Local Unitarity

Building a numerical collider

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In coll. with:

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CERN RAS

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Overview of Local Unitarity

One-dimensional toy example

- Toy expression with $\mathcal{J}$ a measurement function, over $x \in [0, 10]$

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) \qquad\qquad \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$

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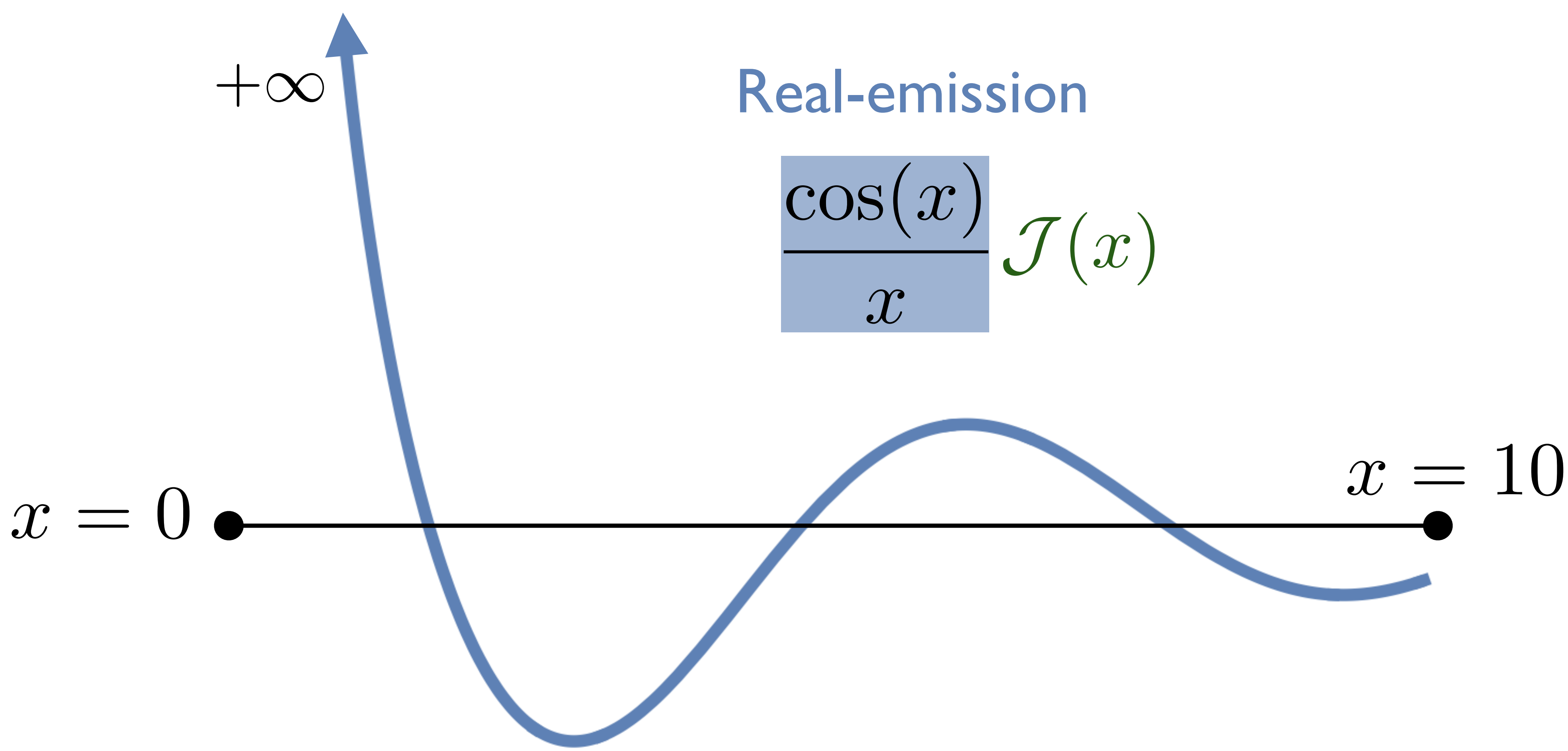

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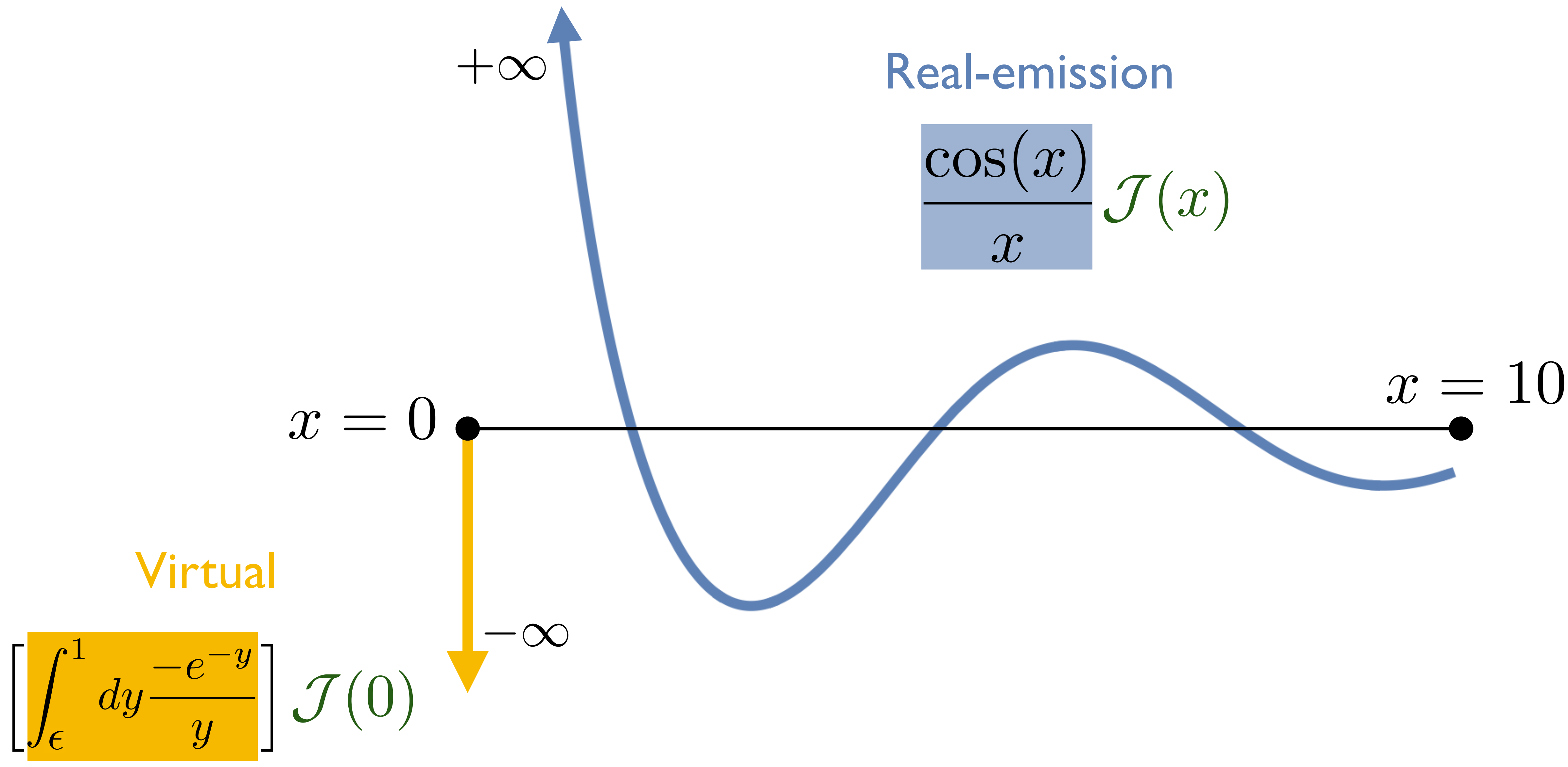

One-dimensional toy example

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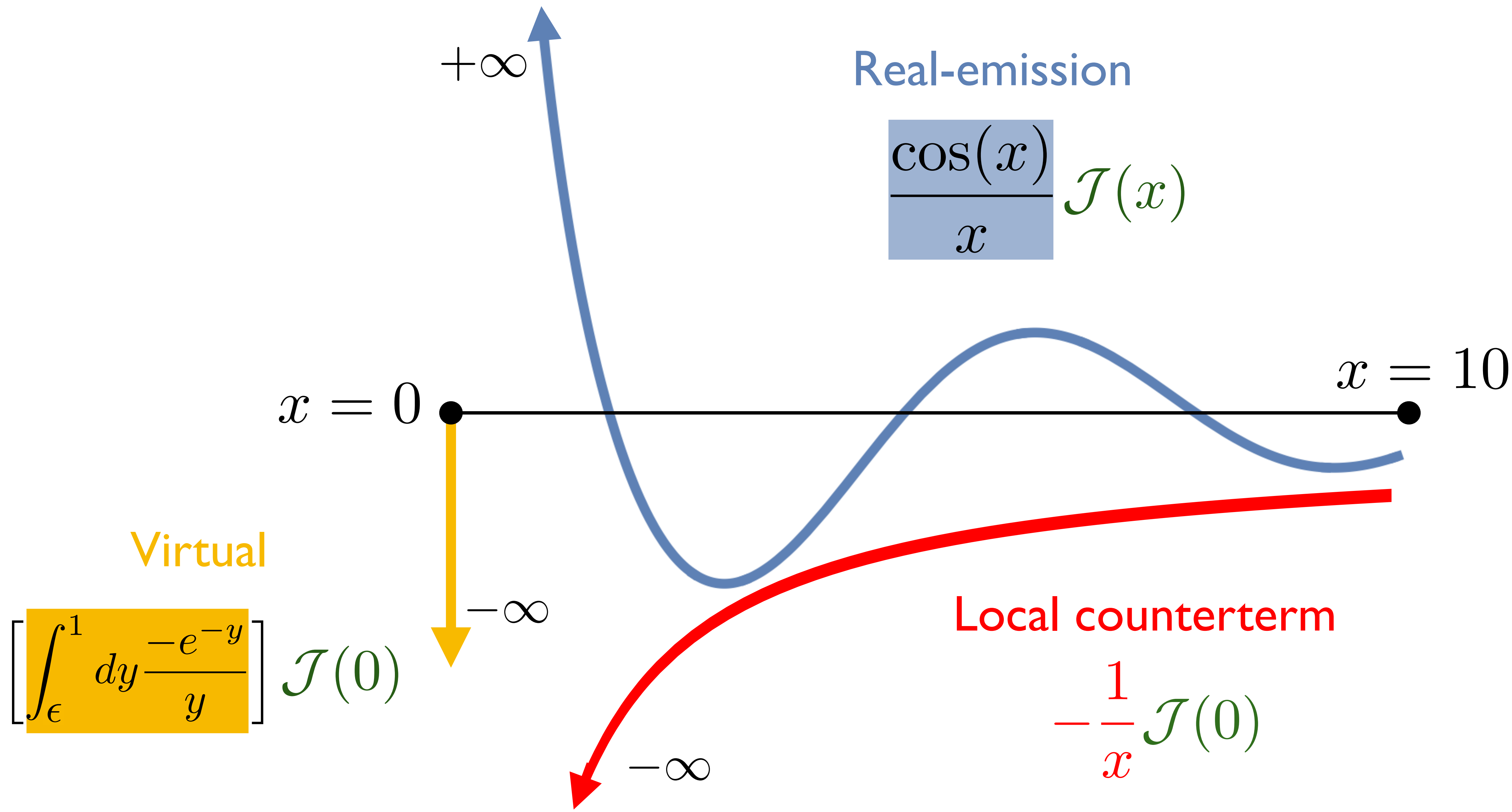
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x} \mathcal{J}(x) - \frac{1}{x} \mathcal{J}(0) \right] + \left[\int_0^1 dy \frac{-e^{-y}}{y} \right] \mathcal{J}(0)$$



One-dimensional toy example

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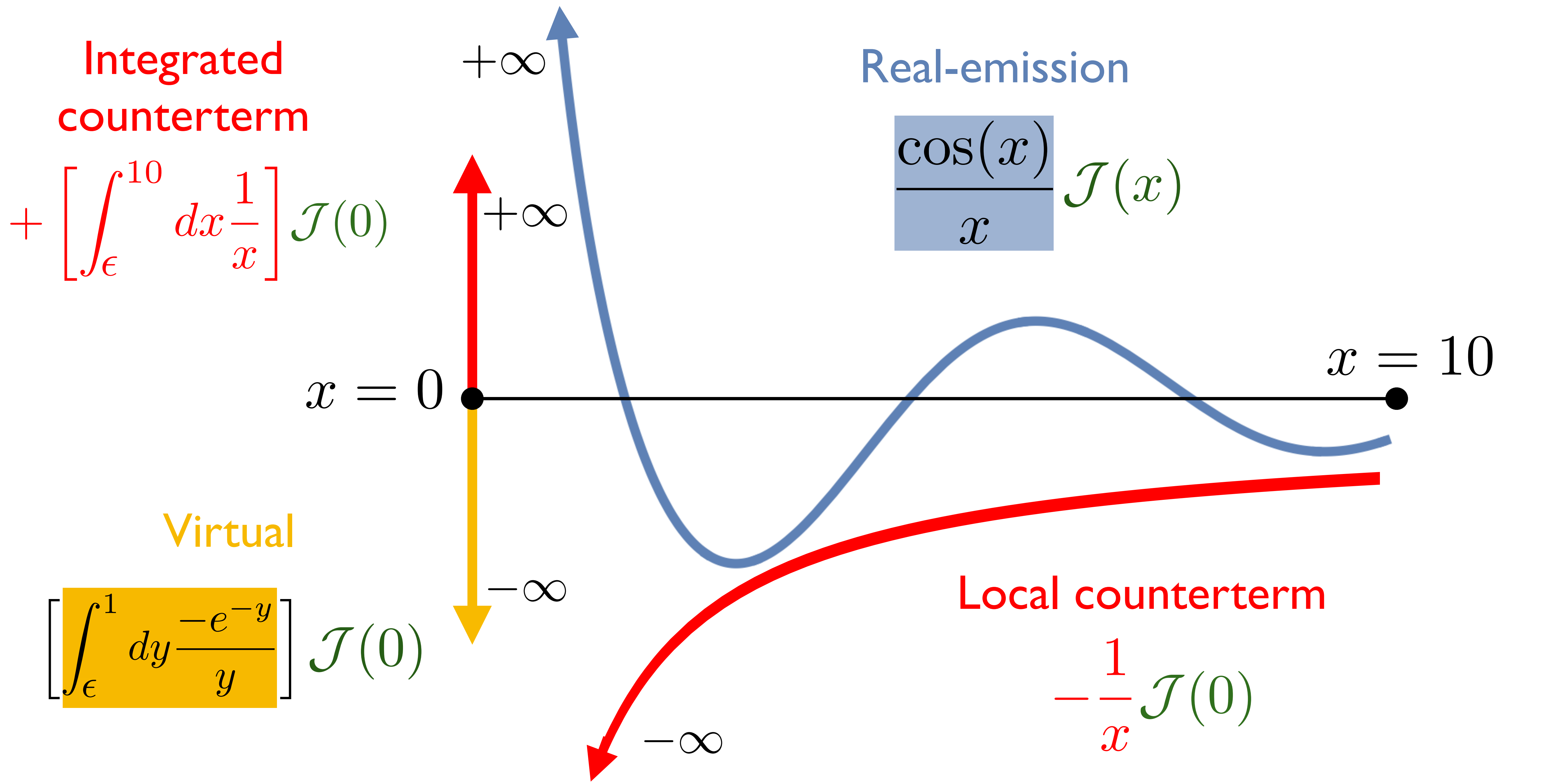
$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left[\int_0^1 dy \frac{-e^{-y}}{y}\right]\mathcal{J}(0) + \left[\int_0^{10} dx \frac{1}{x}\right]\mathcal{J}(0)$$



One-dimensional toy example

- Local unitarity would align the measure between real and virtual

$$\left[\int_0^1 dy \frac{-e^{-y}}{y}\right]$$

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“ LU ”

$$y = x/10$$

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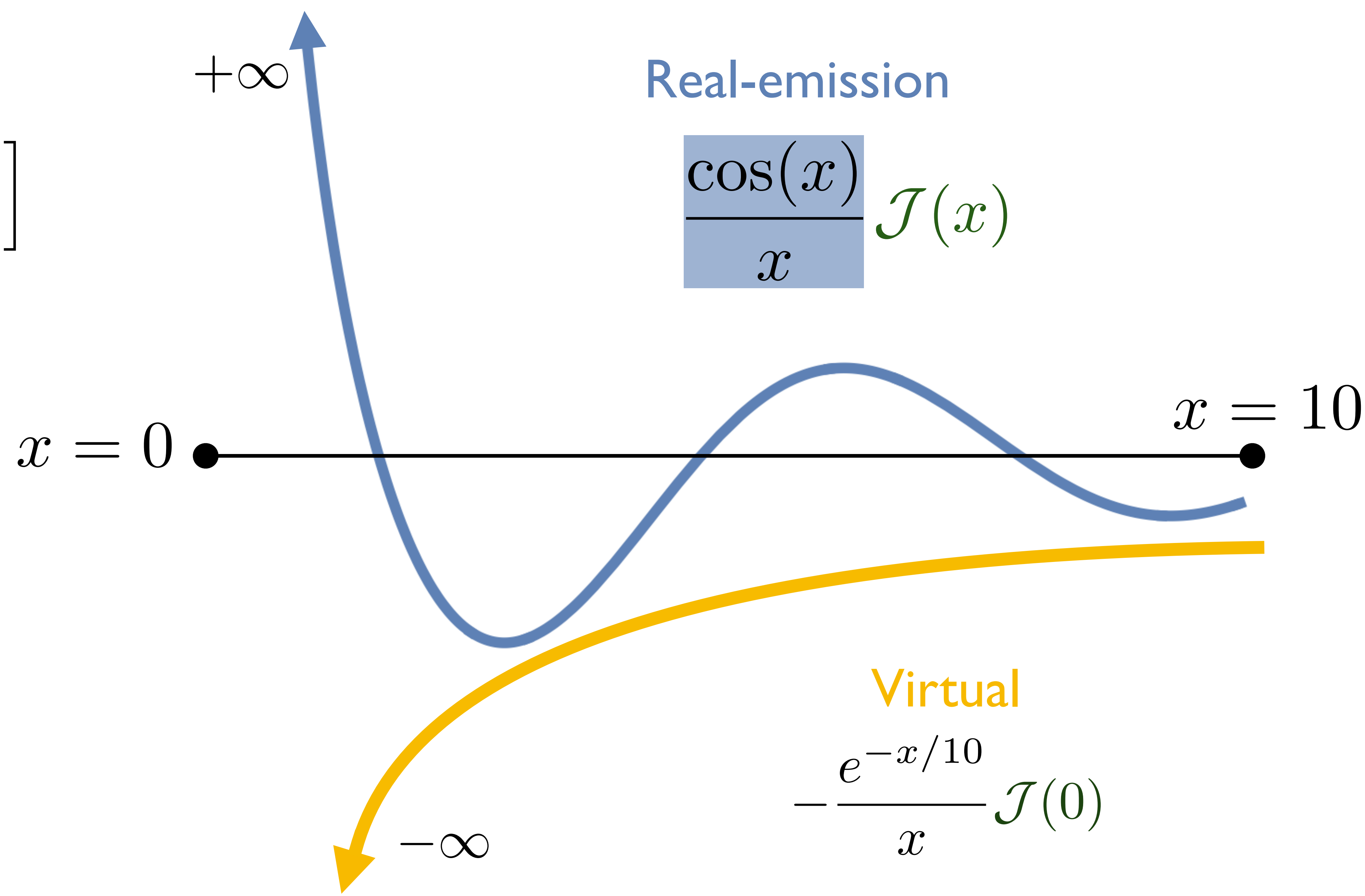
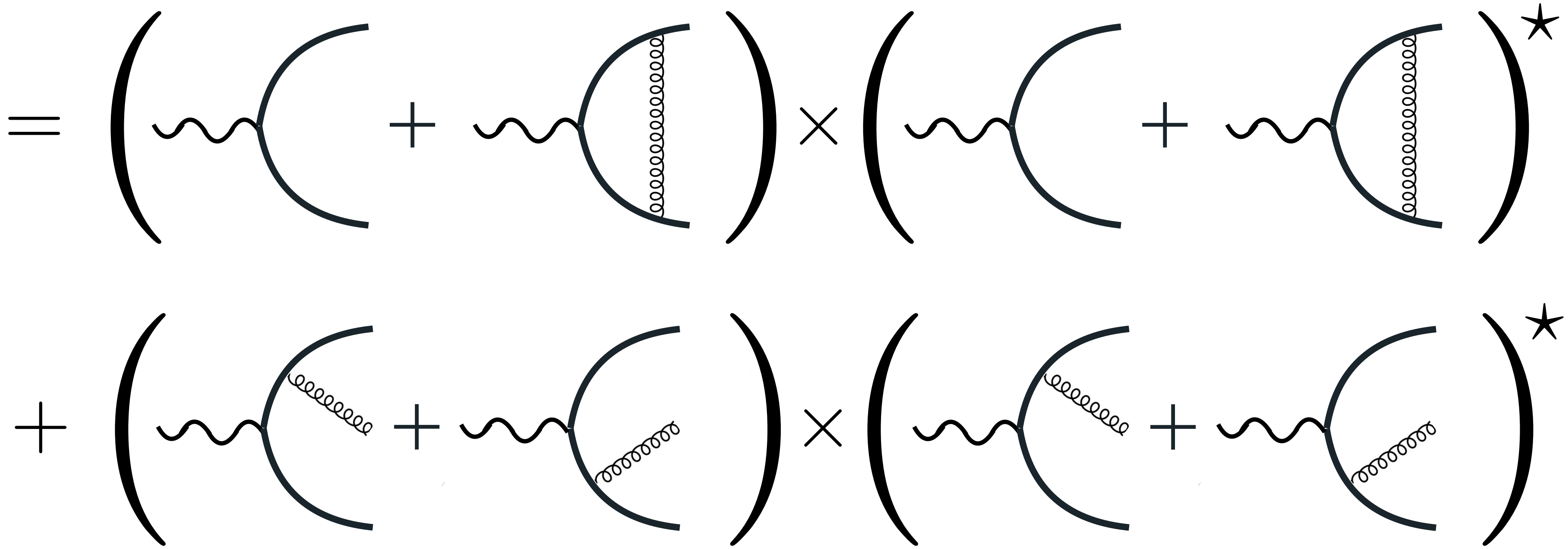


Remedy: forward-scattering graphs

$$\sigma_{\gamma^\star \to d\bar{d}} =$$



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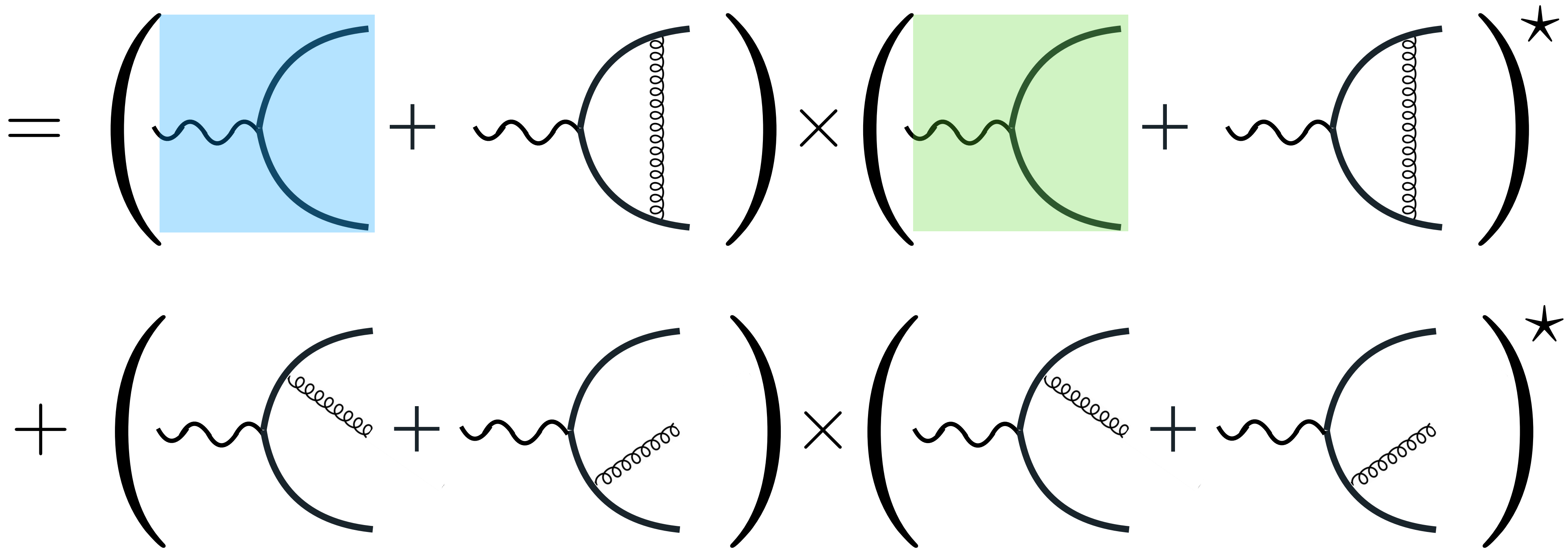

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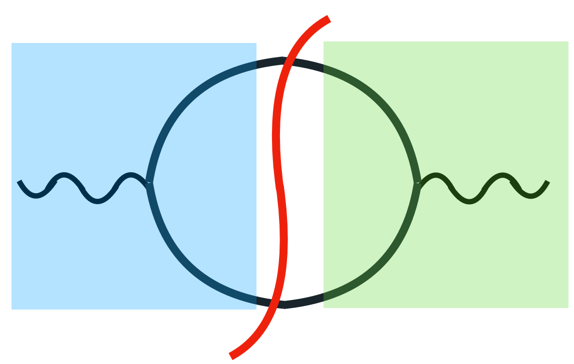

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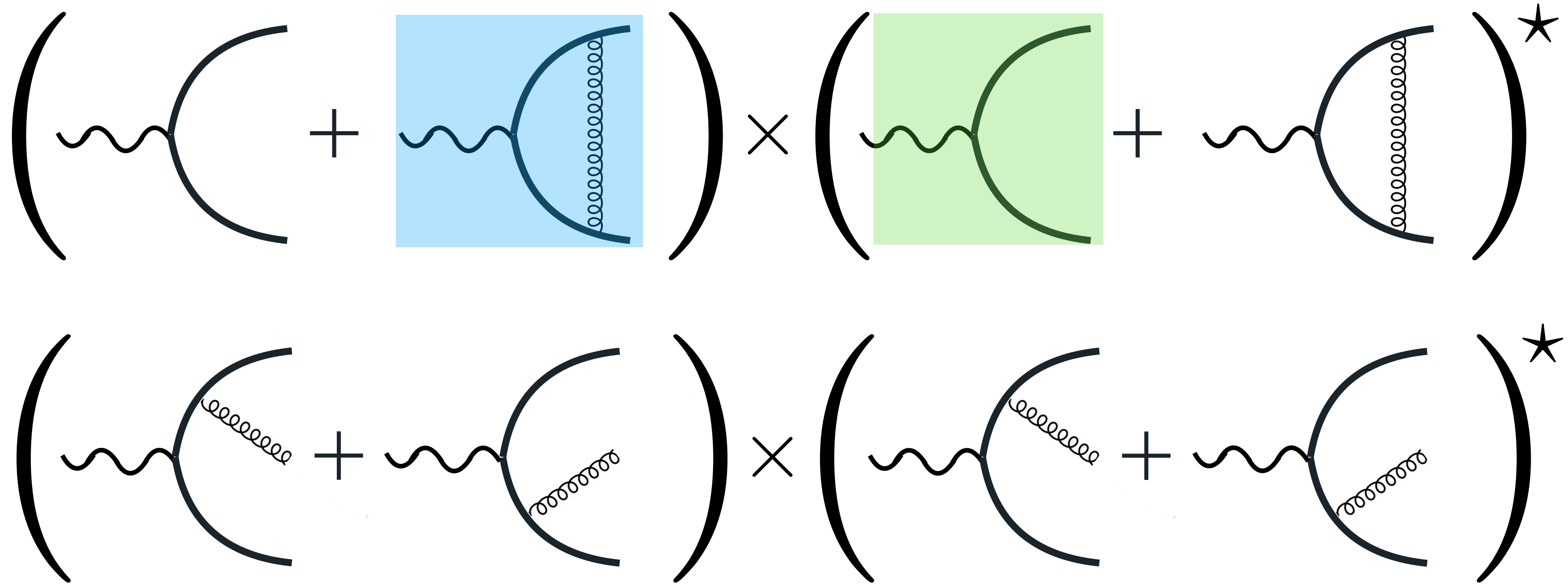

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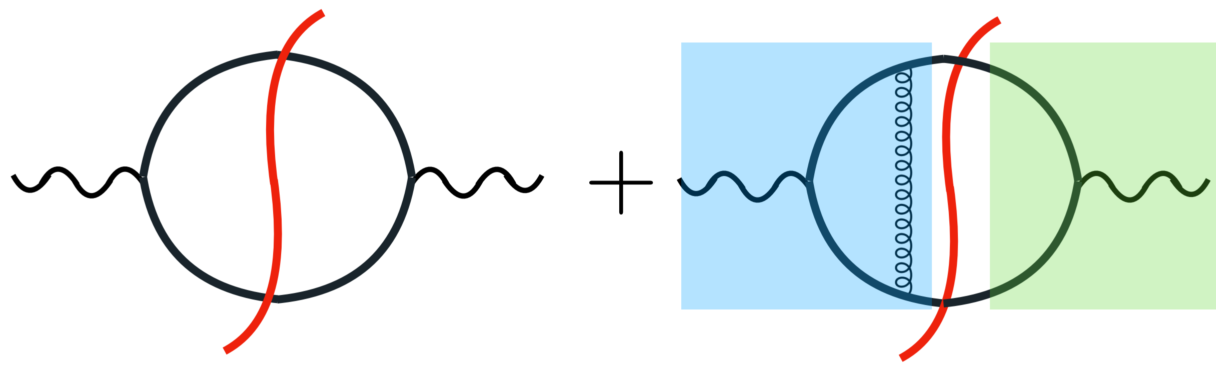

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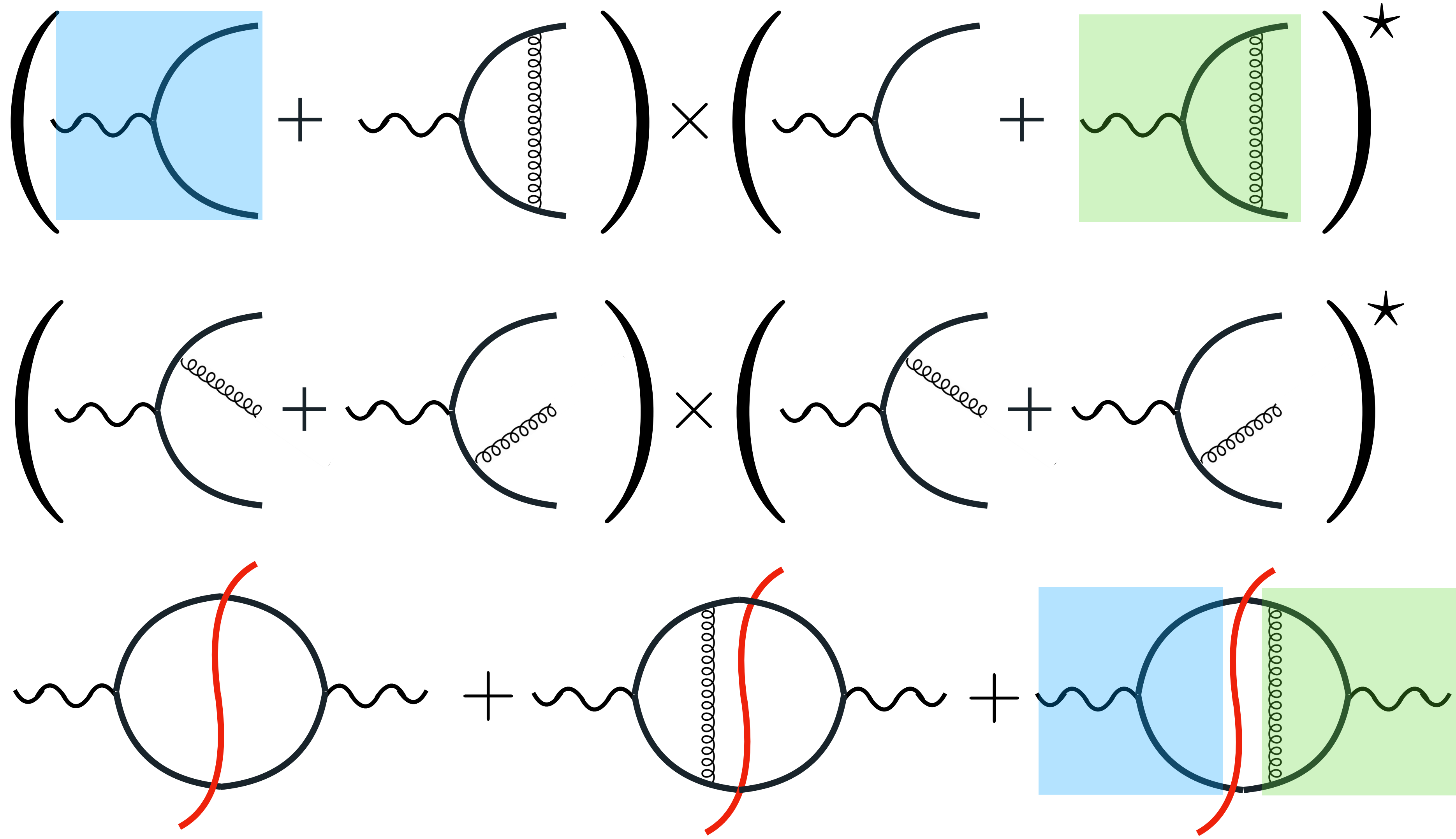

