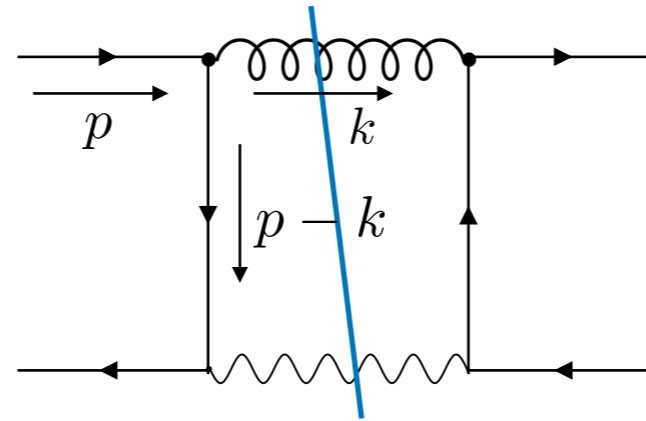
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Also has collinear
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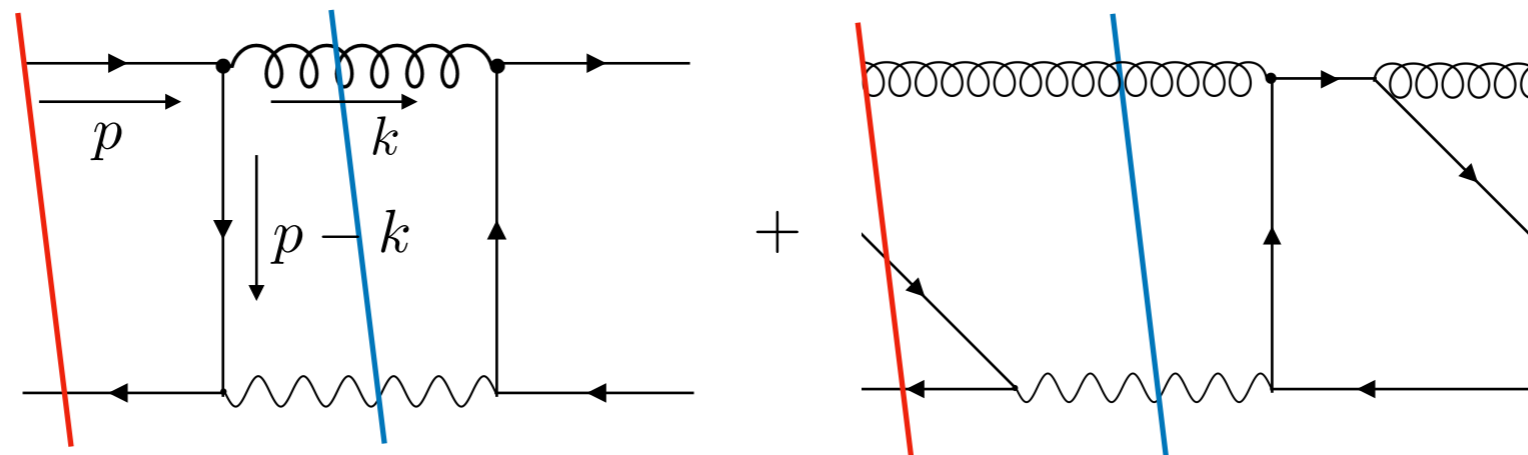
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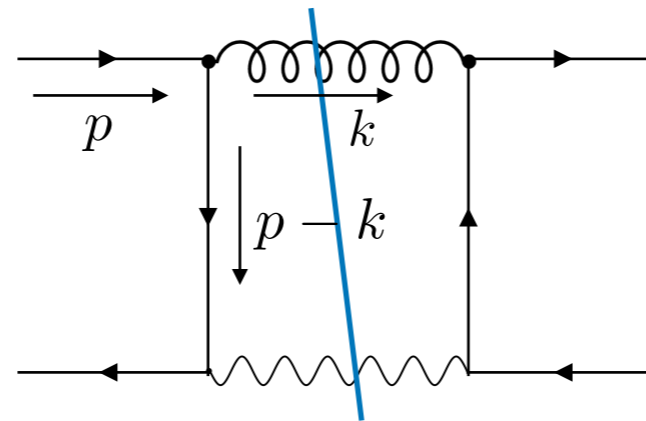
In this case, the cancelling partner is



Higher multiplicity initial states, but also **disconnected!** **Free travelling gluon!**

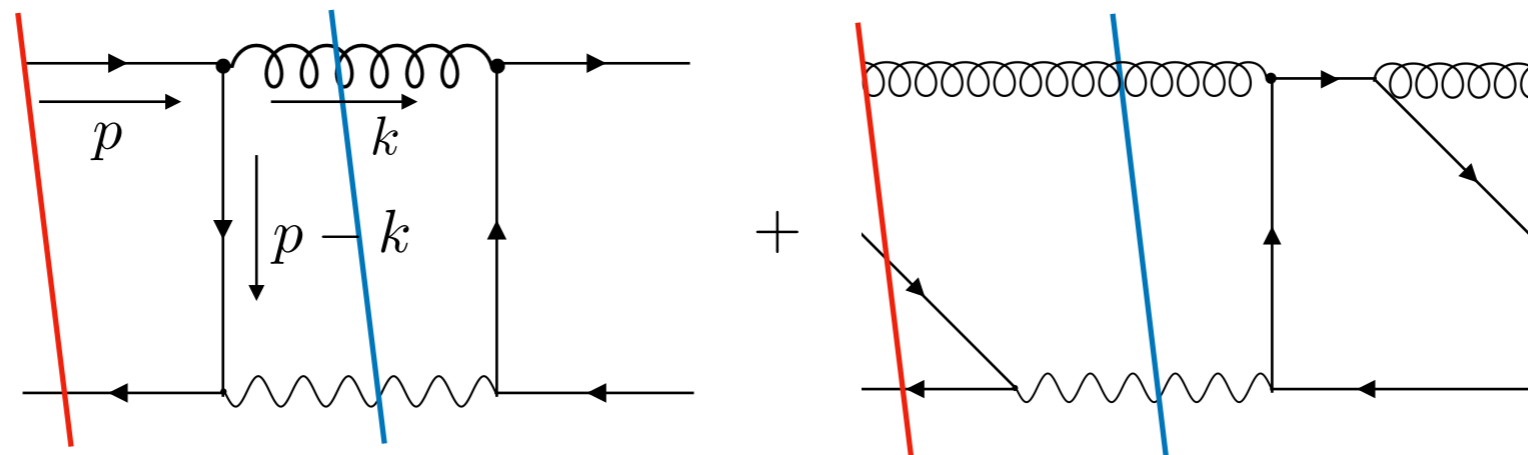
INITIAL-STATE SINGULARITIES: IDEA

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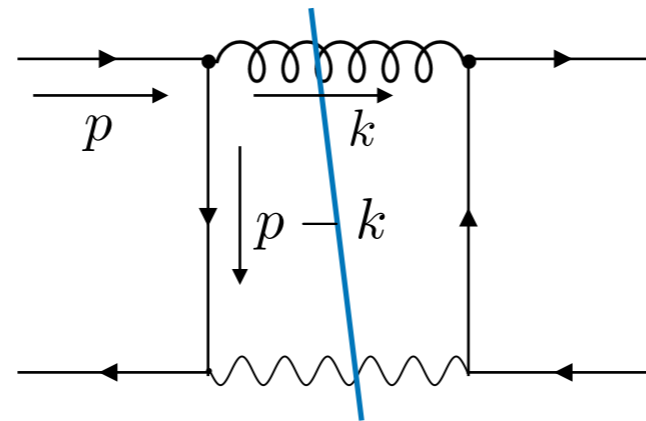


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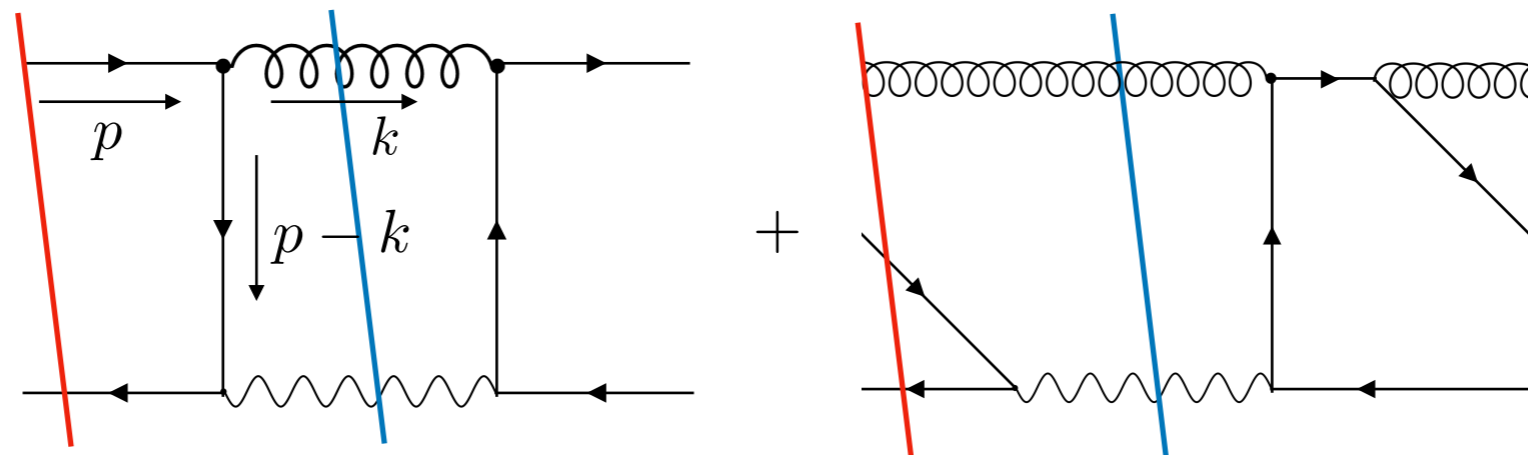
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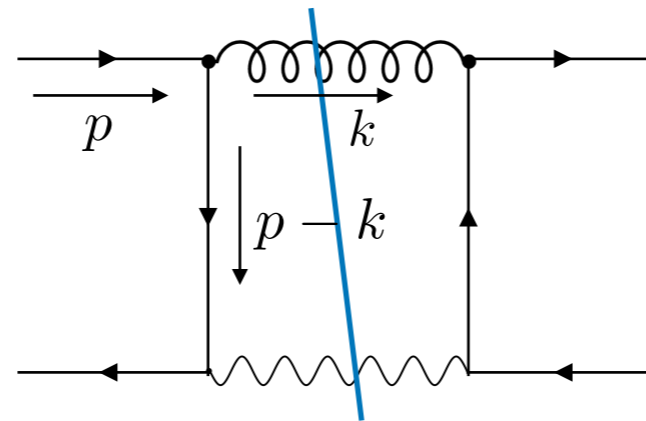
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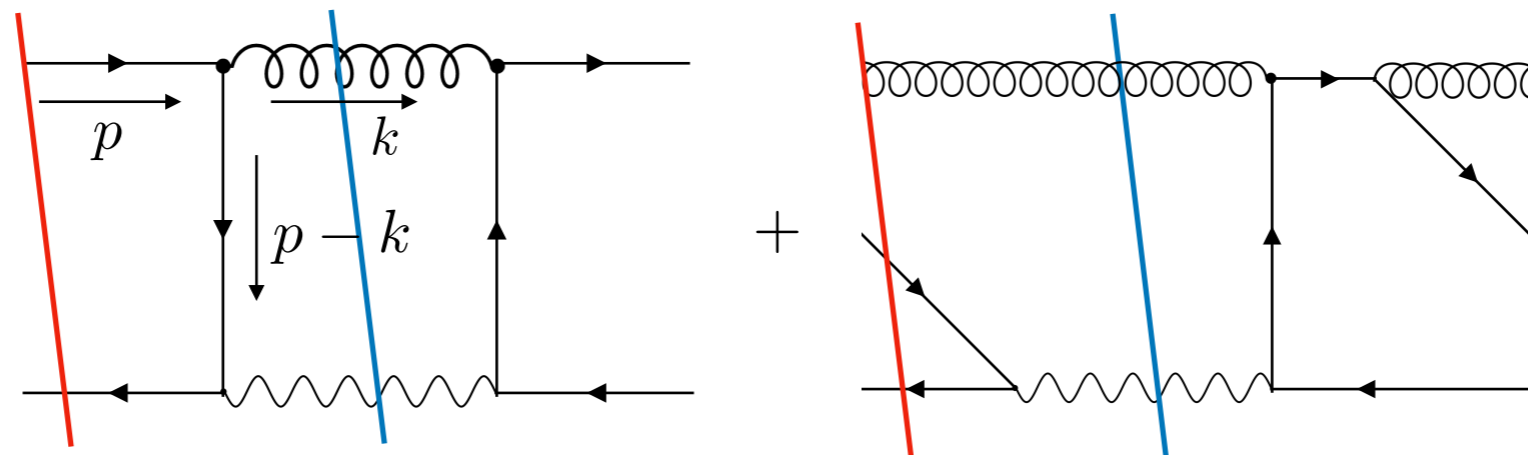
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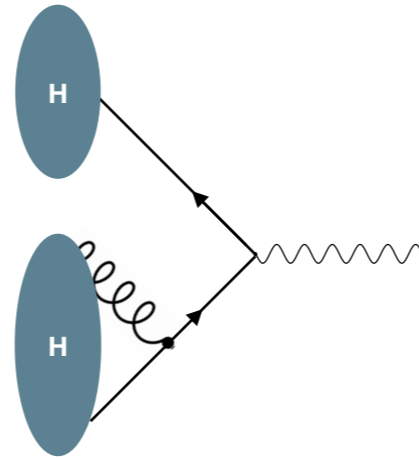
But also more recently, they were studied in:

Frye, Hannesdottir, Paul, Schwartz, Yan
arXiv:1810.10022 (2019)

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

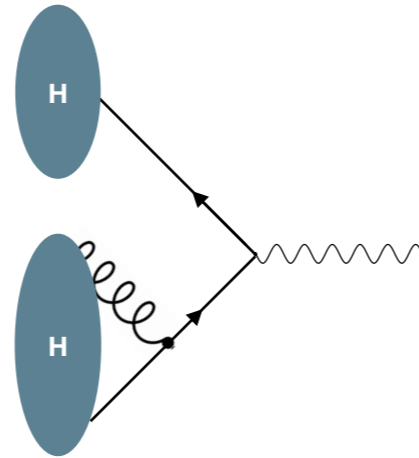
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This argument suggests that, in order to maintain IR-finiteness, one requires more than two initial state partons

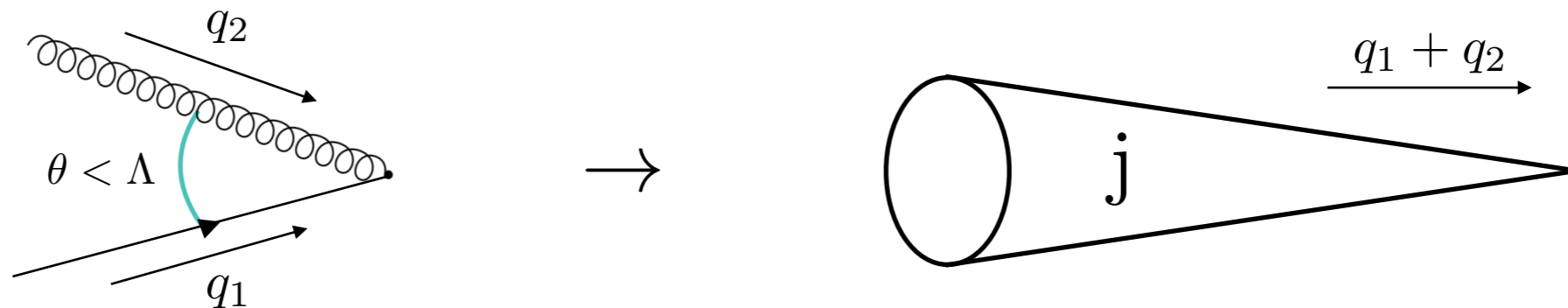


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and that the multiple partons should be clustered into **two jet-like objects** that resemble boosted hadrons



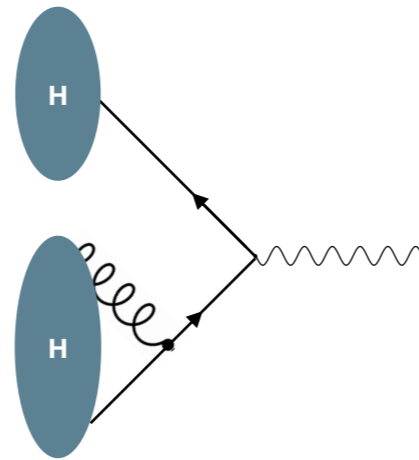
After clustering, we get two jets with momenta

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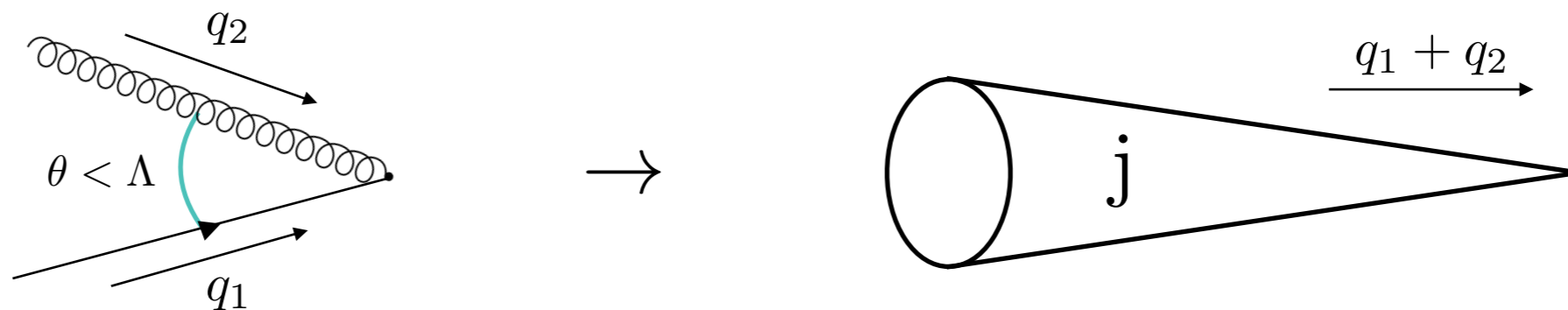
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Cluster initial states analogously to final states: symmetry initial-final state

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

There are two relevant scales for the two initial state jets reconstructed:

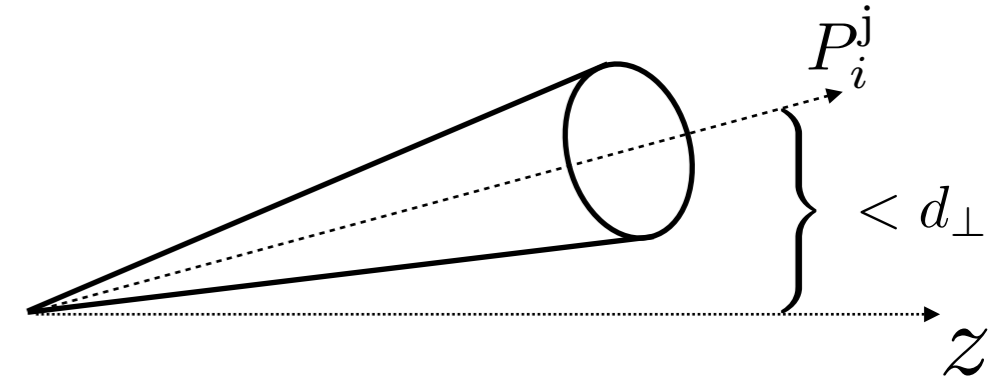
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There are two relevant scales for the two initial state jets reconstructed:

- One measuring the allowed phase space for the **total momentum** of the jet

$$(P_i^j)_\perp < d_\perp$$

If the scale is zero, the jet lies **exactly** on the z axis



If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

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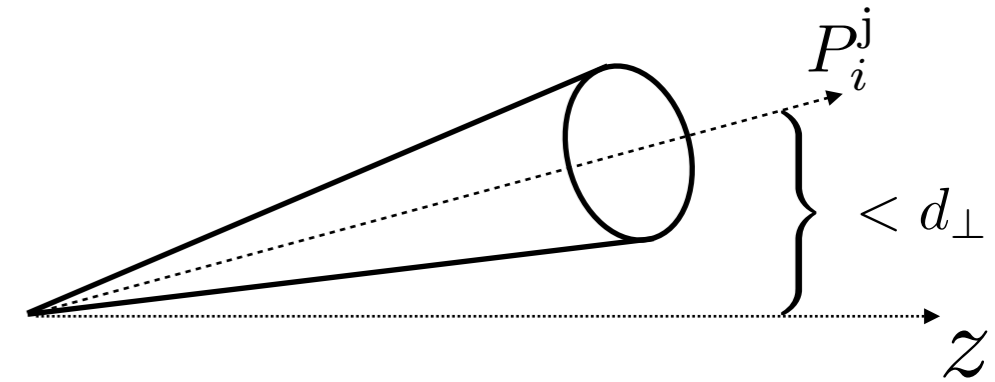
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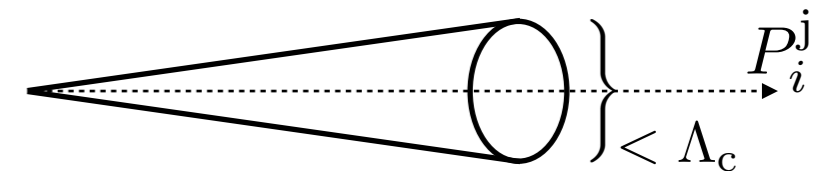


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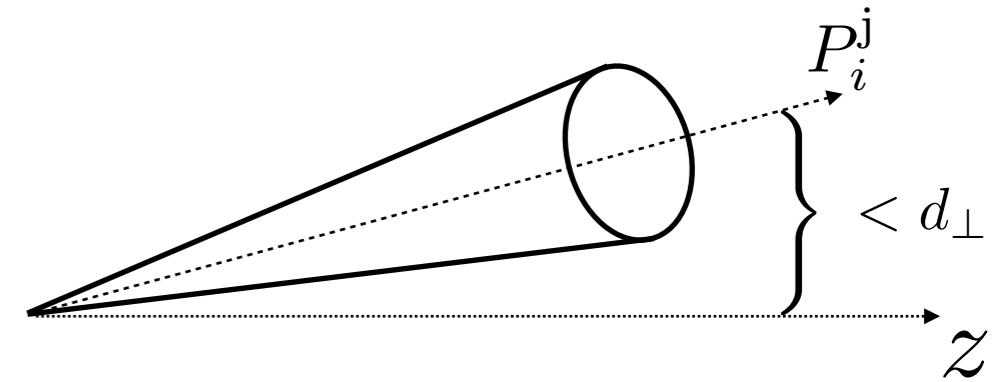
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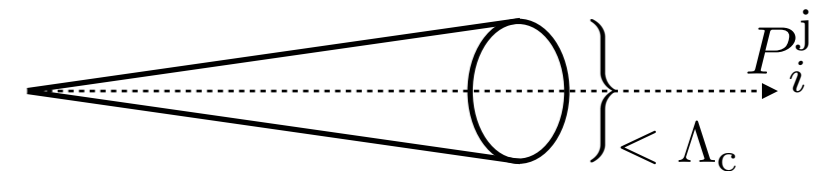


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Λ_c is the equivalent of the factorisation scale! $\approx \log(\Lambda_c)$

INITIAL-STATE SINGULARITIES: PRELIMINARY TESTS

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This allows us to define **Bjorken variables**

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- Bin the distribution in the Bjorken variables \rightarrow **Fit PDFs!**
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- Vary the factorisation scale Λ_c and interpolate the dependence on the factorisation scale **Numerical resummation?** [Banfi, Salam, Zanderighi, arXiv:0407286 \(2004\)](#)

INITIAL-STATE SINGULARITIES: “PDFs”

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We started with a very generic formalism for scattering

$$\sigma(HH \rightarrow X + nj) = \sum_m \int \left[\prod_{i=1}^m d^3 \vec{p}_i \right] f(p_1, \dots, p_m) \frac{d^m \sigma}{dp_1 \dots dp_m}(p_1, \dots, p_m \rightarrow X + nj)$$

Sum over number of initial state partons

Integration over initial state partons momenta

Weight

Cross-sections for m initial state partons

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The diagram shows four labels with arrows pointing to specific parts of the equation:

- Sum over number of initial state partons**: points to the summation index m .
- Integration over initial state partons momenta**: points to the product of momenta $\prod_{i=1}^m d^3 \vec{p}_i$.
- Weight**: points to the function $f(p_1, \dots, p_m)$.
- Cross-sections for m initial state partons**: points to the differential cross-section $\frac{d^m \sigma}{dp_1 \dots dp_m}(p_1, \dots, p_m \rightarrow X + nj)$.

And we “forced” the initial-state observable to reproduce the usual factorised structure:

$$\sigma(HH \rightarrow X + nj) = \int dx_1 dx_2 f(x_1, \Lambda_c) f(x_2, \Lambda_c) \frac{d^2 \sigma_p}{dx_1 dx_2}(2j \rightarrow X + nj, \Lambda_c)$$

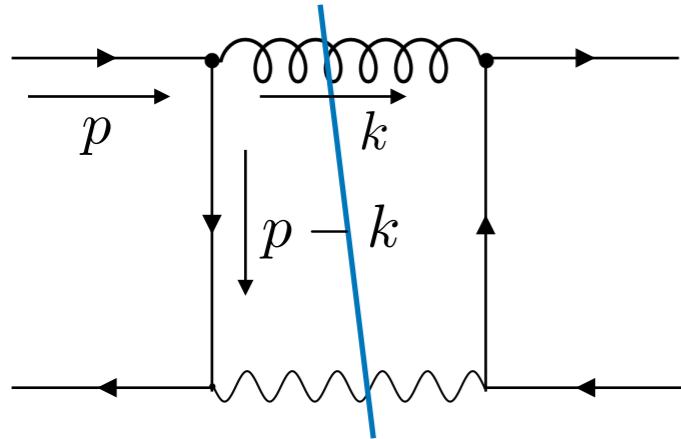
But we did not need to start from this factorised ansatz!

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Numerical example result for this finite sum of two interference diagrams:

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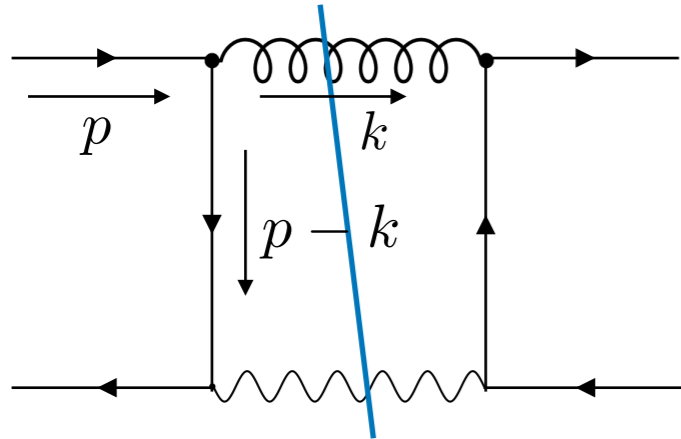


Always included!

This is the usual contribution.

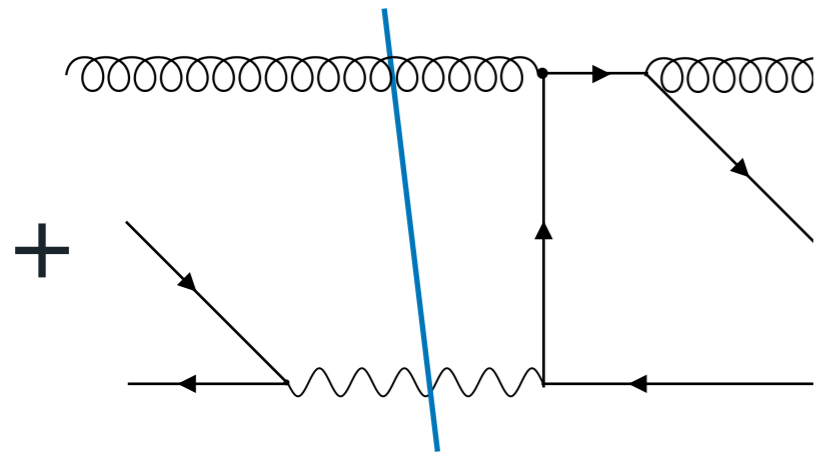
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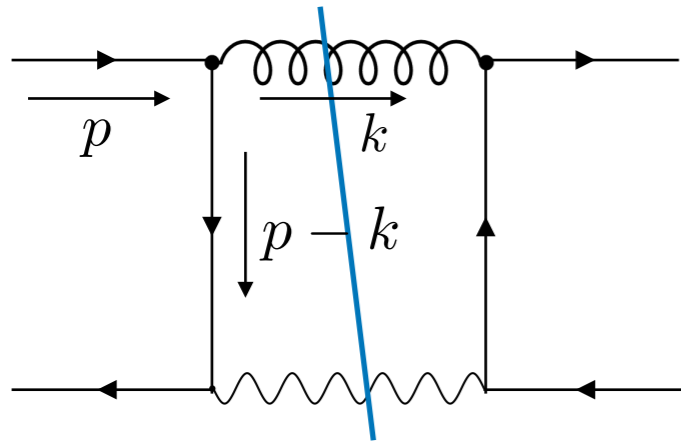
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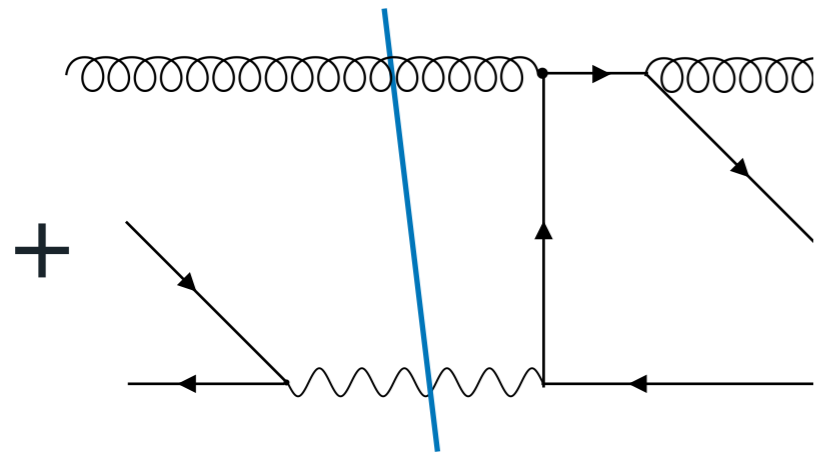
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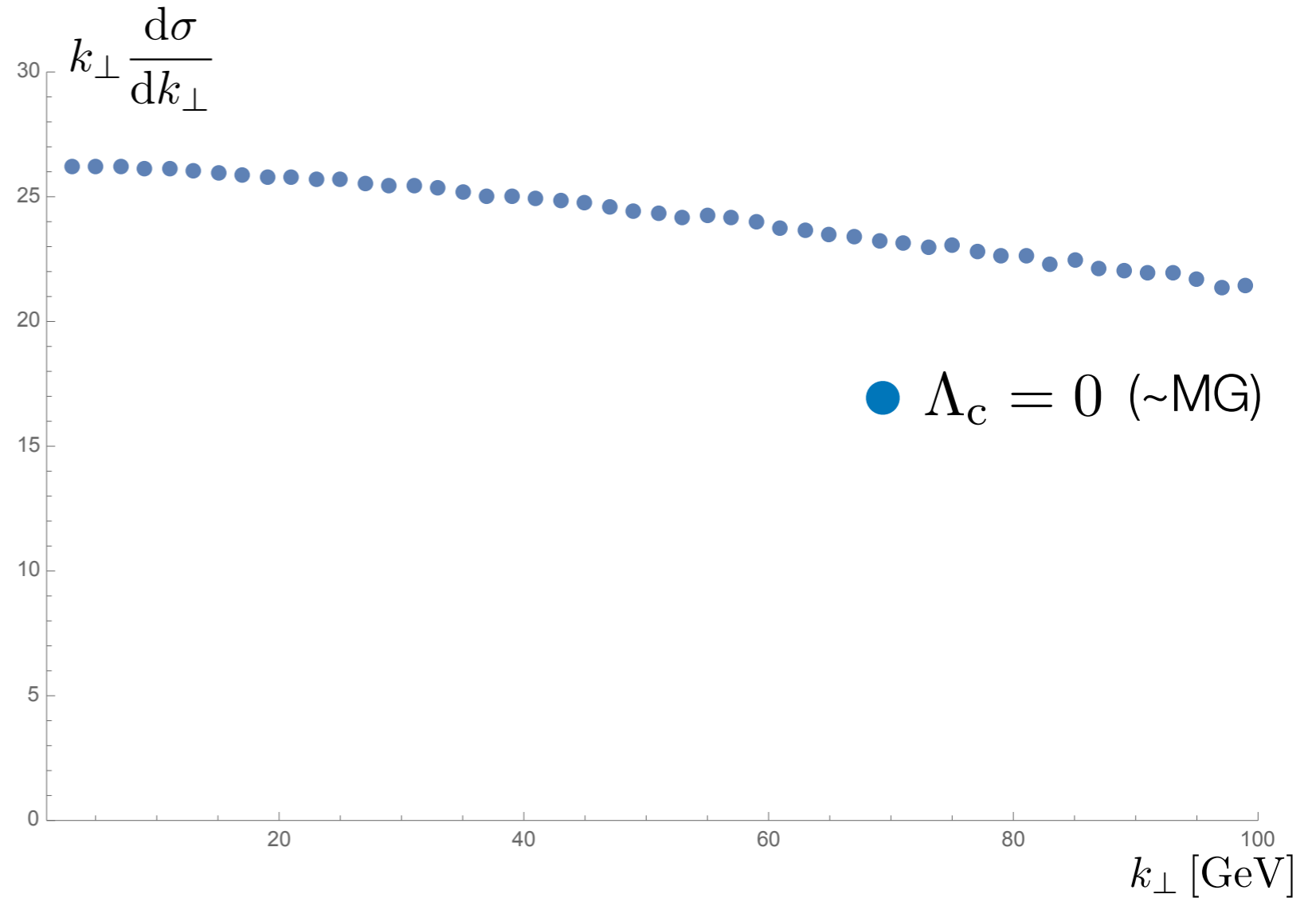
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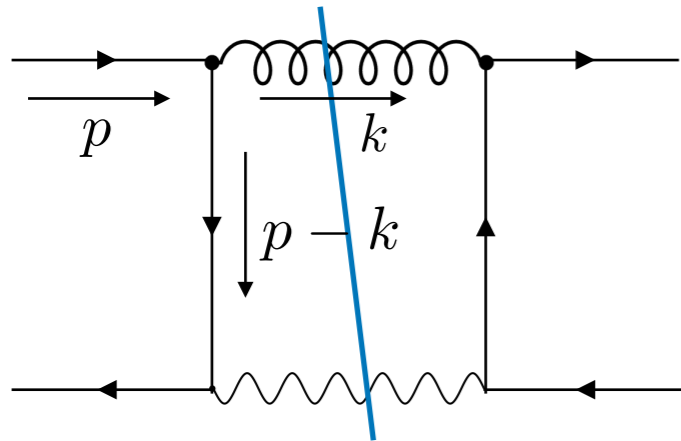
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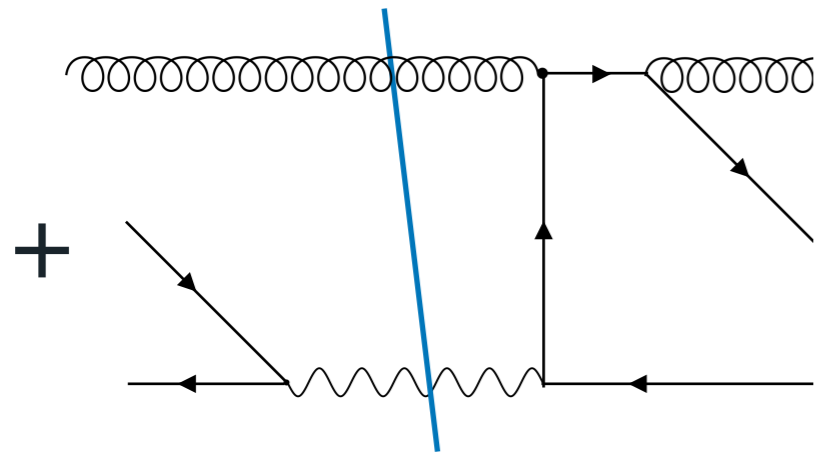
● $\Lambda_c = 0$ (~MG)