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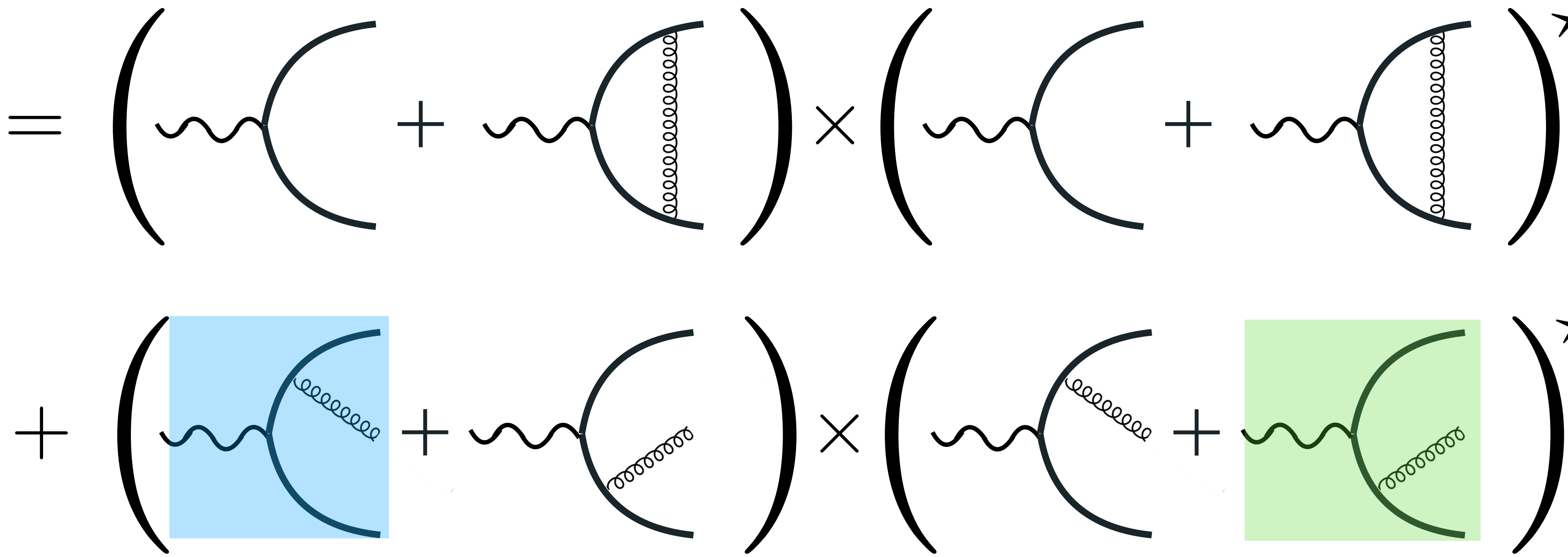

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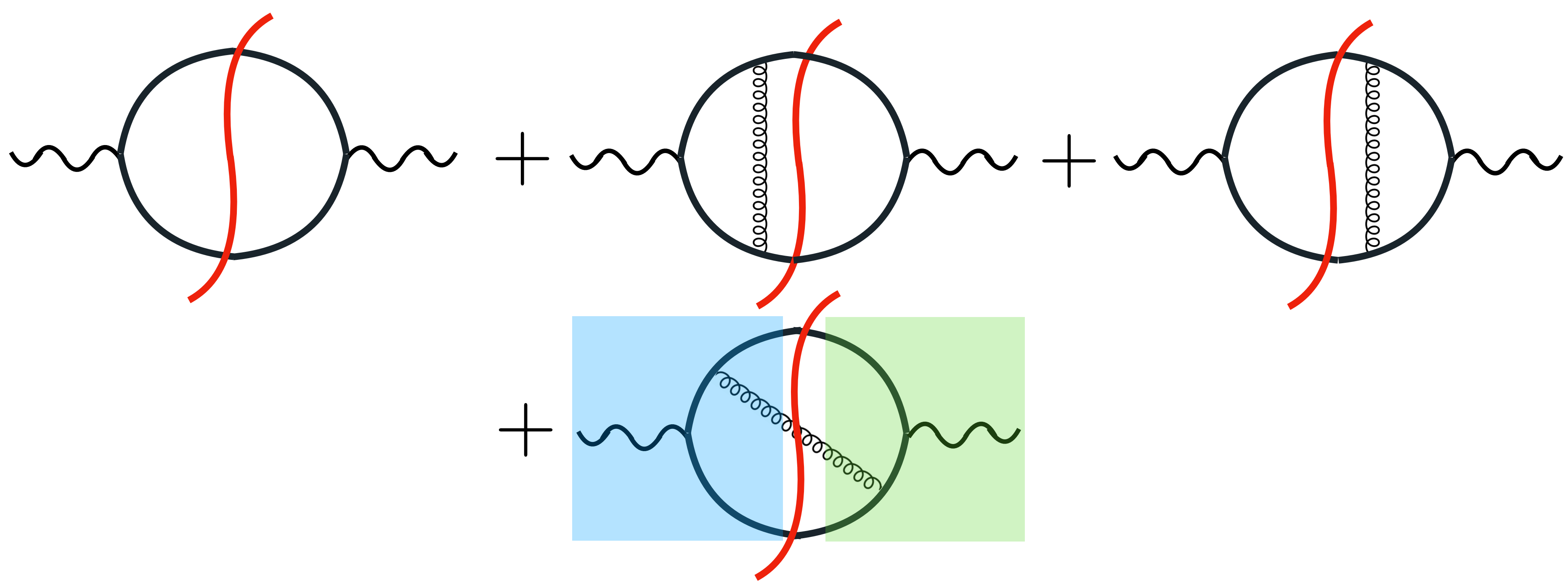

Remedy: forward-scattering graphs

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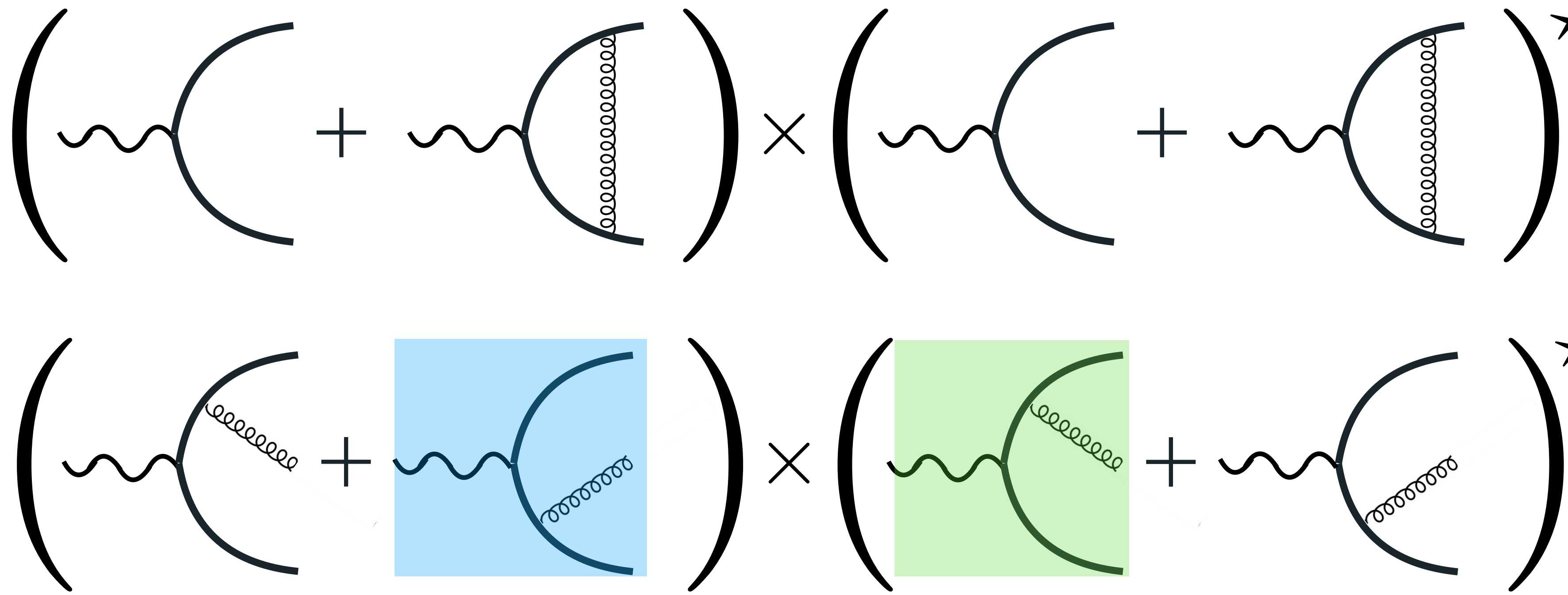

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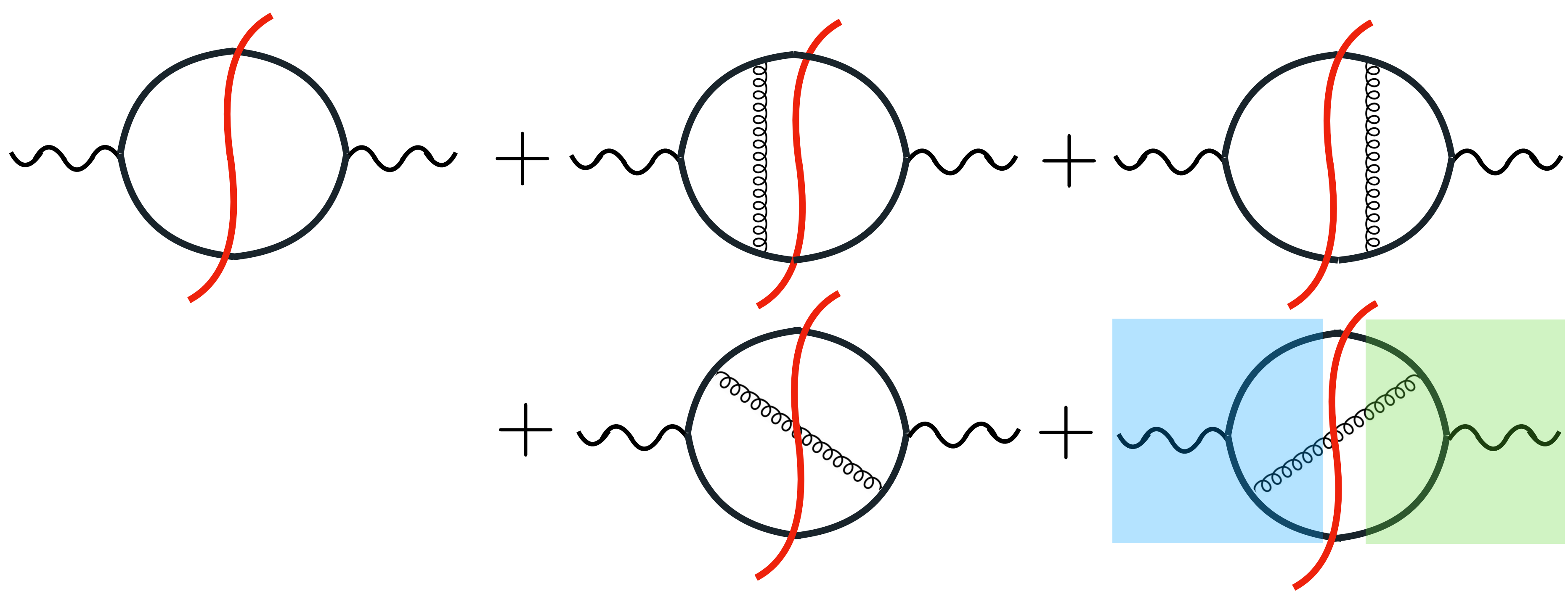

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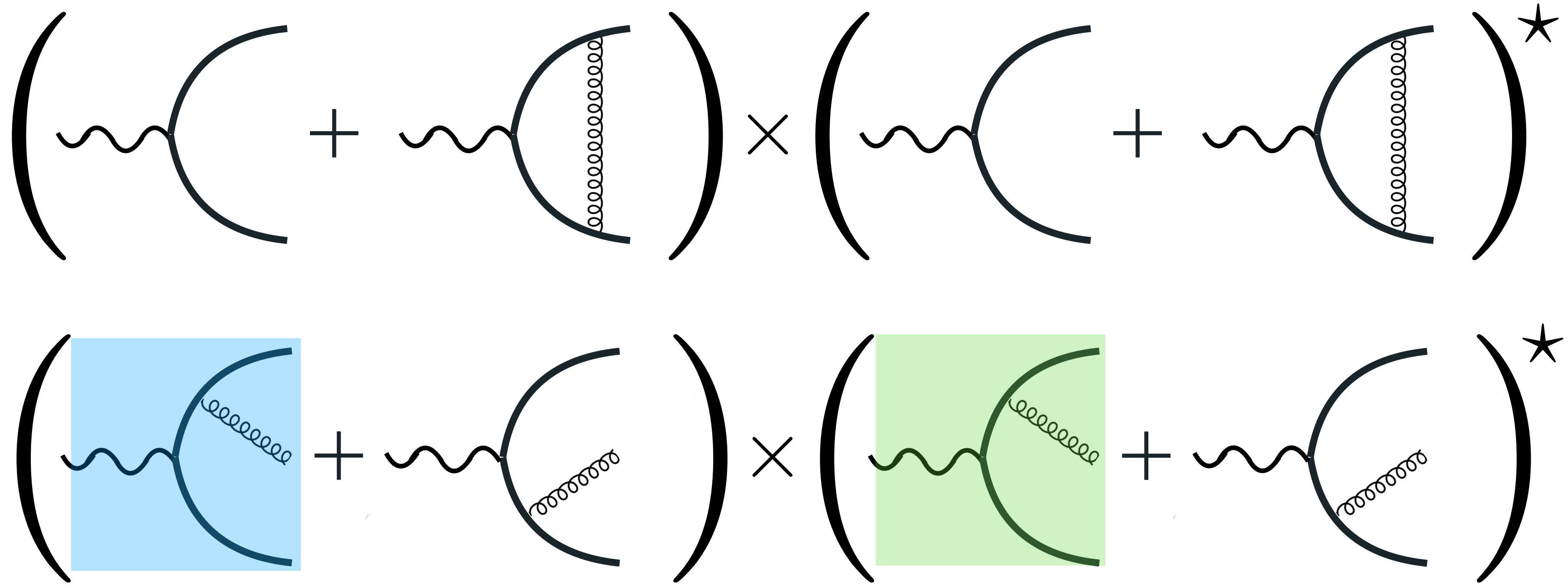

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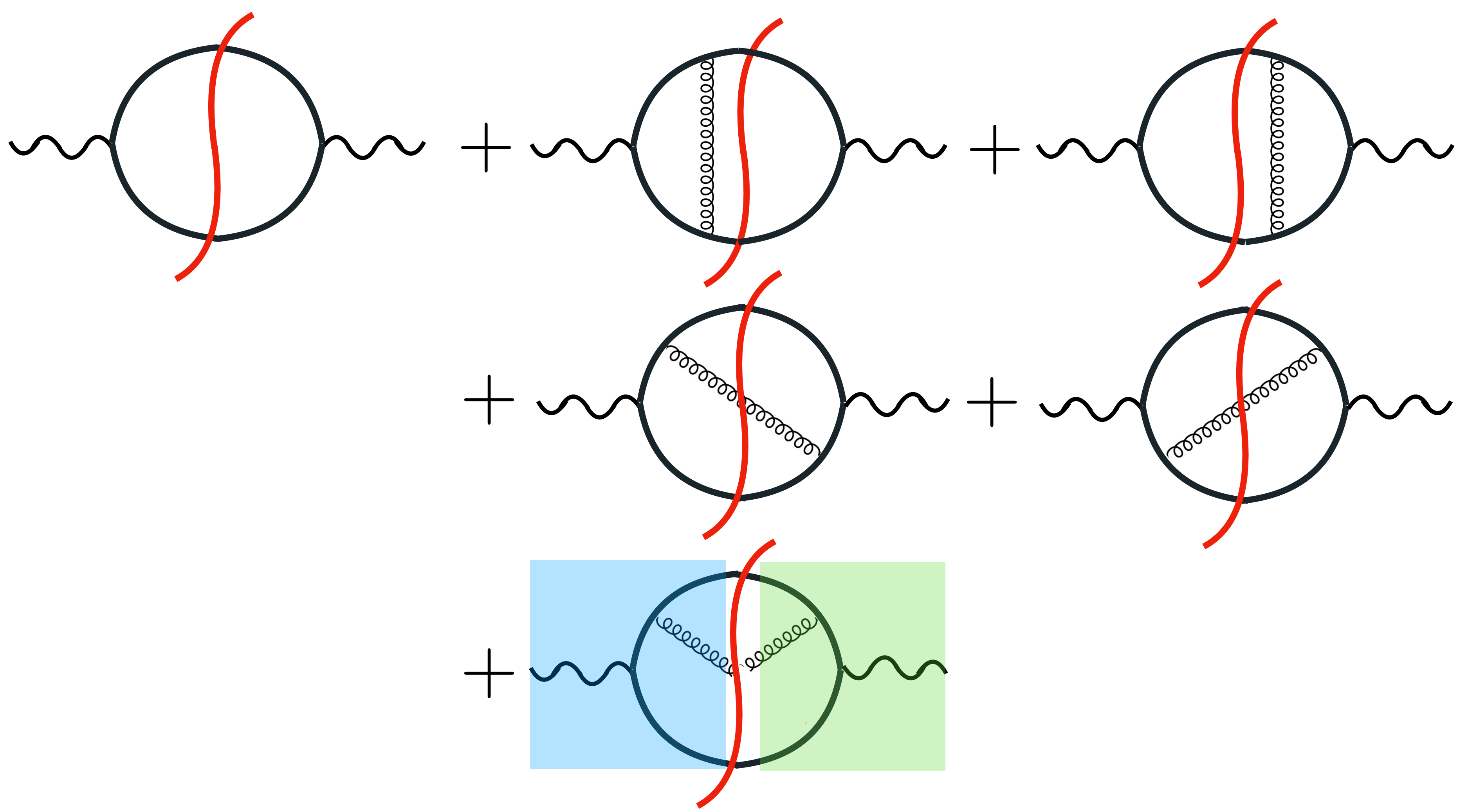

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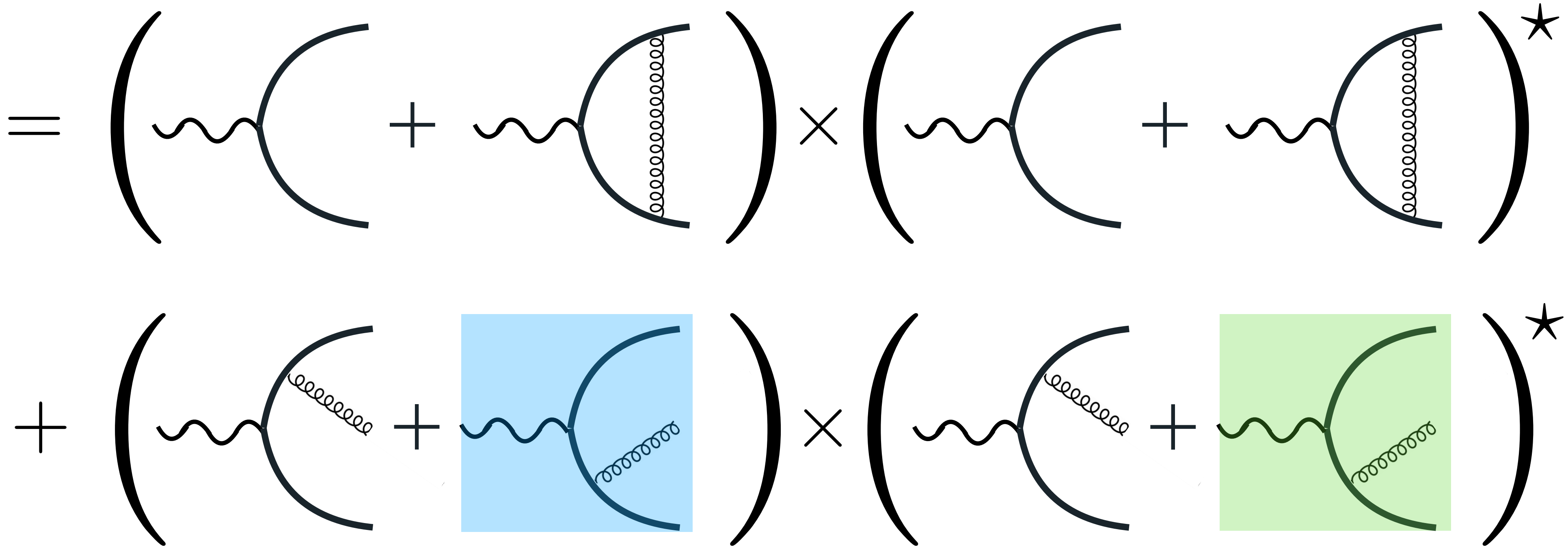

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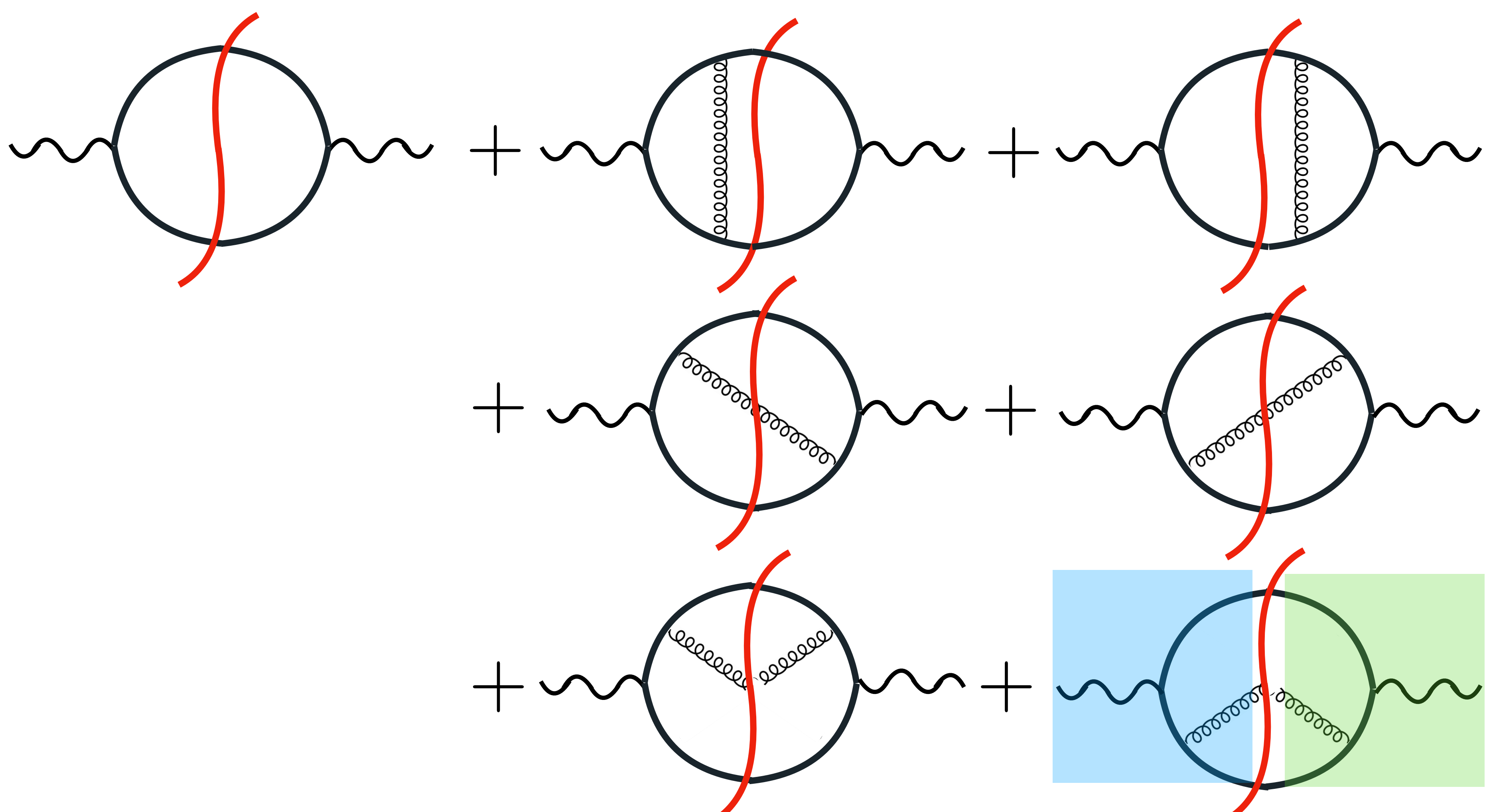

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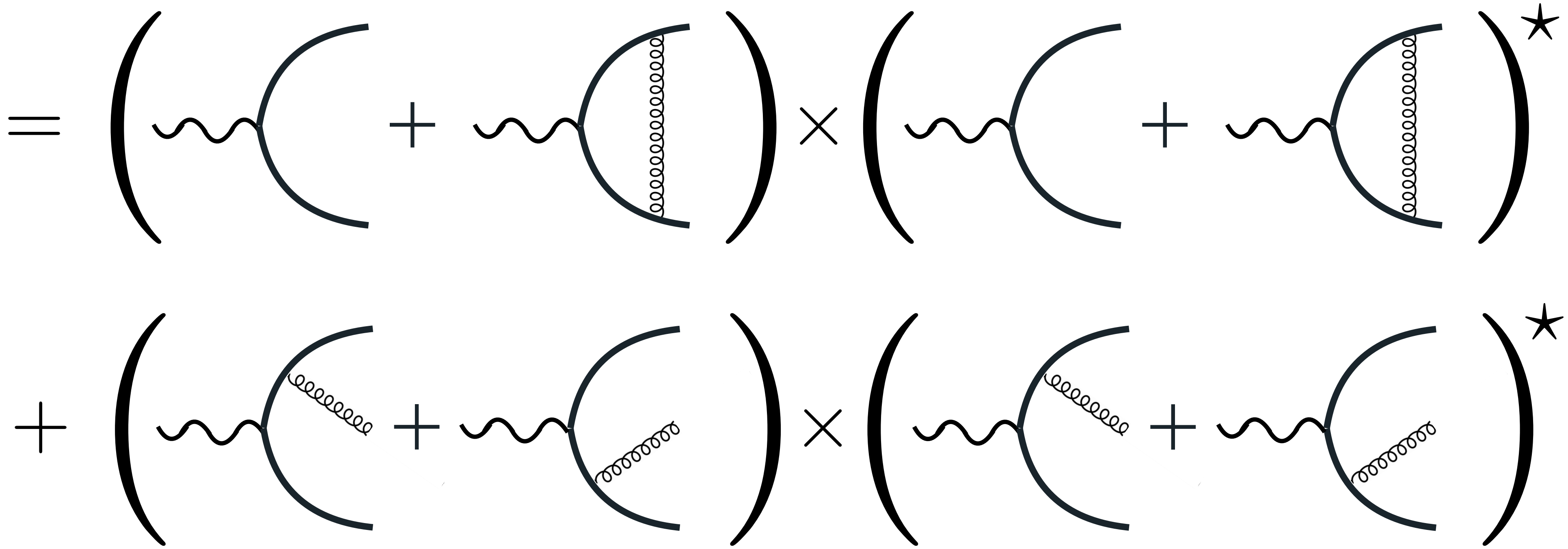

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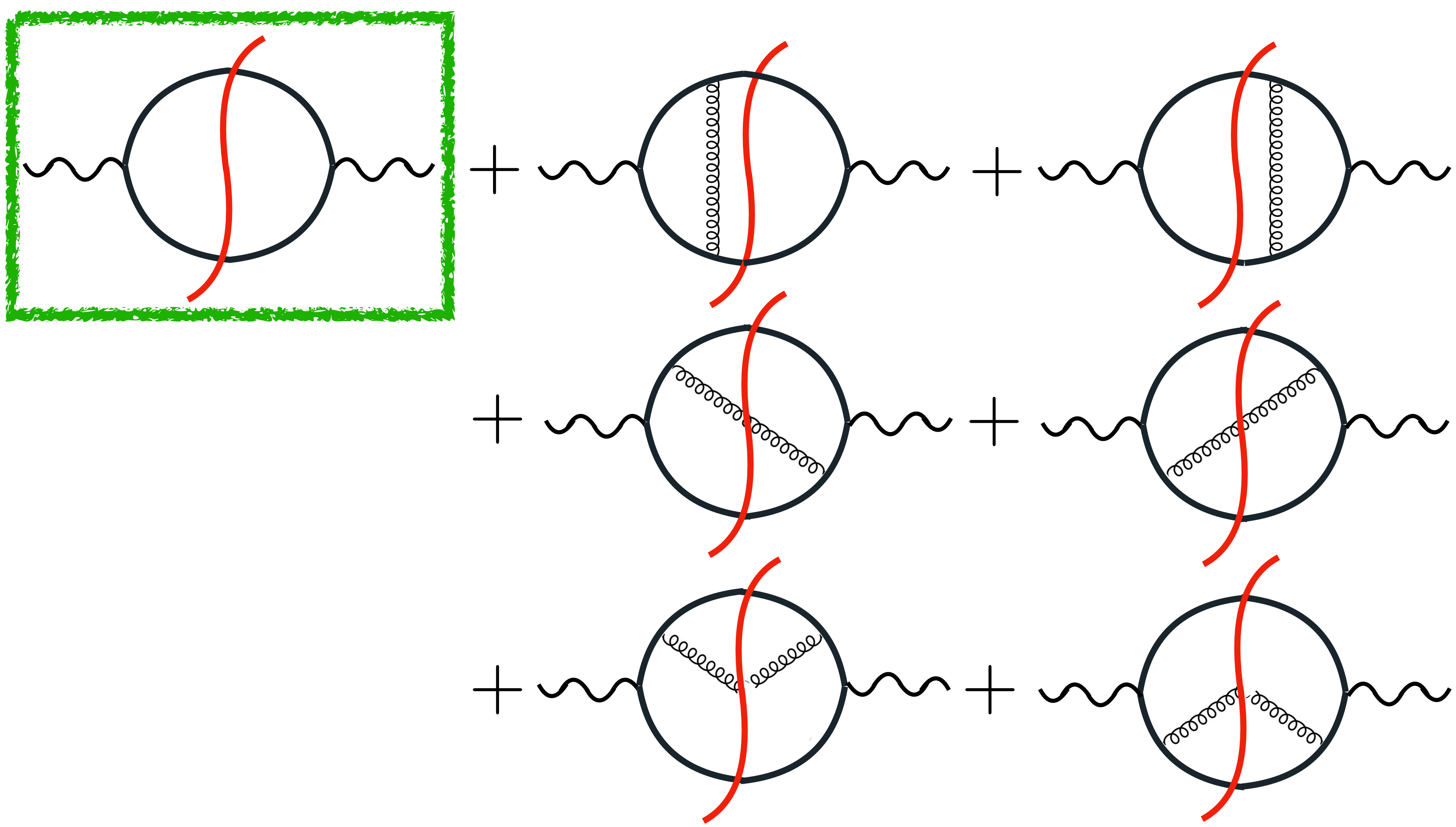
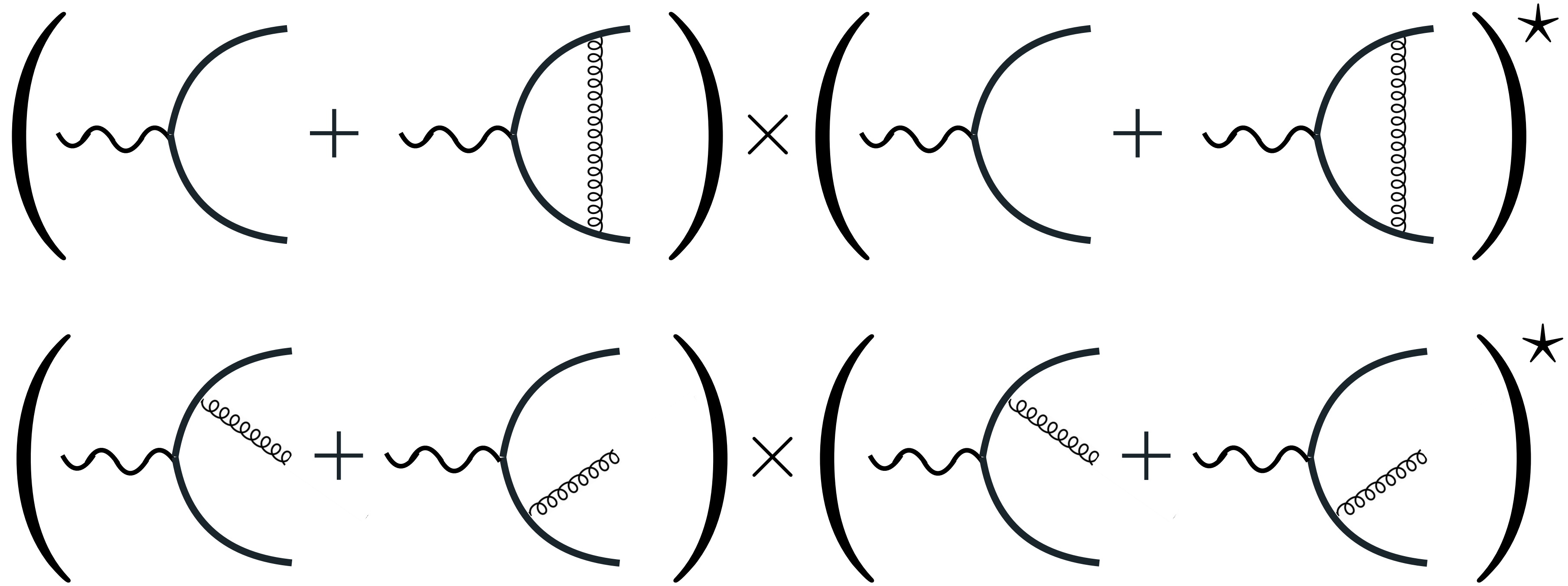


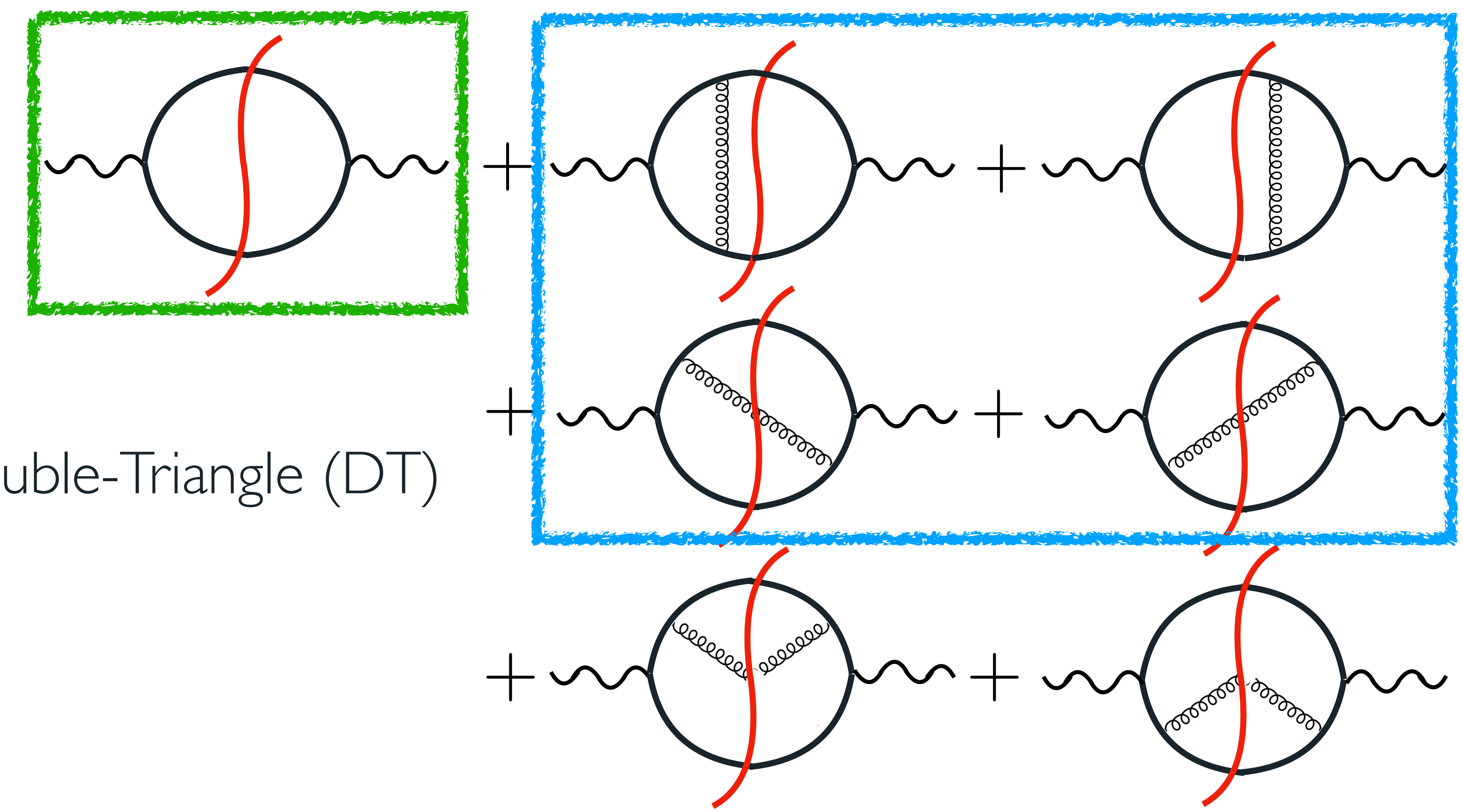
LO

REMEDY: FORWARD-SCATTERING GRAPHS

$\sigma_{\gamma^\star \to d\bar{d}} =$



$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} =$



LO

NLO, Double-Triangle (DT)

Remedy: forward-scattering graphs

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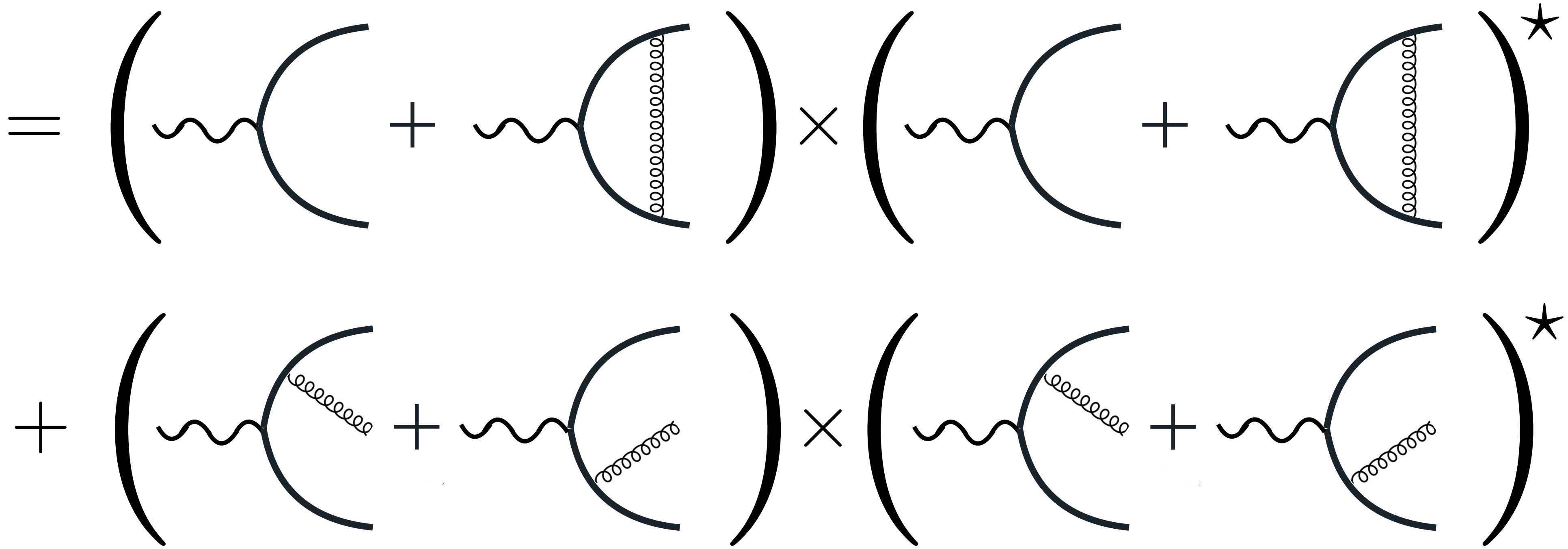

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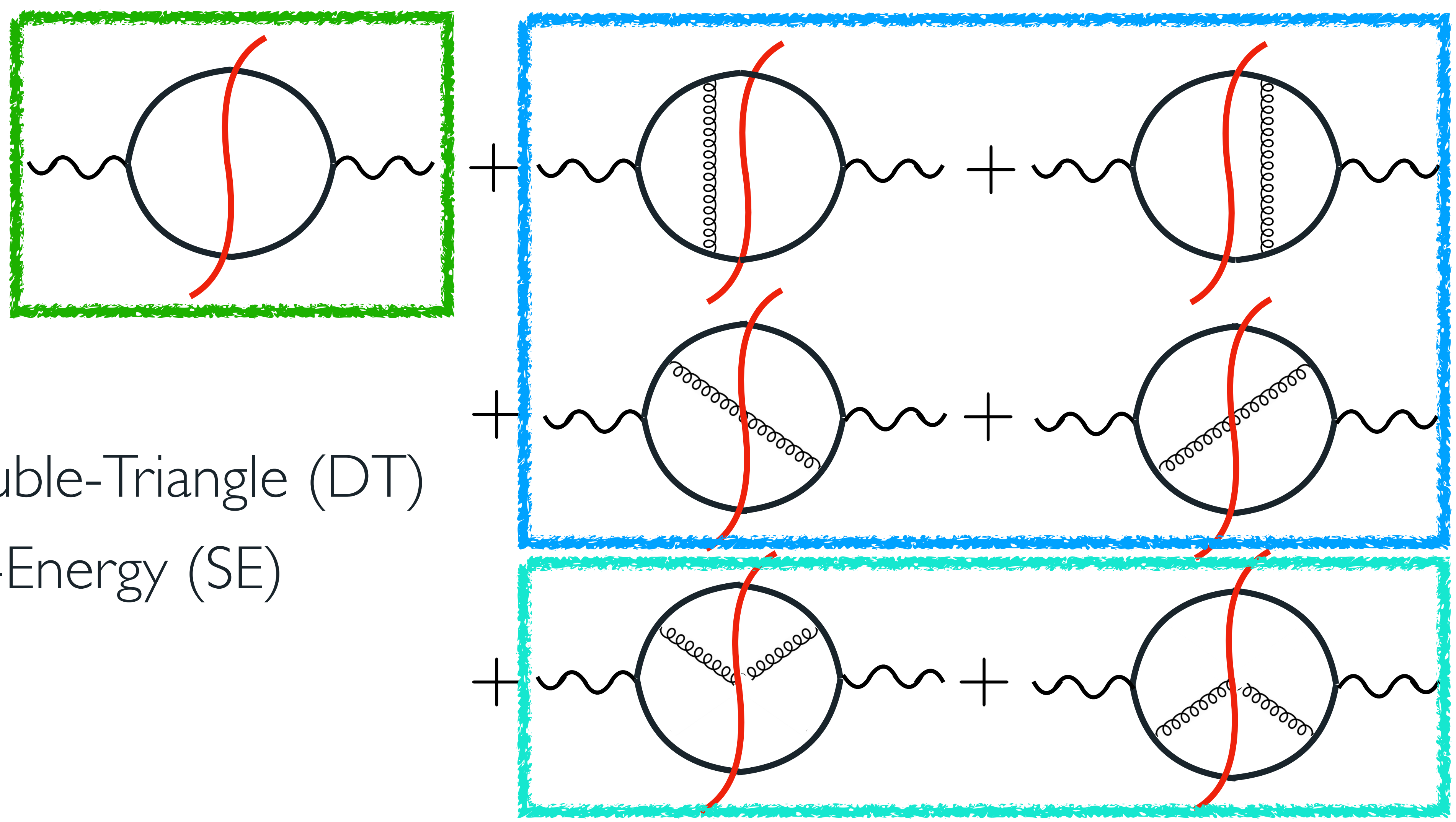


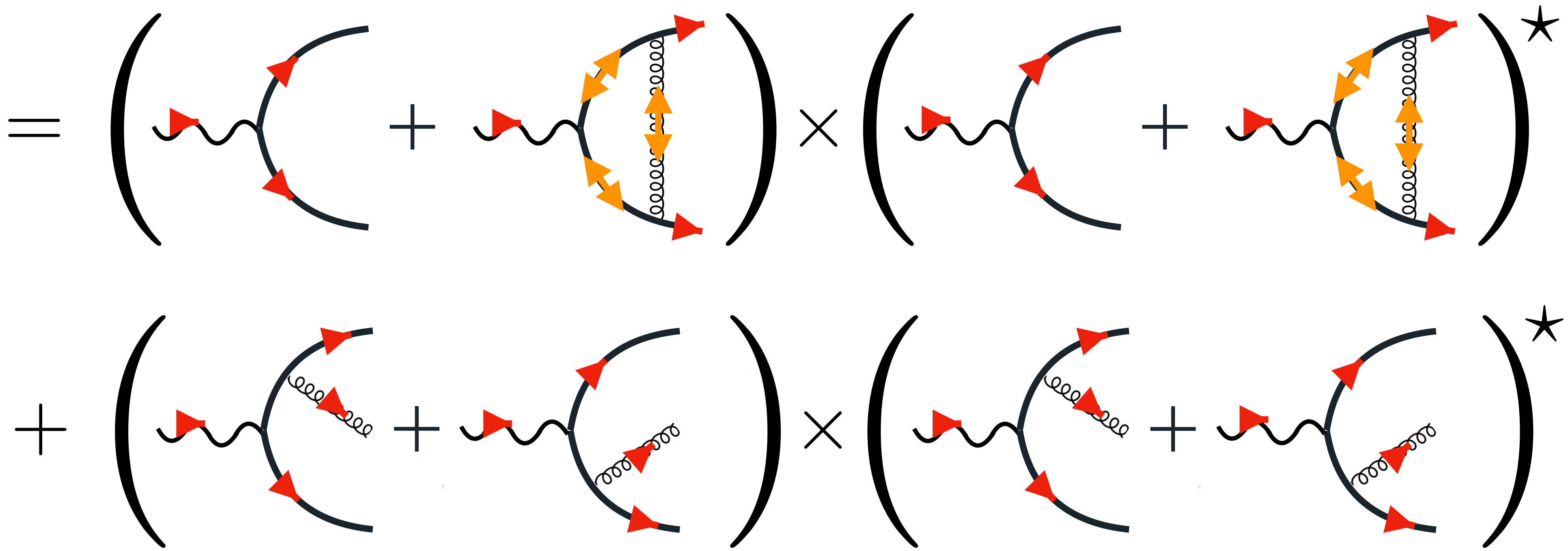
LO

NLO, Double-Triangle (DT)

NLO, Self-Energy (SE)

Remedy: forward-scattering graphs

$\sigma_{\gamma^\star \to d\bar{d}} =$



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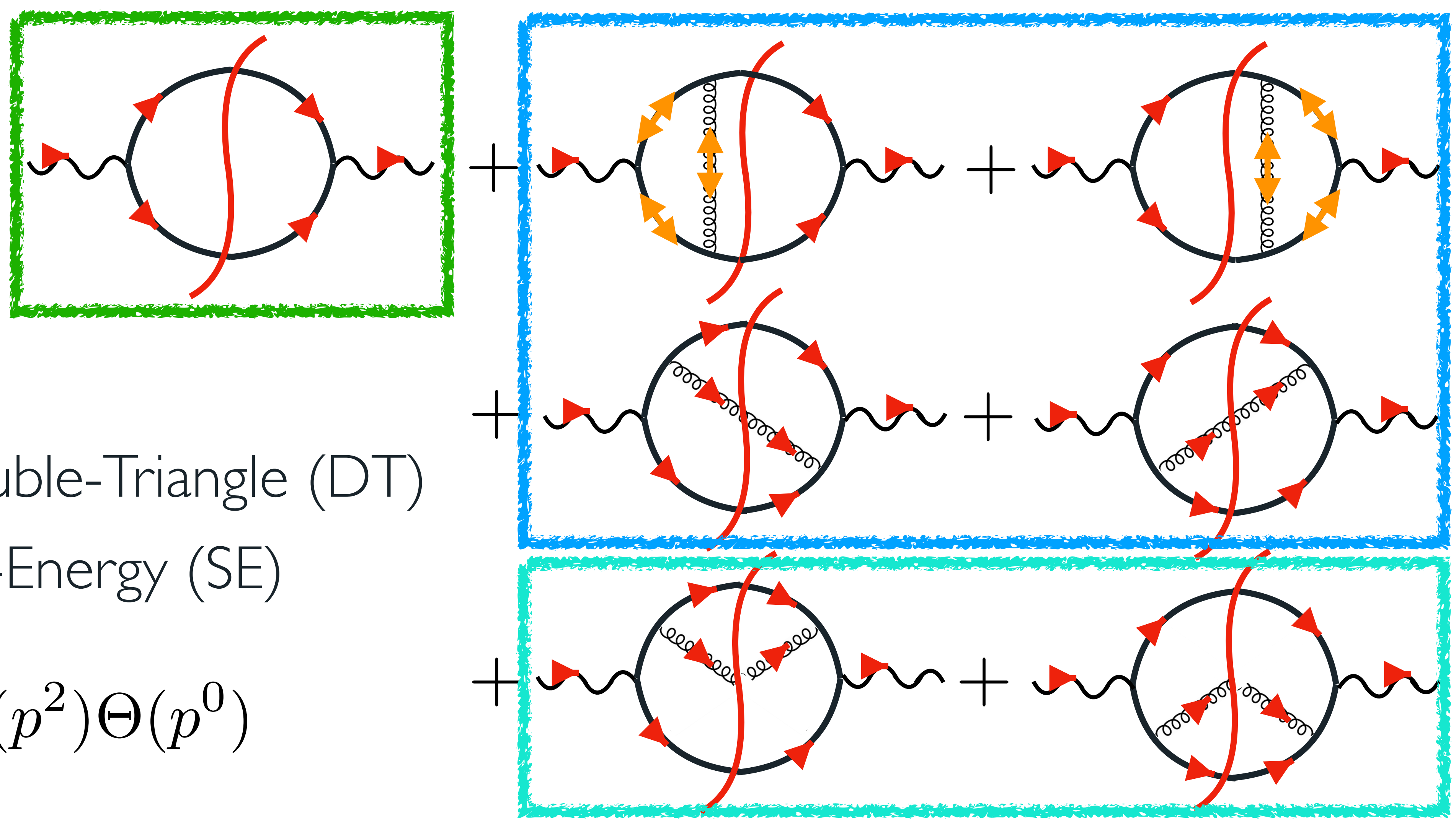
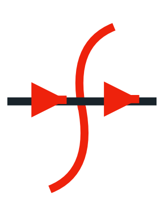
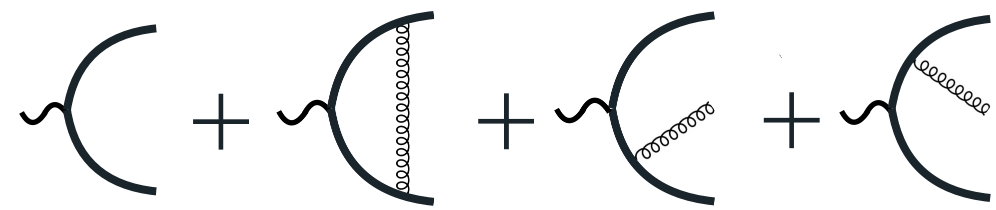


- LO
- NLO, Double-Triangle (DT)
- NLO, Self-Energy (SE)

 $\equiv \dfrac{p^2}{2p^0}\delta(p^2)\Theta(p^0)$

Local Unitarity: A conceptual shift

$$\sigma^{(\text{normal})}_{\gamma^\star \to d\bar{d}} = \int \Pi^{(\text{phase-space})} \left| \quad \right|^2$$



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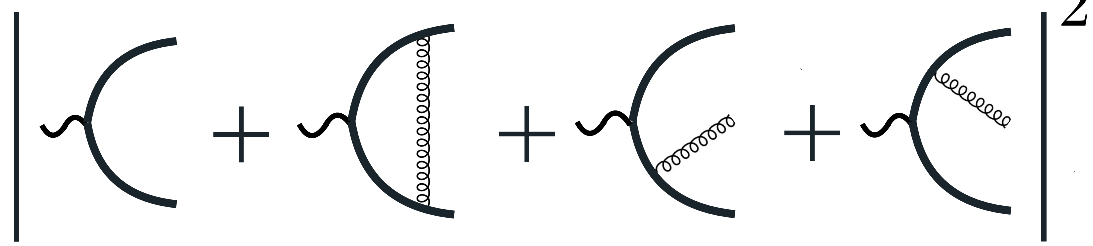
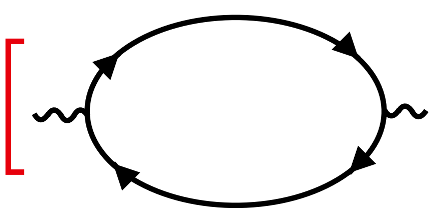
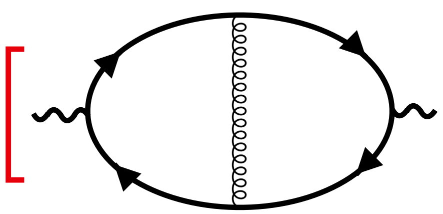


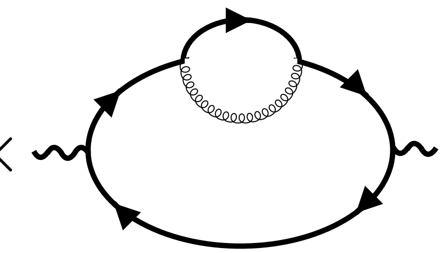
$$\sigma^{(\text{LU})}_{\gamma^\star \to d\bar{d}} = \text{LU}\Big[$$



$$\Big] + \text{LU}\Big[$$



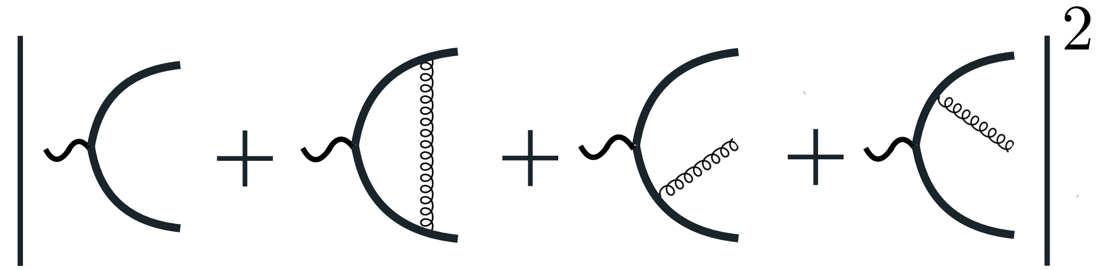
$$\Big] + \text{LU}\Big[\, 2\times$$



$$\Big]$$

Local Unitarity: A conceptual shift

$$\sigma^{(\mathrm{normal})}_{\gamma^\star\to d\bar{d}} = \int \Pi^{(\text{phase-space})}$$



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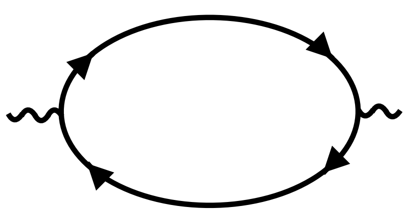

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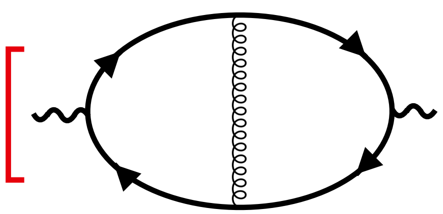

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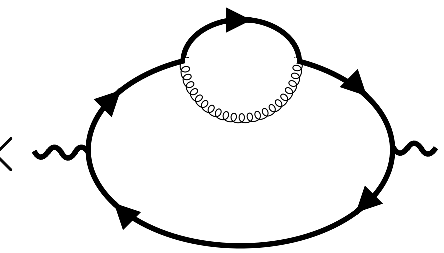


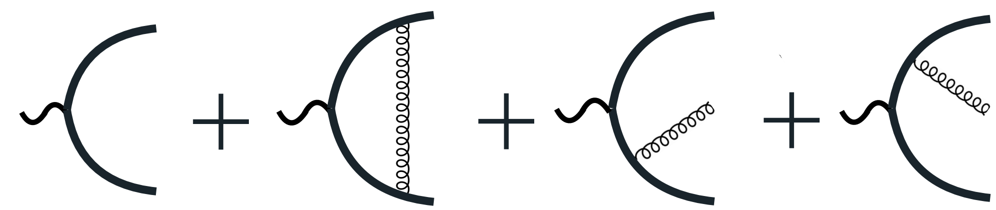
$$\Big]$$

$$\sum_{c\in\{RRR,RRV,RVV,\ldots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\mathrm{amplitudes}}(c)} \int \Pi_{i_c}^{(\mathrm{loop})} \mathcal{A}_{i_c} \right|^2_{\mathrm{truncated}}$$

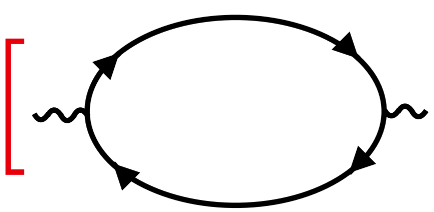
IR-subtraction numerical $d = 4$ analytic $d = 4 - 2\epsilon$

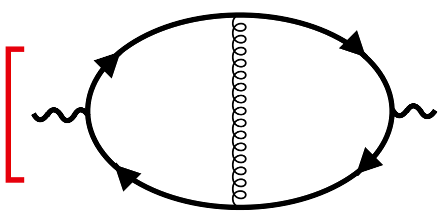
Local Unitarity: A conceptual shift

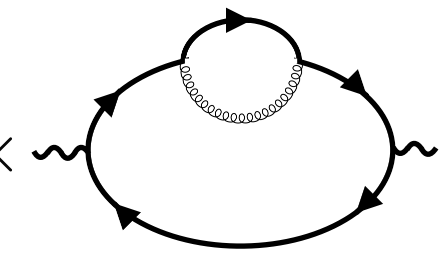
$$\sigma^{(\text{normal})}_{\gamma^\star \to d\bar{d}} = \int \Pi^{(\text{phase-space})} \Bigg| \quad \Bigg|^2$$



$$\sigma^{(\text{LU})}_{\gamma^\star \to d\bar{d}} = \text{LU}\Big[\quad\Big] + \text{LU}\Big[\quad\Big] + \text{LU}\Big[\, 2\times \quad\Big]$$







$$\sum_{c \in \{RRR, RRV, RVV, \dots\}} \int \Pi_c^{(\text{phase-space})} \left| \sum_{i_c=1}^{n_{\text{amplitudes}}(c)} \int \Pi_{i_c}^{(\text{loop})} \mathcal{A}_{i_c} \right|^2_{\text{truncated}}$$

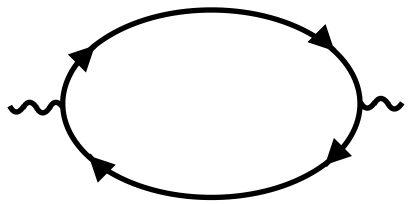
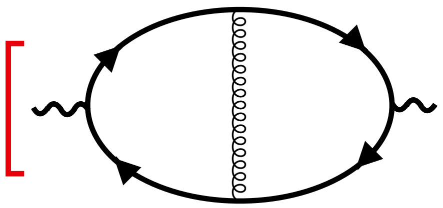
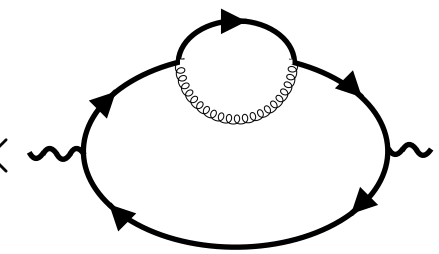
IR-subtraction numerical $d = 4$ analytic $d = 4 - 2\epsilon$

$$\sum_{j=1}^{n_{\text{supergraphs}}} \int \Pi\, g_j^{(\text{LU})}$$

numerical $d = 4$

NO IR-subtraction

So what? It's just the optical theorem no?

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[\, 2\times$  $\Big]$

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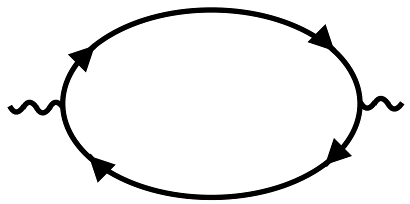
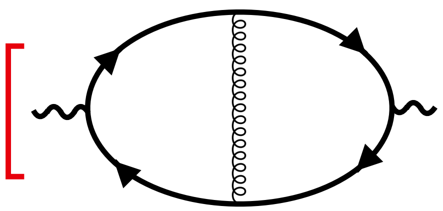
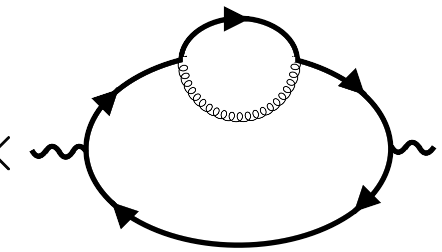
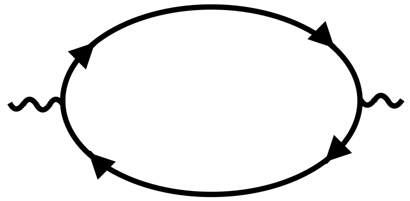
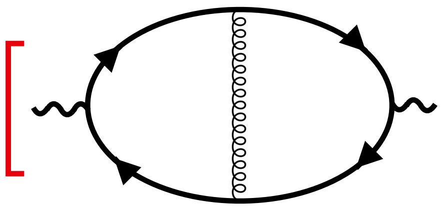
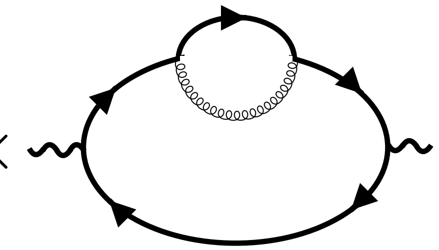
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[\, 2\times$  $\Big]$

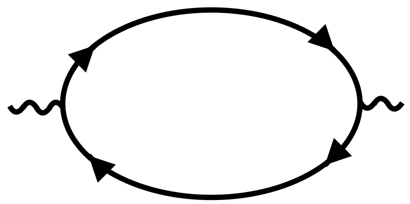
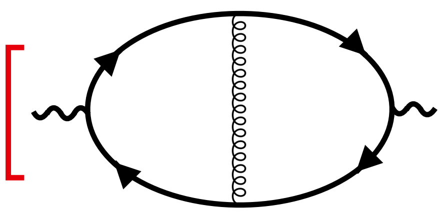
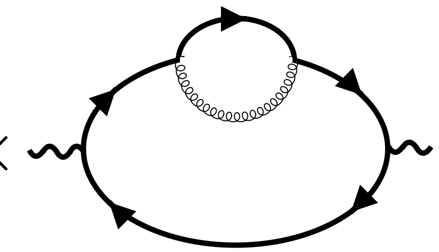
- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$

So what? It's just the optical theorem no?

$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$$  $$\Big] + \mathrm{LU}\Big[$$  $$\Big] + \mathrm{LU}\Big[\, 2\times$$  $$\Big]$$

- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$
- The **differential** version of LU is our new paradigm: **Local Unitarity**

So what? It's just the optical theorem no?

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[\, 2\times$  $\Big]$

- At the **integral** level, the **Optical Theorem** simply gives: $\mathrm{LU}[\cdot] \propto \mathrm{Im}[\cdot]$
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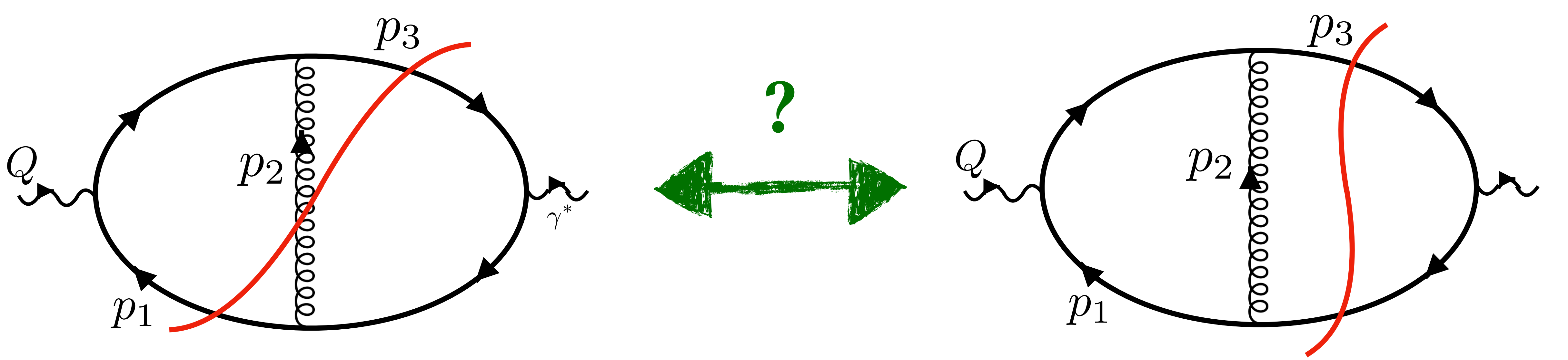
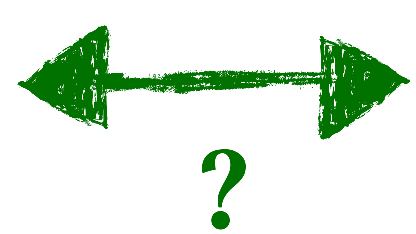
- Each supergraph is **individually locally finite**, how you ask ?