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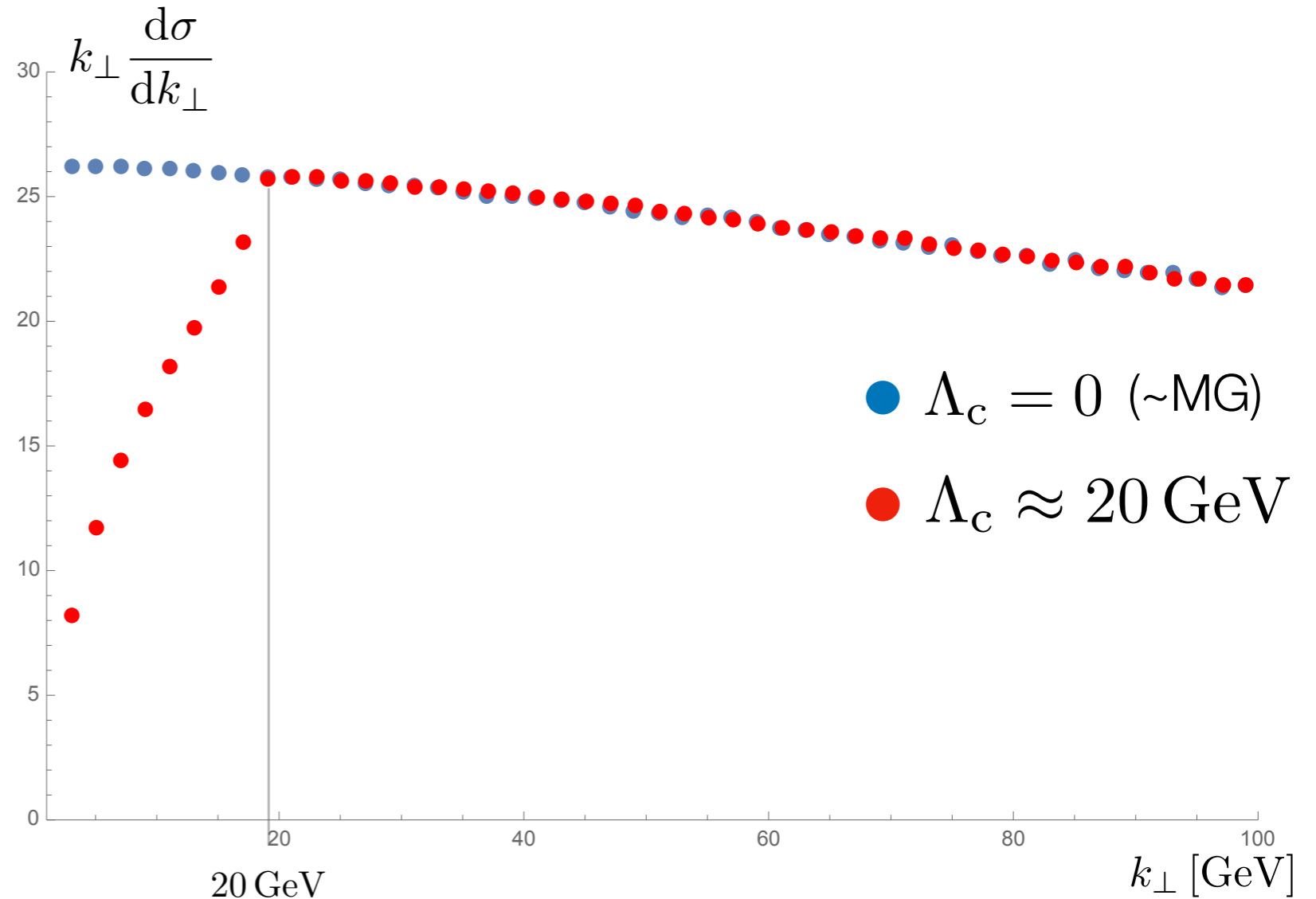
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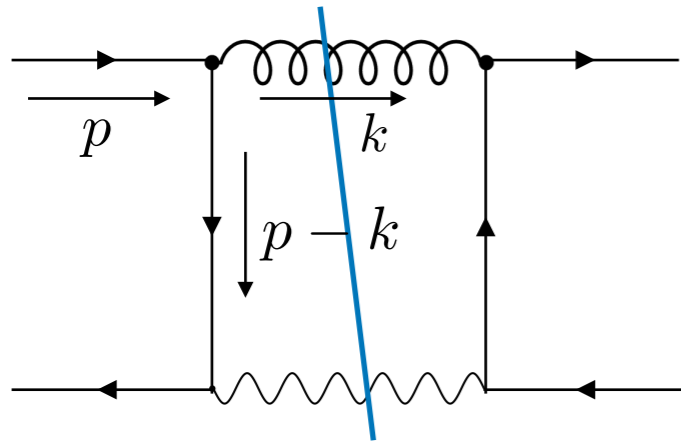


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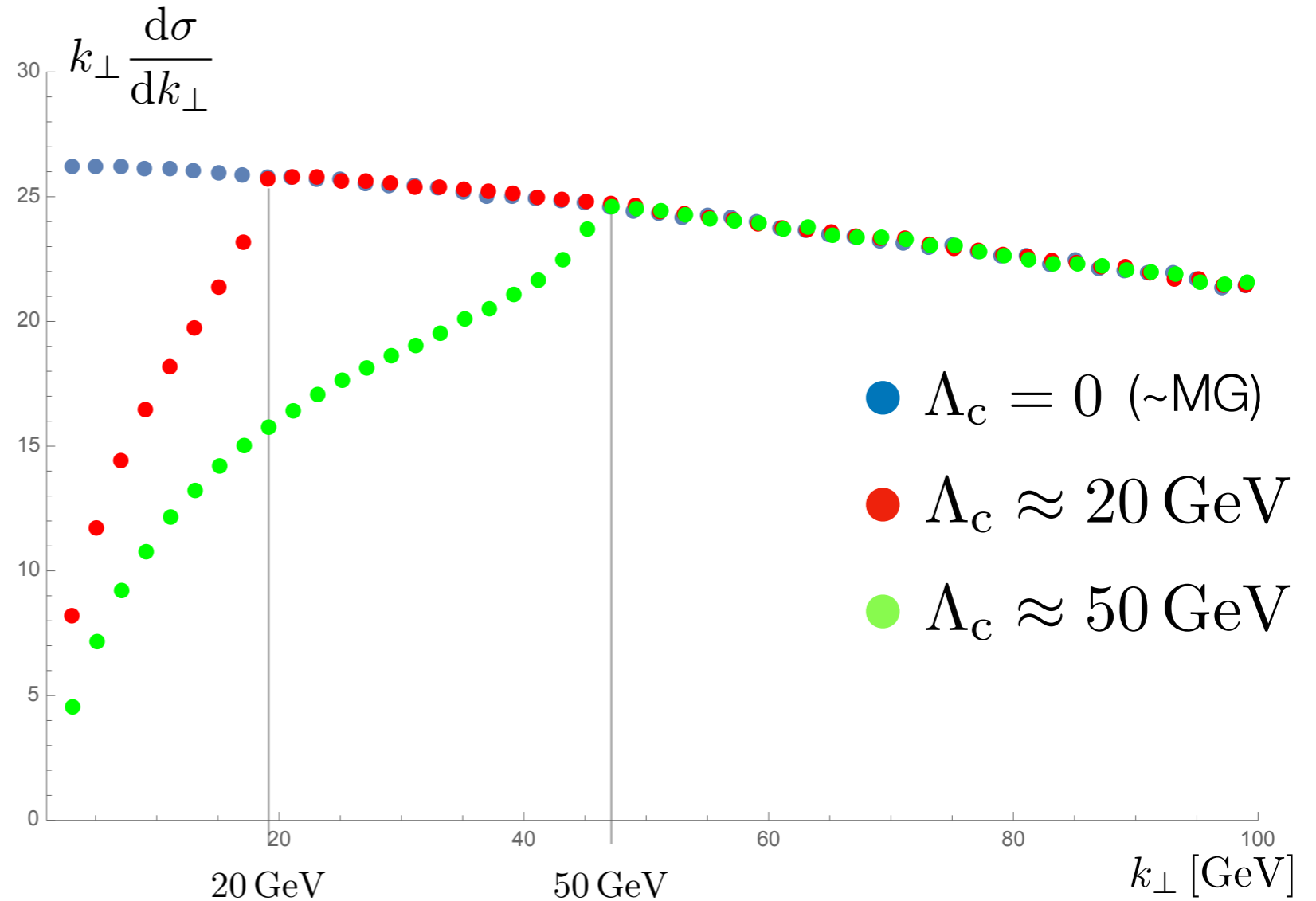
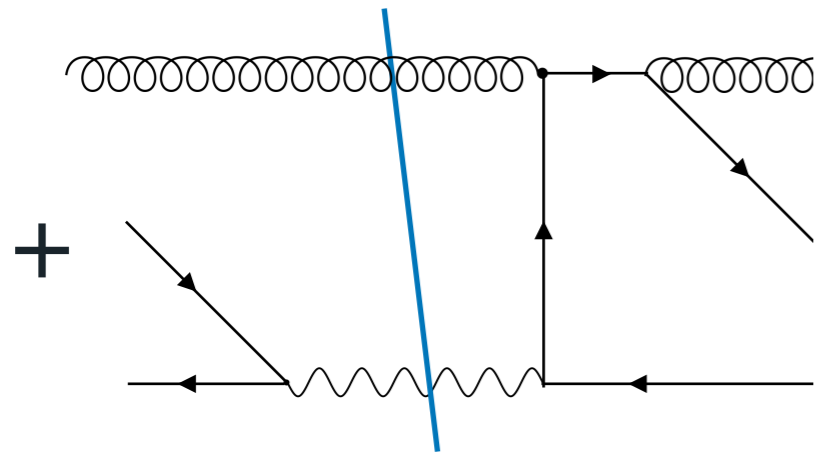


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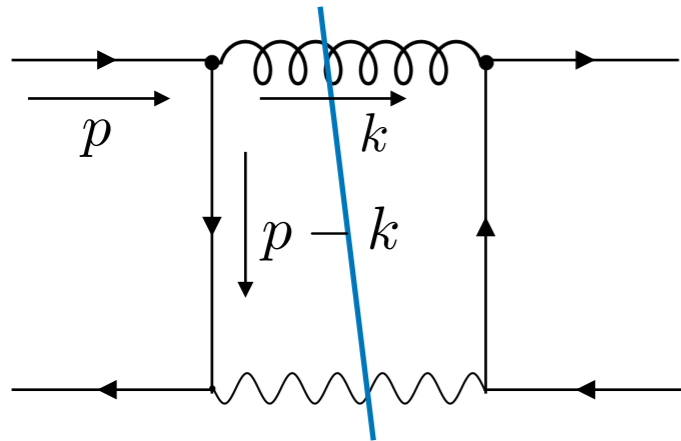
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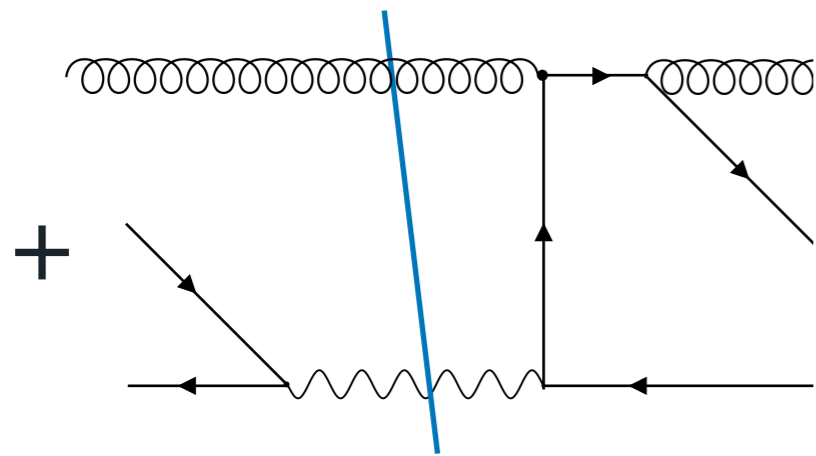
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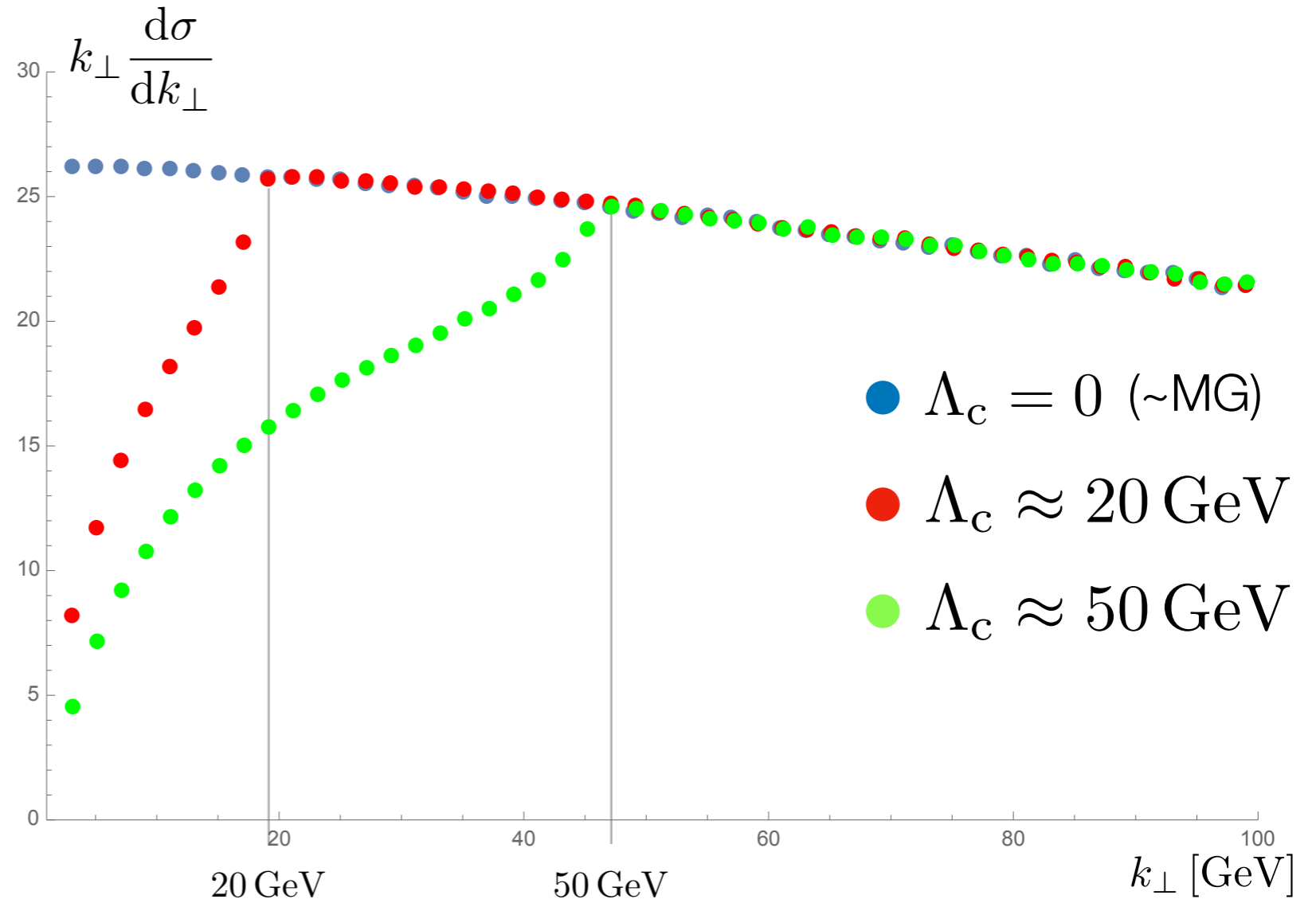
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Note that : for $\Lambda_c > 50$ GeV
the distribution does not change anymore
because highest separation of two partons
in a jet is of **order of Z mass**

INITIAL-STATE SINGULARITIES: SCALE DEPENDENCE

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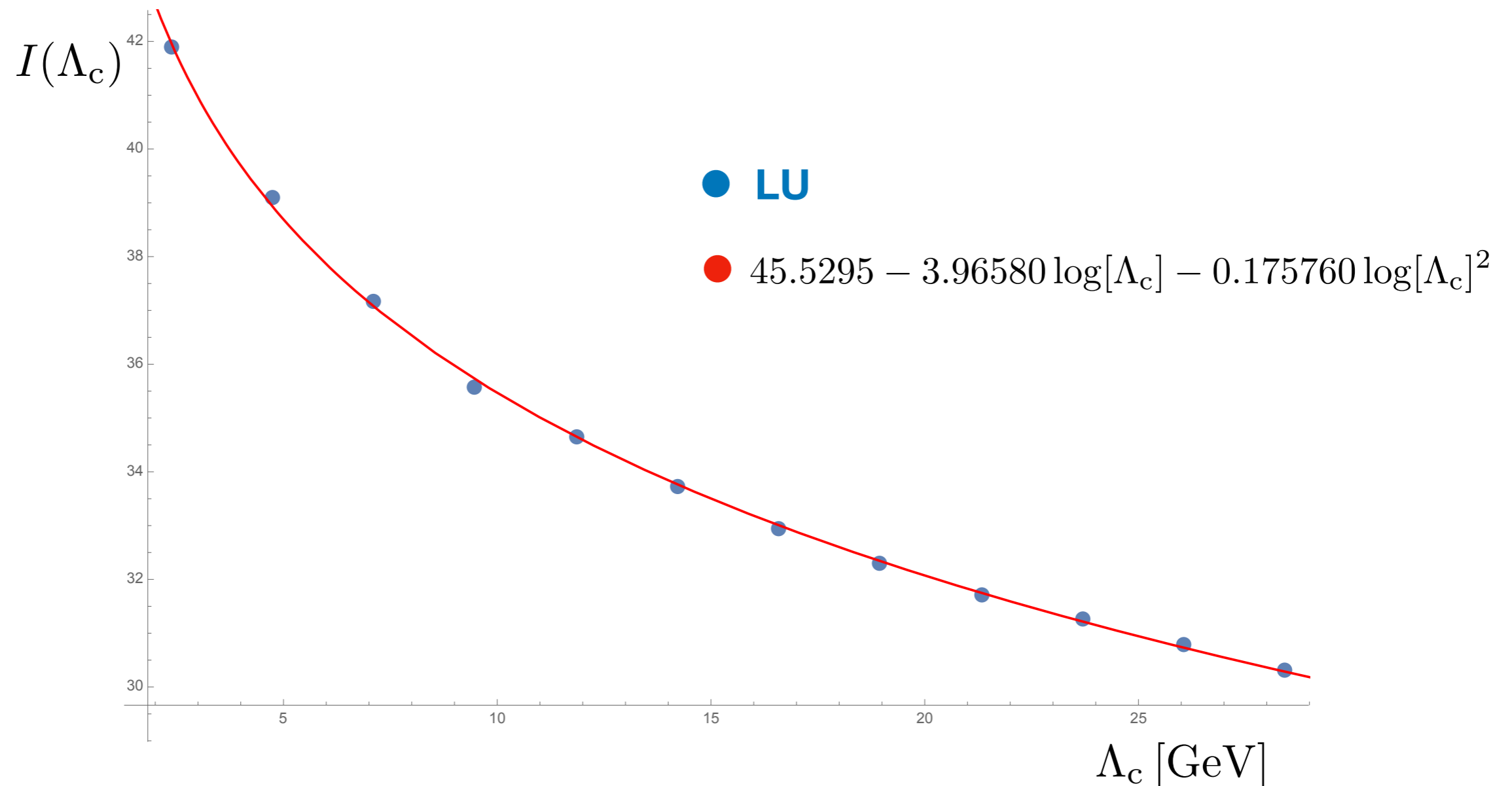
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$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{1}{x} \mathcal{J}(0) \right] + \left(\left[\int_0^1 dy \frac{-e^{-y}}{y} \right] + \left[\int_0^{10} dx \frac{1}{x} \right] \right) \mathcal{J}(0)$$