$\propto \delta^{(+)}(p_3^2)\delta^{(-)}(p_1^2)\delta^{(-)}(p_2^2)$  $\propto \delta^{(+)}(p_3^2)\delta^{(-)}((p_1+p_2)^2)$

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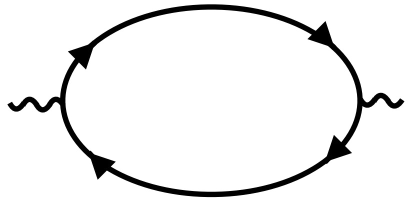
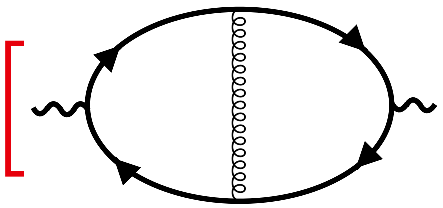
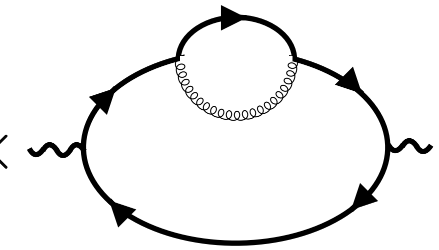
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[$  $\Big] + \mathrm{LU}\Big[\, 2\times$  $\Big]$

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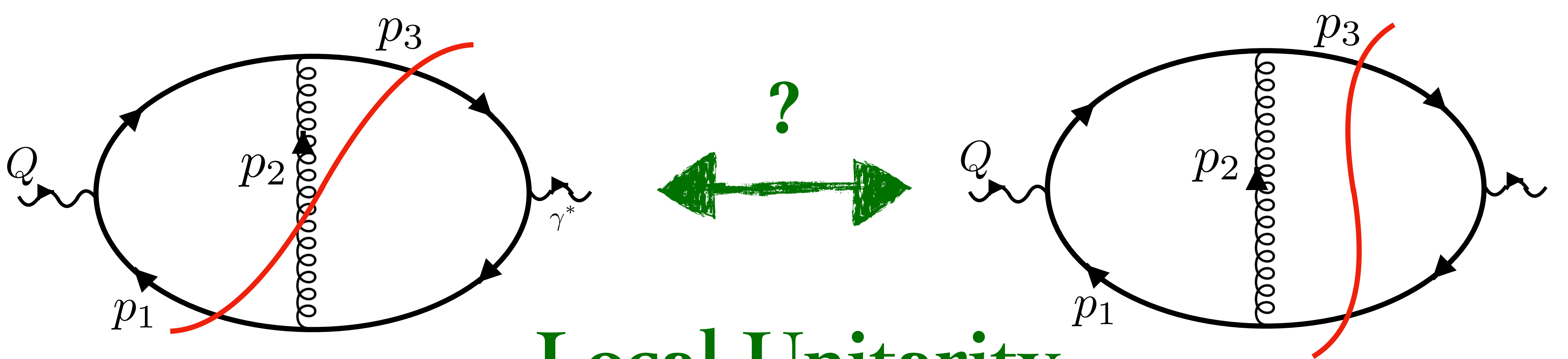
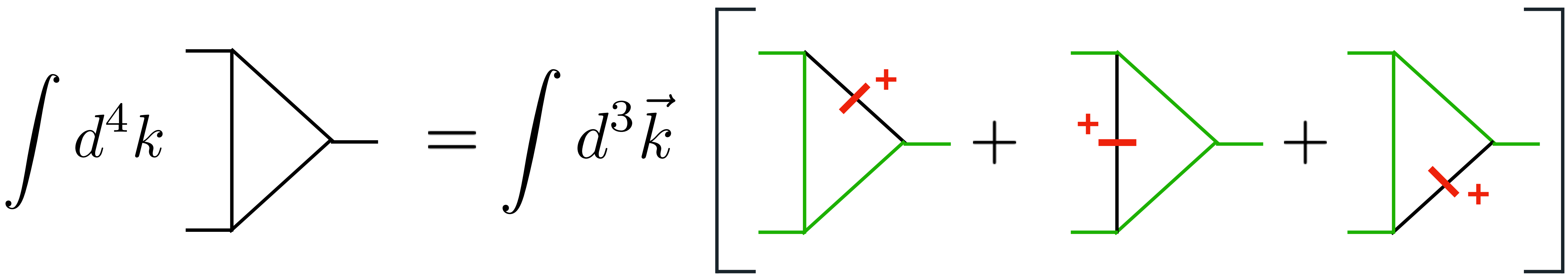


Local Unitarity

$\propto \delta^{(+)}(p_3^2)\delta^{(-)}(p_1^2)\delta^{(-)}(p_2^2)$? $\propto \delta^{(+)}(p_3^2)\delta^{(-)}((p_1+p_2)^2)$

Main ingredient: Loop Tree Duality



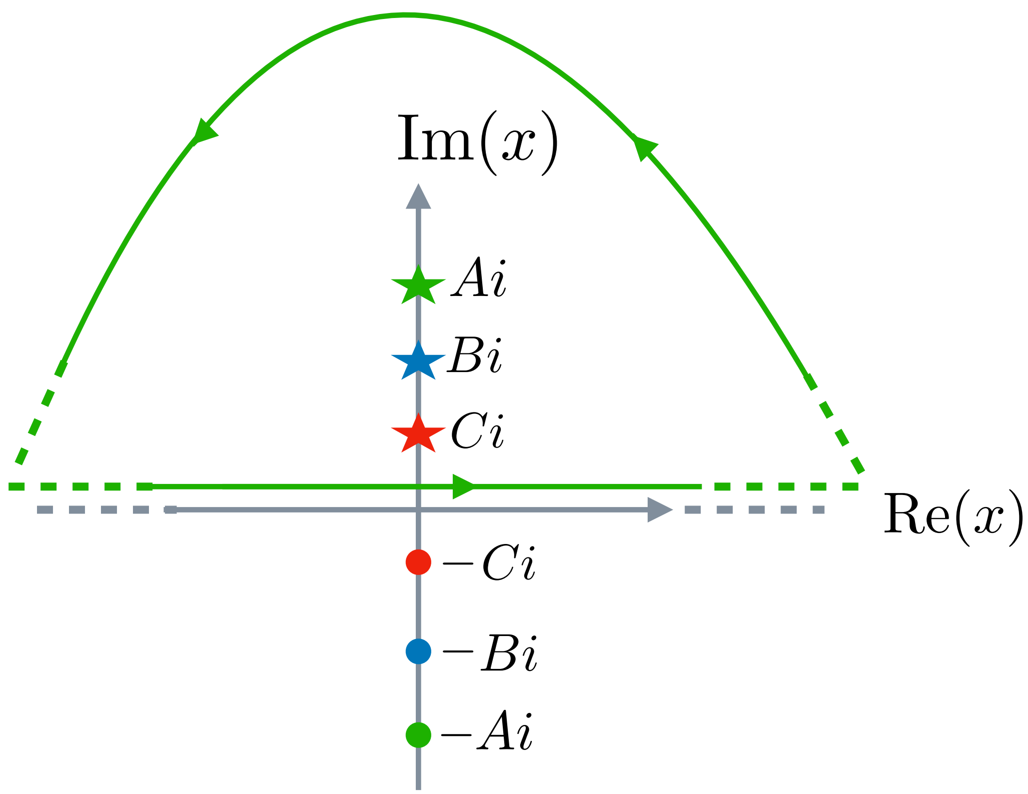
(closely related to the Feynman-tree theorem
And Time-Ordered Perturbation Theory)

(One)-Loop Tree duality mock-up

$$I = \int_{-\infty}^{+\infty} dx F(x) \qquad F(x) = \frac{1}{x^2+A^2}\frac{1}{x^2+B^2}\frac{1}{x^2+C^2}$$

$$F(x) = \frac{1}{(x-Ai)(x+Ai)}\frac{1}{(x-Bi)(x+Bi)}\frac{1}{(x-Ci)(x+Ci)}$$

$(\text{Assumptions} \to \{A>0, B>0, C>0\})$



Cauchy: $\left(R(x^\star) \equiv \mathrm{Res}(F, x = x^\star) \right)$

$$I = (-2\pi i)[R(Ai) + R(Bi) + R(Ci)]$$

What does it correspond to for a one-loop integral?

(ONE-)LOOP TREE DUALITY

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

(One-)Loop Tree duality

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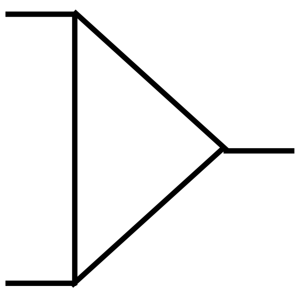
Pole selected for each propagator

Then integrate the energy component using residue theorem



$$= \int \mathrm{d}^3\vec{k} \left[\mathrm{Res}_1\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_2\left[\frac{N}{D_1 D_2 D_3}\right] + \mathrm{Res}_3\left[\frac{N}{D_1 D_2 D_3}\right]\right]$$

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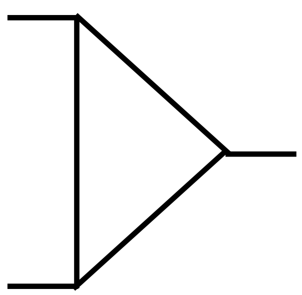

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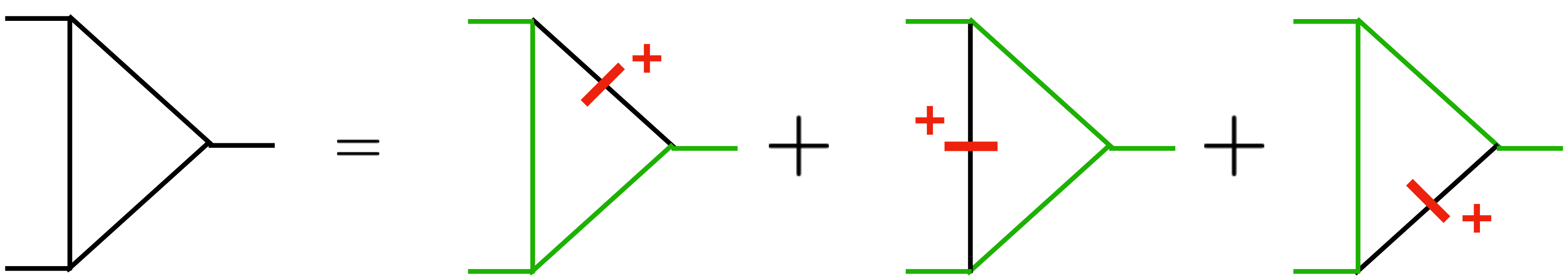
Residues can be represented as cuts:



(One-)Loop Tree duality

$$\frac{1}{k^2 - M^2 + i\delta} = \frac{1}{(k^0)^2 - |\vec{k}|^2 - M^2 + i\delta} = \frac{1}{\boxed{\left(k^0 - \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}\left(k^0 + \sqrt{|\vec{k}|^2 + M^2 - i\delta}\right)}$$

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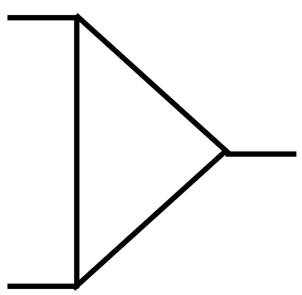

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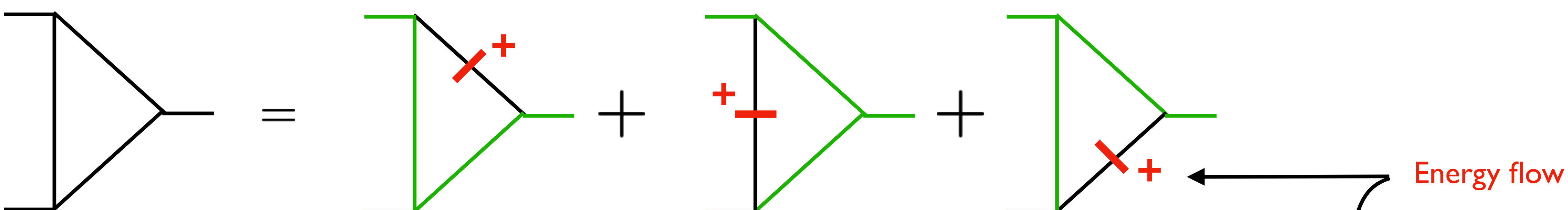
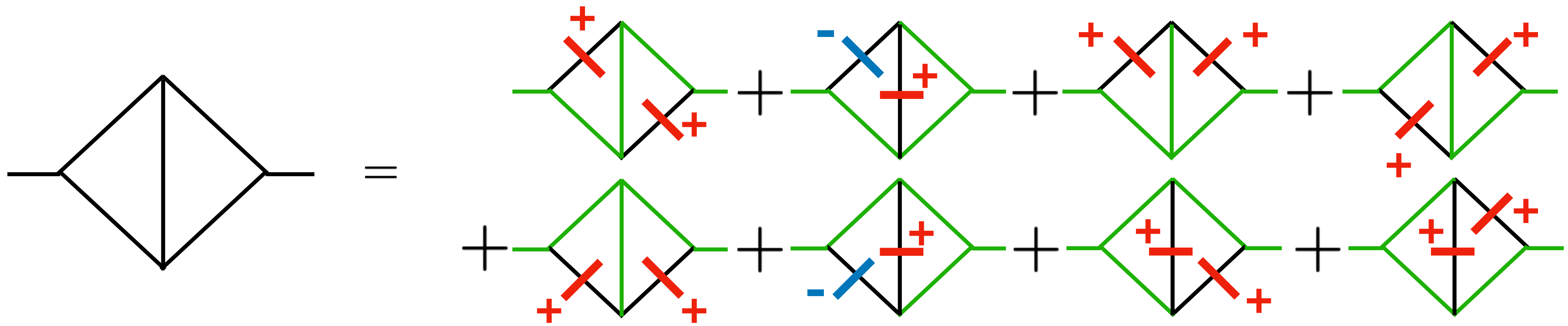


$$= \int \mathrm{d}^4 k \frac{N}{D_1 D_2 D_3}\left(D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3)\right)$$

(MULTI-)LOOP TREE DUALITY

Applying LTD to a two-loop double-triangle: one residue per spanning tree



Interplay of momentum conservation and causal prescription is key to obtain the energy flow

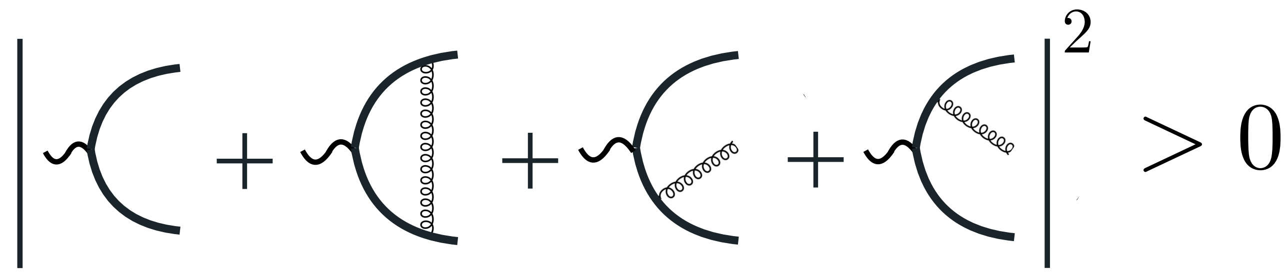
- **Distributional identities:** [Bierenbaum, Catani, Draggiotis, Rodrigo,
- **Averaging procedure:** [Runkel, Scór, Vesga, Weinzierl,
- **Iterative procedure:** [Capatti, VH, Kermanschah, Ruijl,
- **Manifestly causal:** [Capatti, VH, Kermanschah, Pelloni, Ruijl,
- **Cross-Free Family** [Capatti, arxiv: 2211.09653]
  (the best 3D repr.!)

Codes : [https://github.com/apelloni/cLTD]

[https://bitbucket.org/wjtorresb/lotty]

{ Pick your candy: cannot all be manifest }

Positivity



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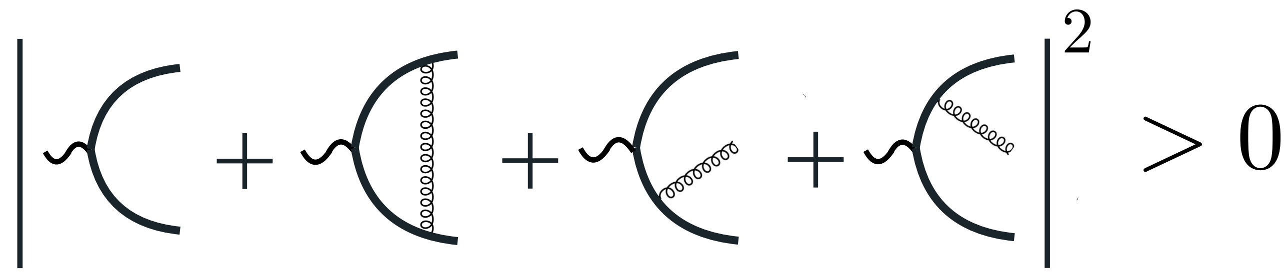

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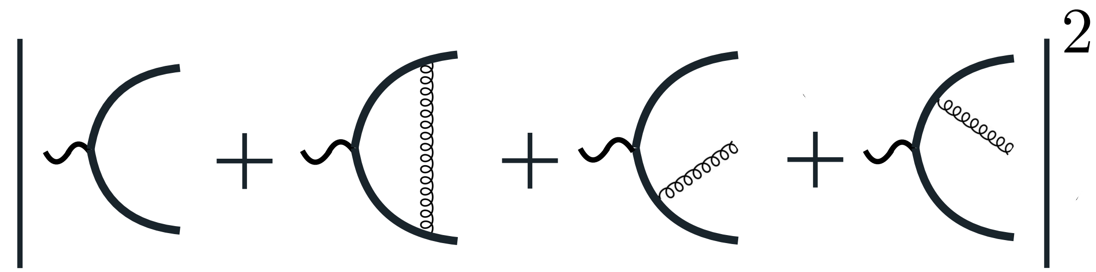

> 0

Lorentz Invariance

$$\mathcal{M}^{\mu}\left(\{p_i^{\mu}\}\right) = \Lambda^{\mu}_{\nu}\mathcal{M}^{\nu}\left(\{\Lambda^{\mu}_{\nu}p_i^{\nu}\}\right)$$

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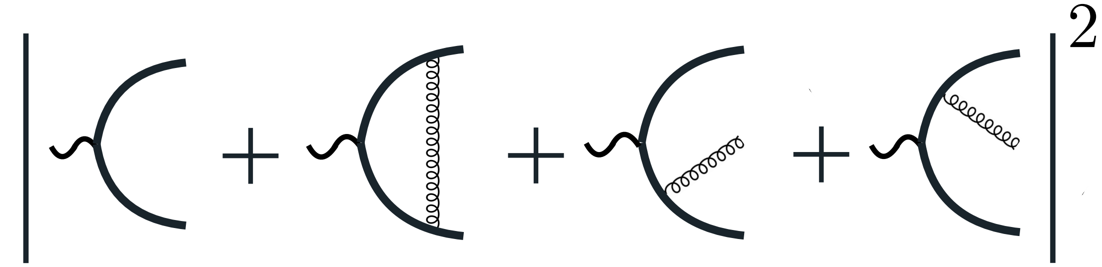

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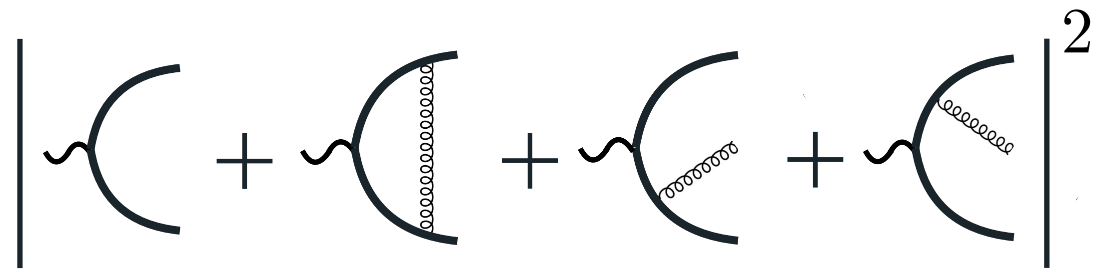

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$$k^{\mu}\mathcal{M}_{\mu} = 0$$

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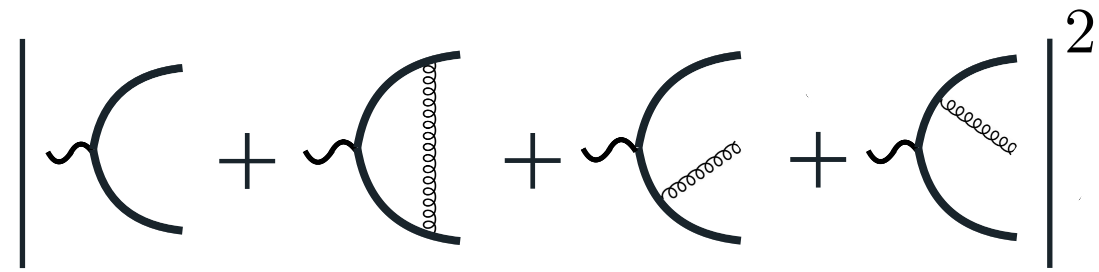

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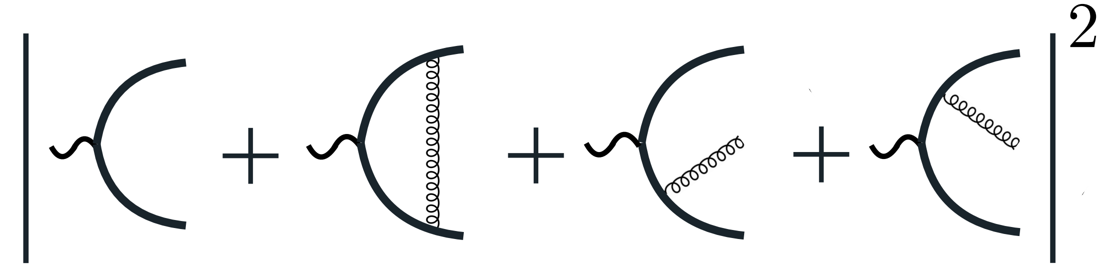

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$$\mathrm{i}(T^{\dagger} - T) = T^{\dagger}T$$

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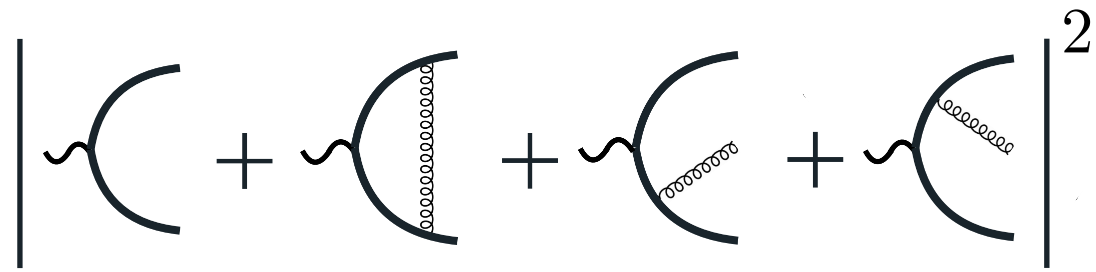

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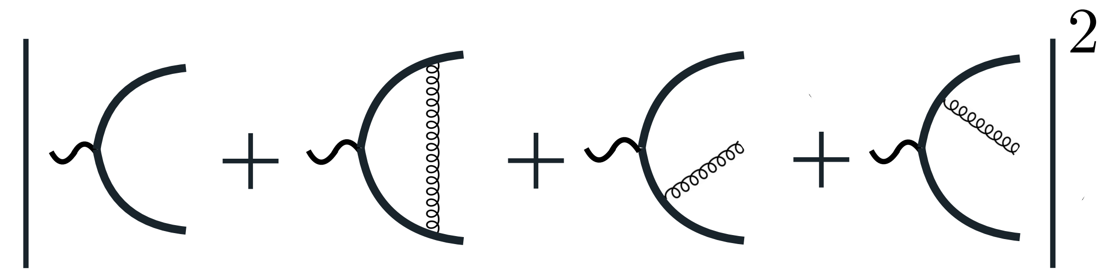

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Locally finite

$$\lim_{k \to \text{soft,colli,UV}} I(k) = \mathcal{O}(1)$$

Four types of singularities :

Thresholds | Contour-deformation
or
thres. subtraction

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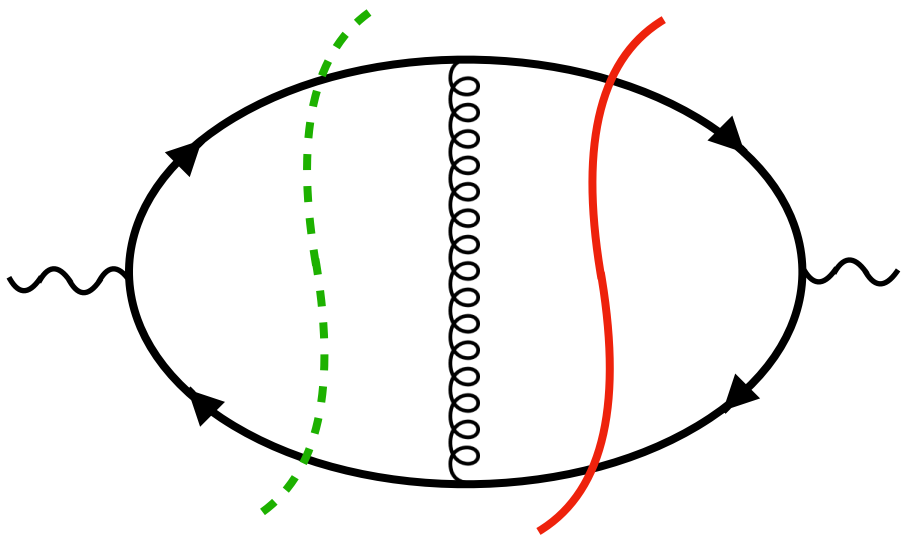
Ultraviolet	Local BPHZ

Thresholds

$$E_1 = \sqrt{|\vec{k}|^2 + m^2 - i\epsilon}$$

$$q = (Q^0, \vec{q})$$



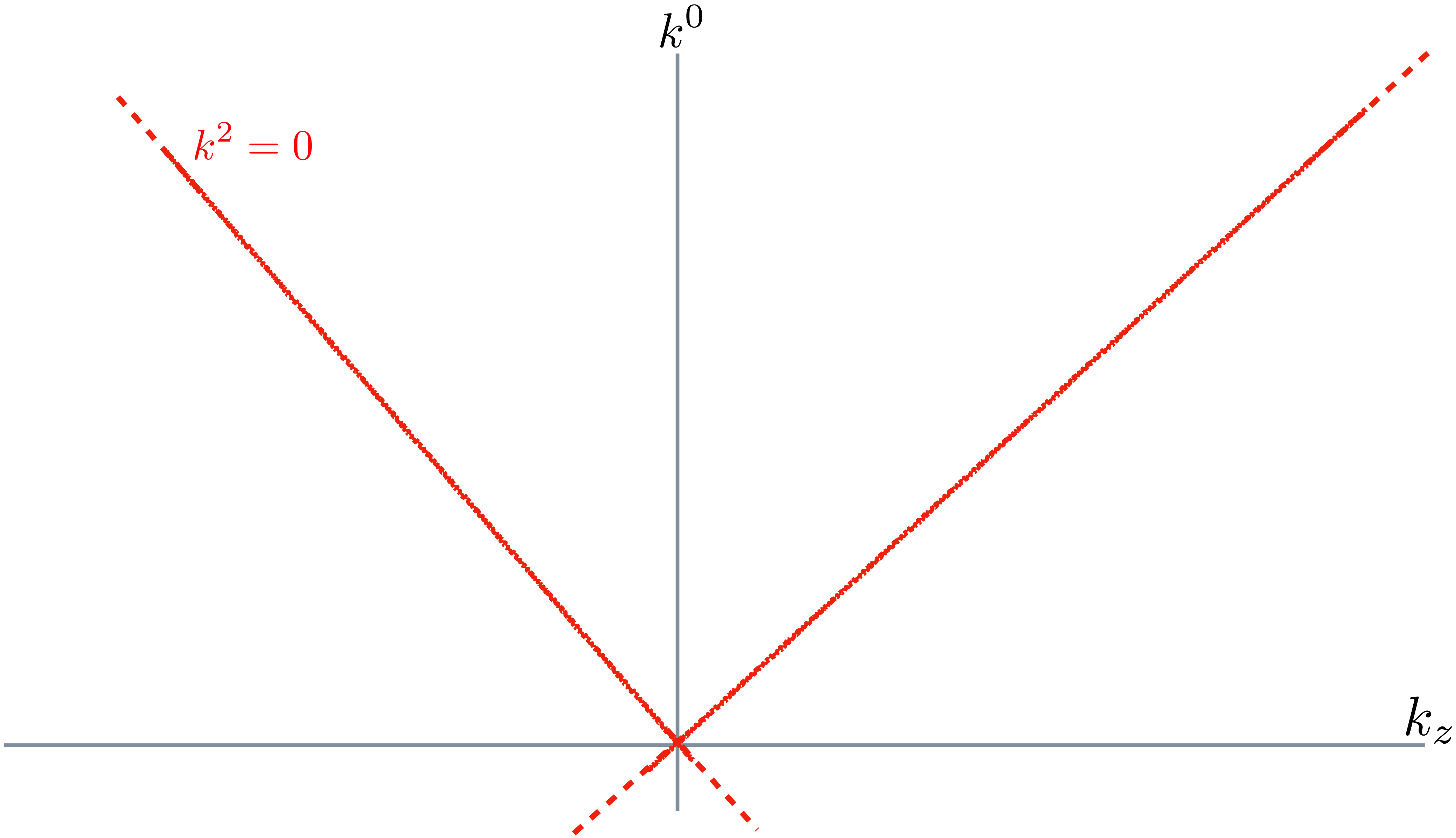
$$E_2 = \sqrt{|\vec{k} - \vec{q}|^2 + m^2 - i\epsilon}$$

$$\int d^3\vec{k}\; I^{(\text{Local Unitarity})} \supset \int d^3\vec{k}\; \frac{1}{E_1 E_2 E_3} \left(\frac{1}{(E_1 + E_2 - Q^0)(E_1 + E_2 + Q^0)} \right)$$

$$\eta(\vec{k}) = E_1 + E_2 - Q^0 \overbrace{=}^{\vec{Q}=0\;\; m=0} 2|\vec{k}| - Q^0$$

Singular surfaces in Minkowski space

$$\int d^4k \frac{1}{k^2 (k-p_1)^2 (k-p_1-p_2)^2}$$



The integrand is singular along each of the coloured surface

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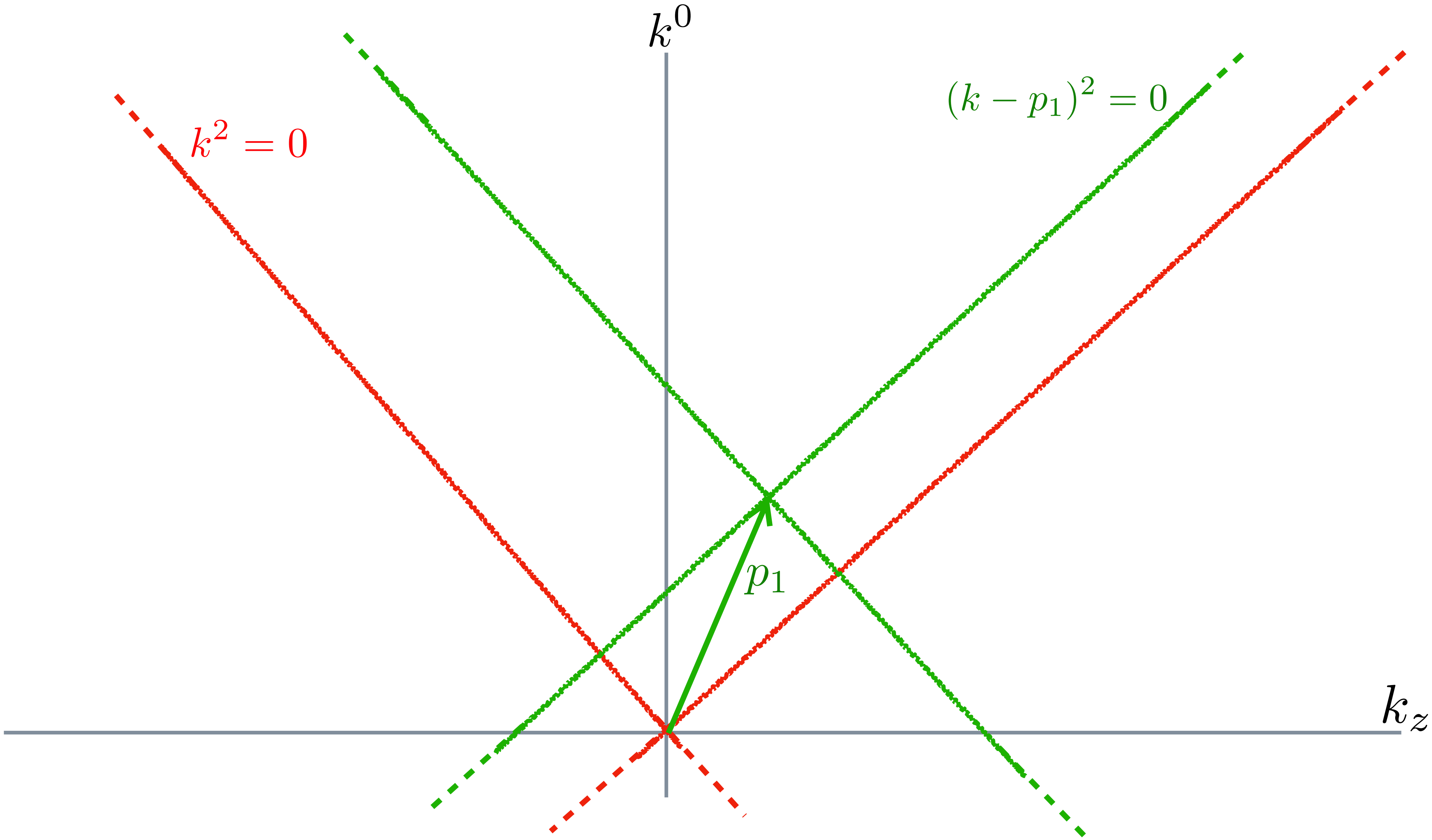

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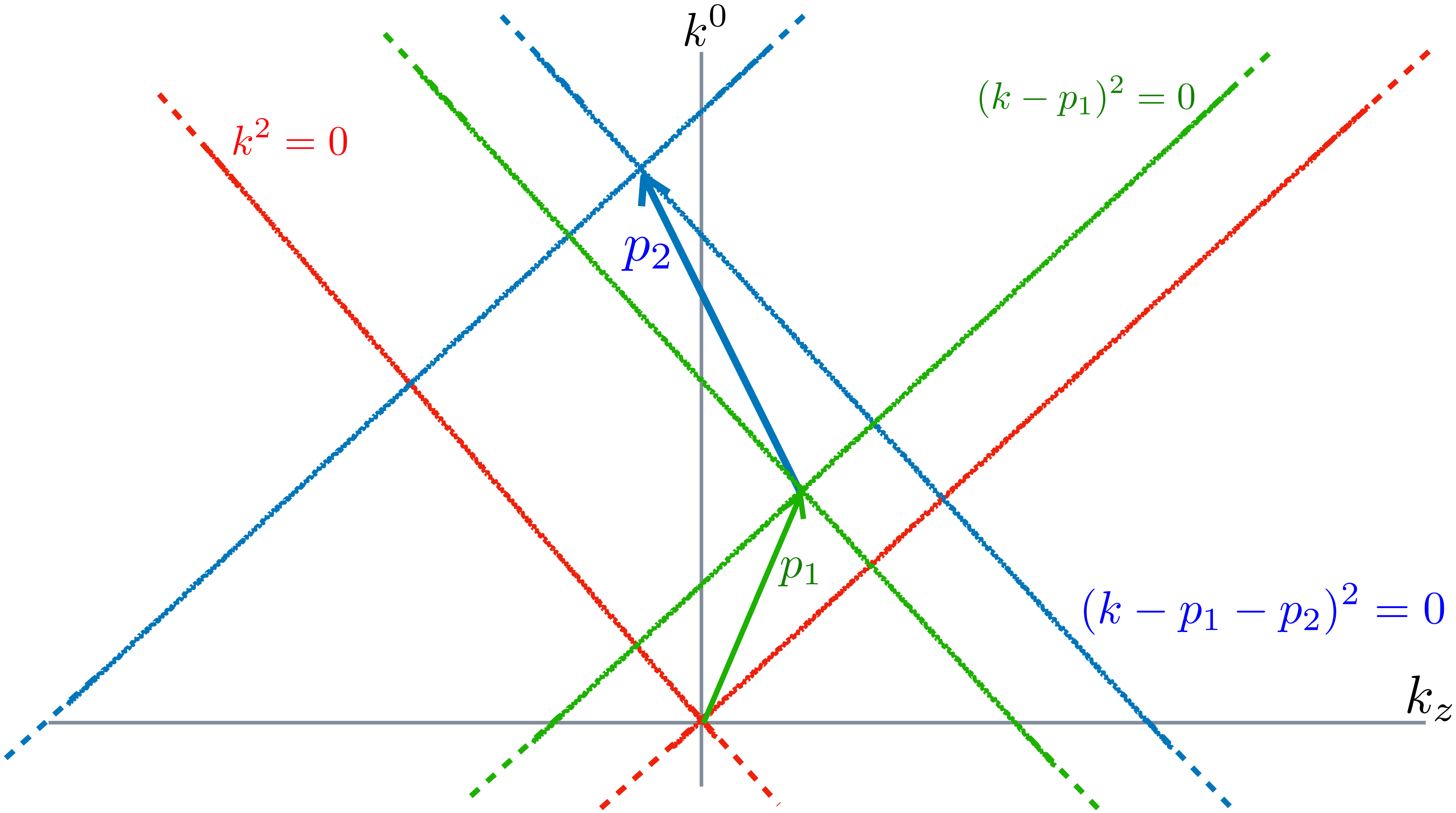

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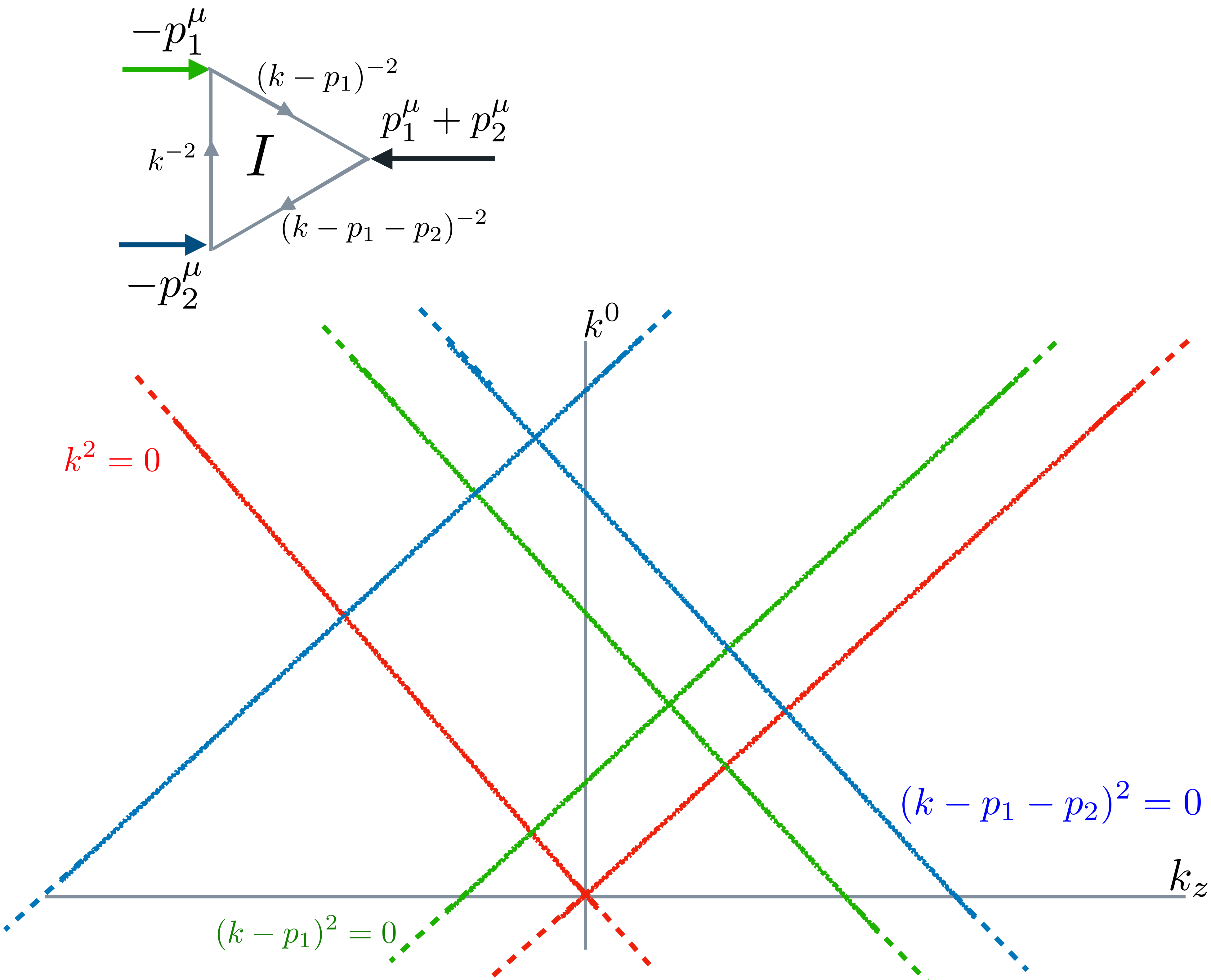
Singular surfaces of the LTD representation

Analytically integrate over the loop energies using Cauchy's theorem (LTD):



$p_i{}^2 > 0 \quad \forall i$

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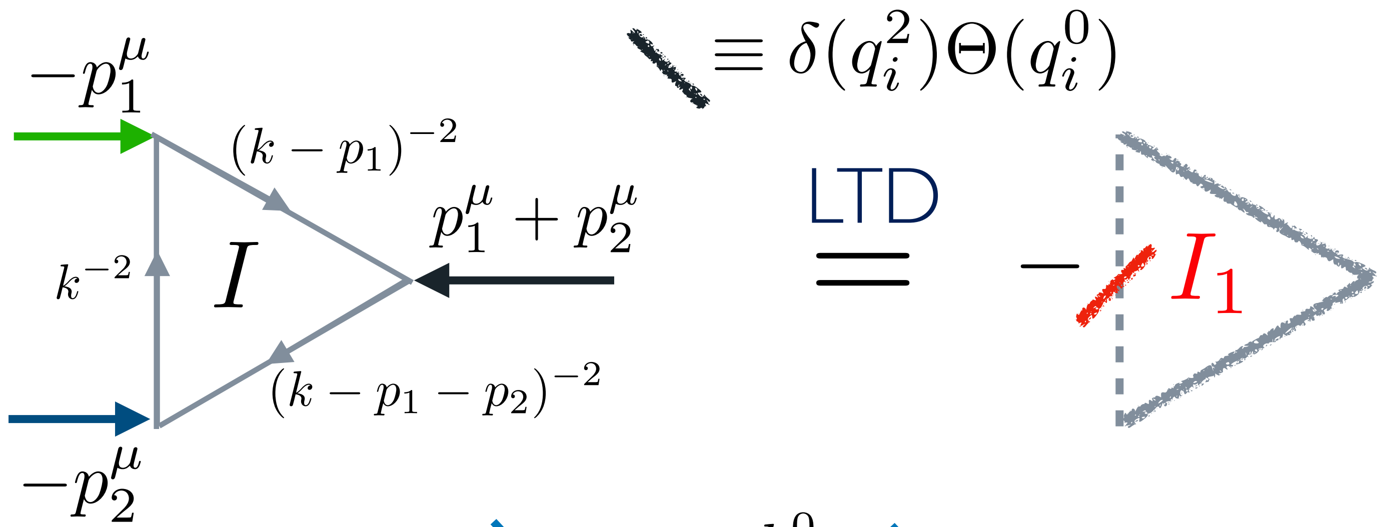

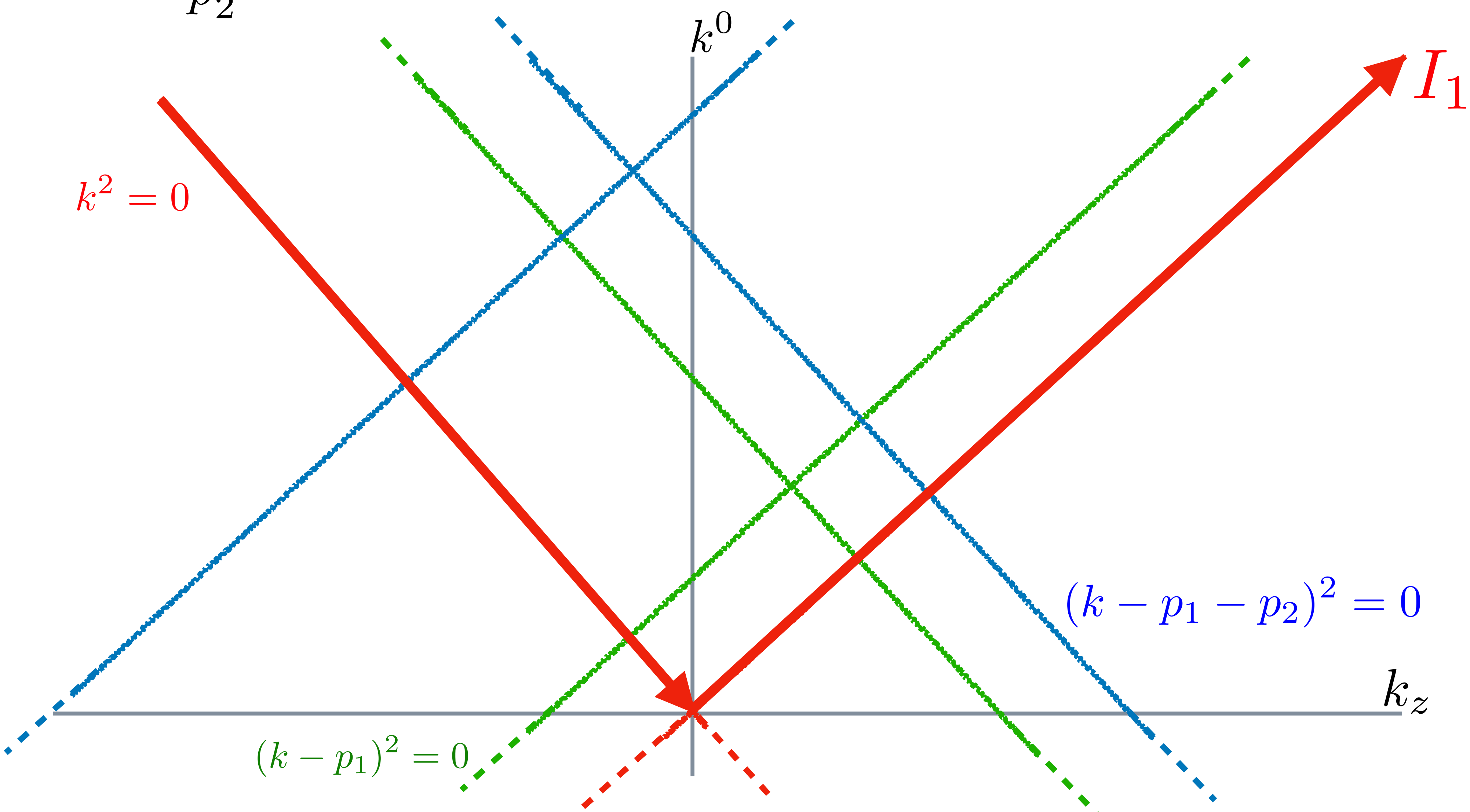

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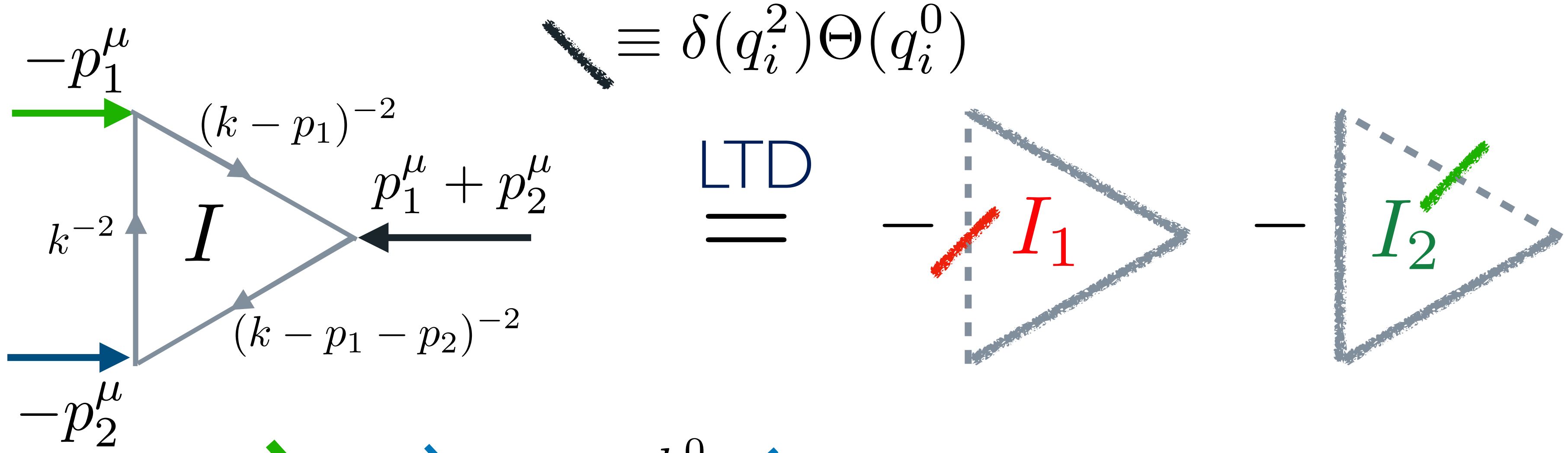

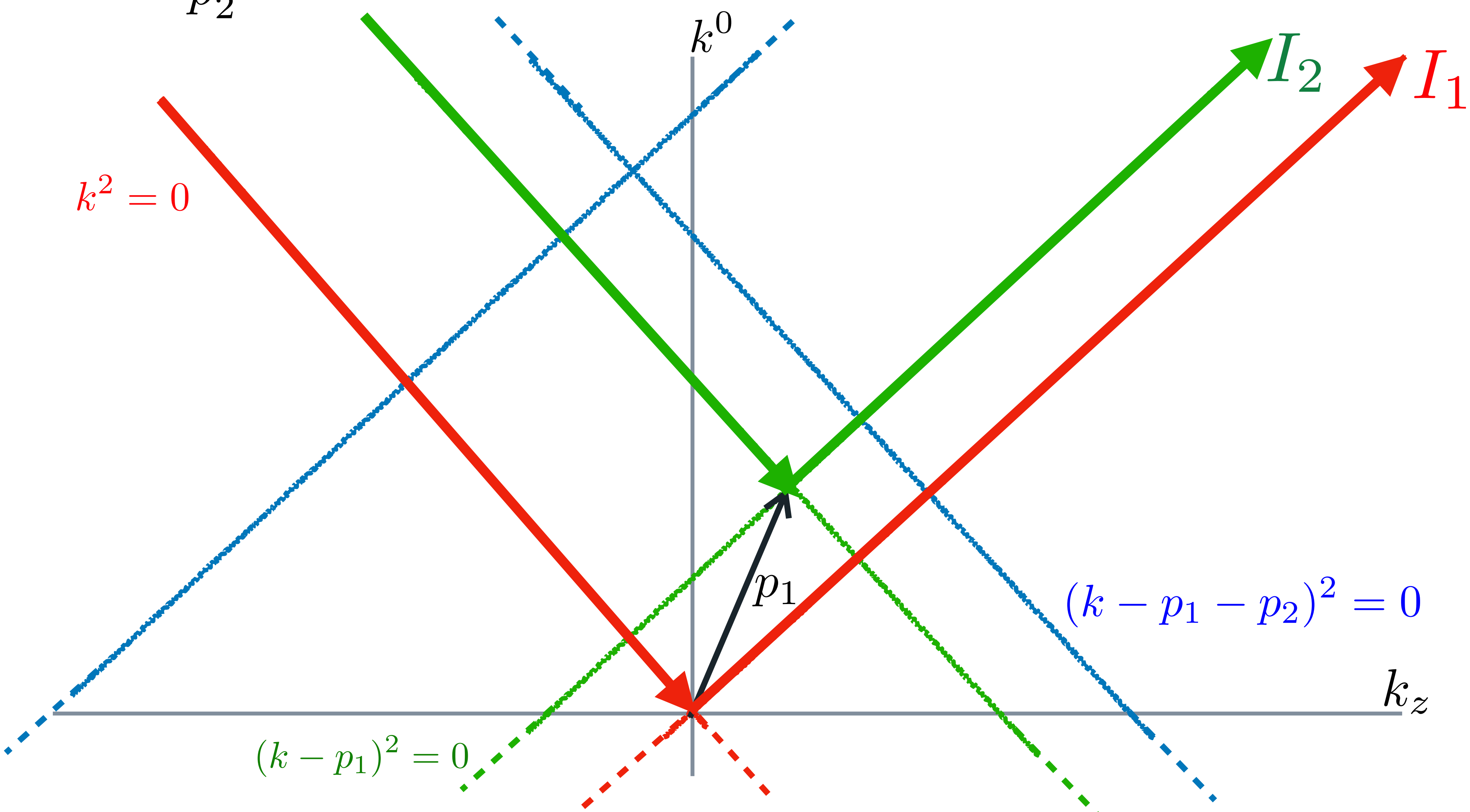

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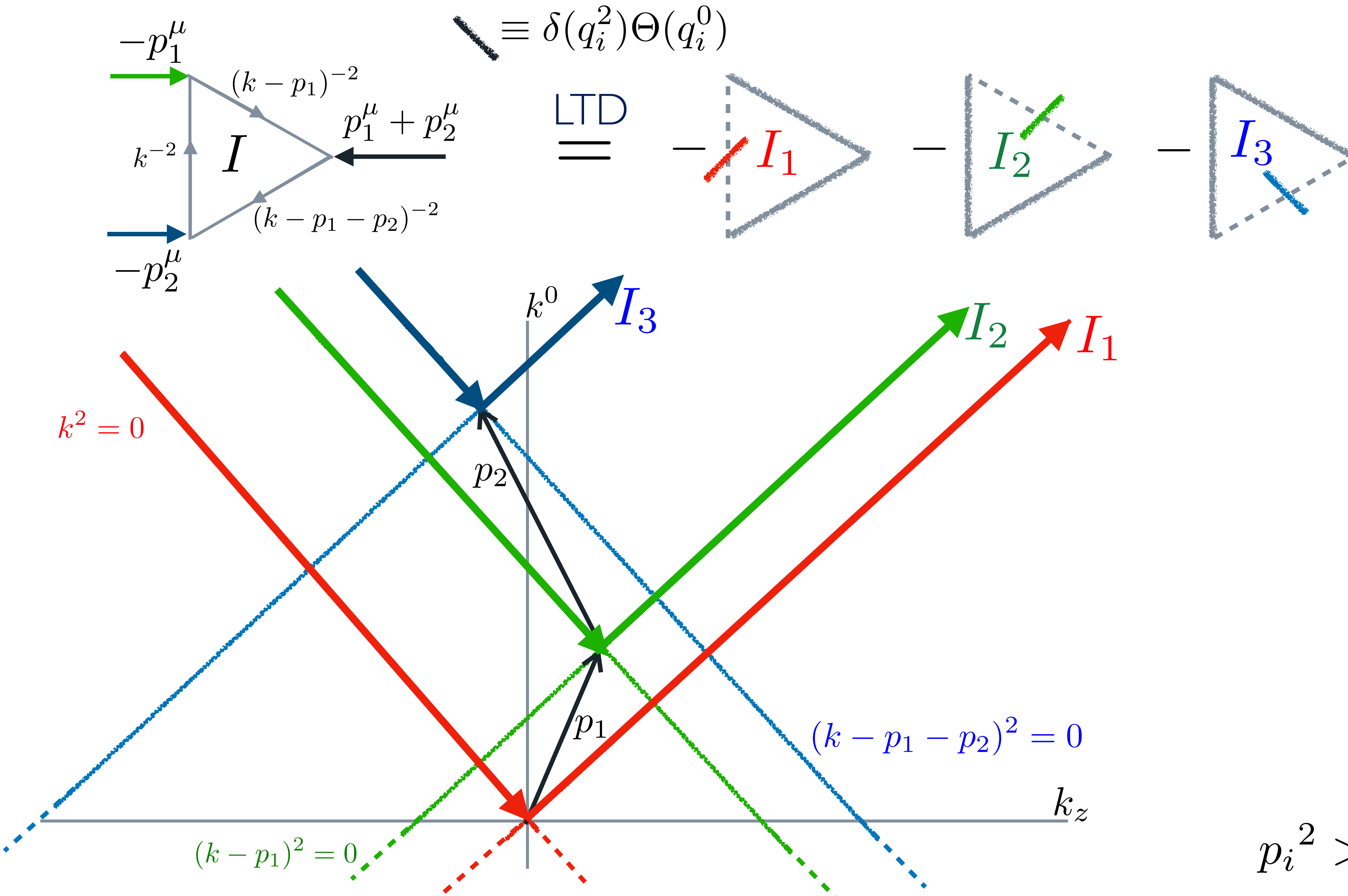

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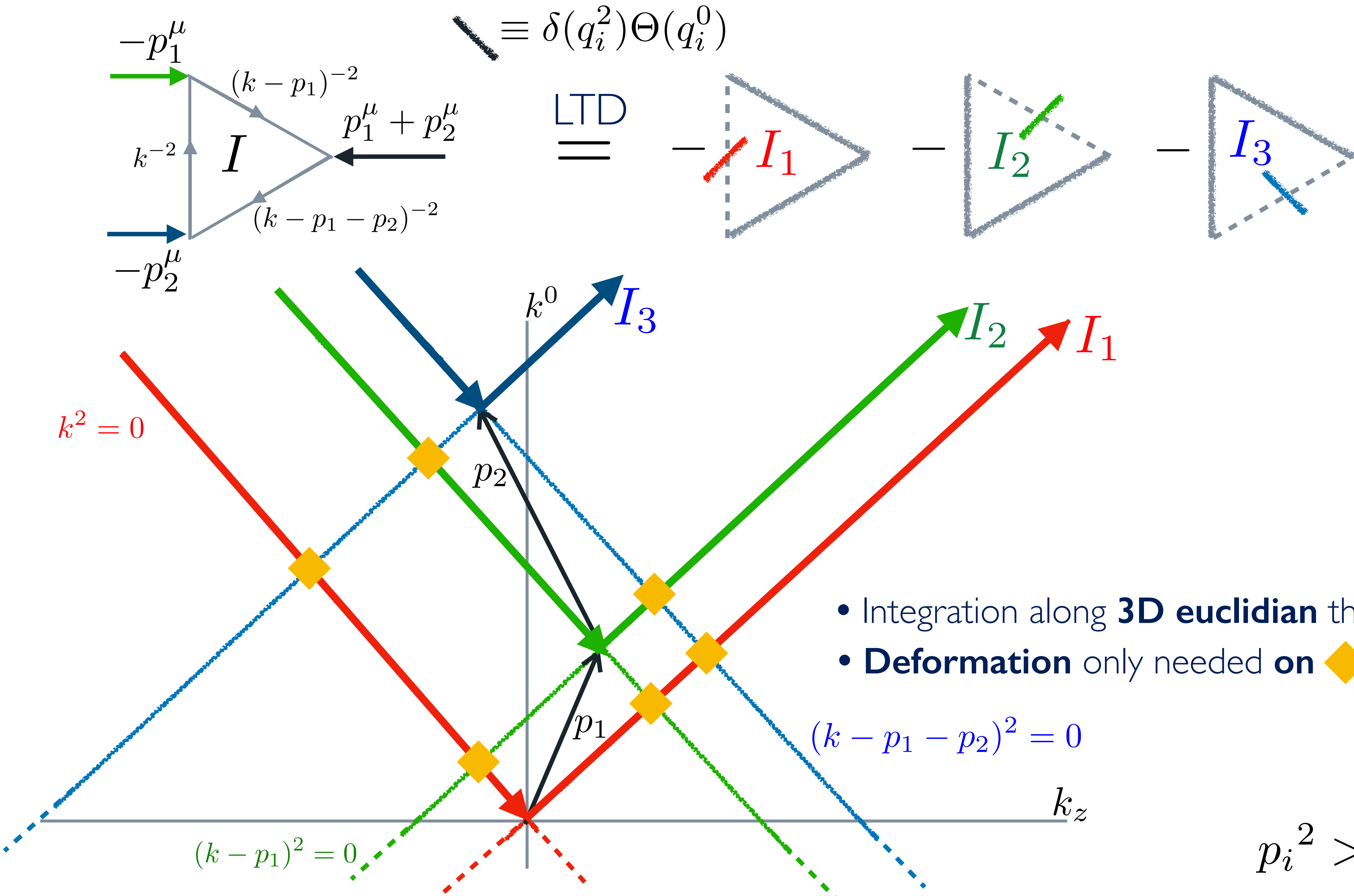

- Integration along **3D euclidian** thick lines
- **Deformation** only needed **on** ◆ !

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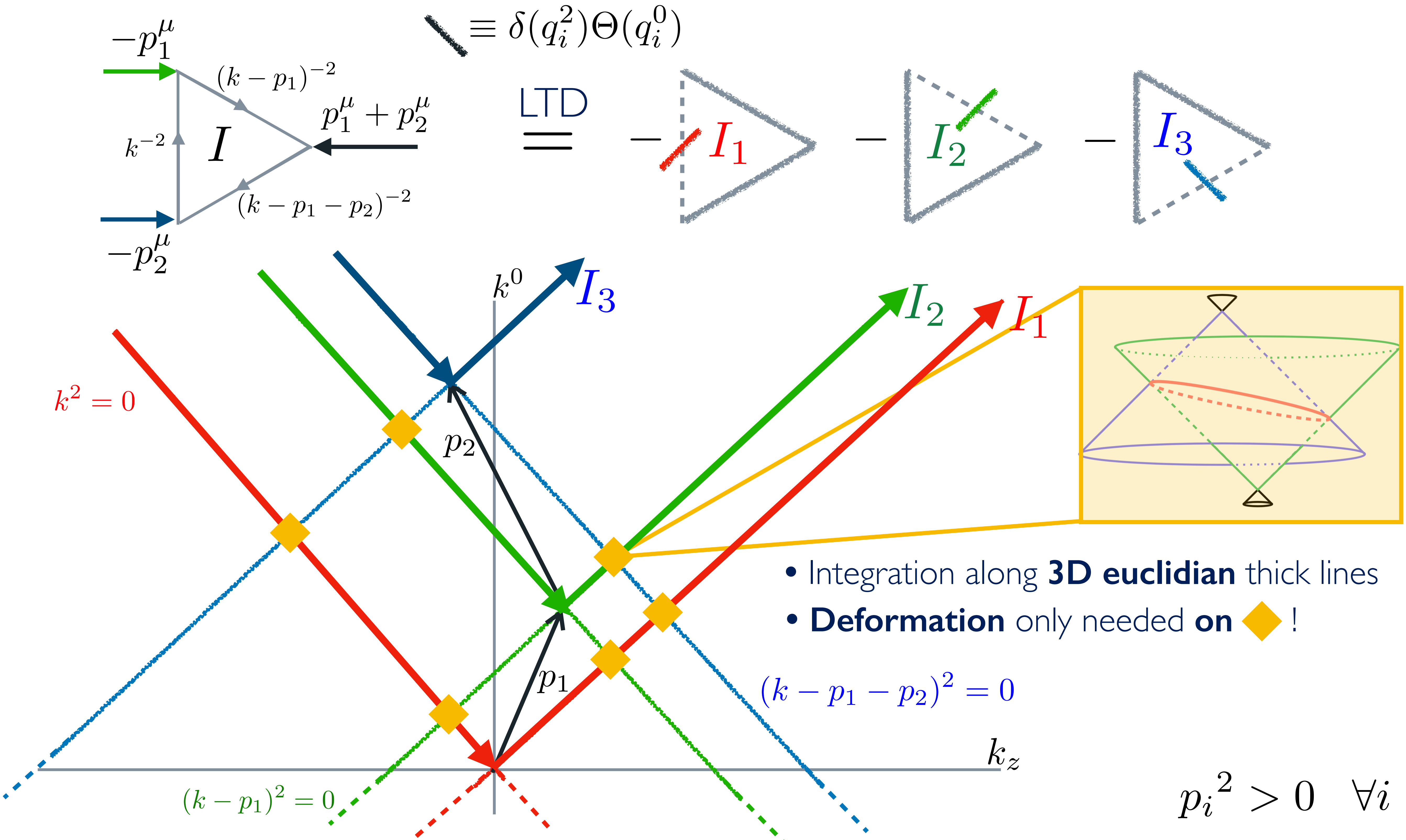

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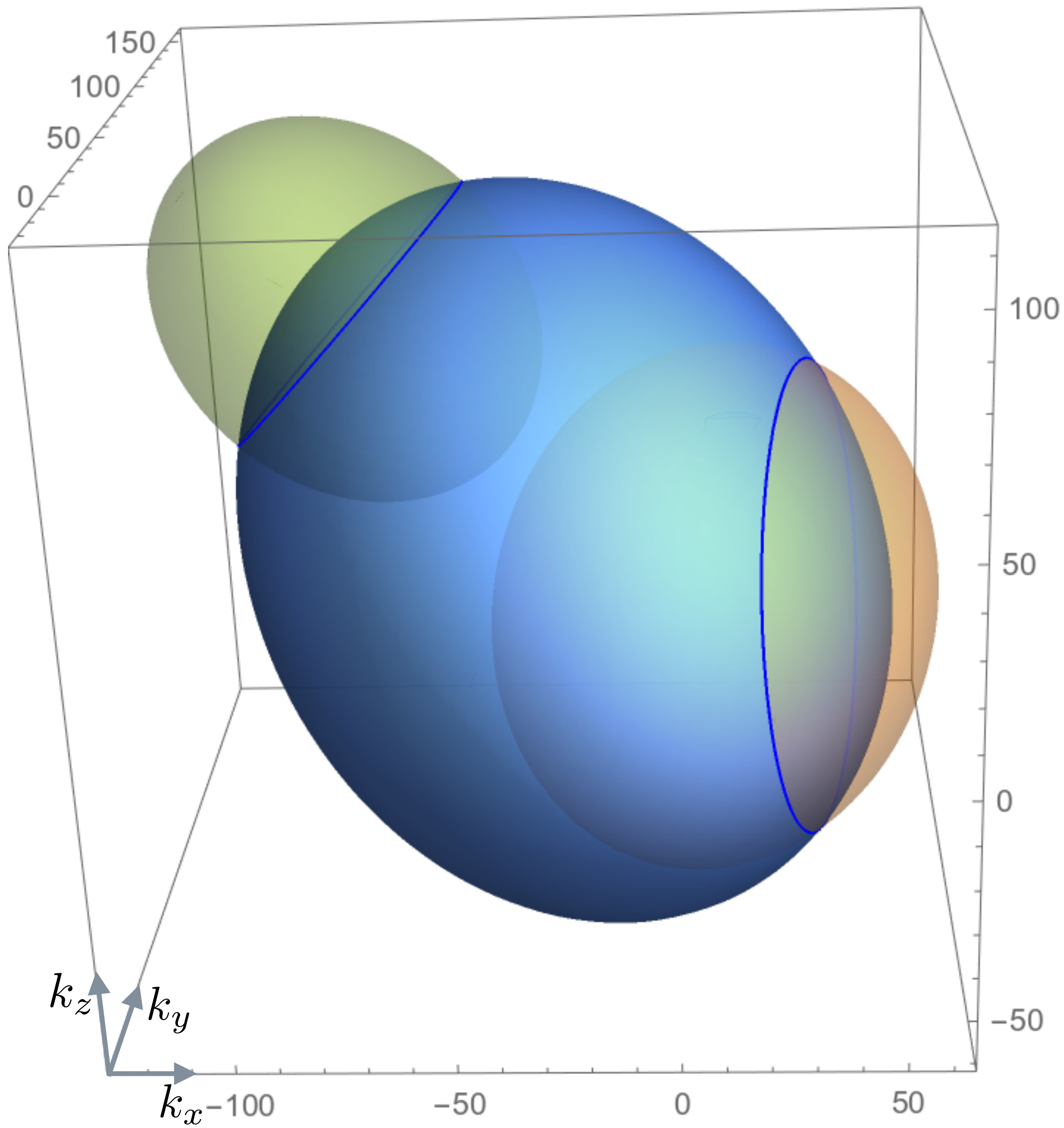
Singular surfaces - 2D ellipsoids