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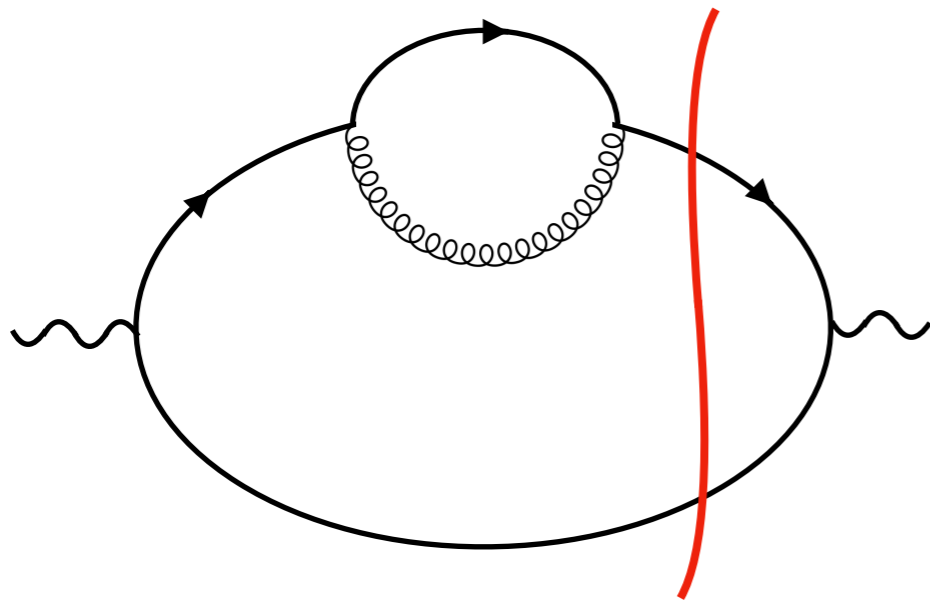
- Local Unitarity** aligns the measure and combines “real and virtual”:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) + \frac{-e^{-x}}{x} \mathcal{J}(0) \Theta(1-x) \right]$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

In **LU**, we cannot consider *truncated* amplitudes only :



Traditional Cutkosky rule

$$\int_{\vec{p}} = -2\pi i \frac{\delta(p^0 - E(\vec{p}))}{2E(\vec{p})}$$

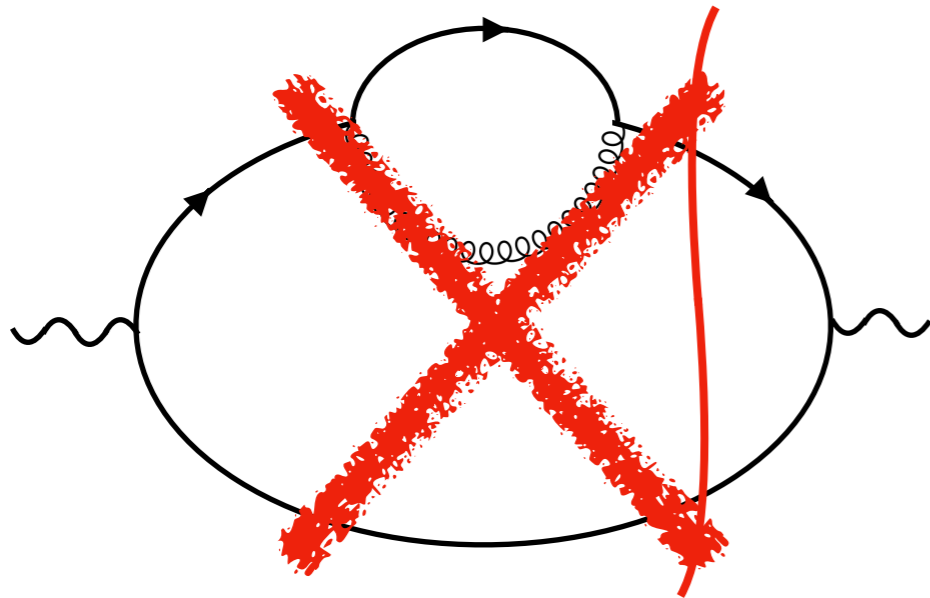
$$E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$$

would not apply here !

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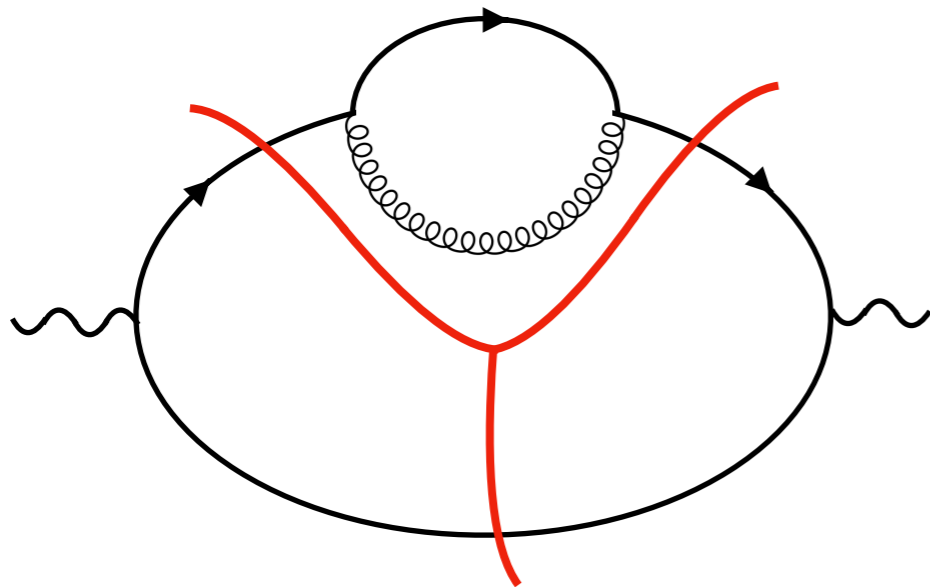
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would not apply here !

So consider this Cutkosky cut as a **higher-order residue** → **Generalised cutting rule**



$$\dots \int \dots = -2\pi i \frac{\delta^{(n)} [p^0 - E(\vec{p})]}{(p^0 + E(\vec{p}))^2}$$

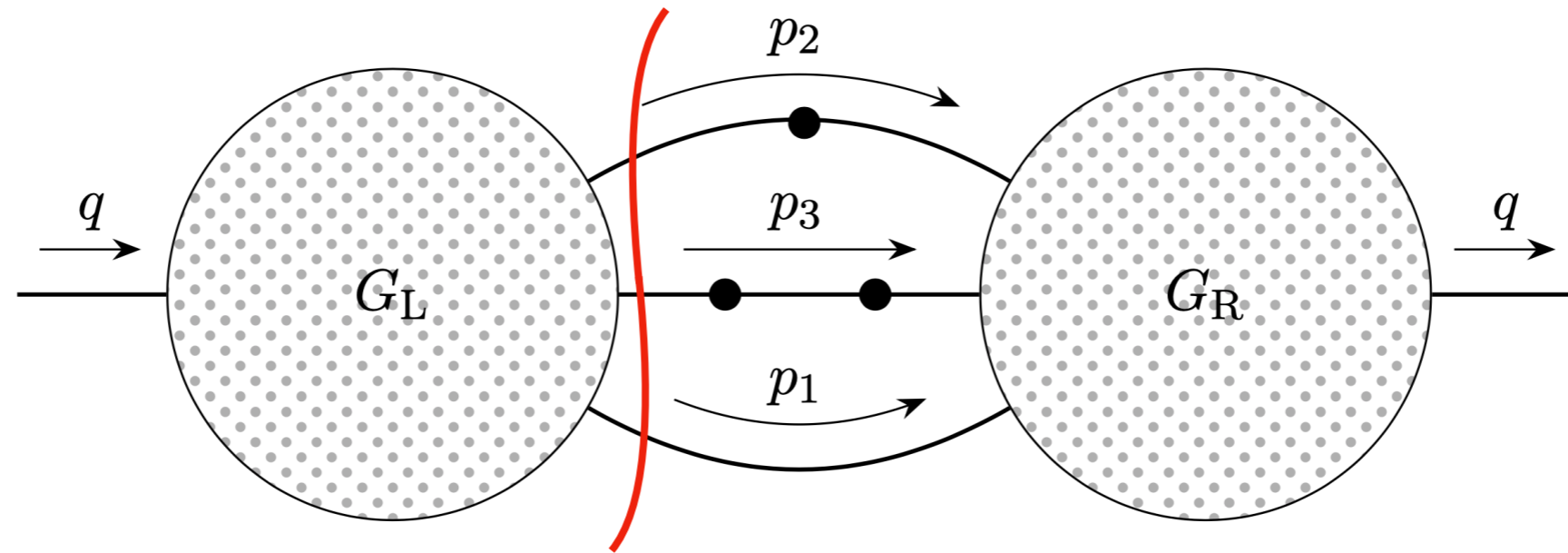
n - times

$$\int dx \delta^{(n+1)} [x] f(x) = \frac{1}{n!} \frac{d^n f}{dx^n} \Big|_{x=0}$$

LOCALITY UNITARITY: RAISED PROPAGATORS

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :

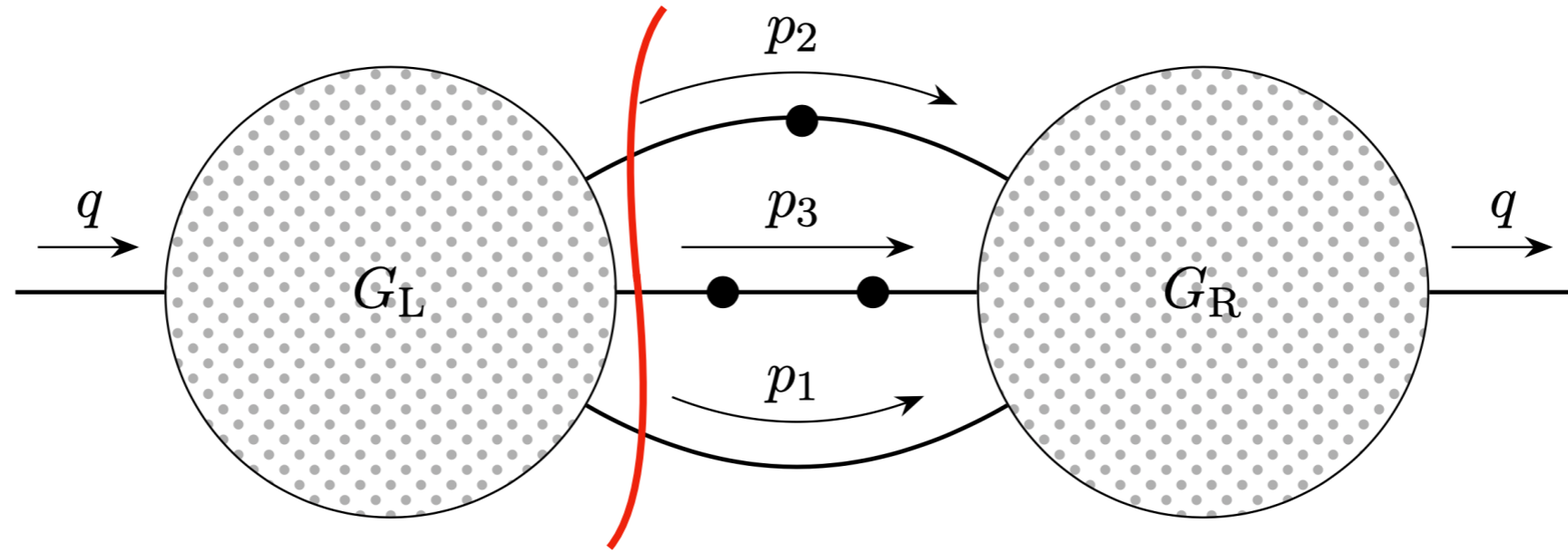


$$\propto \delta^{(1)} [p_1^0 - E(\vec{p}_1)] \delta^{(2)} [p_2^0 - E(\vec{p}_2)] \delta^{(3)} [p_3^0 - E(\vec{p}_3)]$$

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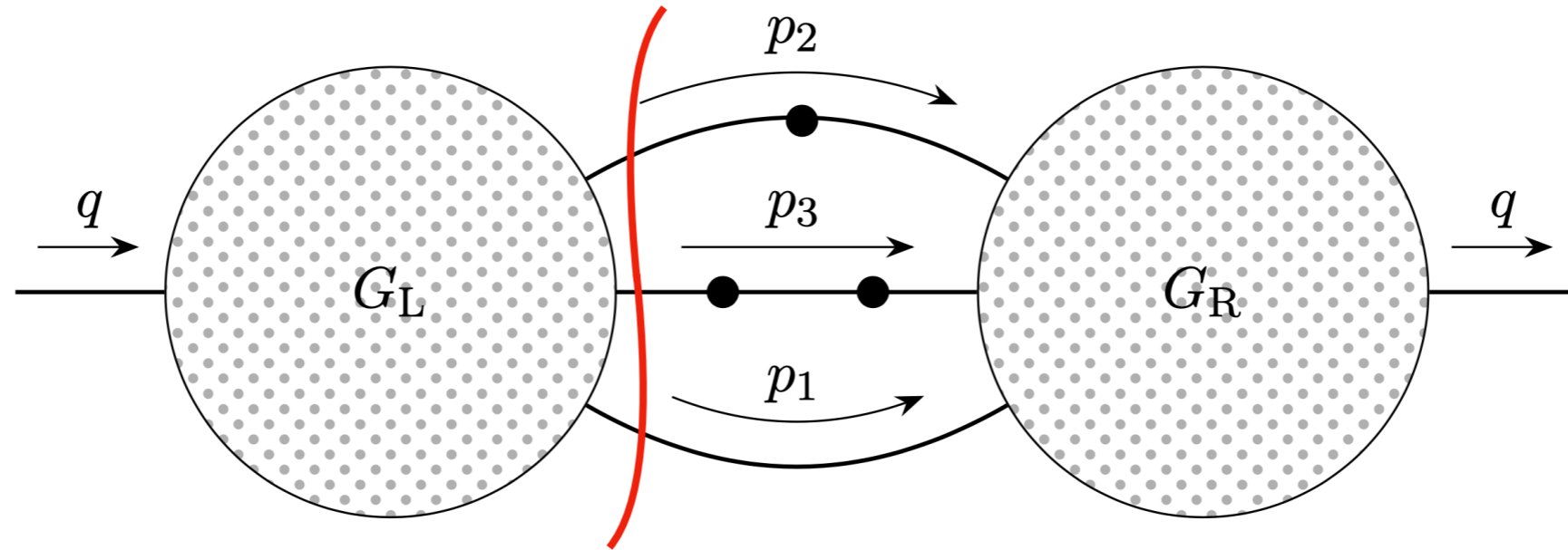
Derivatives can act on each other because: $p_3 = q - p_1 - p_2$