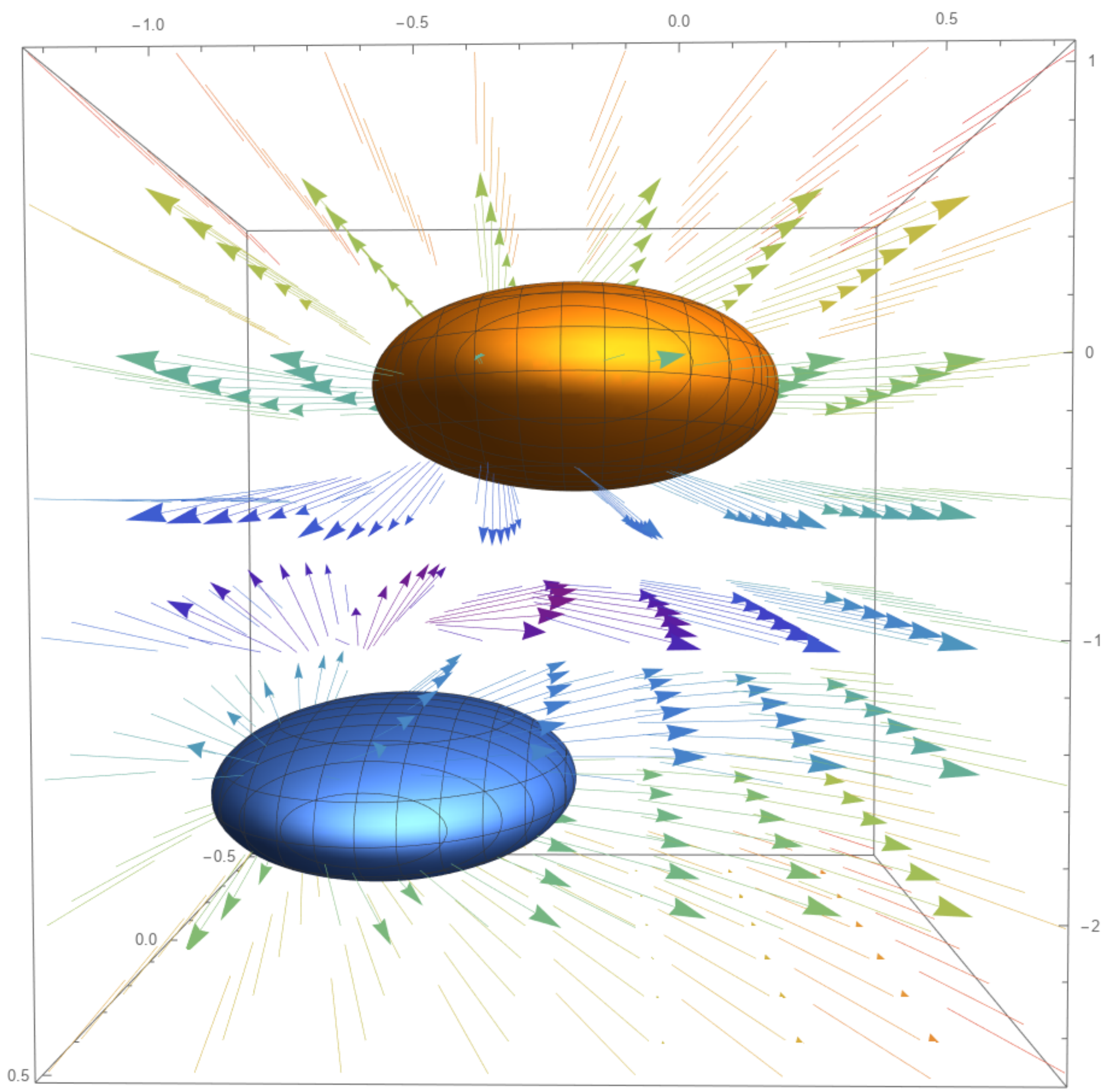
- General **one-loop** ellipsoid **expression**:

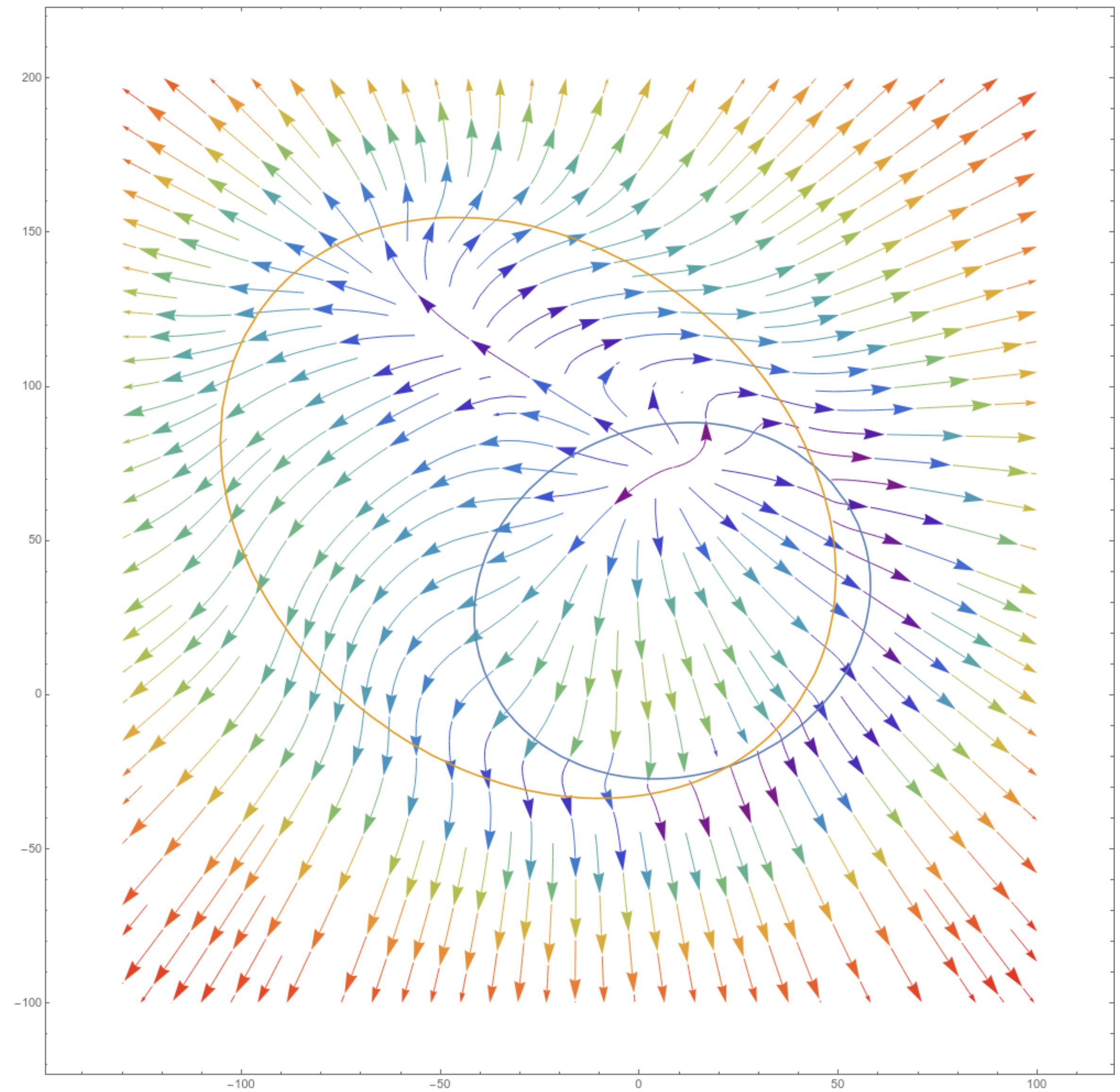
$$E_{ij}(\vec{k}) = \sqrt{\left(\vec{k}+\vec{p_i}\right)^2 + m_i^2 - i\delta} + \sqrt{\left(\vec{k}+\vec{p_j}\right)^2 + m_j^2 - i\delta} - p_i^0 + p_j^0$$

Deforming around singular 2D-ellipsoids

Deformation: $\vec{k} \to \vec{k} - i\vec{\kappa}$

Causal prescription imposes: $\vec{\kappa} \cdot \vec{n}_{E_{ij}} > 0$





Deforming around singular 2D-ellipsoids

How to construct such a field? For example for this case:

[Capatti, VH, Kermanschah, Pelloni, Ru
[arxiv:1906.06138]

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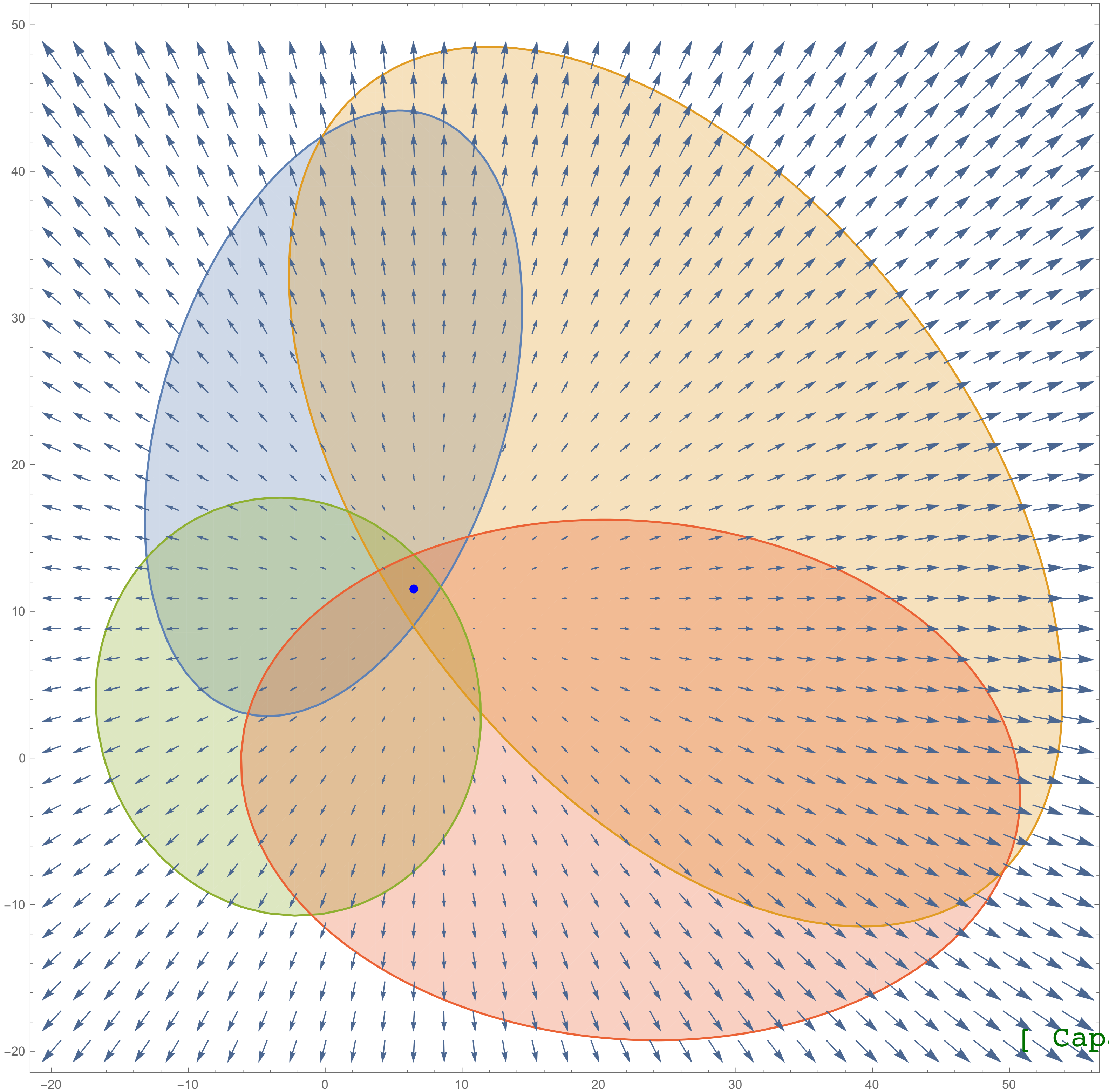
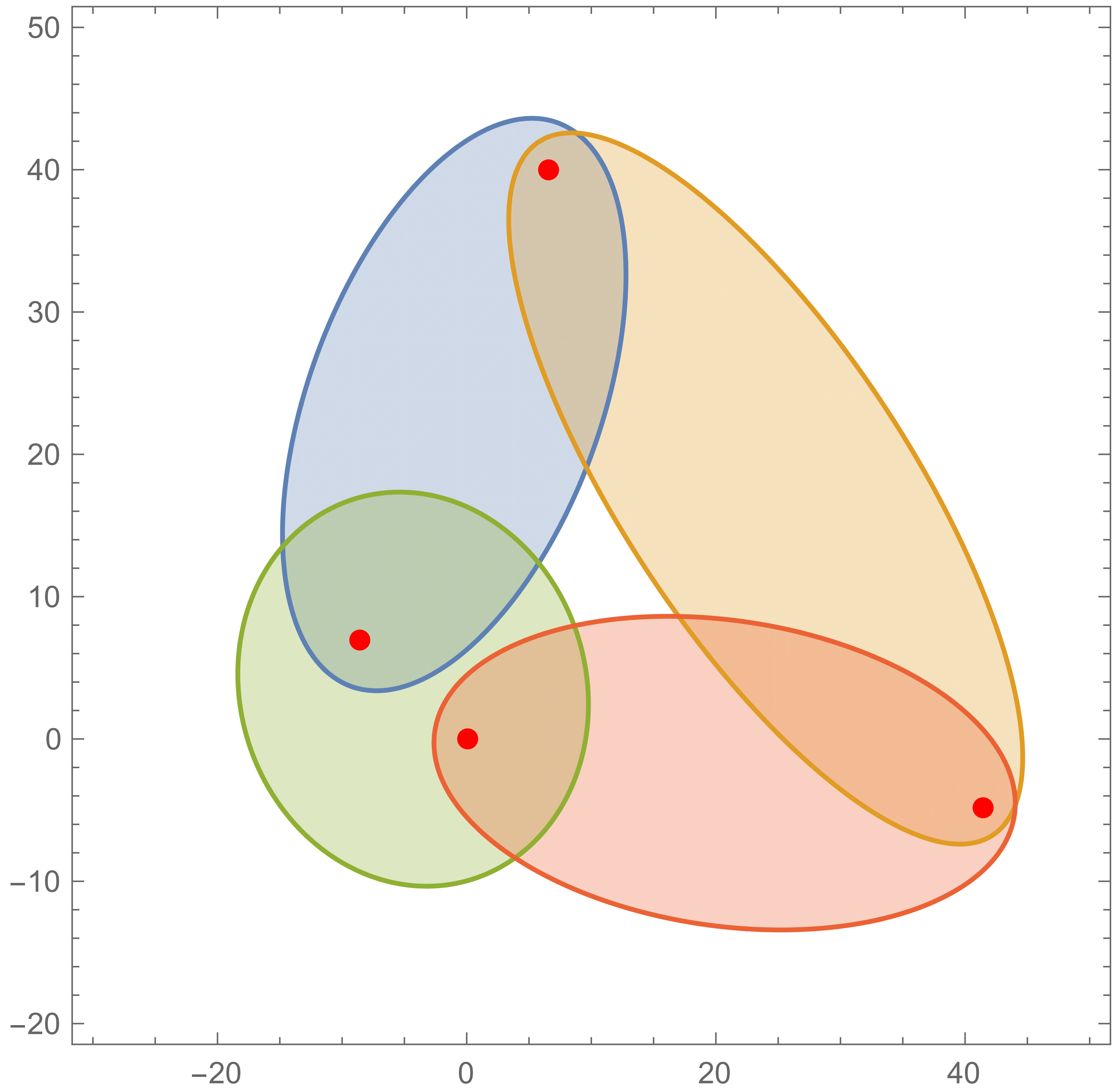


A **radial field** centered in the inside of all ellipsoids!

[Capatti, VH, Kermanschah, Pelloni, Ru
[arxiv:1906.06138]

Deforming around singular 2D-ellipsoids

But then what if there is no point in the inside of **all ellipsoids** (`Box4E` example) ?



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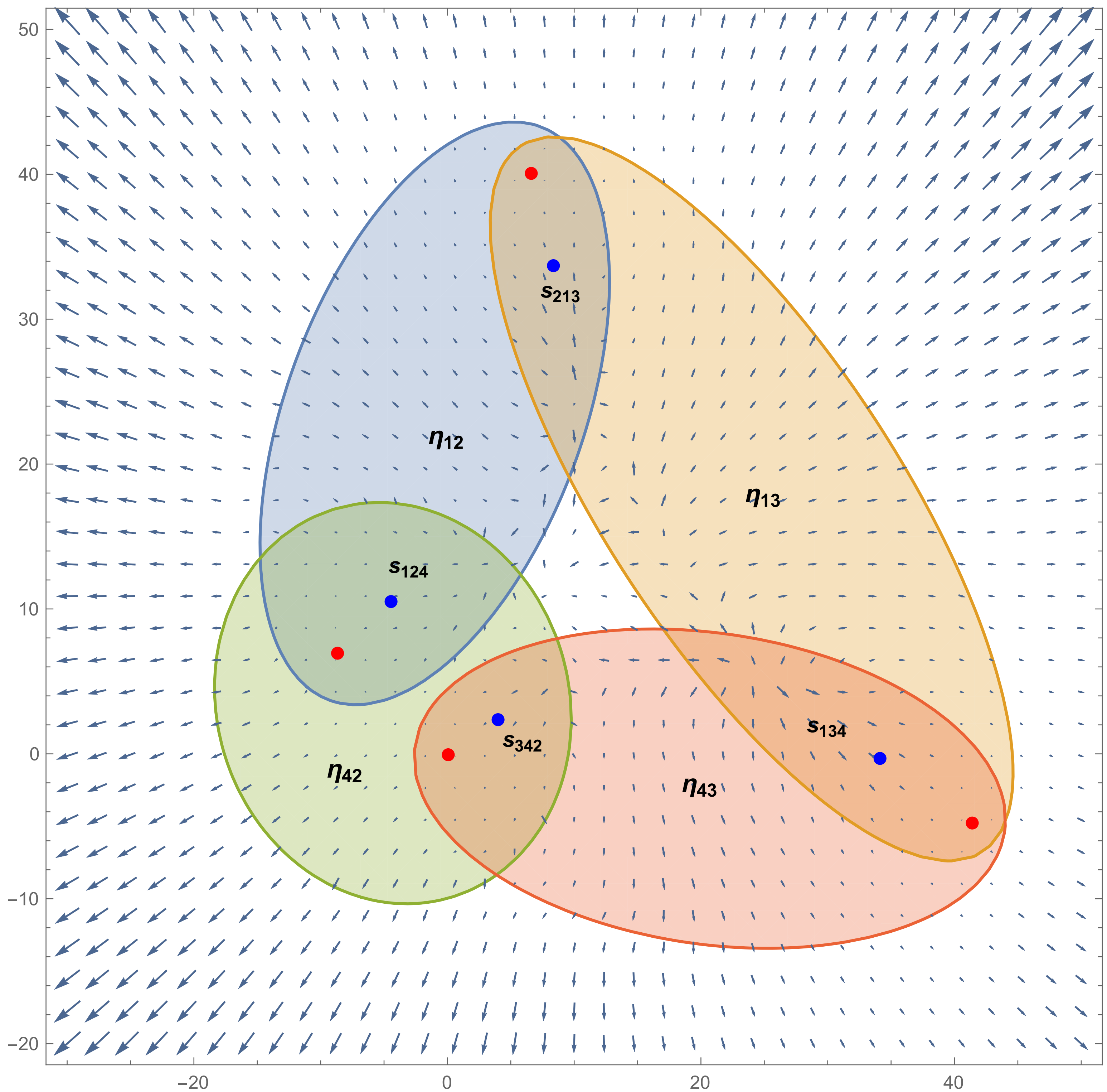

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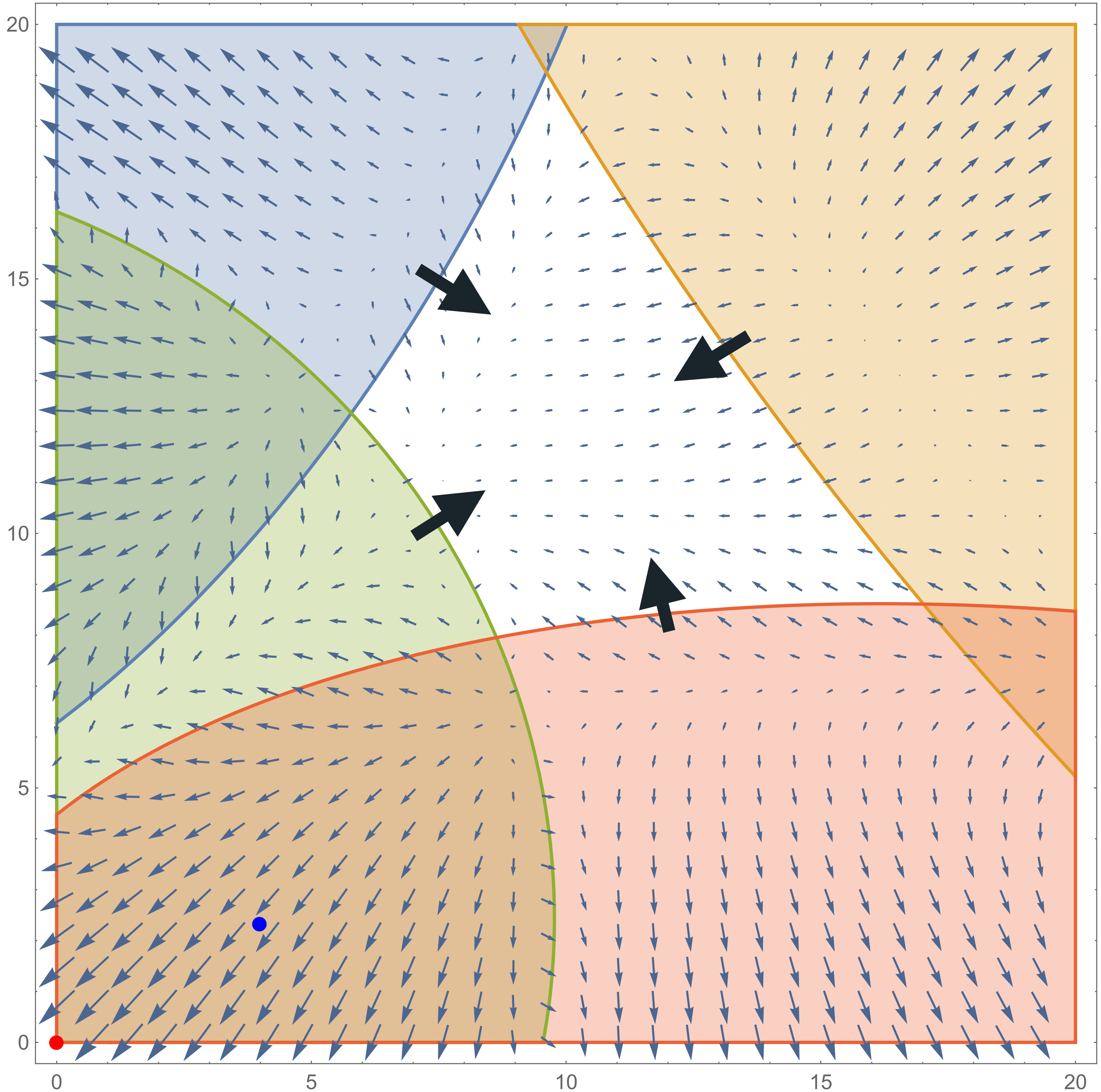

THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

[D. Kermanschah, arXiv : 2110.06869]

$$\frac{1}{E_1 + E_2 - p_1^0} = \frac{1}{|\vec{k}| + |\vec{k} - \vec{p}_1| - p_1^0}$$

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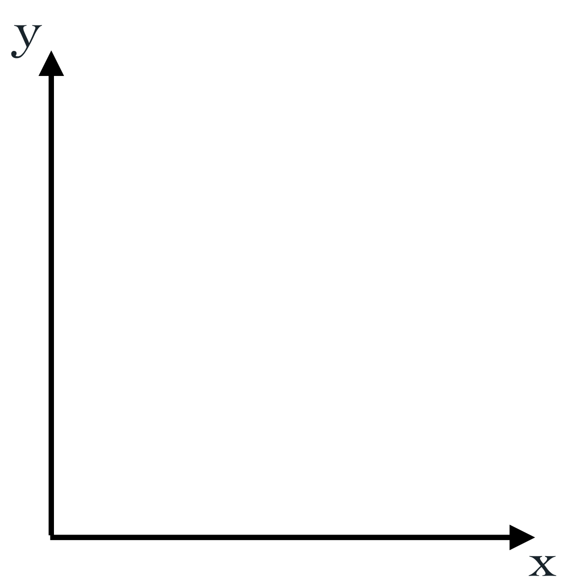

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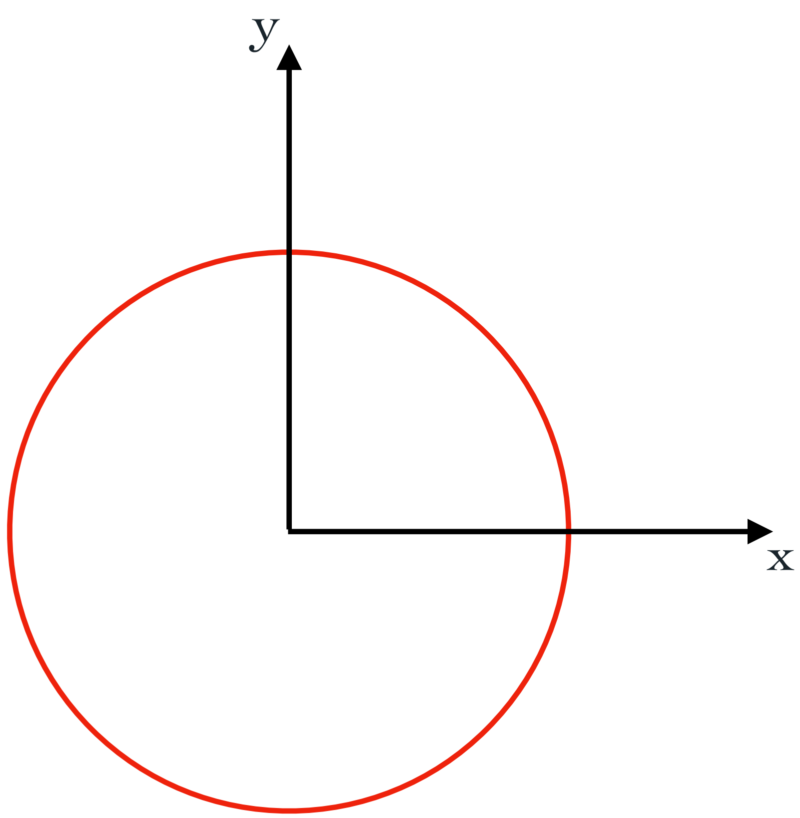

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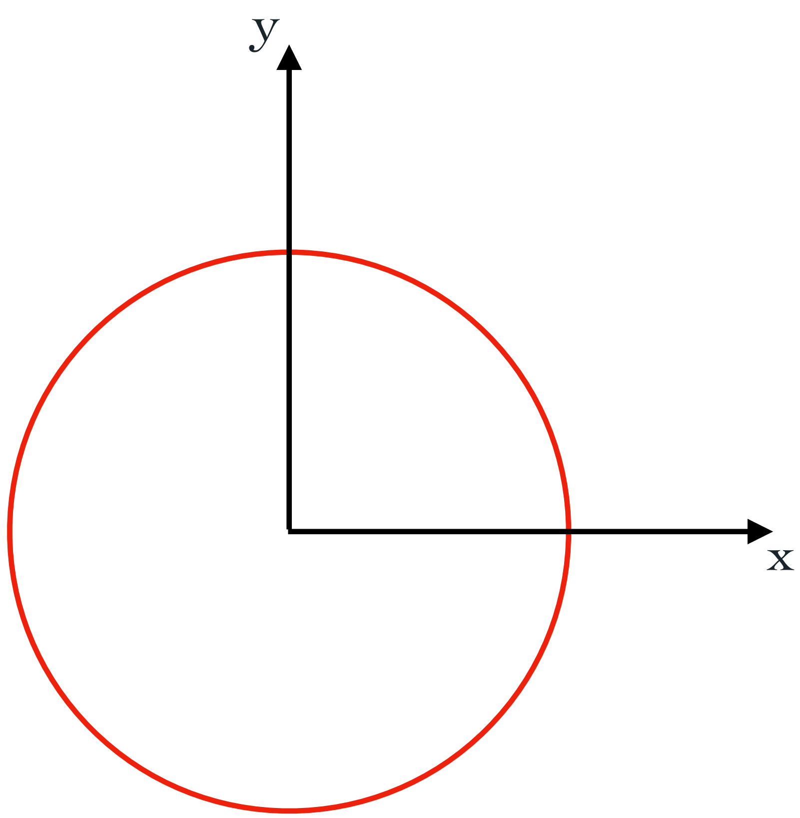

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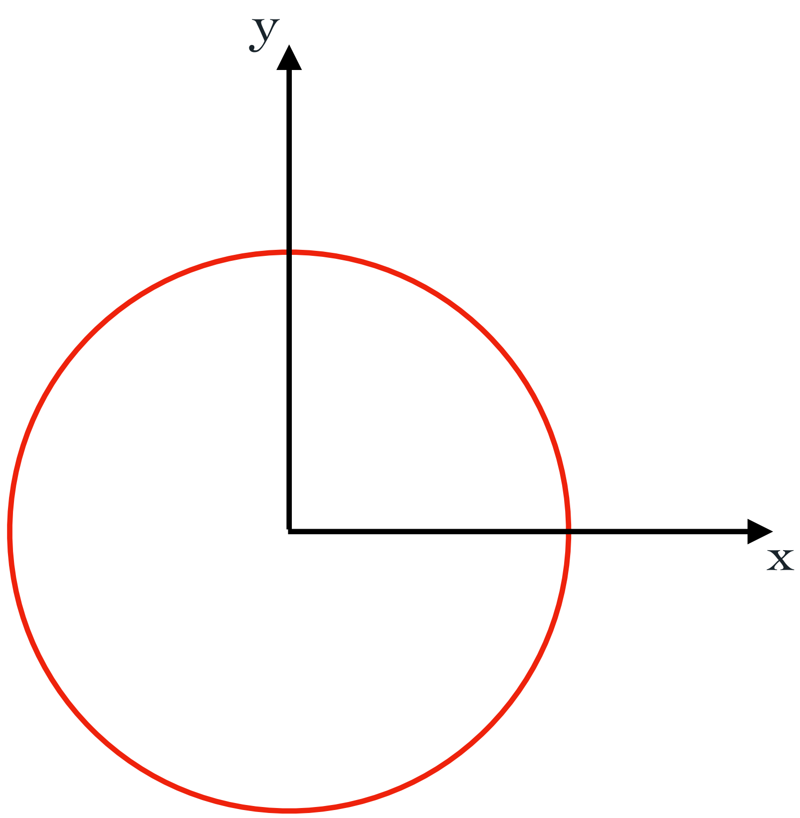

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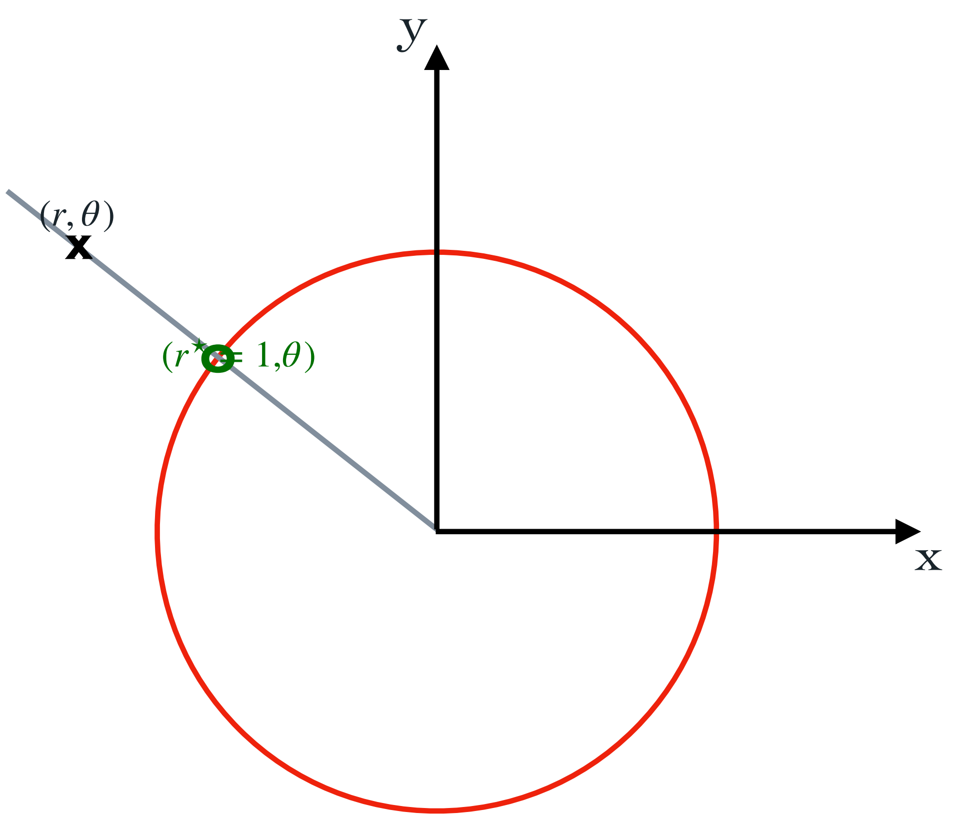