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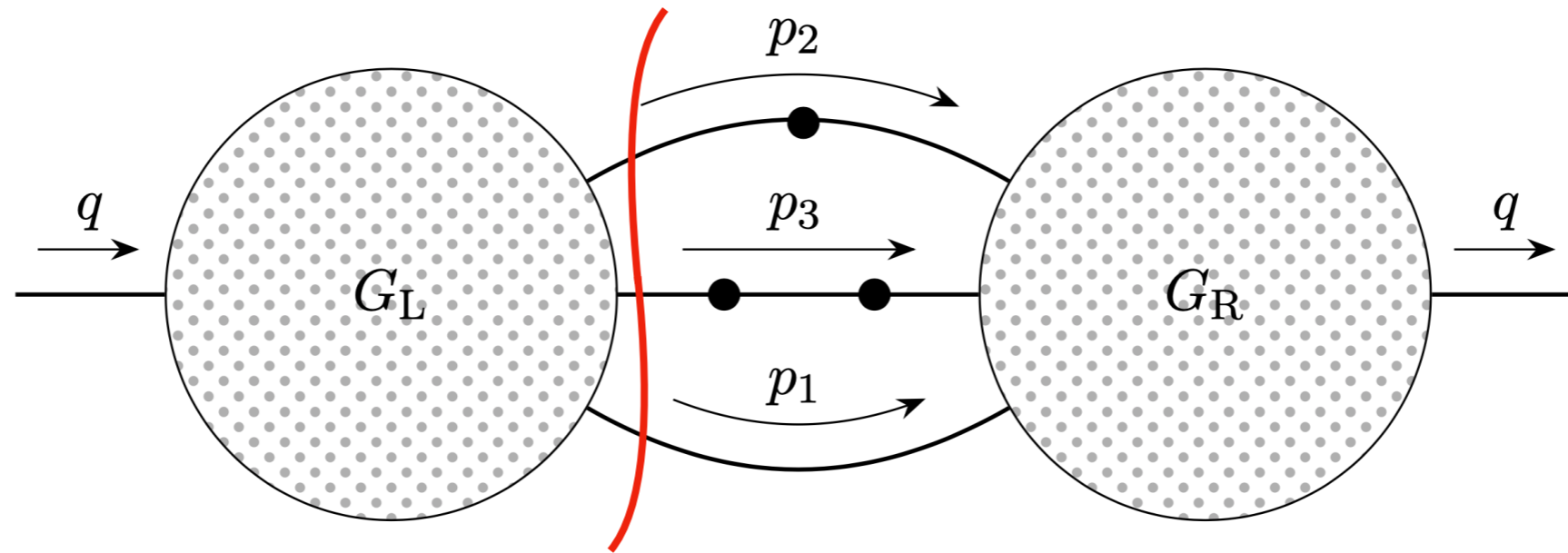
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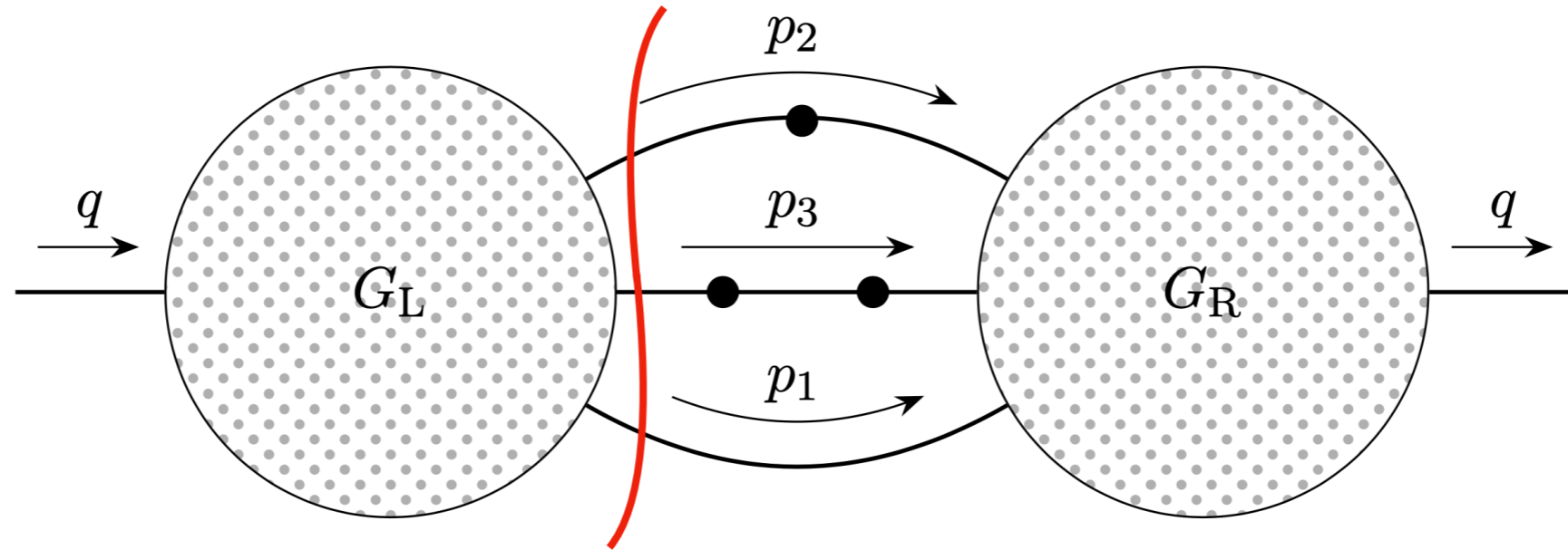
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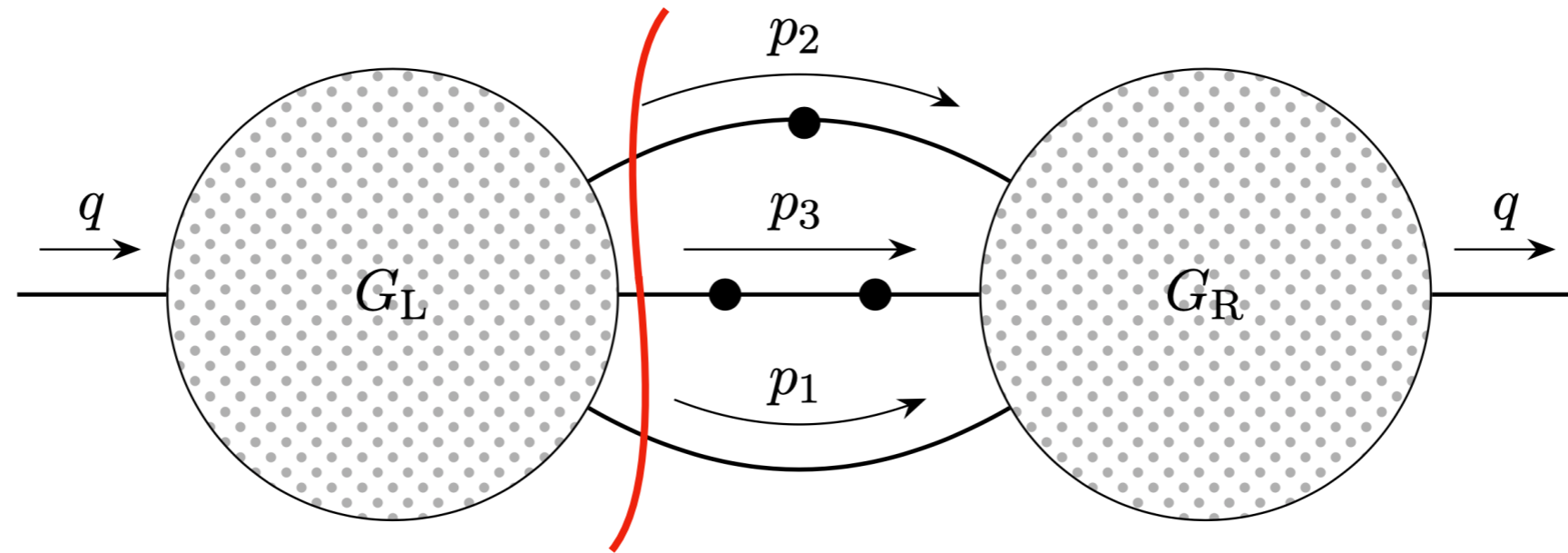
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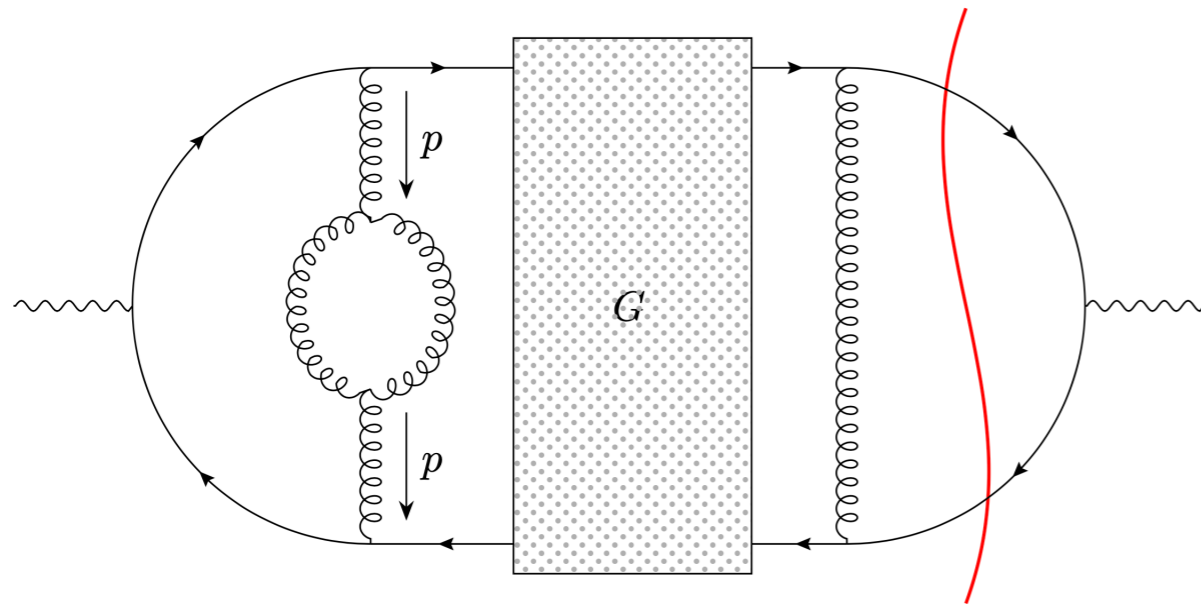
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Use **multivariate dual numbers** (auto-differentiation) in order to **efficiently compute amplitude derivatives** of G_L and G_R in p_2^0 and t (in this example)

SPURIOUS SOFT SINGULARITIES

[Capatti, VH, Ruijl, arxiv : 2203.11038]



$$\propto \frac{1}{(p^2)^2}$$

For $p = 0$
 this induces a spurious
 soft divergence whose
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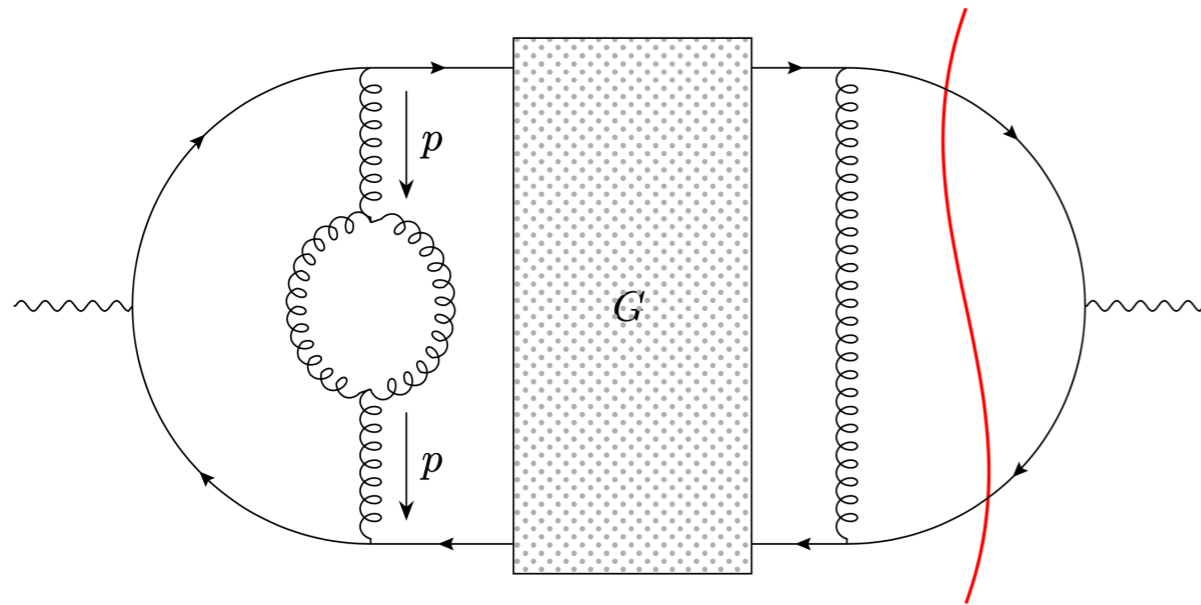
Only beyond NLO (needs a soft propagator dressed with a self-energy correction)

At the integrated level, we have

$$\bullet \text{---} \text{loop} \text{---} \bullet \propto \frac{1}{p^2} (p^2 g^{\mu\nu} - p^\mu p^\nu) \frac{1}{p^2}$$

SPURIOUS SOFT SINGULARITIES

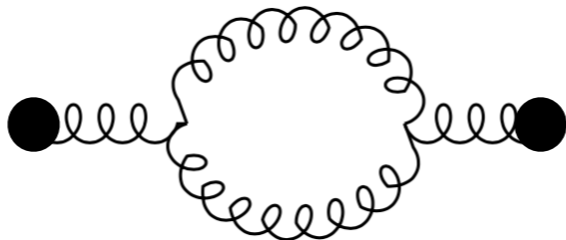
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Only beyond NLO (needs a soft propagator dressed with a self-energy correction)

At the integrated level, we have  $\propto \frac{1}{p^2} (p^2 g^{\mu\nu} - p^\mu p^\nu) \frac{1}{p^2}$

but not at the local level; we must introduce **spurious soft counterterms** :

$$\text{gluon loop} - \tilde{T}_1 \left(\text{gluon loop} \right), \quad \tilde{T}_{\text{soft_dod}}(\gamma) = \sum_{j=0}^{\text{soft_dod}(\gamma)} \frac{1}{j!} \frac{d^j}{d\lambda^j} \gamma(\lambda p) \Big|_{\lambda=0}, \quad [\tilde{T}] = 0$$

COMBINED UV AND SPURIOUS IR FOREST

Eureka moment:

- Remarkably, we always have : $\text{soft_dod} = \text{UV_dod} - 1$
- Spurious soft expansion also valid as UV counter term.
- Spurious soft IR forest similar to the one produced by the R-operation

so that we can combine the UV and spurious soft subtraction as one !

$$\hat{T}_{\text{dod}} = T_{\text{dod}} + \tilde{T}_{\text{dod}-1} - T_{\text{dod}}\tilde{T}_{\text{dod}-1}$$

until we realised that we had just re-invent[[J. H. Lowenstein, 1976](#)]

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Novelty though: **automatic renormalisation** of fermion masses in the OS scheme:

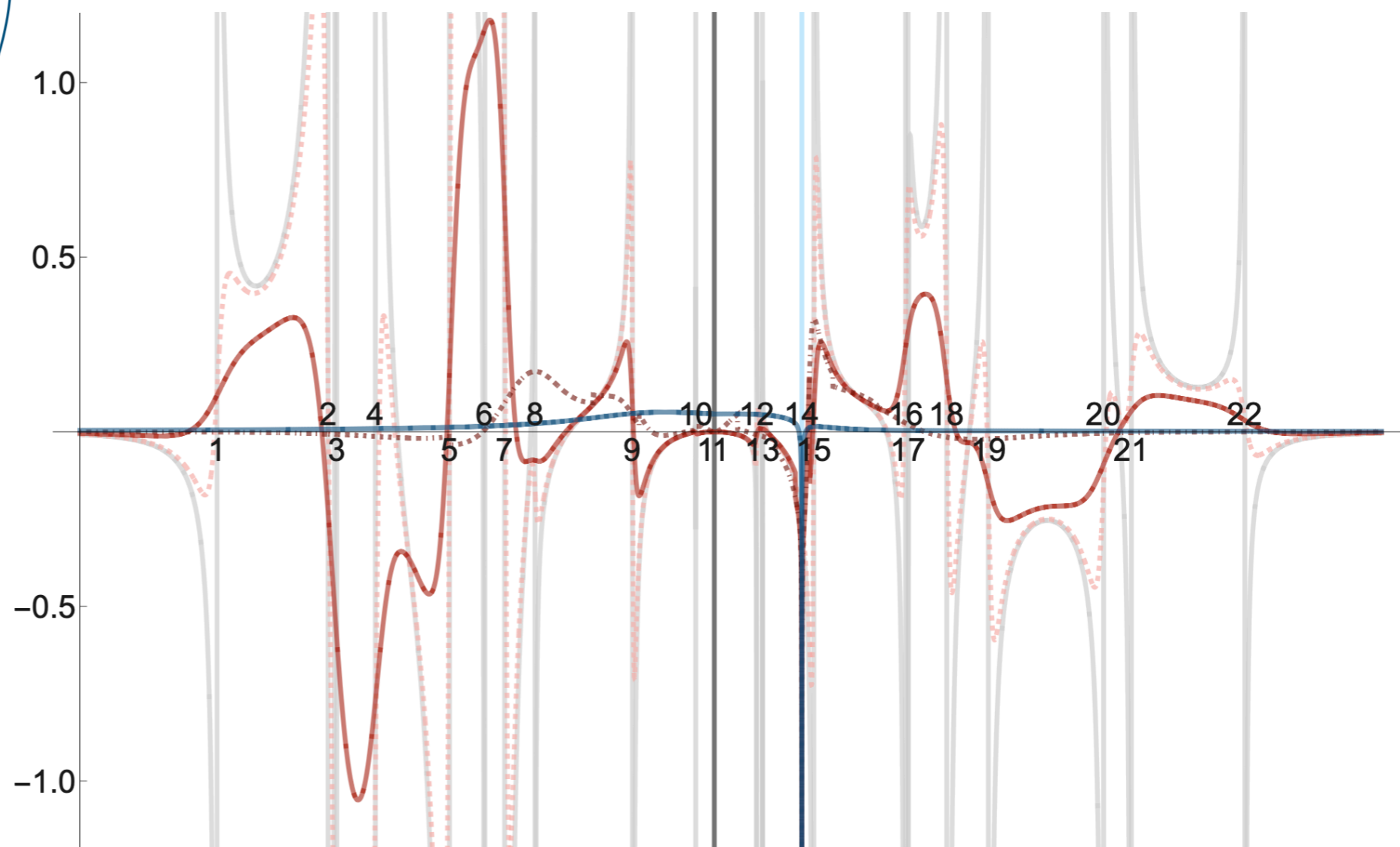
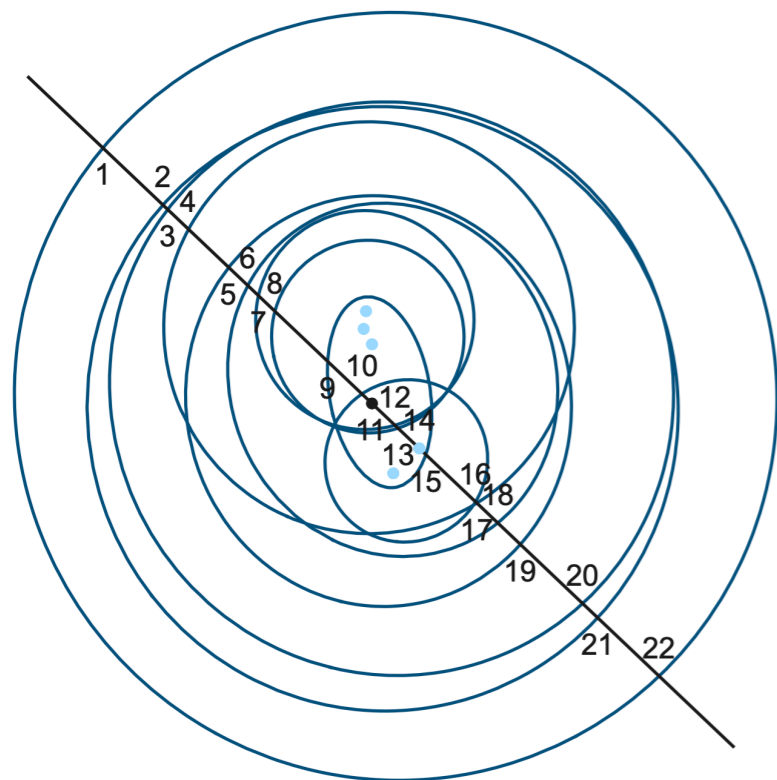
$$T^{\text{os}\pm} \left(\Sigma = \overset{p}{\rightarrow} \bullet \text{---} \right) = (1 \pm \gamma^0) \Sigma(p = \pm p^{\text{os}}), \quad p^{\text{os}} = (m, 0, 0, 0)$$

$$\frac{1}{2} \left([T^{\text{os}+}(\Sigma)] + [T^{\text{os}-}(\Sigma)] \right) = \delta m^{\text{os}}$$

Implying that our local UV counterterm T^{os} automatically generates the OS mass renormalisation counterterm !

THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kormanschah, arXiv : [2110.06869](https://arxiv.org/abs/2110.06869)]



— $I_{\epsilon=0}$

— $I_{\text{subtracted}}$

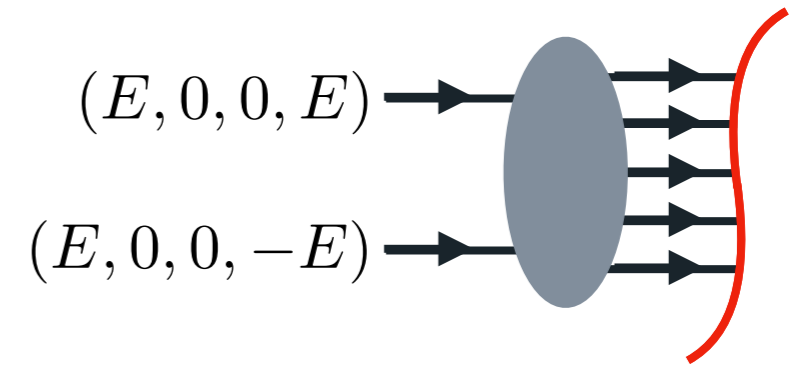
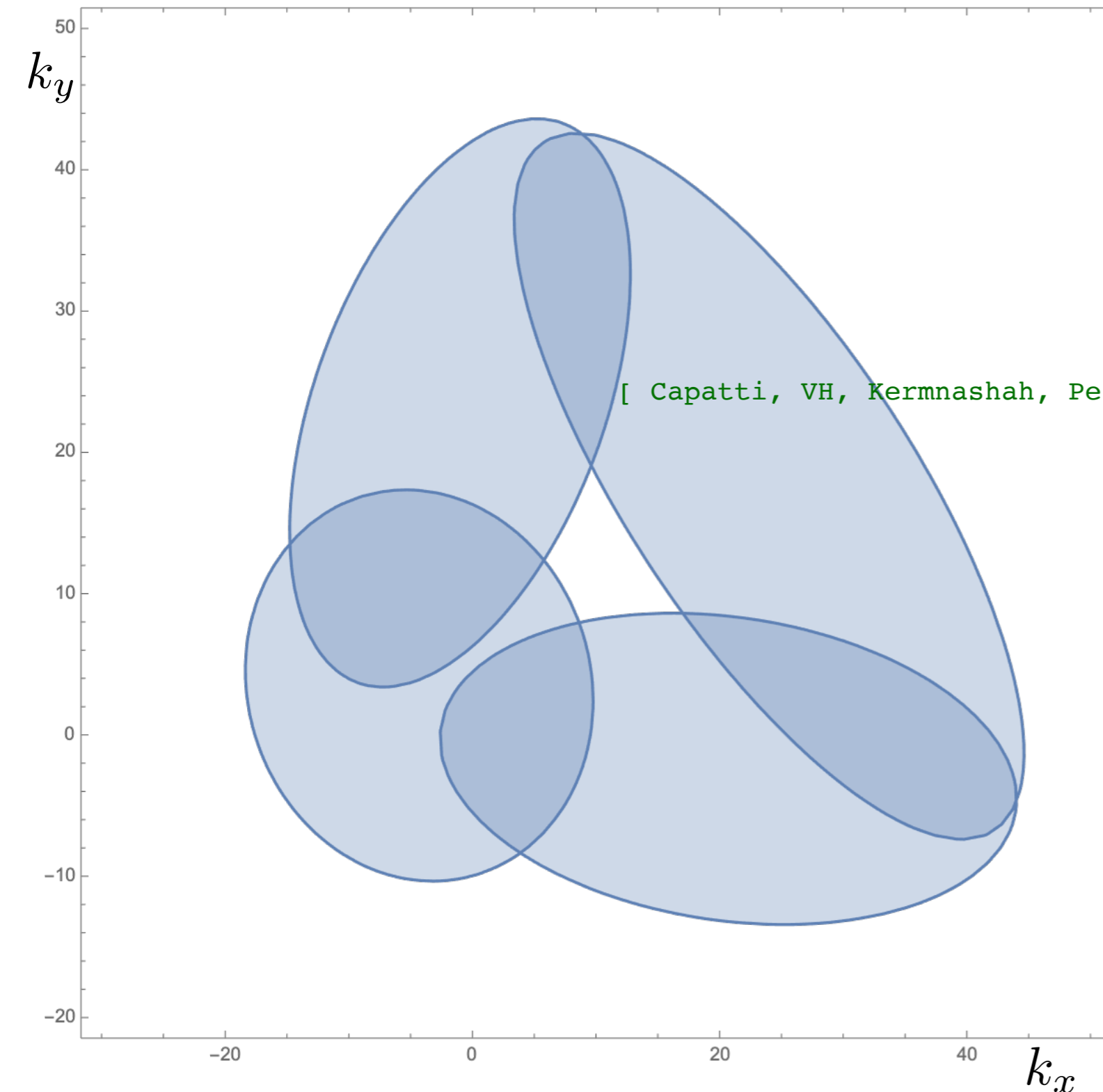
⋯⋯ $\text{Re } I_{\text{deformed}}(\lambda_{\text{max}} = 3)$

— $\text{Re } I_{\text{deformed}}(\lambda_{\text{max}} = 10)$

⋯⋯ $\text{Re } I_{\text{deformed}}(\lambda_{\text{max}} = 300)$

LOCALITY UNITARITY: CAUSAL FLOW

The **rescaling** change of variables is however **not general** : (**but always sufficient in practice!**)

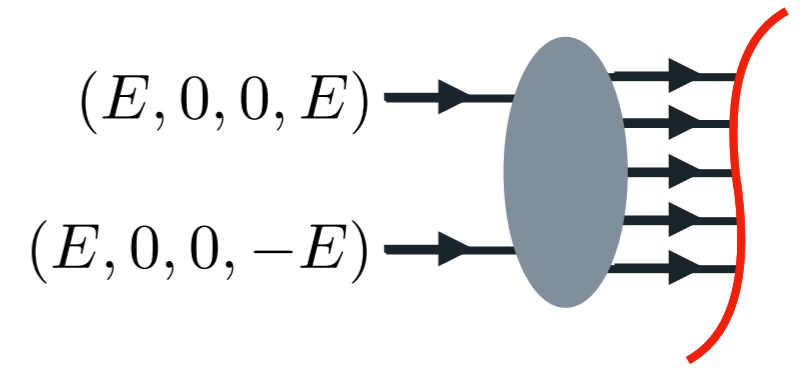
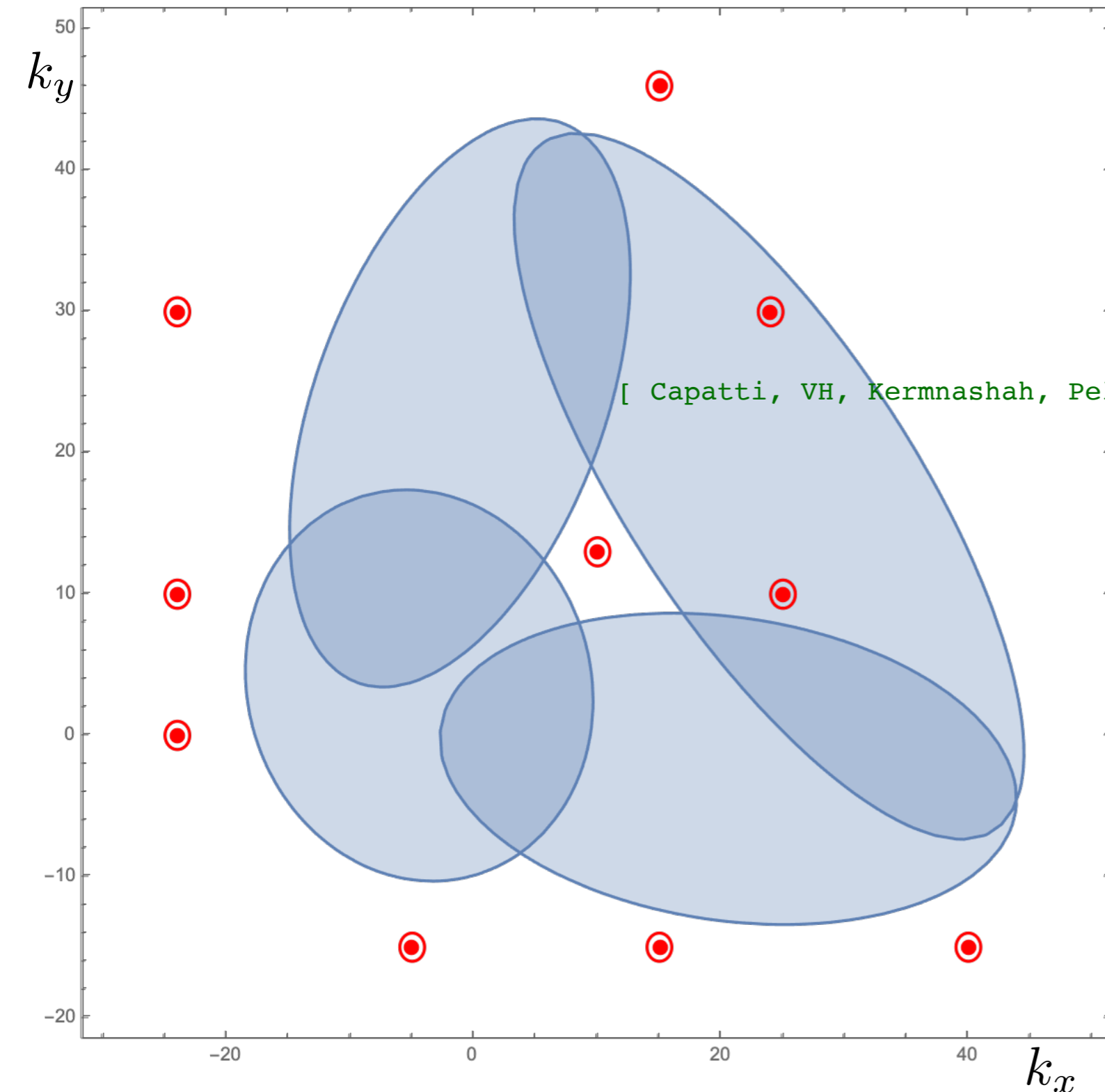


Ex: **Box_4E** from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

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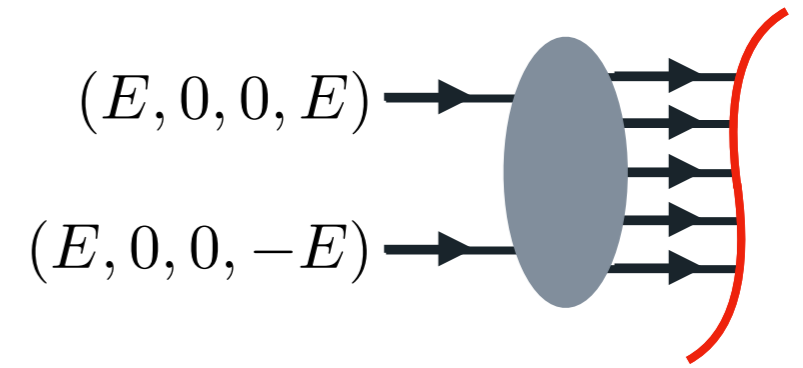
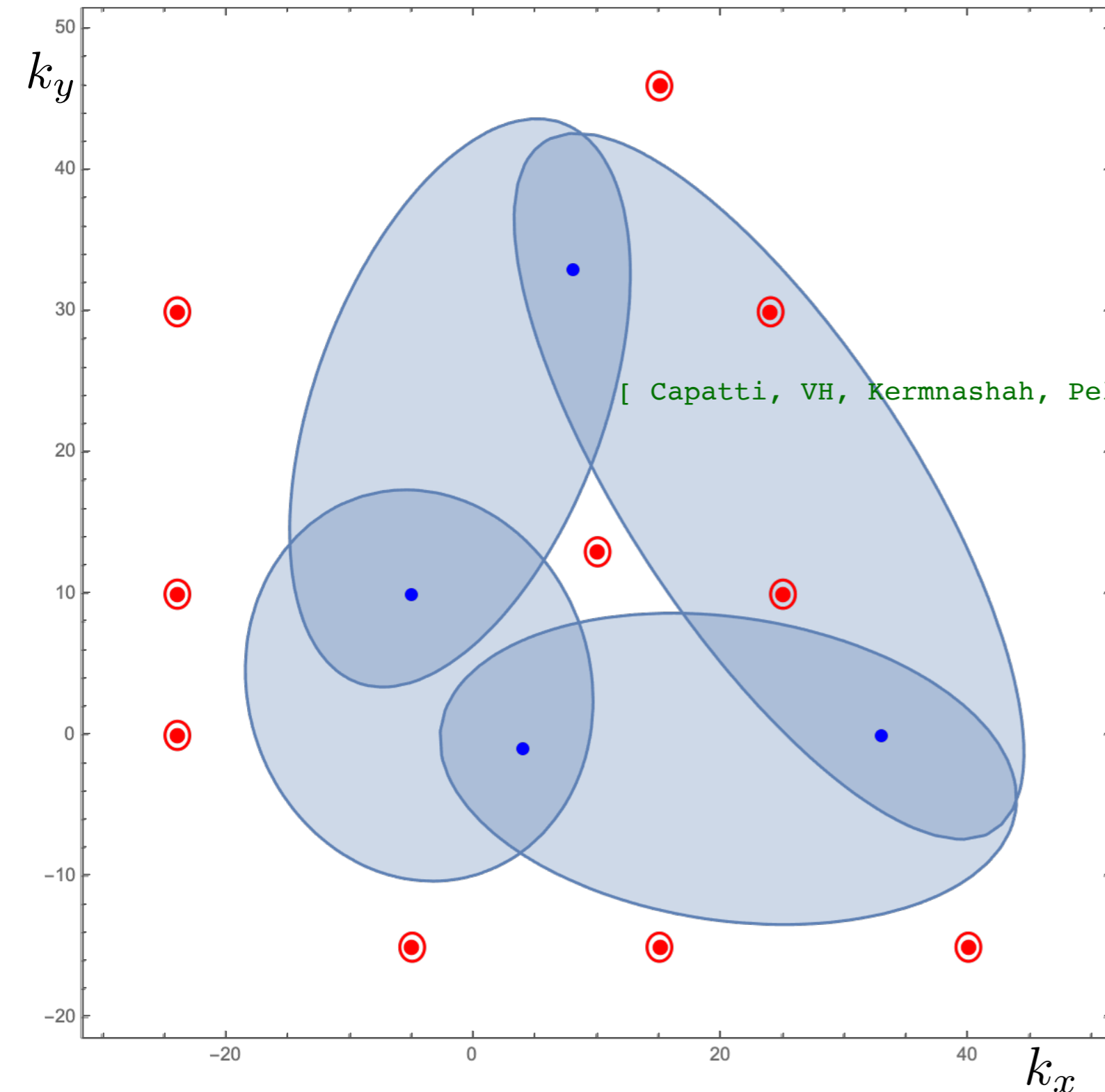


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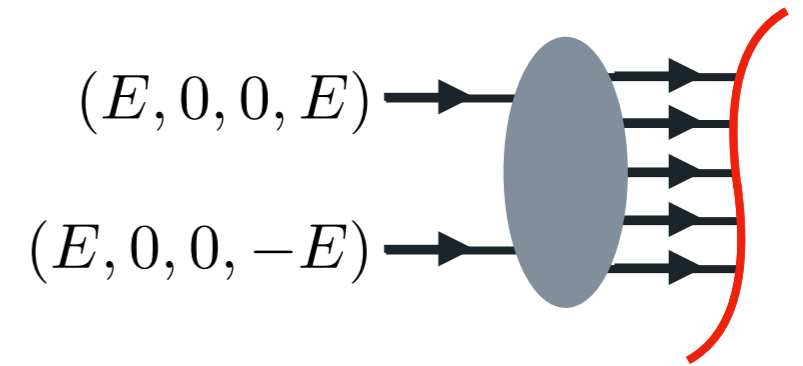
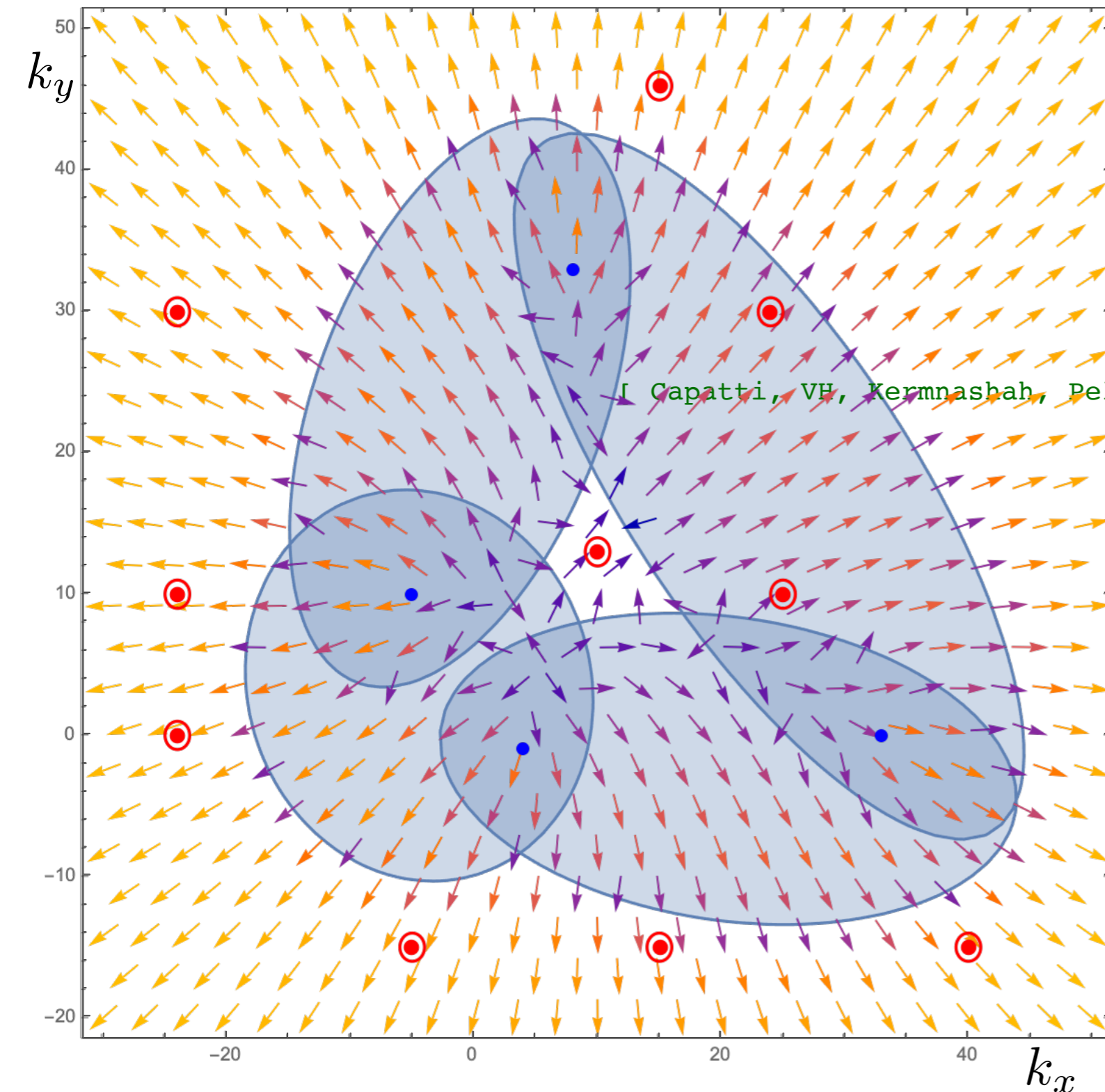