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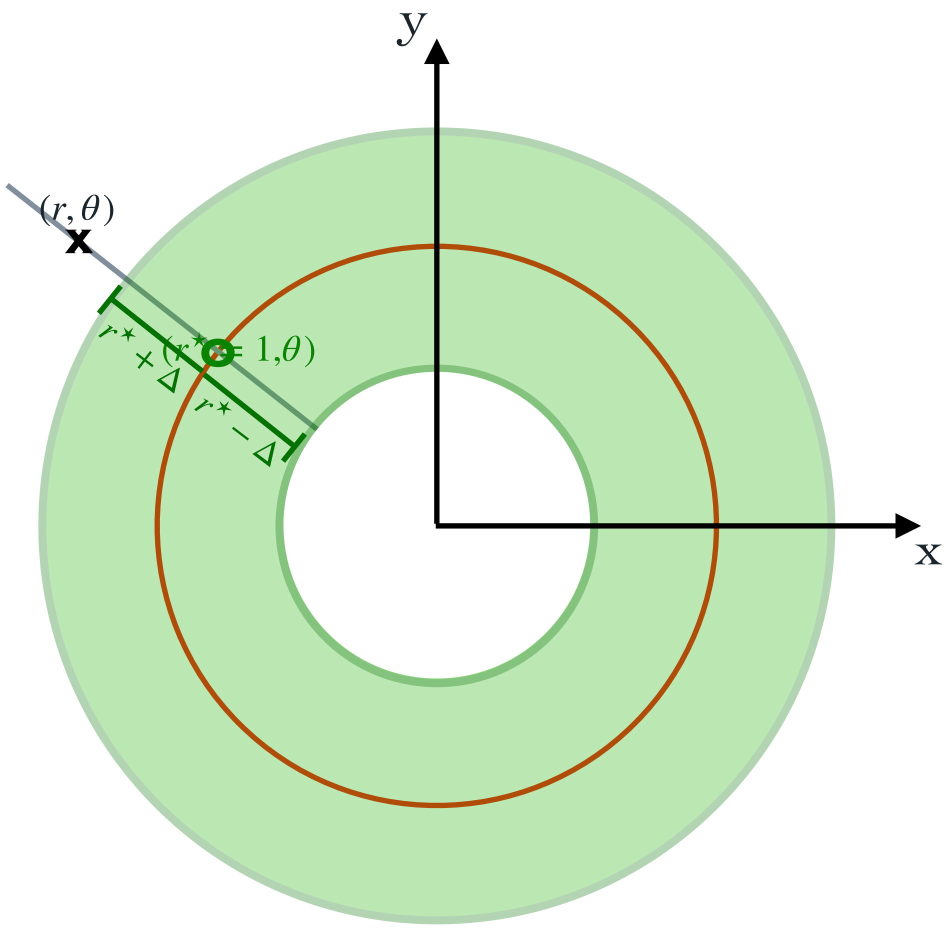

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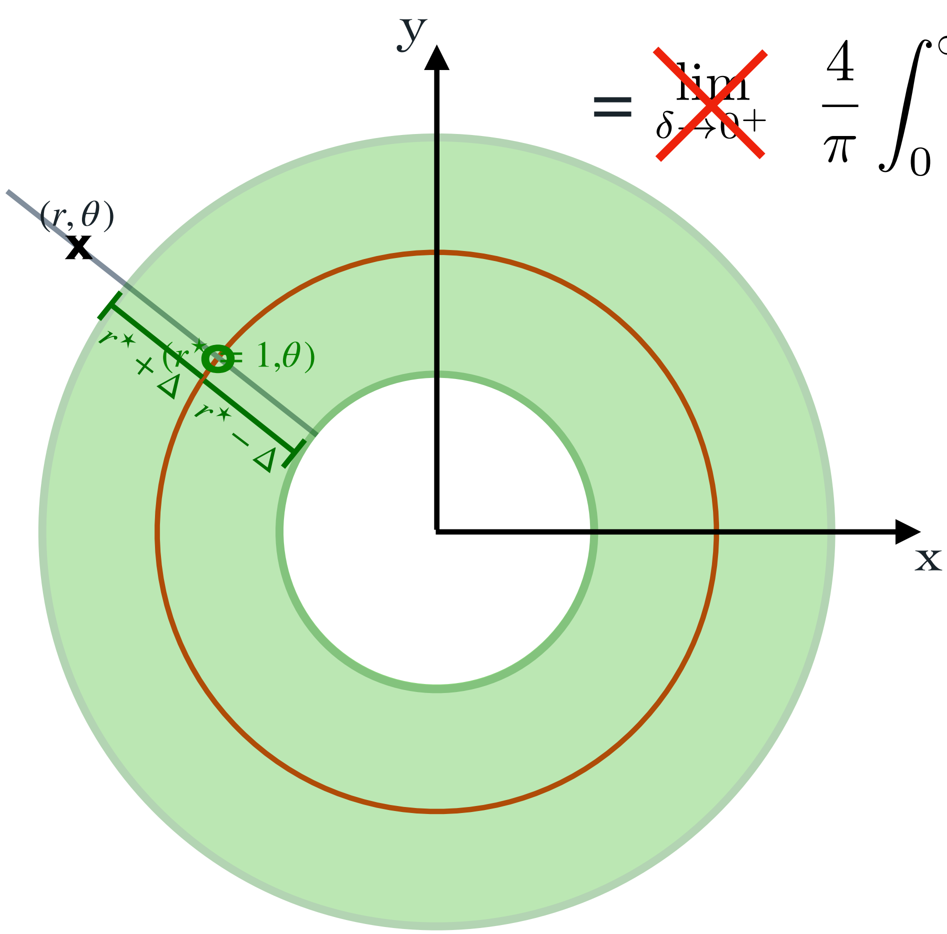
$\delta = 0$ can be taken safely !



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Integrated threshold counterterm

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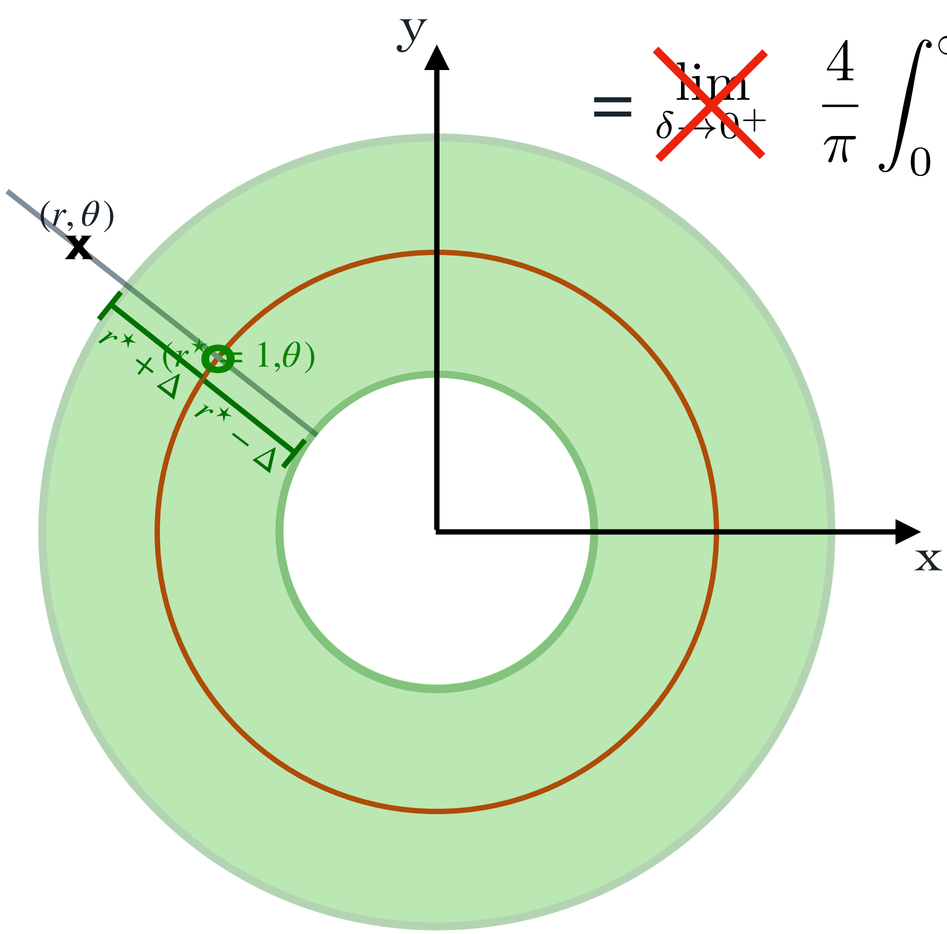

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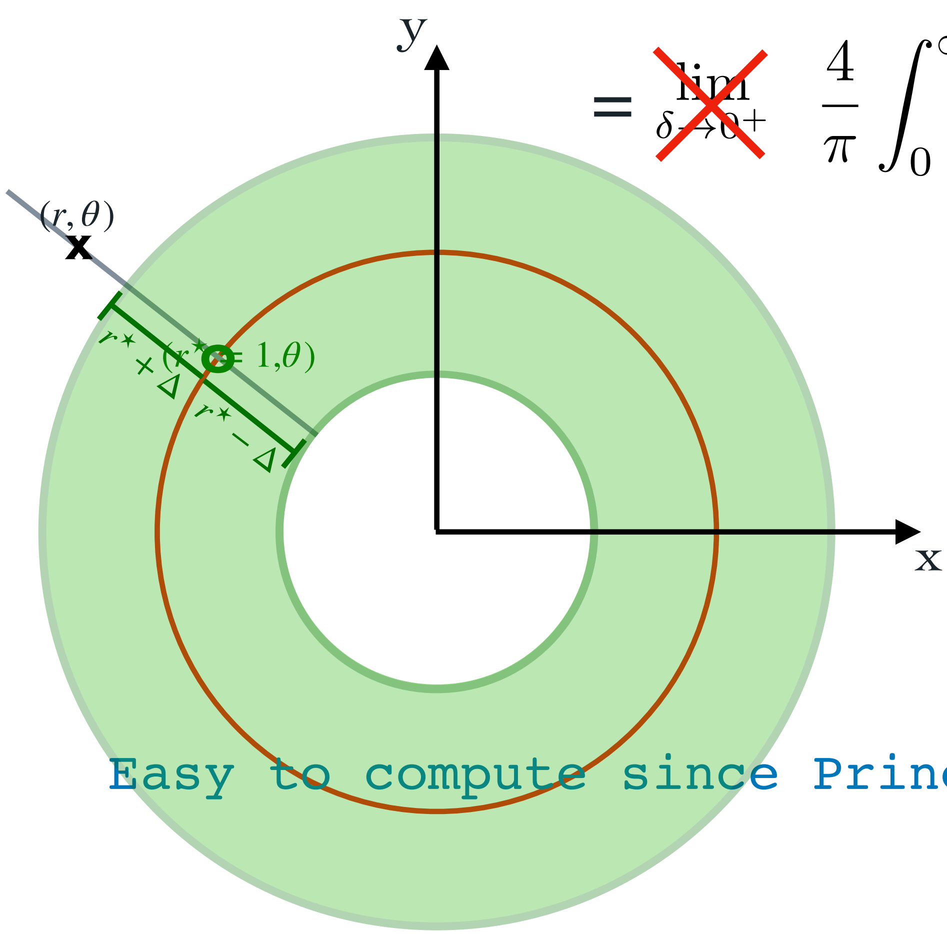
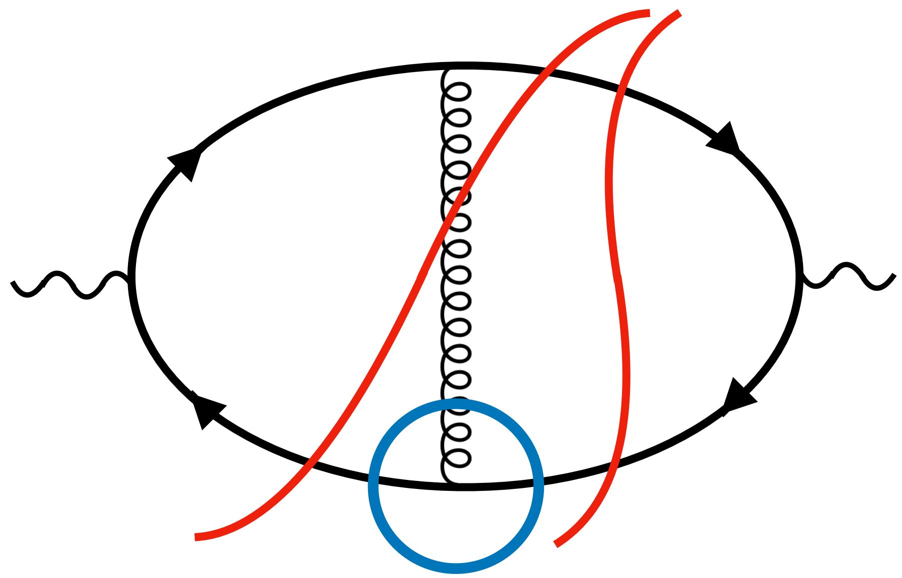


$$+ \lim_{\delta\to 0^+}\int_{1-\Delta}^{1+\Delta} dr \frac{4}{\pi}\frac{1}{2}\frac{1}{r-1\pm \mathrm{i}\delta} \overset{\Delta\leq 1}{=} \underbrace{\frac{2}{\pi}\mathrm{PV}\left[\frac{1}{r-1\pm \mathrm{i}\delta}\right]}_{0}\boxed{\mp 2\,\mathrm{i}}$$

Easy to compute since Principal Value is zero by construction !

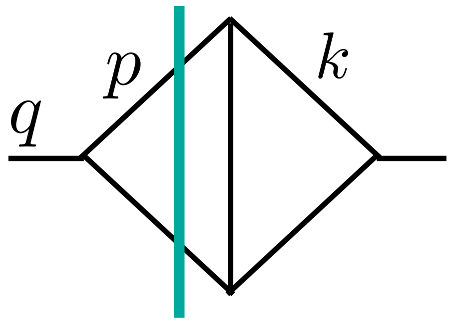
Local Unitarity



Locality Unitarity

We convert the four-dimensional Minkowski loop integration measure into a three-dimensional Euclidean phase-space measure:

$$\frac{\mathrm{d}^3\vec{p}}{2|\vec{p}|}\mathrm{d}^4k\,\delta(|\vec{p}| + |\vec{p} - \vec{q}| - Q_0)$$



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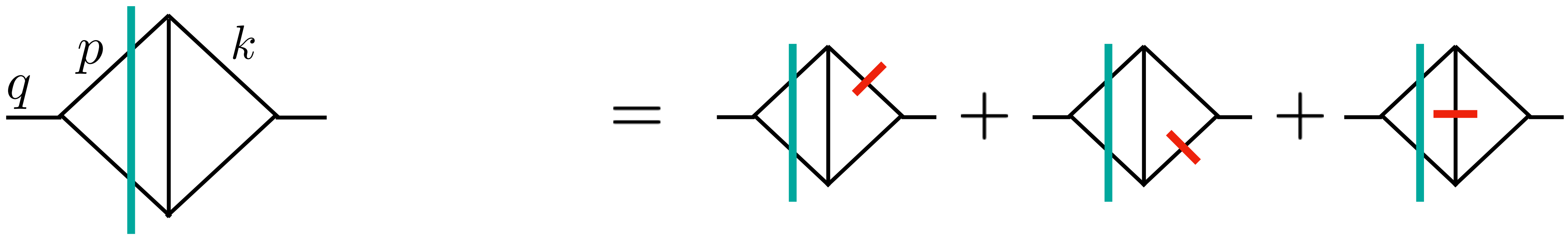

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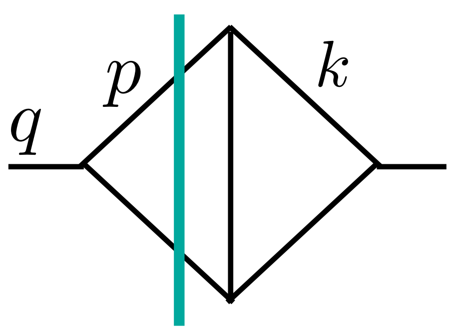

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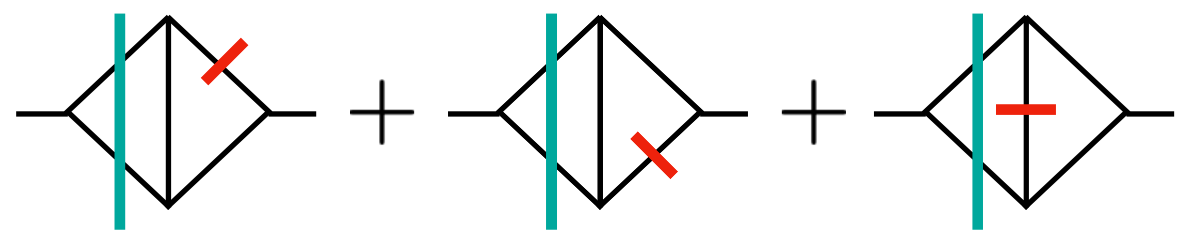

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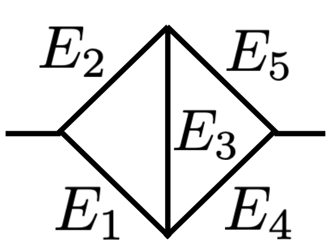

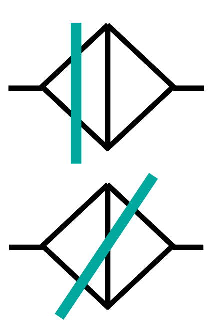

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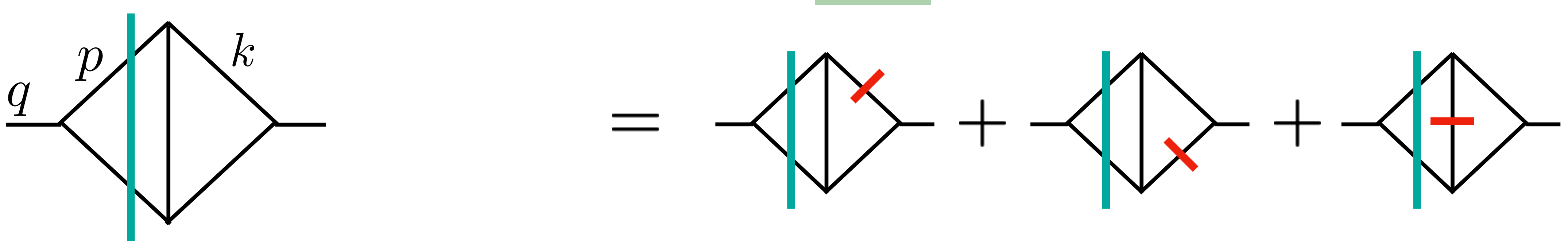

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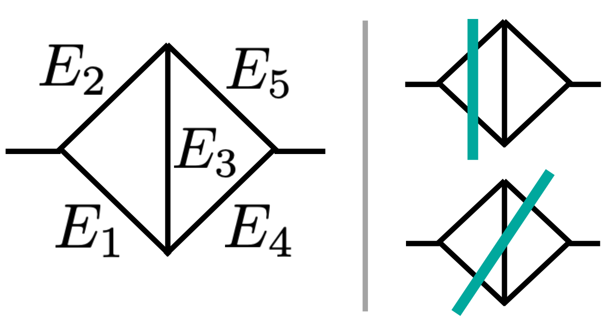

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(on-shell energies: $E_i(\vec{k}_i)=\sqrt{\vec{k}_i^2+m_i^2-i\delta}$)

Causal flow

Causal flow

The measure now differs only in the **delta enforcing on shell energy conservation**

 $\sim \delta(E_1 + E_2 - Q_0)$

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Objective: find a common variable to solve both deltas.

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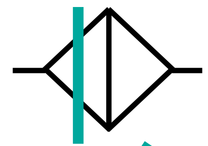

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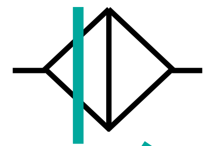
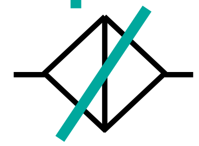
 $\sim \delta(E_1 + E_2 - Q_0)$

 $\sim \delta(E_1 + E_3 + E_5 - Q_0)$

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Soper,
arXiv: 9804454 (1998)

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ZC, Hirschi, Pelloni, Ruijl
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General FSR cancellations
For N to M N^{k}LO processes

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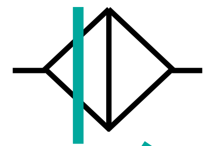
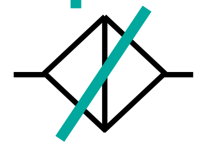
 $\sim \delta(E_1 + E_2 - Q_0)$

 $\sim \delta(E_1 + E_3 + E_5 - Q_0)$

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A toy example:

$$\int \mathrm{d}^3\vec{k}\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$

Causal flow : toy integral

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$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0) f(\vec{k}) \qquad \text{using} \qquad 1 = \int \mathrm{d}t\, h(t)$$

Causal flow : toy integral

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0) f(\vec{k})$$ using $$1 = \int \mathrm{d}t\, h(t)$$

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$$= \int \mathrm{d}^3\vec{k}\, \frac{Q_0^3}{|\vec{k}|^4}\, h(Q_0/|\vec{k}|)\, f(Q_0\vec{k}/|\vec{k}|)$$ with $$t^\star = Q_0/|\vec{k}|$$

Solve all deltas in the common scaling variable. This completes the alignment of the measure!

Causal flow : toy integral

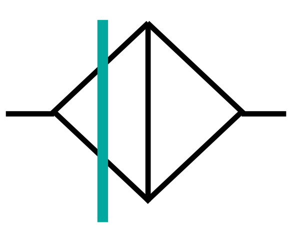
$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, h(t)\, \delta(|\vec{k}| - Q_0)\, f(\vec{k}) \qquad \text{using} \qquad 1 = \int \mathrm{d}t\, h(t)$$

$$= \int \mathrm{d}^3\vec{k} \int \mathrm{d}t\, t^3\, h(t)\, \delta(t|\vec{k}| - Q_0)\, f(t\vec{k}) \qquad \text{using} \qquad \vec{k} \to t\vec{k}$$

$$= \int \mathrm{d}^3\vec{k}\, \frac{Q_0^3}{|\vec{k}|^4}\, h(Q_0/|\vec{k}|)\, f(Q_0\vec{k}/|\vec{k}|) \qquad \text{with} \qquad t^\star = Q_0/|\vec{k}|$$

Solve all deltas in the common scaling variable. This completes the alignment of the measure!

When applying this construction to LU we get:



$$= \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\delta(E_1 + E_2 - Q_0) f_{\mathrm{virt}} = \int \mathrm{d}^3\vec{k}\mathrm{d}^3\vec{p}\, g_{\mathrm{v}}(t_{\mathrm{v}}^\star)$$

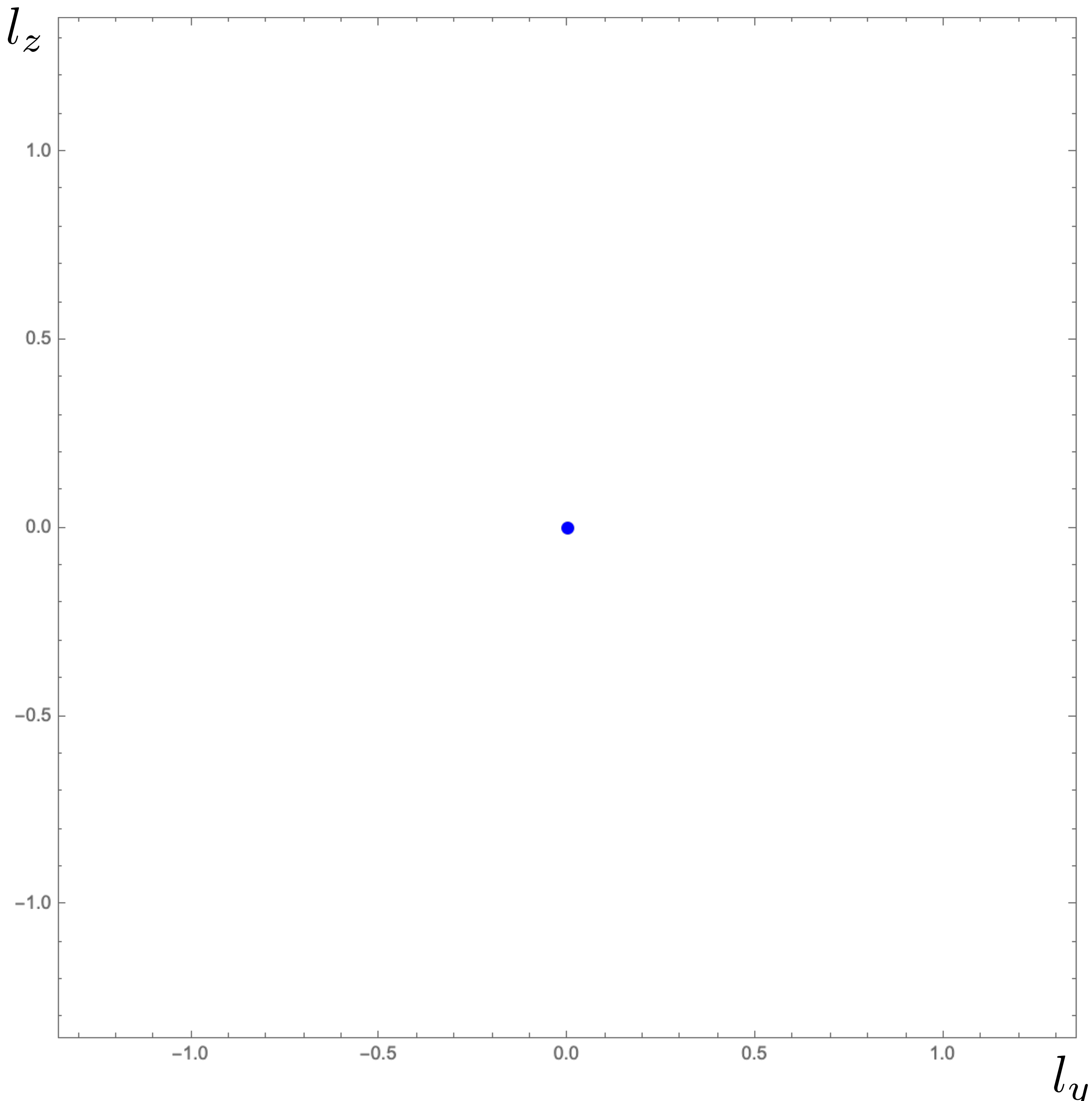
where $\quad t_{\mathrm{v}}^\star = t_{\mathrm{v}}^\star(\vec{k}, \vec{p}) = \dfrac{Q_0}{E_1 + E_2}$

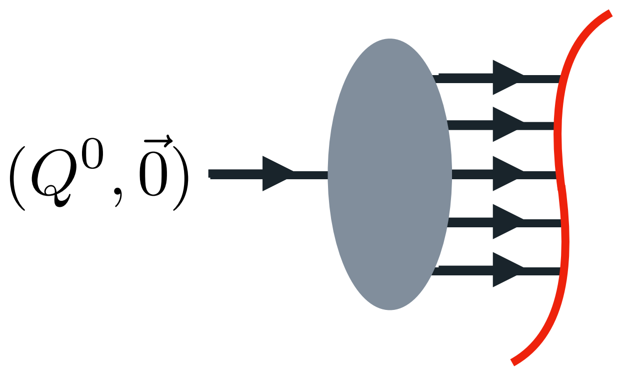
$$(\vec{p}, \vec{k}) \to \vec{\phi}(t, (\vec{p}, \vec{k}))$$

"Causal flow" is called like this because it is the generalisation of the Soper, derive from a contour deformation field satisfying the causal constraints.

$$\begin{cases} \partial_t \vec{\phi} = \vec{\kappa} \circ \vec{\phi} \\ \vec{\phi}(0, (\vec{k}, \vec{l})) = (\vec{k}, \vec{l}) \end{cases}$$

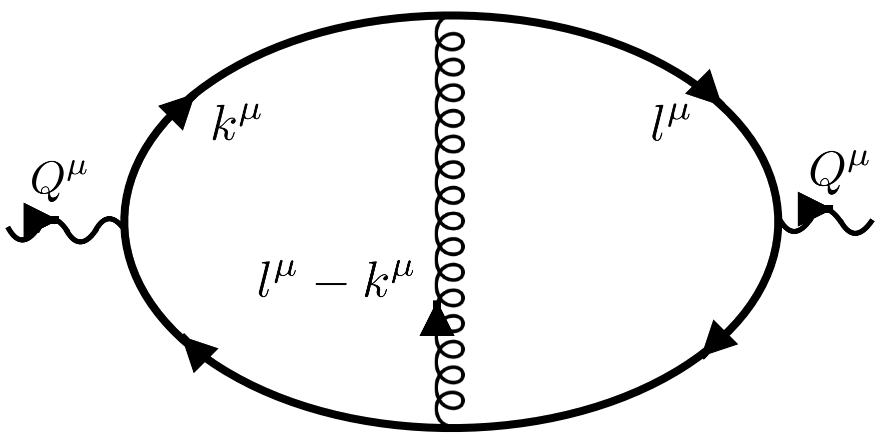
Locality Unitarity: visualisation





$Q^\mu = (2, 0, 0, 0)$

$(\vec{k}, \vec{l}) = (\ (0, 0.5, 0.5),\ (0, l_y, l_z)\ )$



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— = Cutkosky cut ≡ threshold

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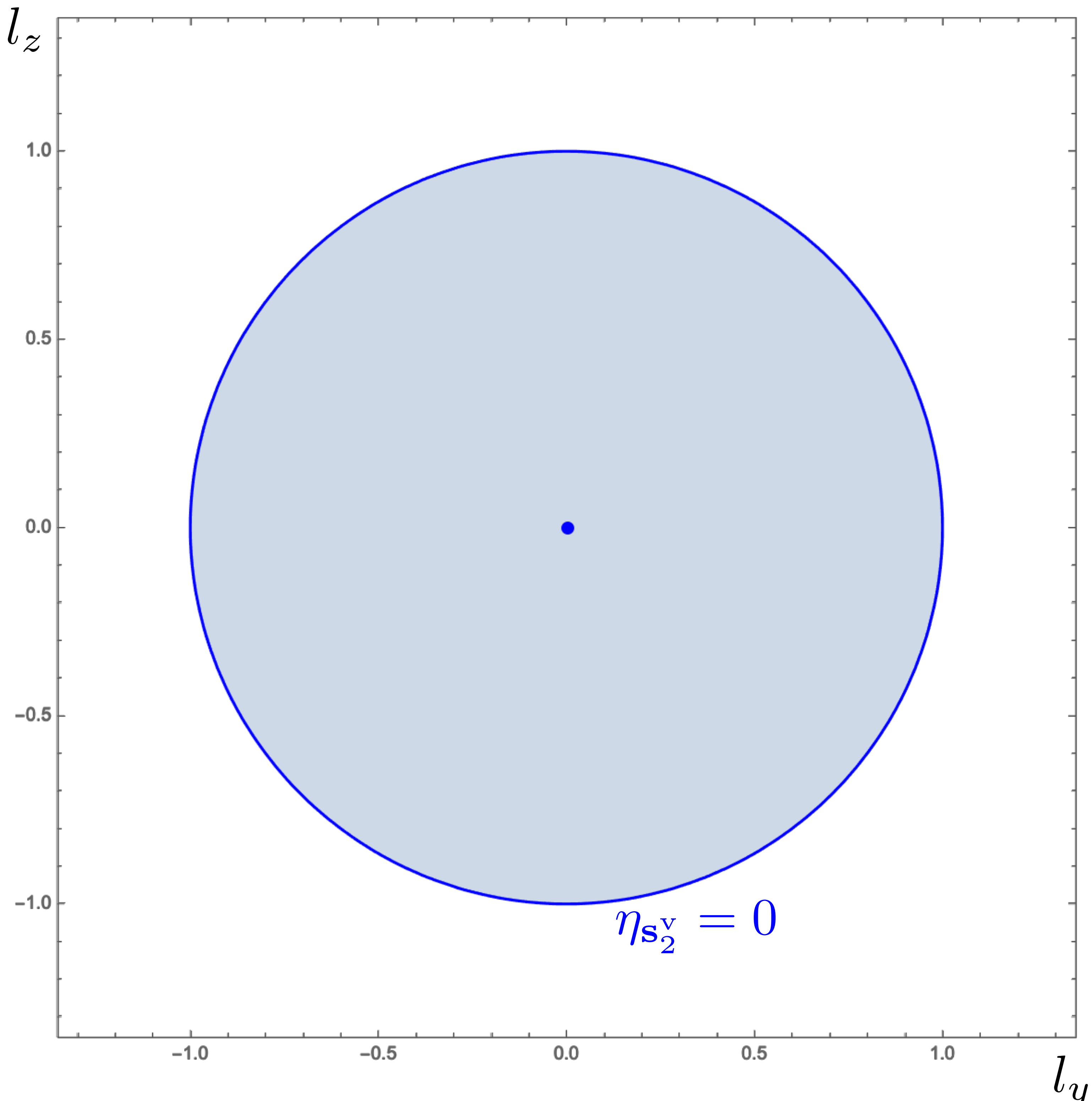