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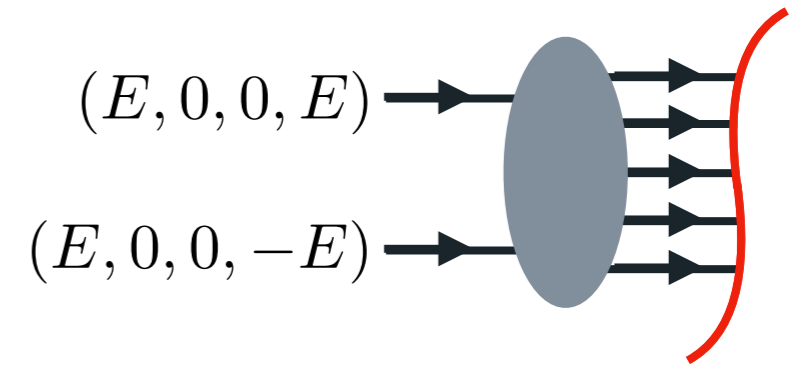
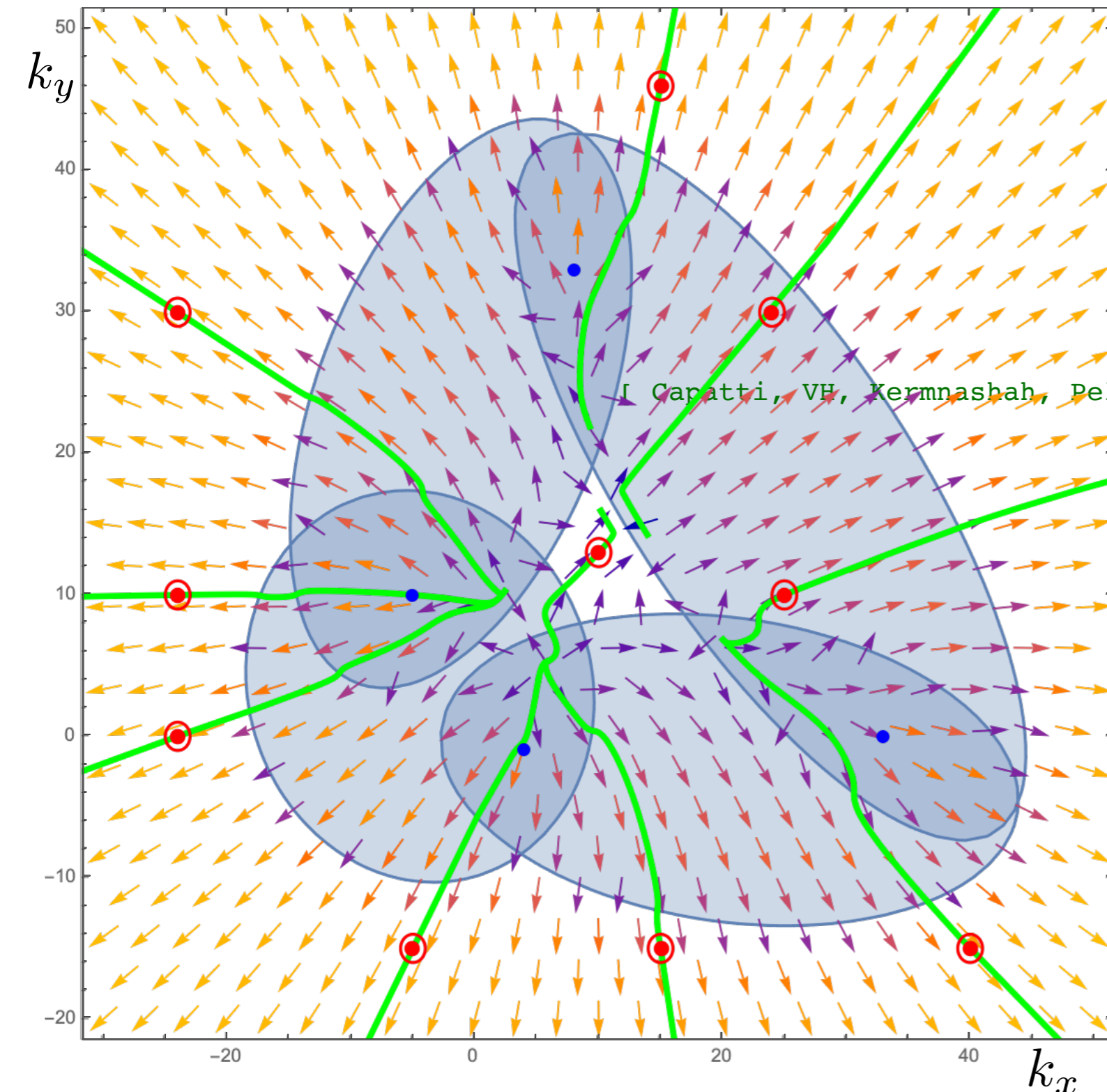


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Compute a **causal flow** $\vec{\phi}$ from our existing construction of a **deformation field** $\vec{\kappa}$:

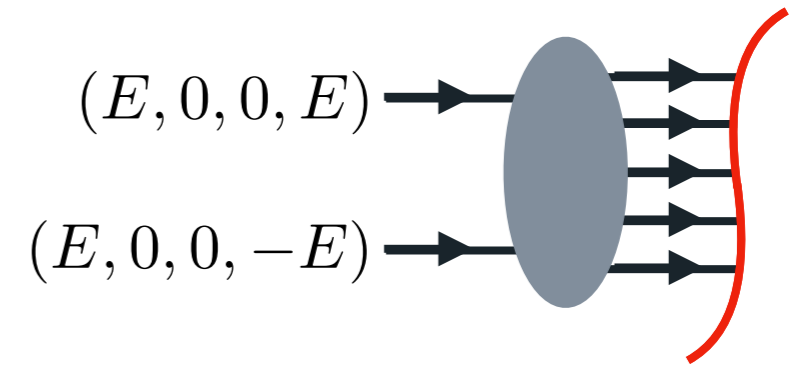
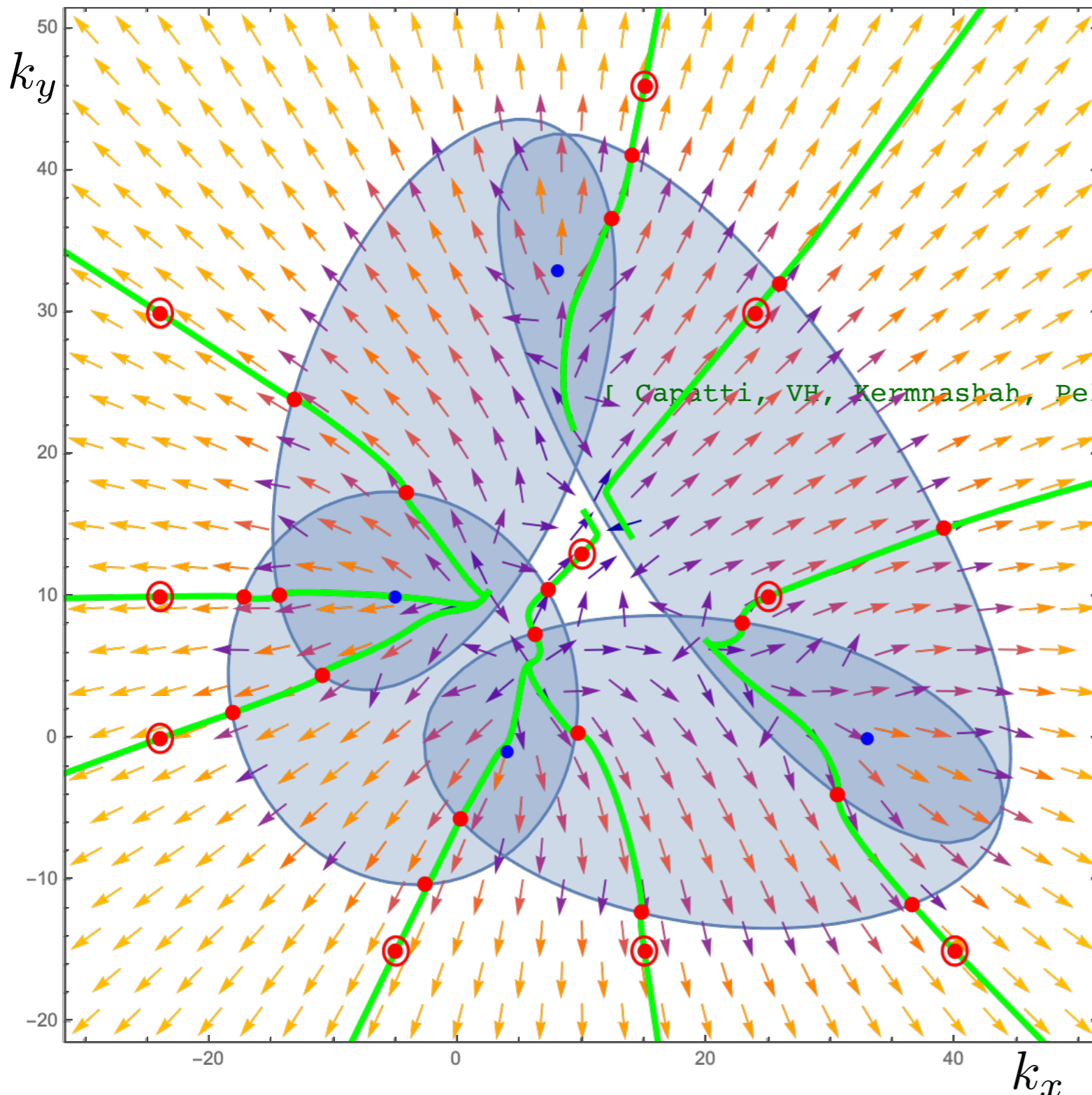
$$\partial_t \vec{\phi}(t, \vec{k}) = \vec{\kappa}(\vec{\phi}(t, \vec{k}))$$

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In general, this **ODE** can be **solved numerically**.

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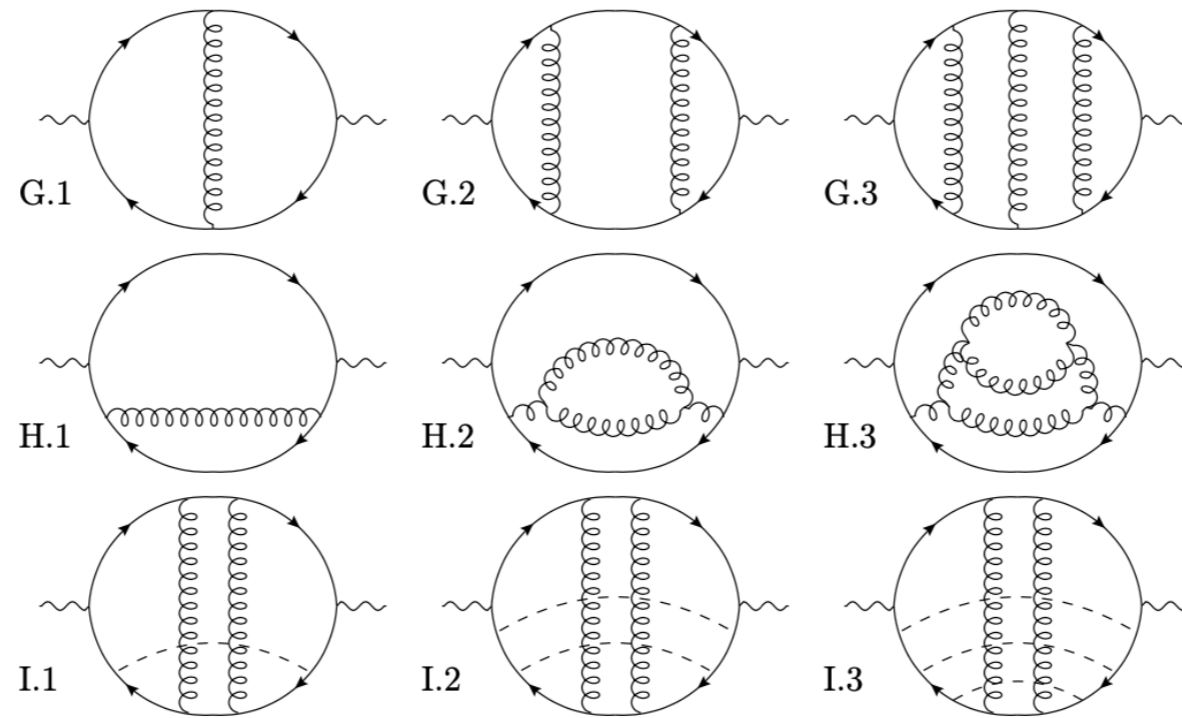
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IMPLEMENTATION RUN-TIME PERFORMANCE



SG	proc.	order	t_{gen} [s]	M_{disk} [MB]	N_{sg} [-]	N_{cuts} [-]	t_{eval} [ms]	$t_{\text{eval}}^{(\text{f128})}$ [ms]
G.1	1 → 2	NLO	0.1	0.13	2	4	0.004	0.13
G.2	1 → 2	NNLO	4.7	3.0	17	9	0.04	2.1
G.3	1 → 2	N3LO	36K	509	220	16	17.6	281
H.1	1 → 2	NLO	0.07	0.12	2	2	0.006	0.14
H.2	1 → 2	NNLO	1.5	1.3	17	3	0.056	1.9
H.3	1 → 2	N3LO	255	43	220	4	2.35	56
I.1	1 → 3	NNLO	126	22	266	9	0.32	12.4
I.2	1 → 4	NNLO	1.9K	120	4492	9	4.4	67
I.3	1 → 5	NNLO	36K	20K	$\mathcal{O}(100\text{K})$	9	3.6K	17.3K

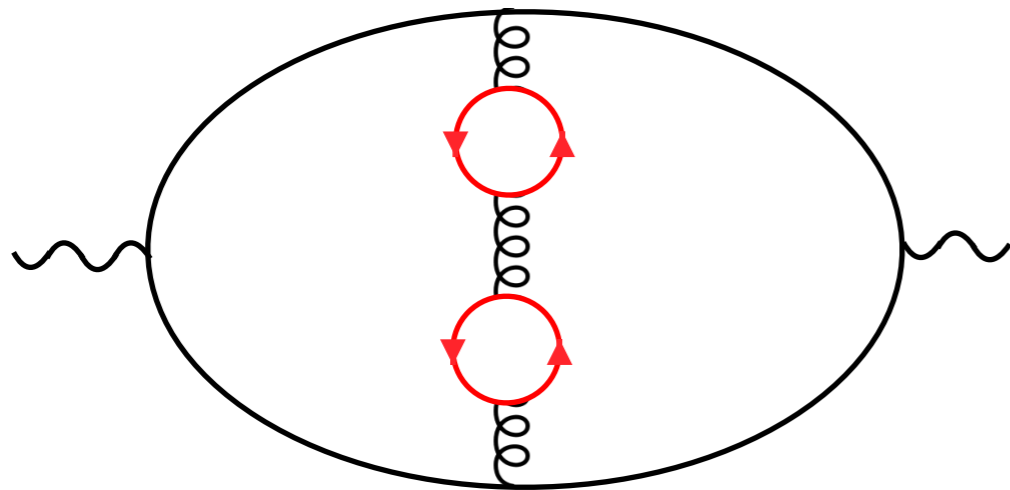
NB: these are **integrand** performance.

(Note: we recently found massive speedup w.r.t the above)

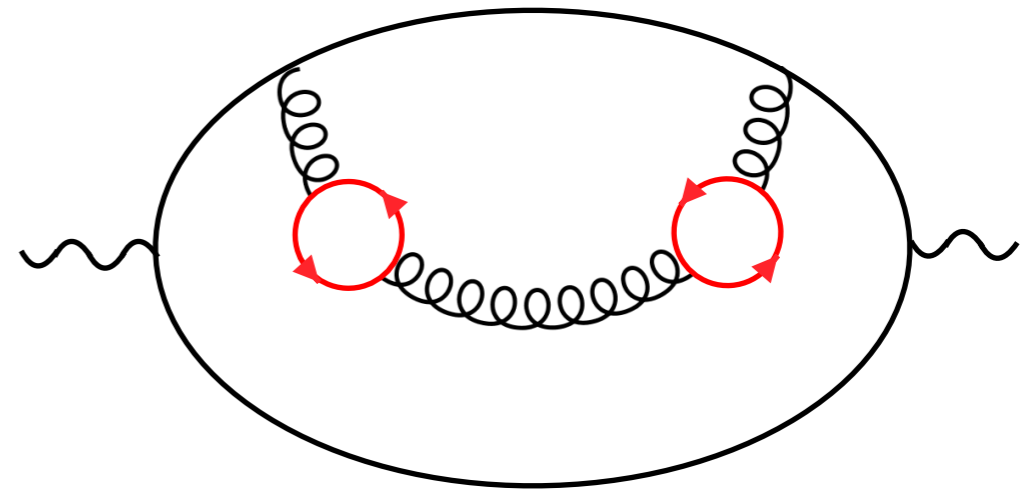
Integration (sampling) not optimised yet.
so we do not report quantitatively on it yet.

PARTIAL N3LO RESULTS

n_f^2 contributions :



$$K_{jj}^{(\text{MC LU}) \text{ I}} = 24.45(10)$$



$$K_{jj}^{(\text{MC LU}) \text{ II}} = -24.80(22)$$

(Large accidental cancellation between the two graphs, but validation otherwise successful)

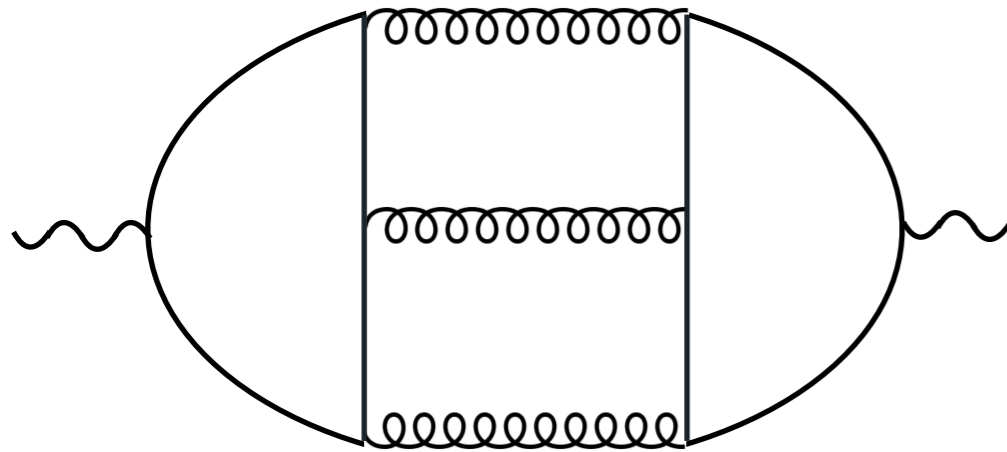
$$K_{jj}^{(\text{MC LU}) \text{ I+II}} = -0.35(24)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f^2)} = C_F \left(\frac{1208}{27} - \frac{8}{3} \zeta_2 - \frac{304}{9} \zeta_3 \right) = -0.331415$$

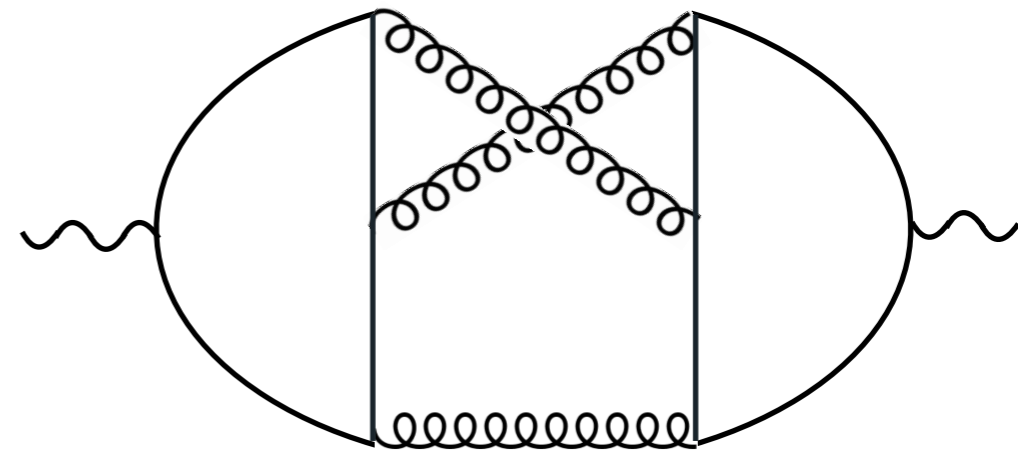
[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

PARTIAL N3LO RESULTS

Singlet contributions : (Results for low Monte-Carlo statistics here)



$$K_{jj}^{(\text{MC LU})\text{I}} = 48.4(1.0)$$



$$K_{jj}^{(\text{MC LU})\text{II}} = -74.0(1.1)$$

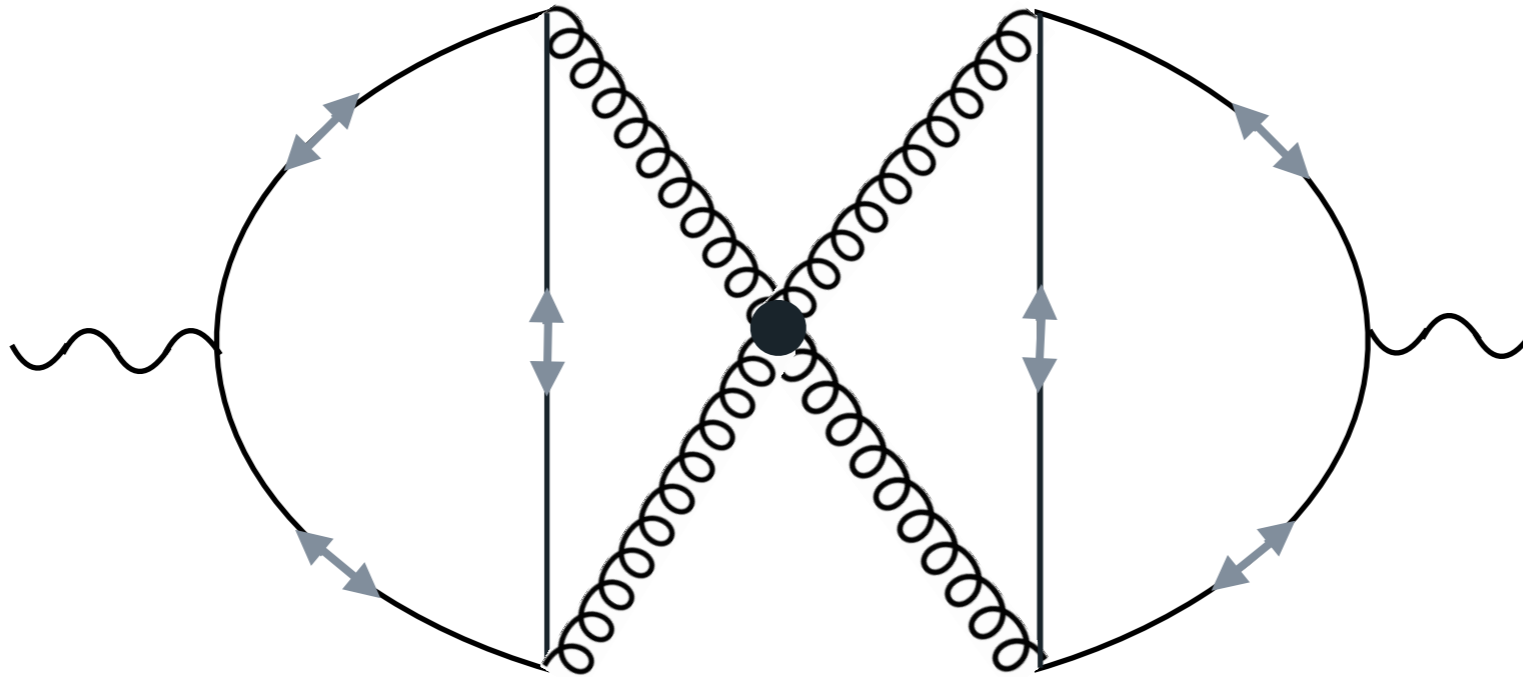
$$K_{jj}^{(\text{MC LU})\text{I+II}} = -25.6(1.5)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3), \text{singlet}} = \frac{d_F^{abc} d_F^{abc}}{N_R} \left(\frac{176}{3} - 128\zeta_3 \right) = -26.4435$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

CURIOSITY

Singlet contributions : A non-obvious zero contribution...

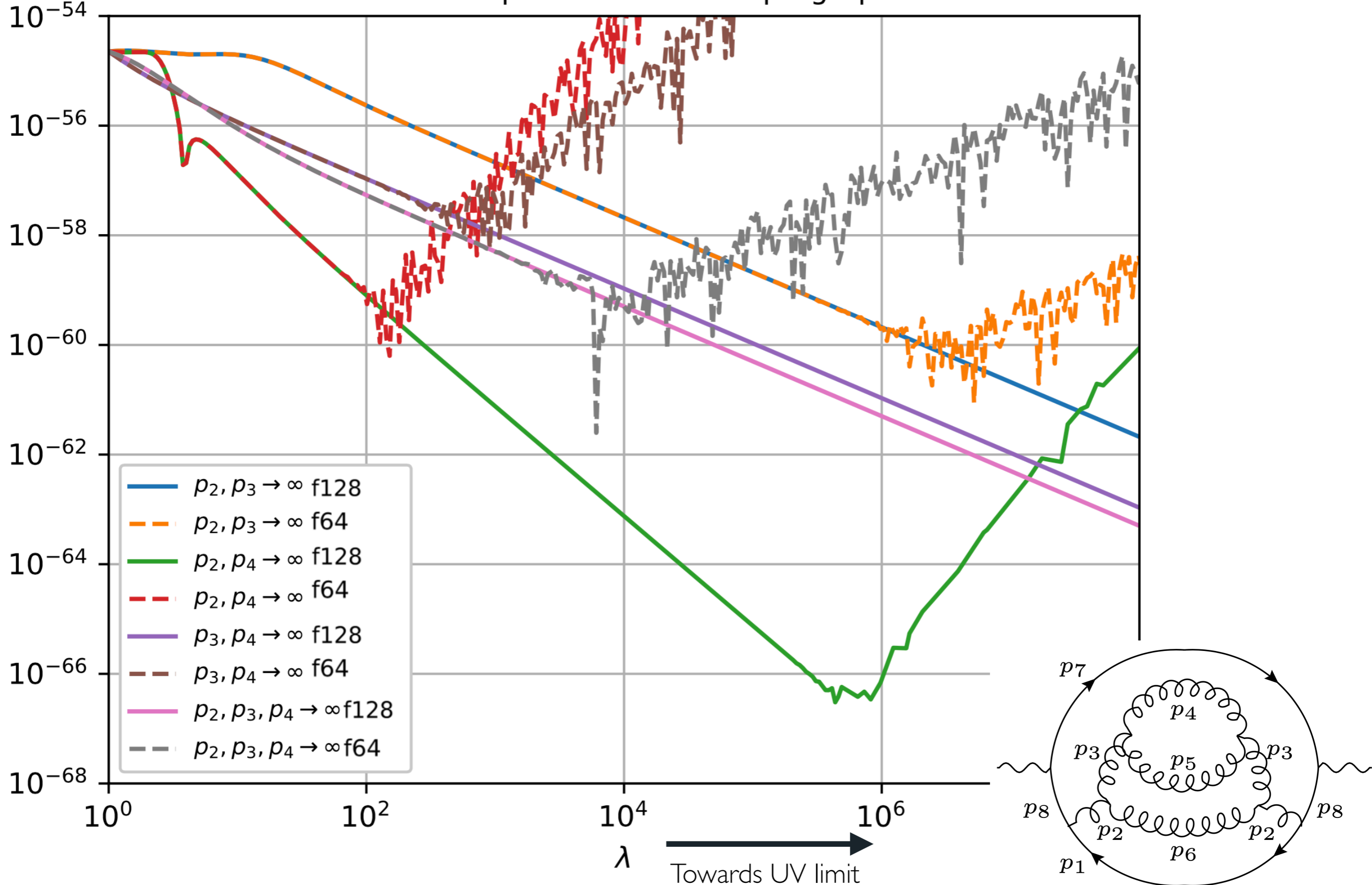


This graph gives no contribution inclusively.
I have not looked into it with any depth, but I don't see
an obvious reason as to why it should be zero...

TESTING N3LO UV LIMITS

[arbitrary units]

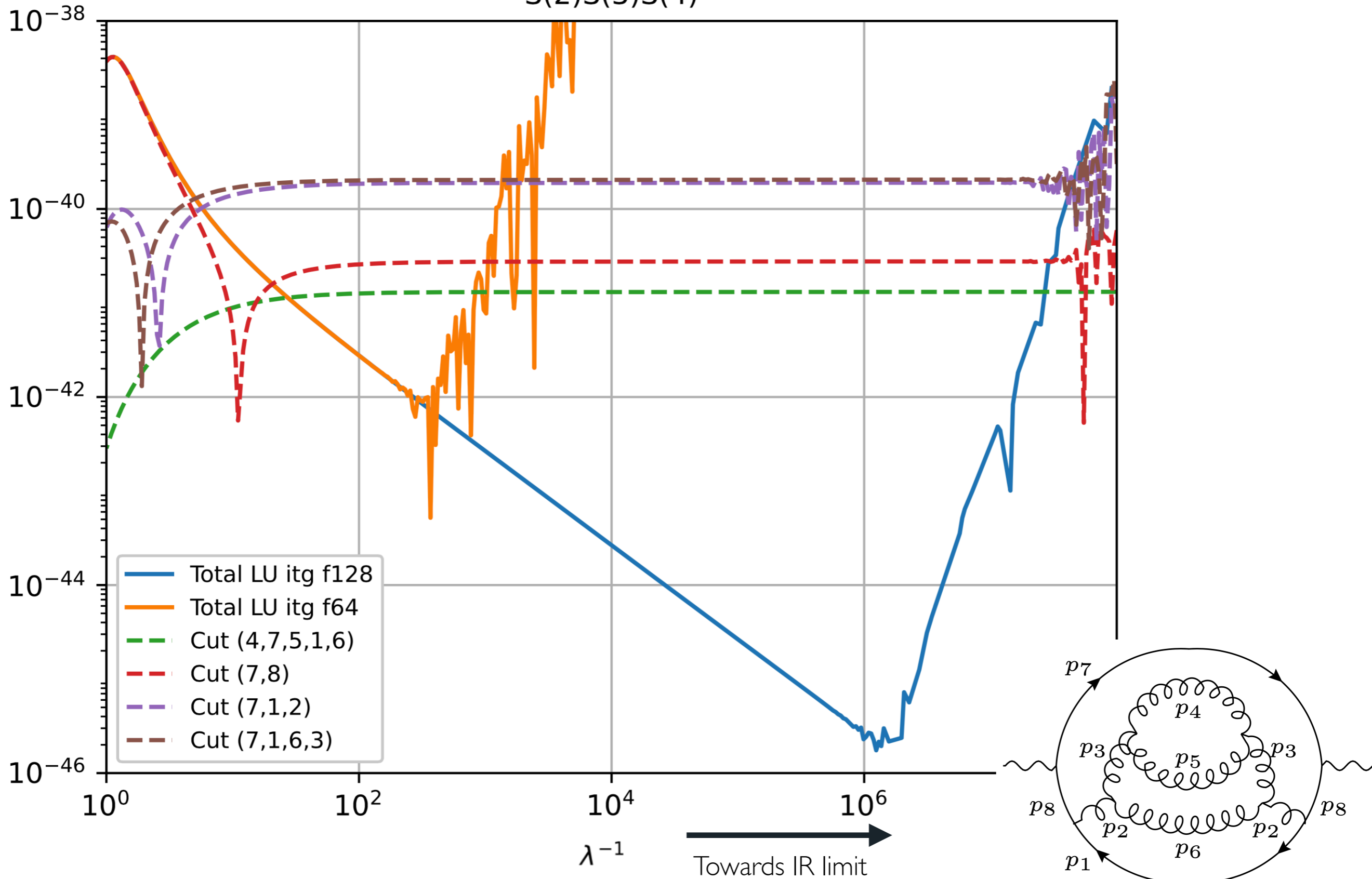
Double and triple UV limits of supergraph H.3



TESTING IR SOFT LIMITS

[arbitrary units]

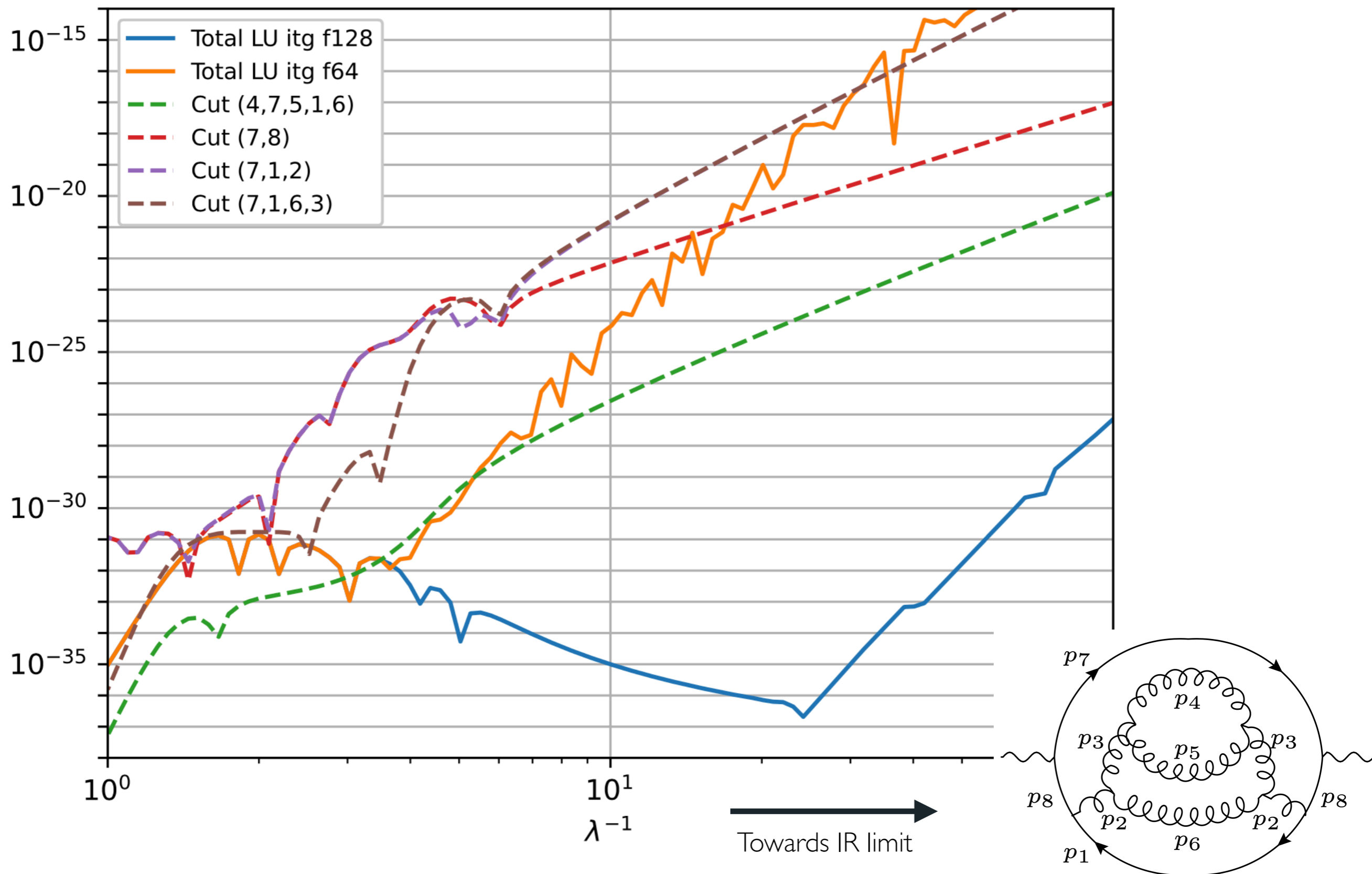
S(2)S(3)S(4)



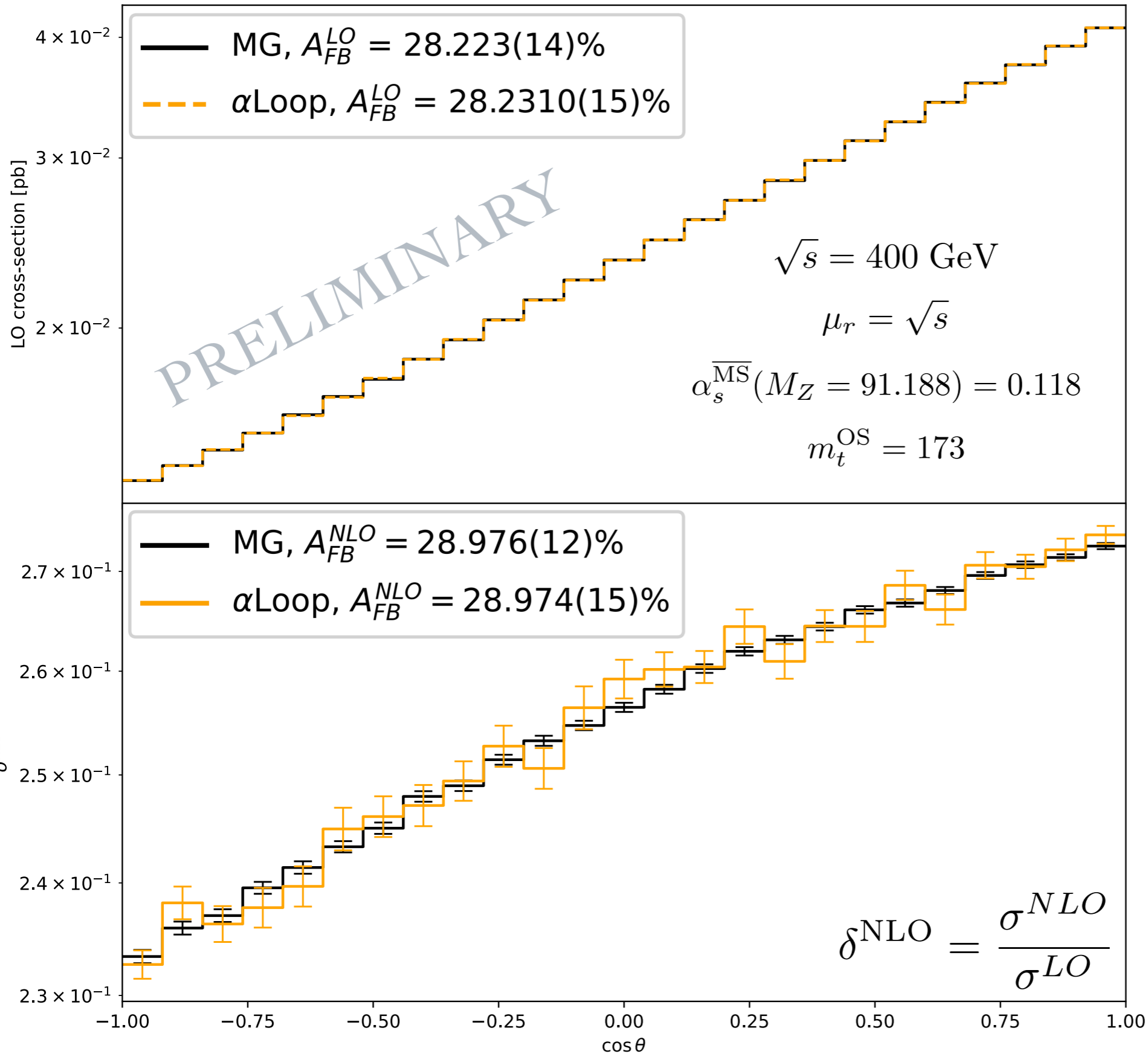
TESTING IR SOFT-COLLINEAR LIMITS

[arbitrary units]

C[1,2,S(3),S(4)]



EXAMPLE II : NLO AFB FOR $e^+e^- \rightarrow \gamma^*/Z \rightarrow t\bar{t}$



First result in LU with γ^5 and EW-boson

Contour deformation well-behaved in this case

Credits to ETHZ student

Max Hofer