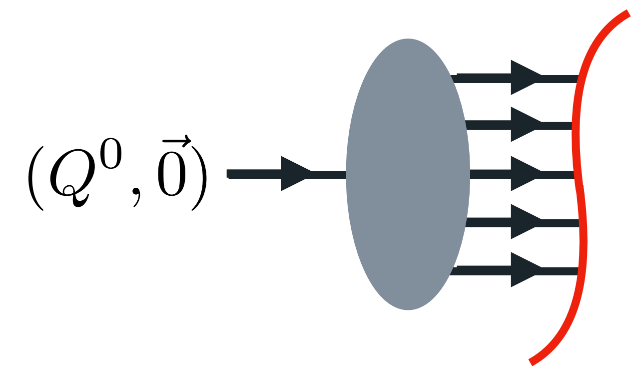

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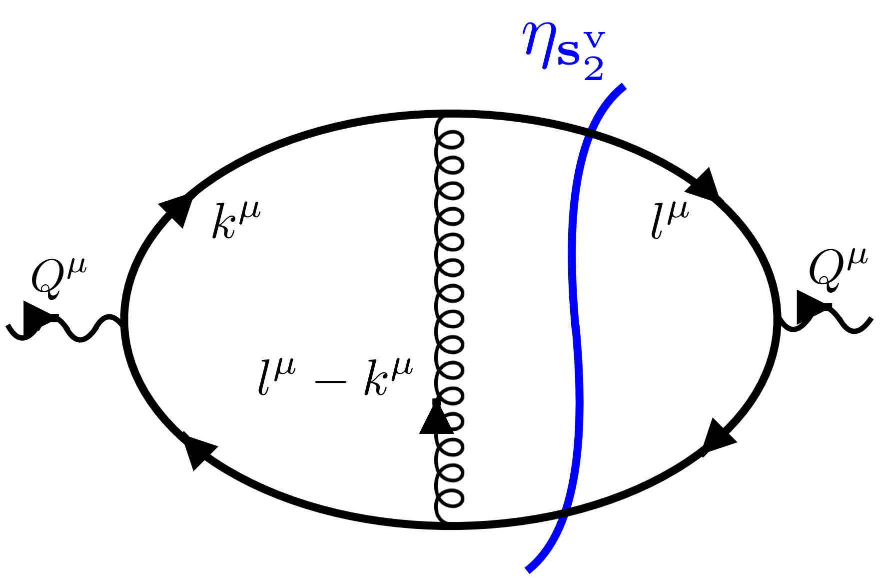

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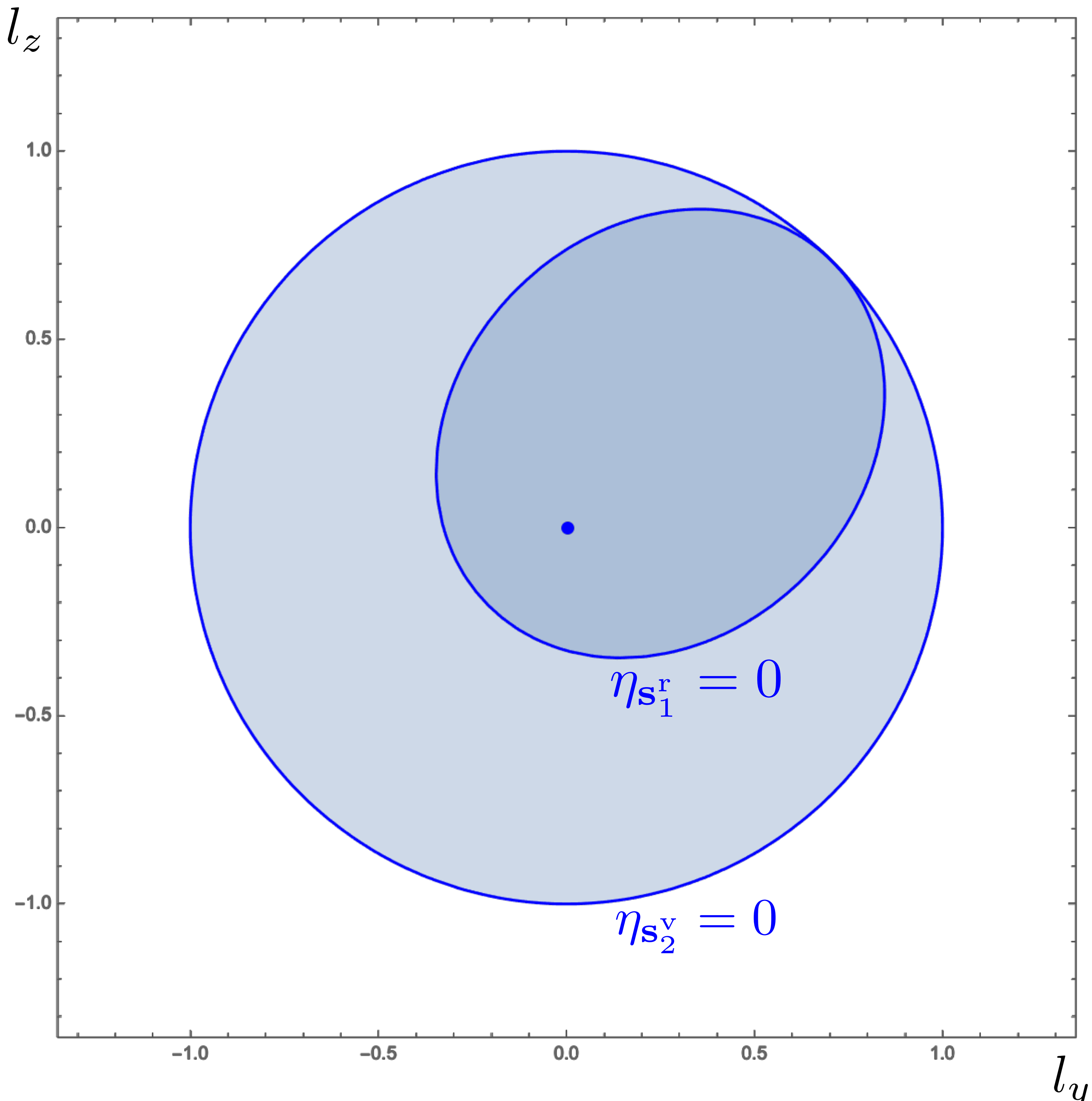

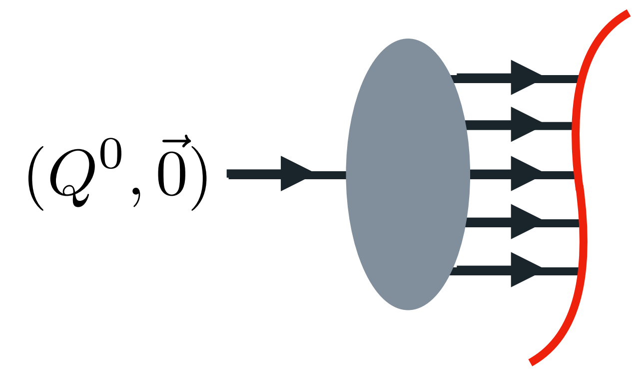

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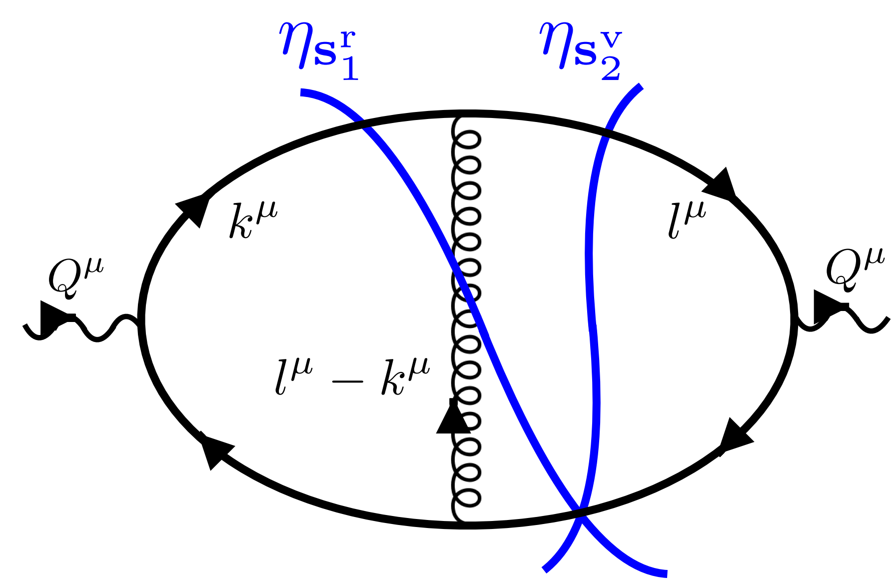

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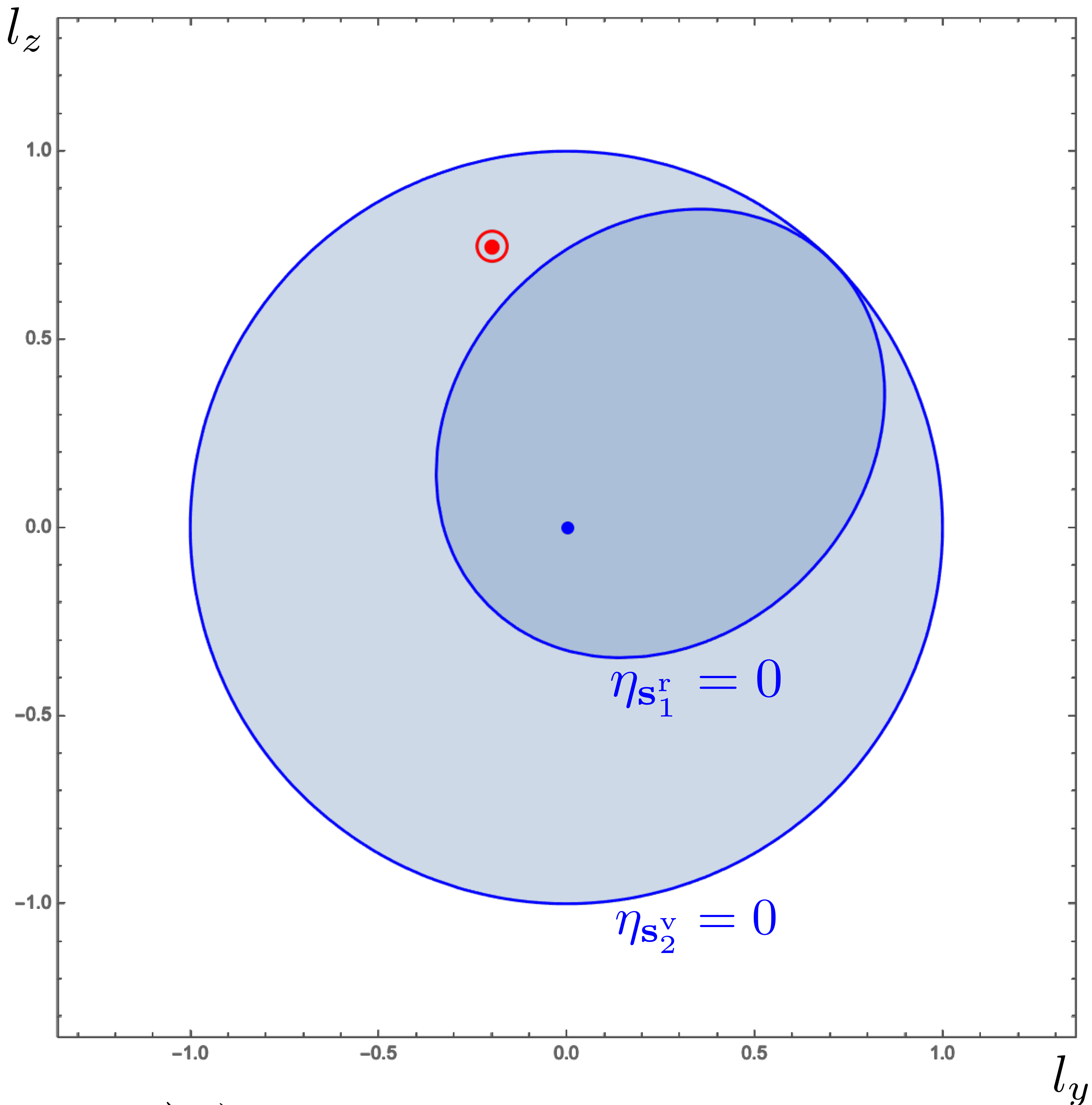

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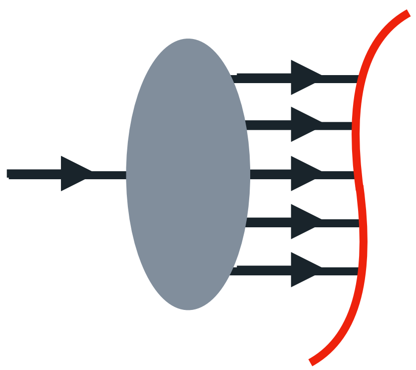

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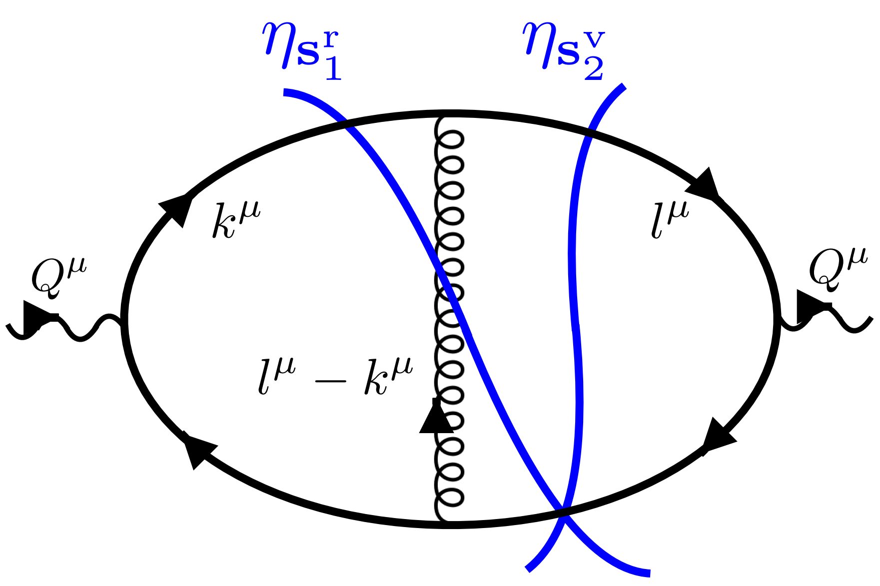

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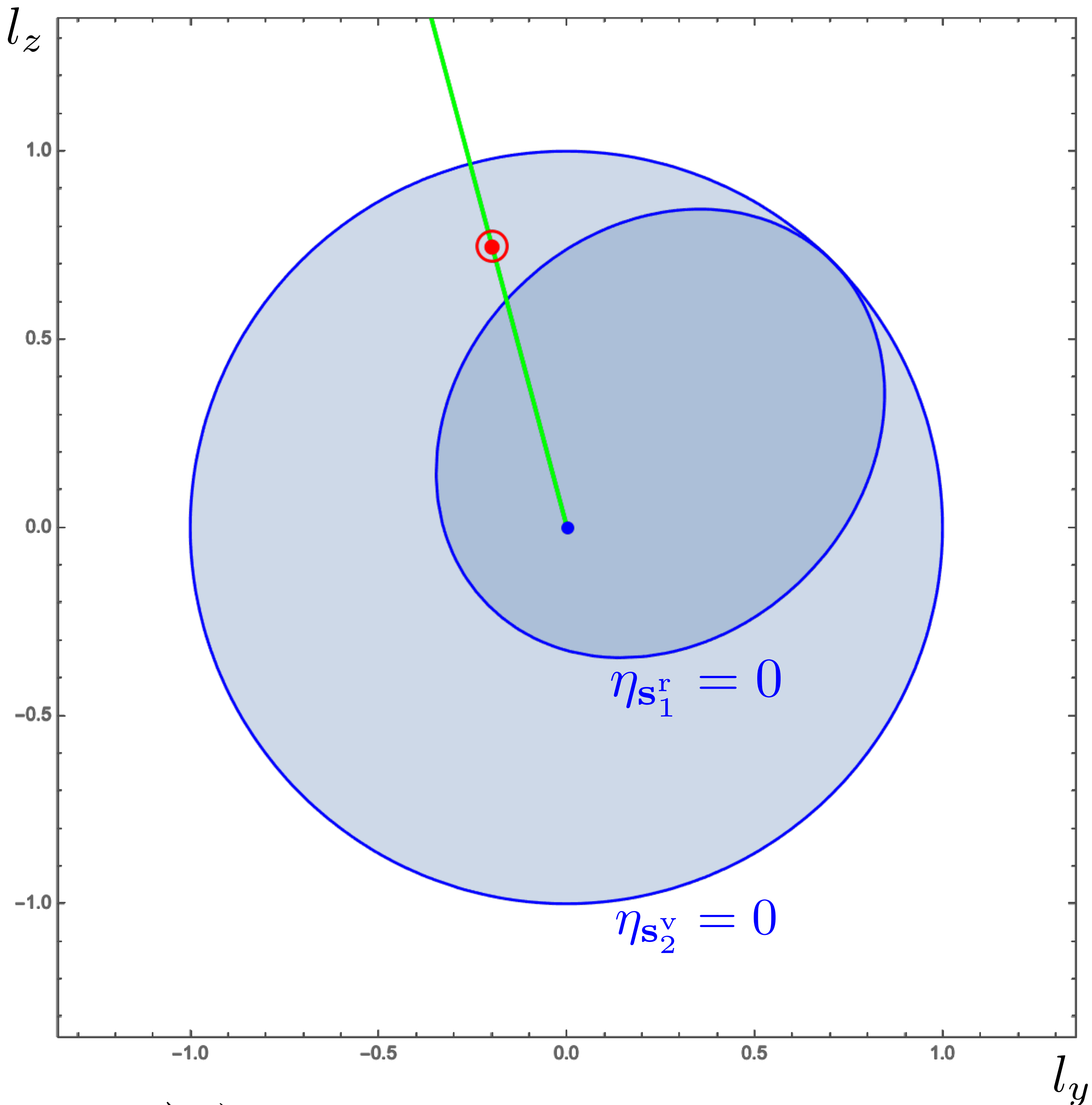

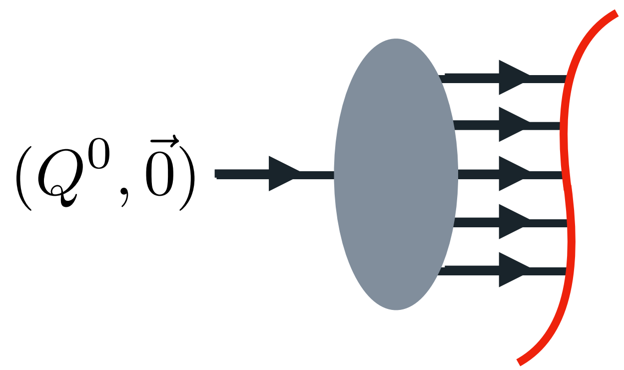

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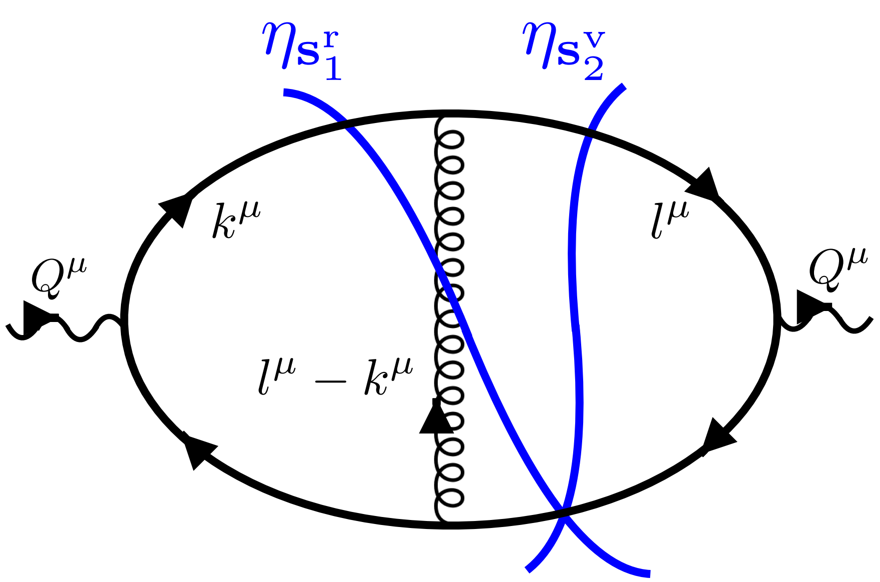

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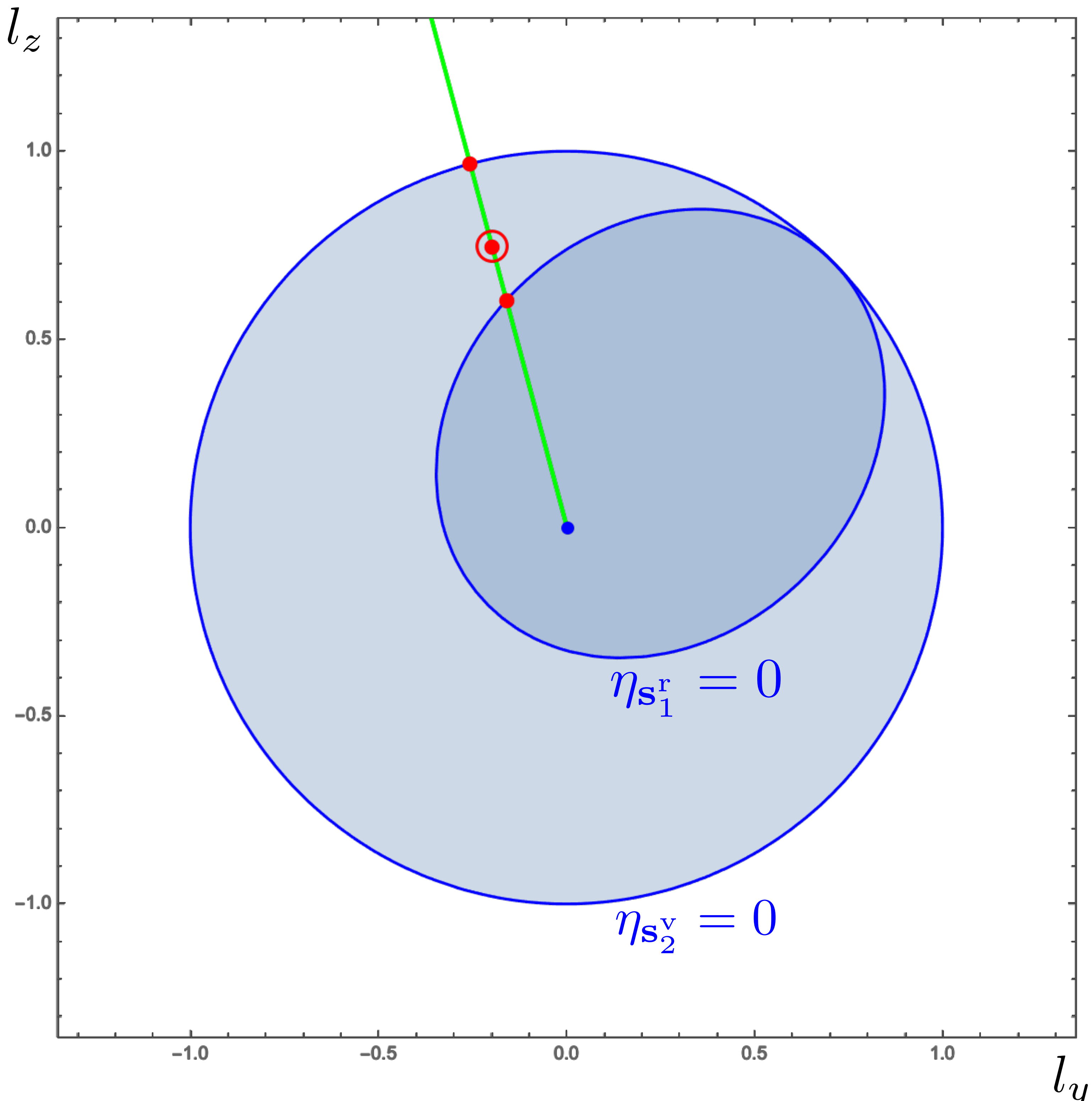
Locality Unitarity: visualisation



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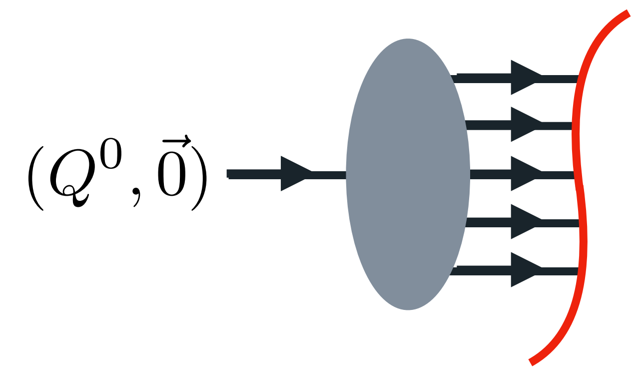

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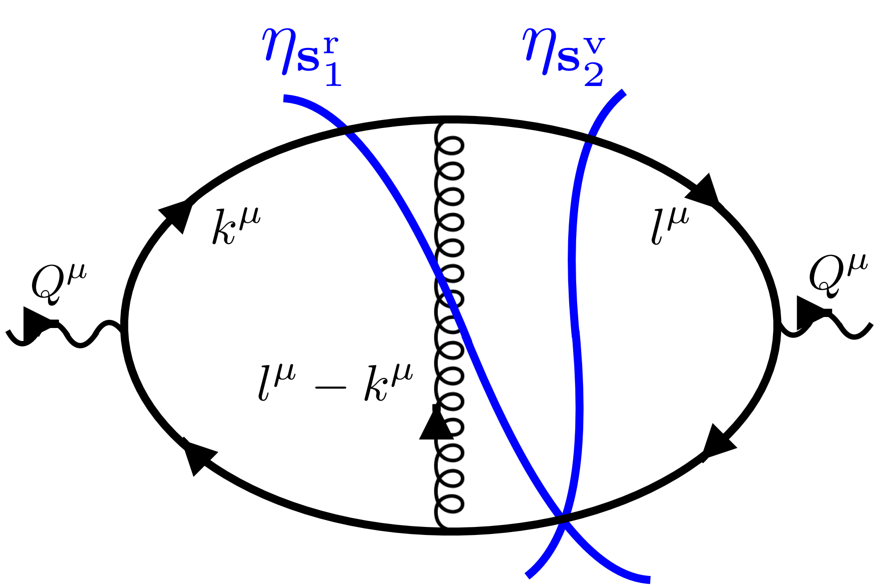

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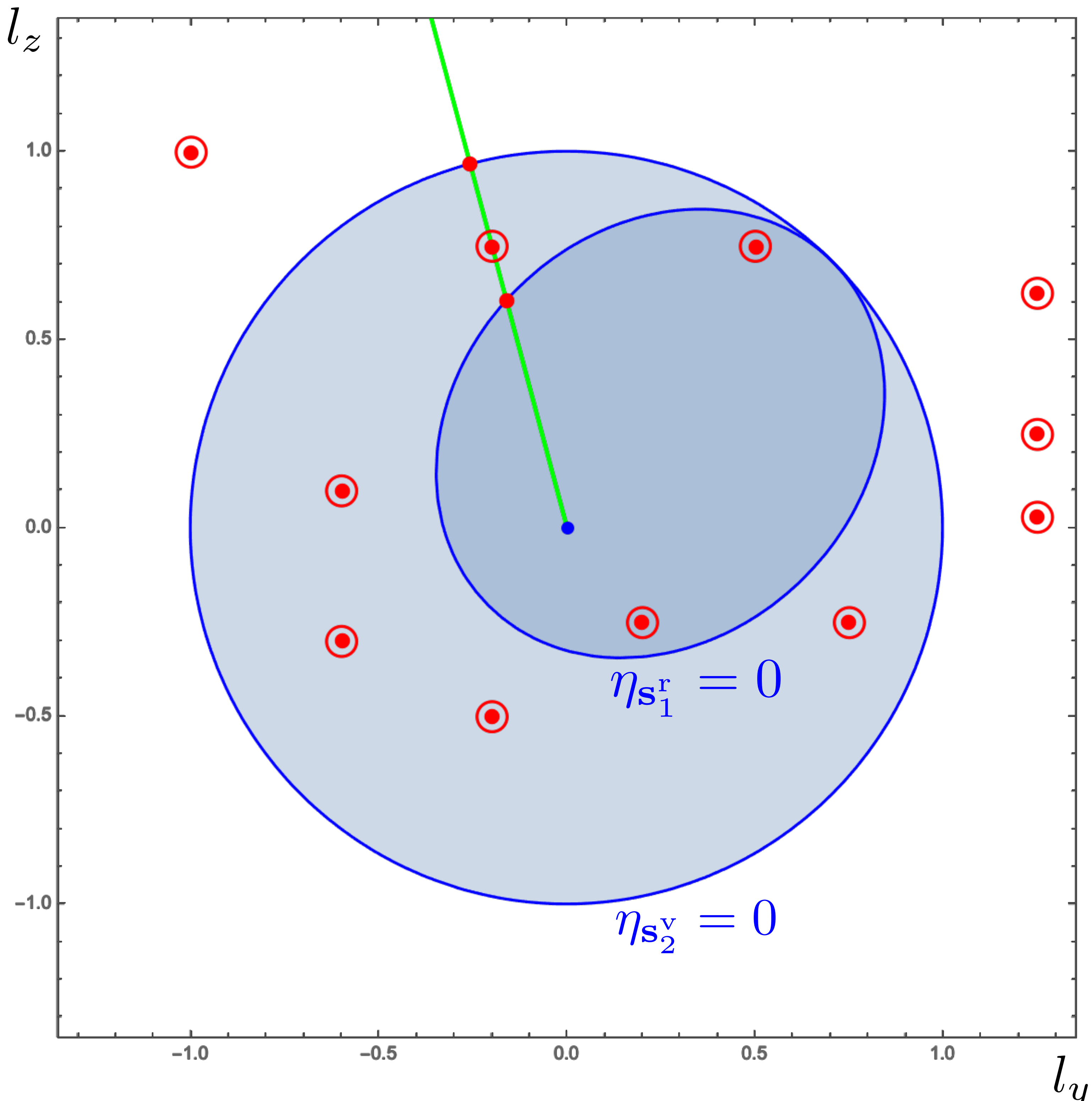
Locality Unitarity: visualisation



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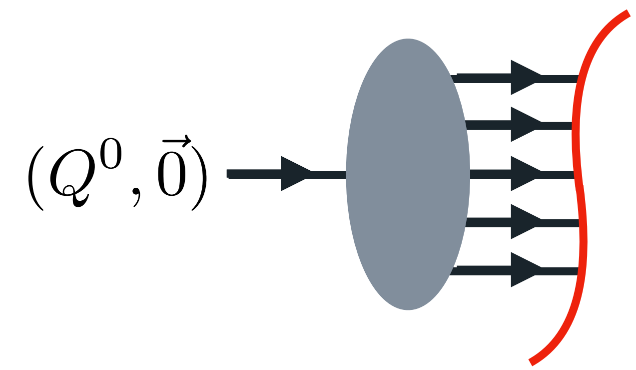

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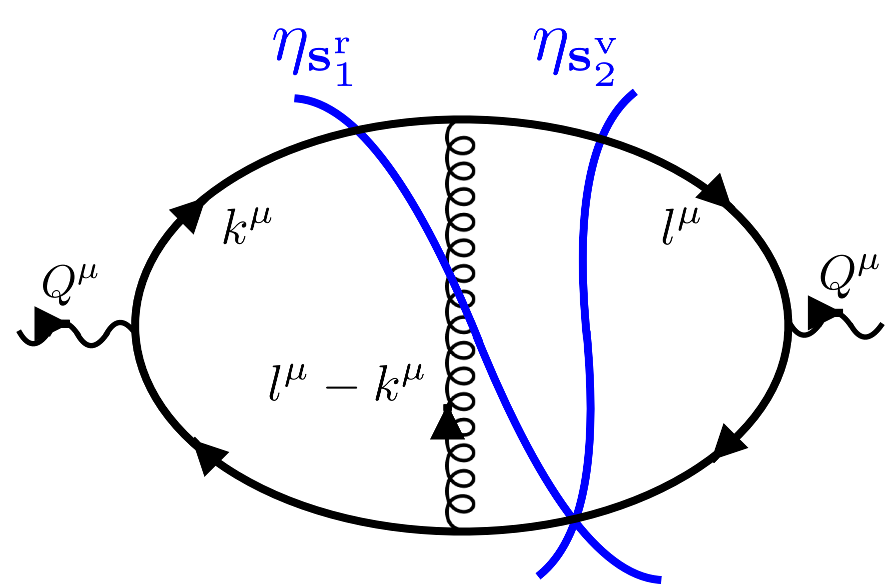

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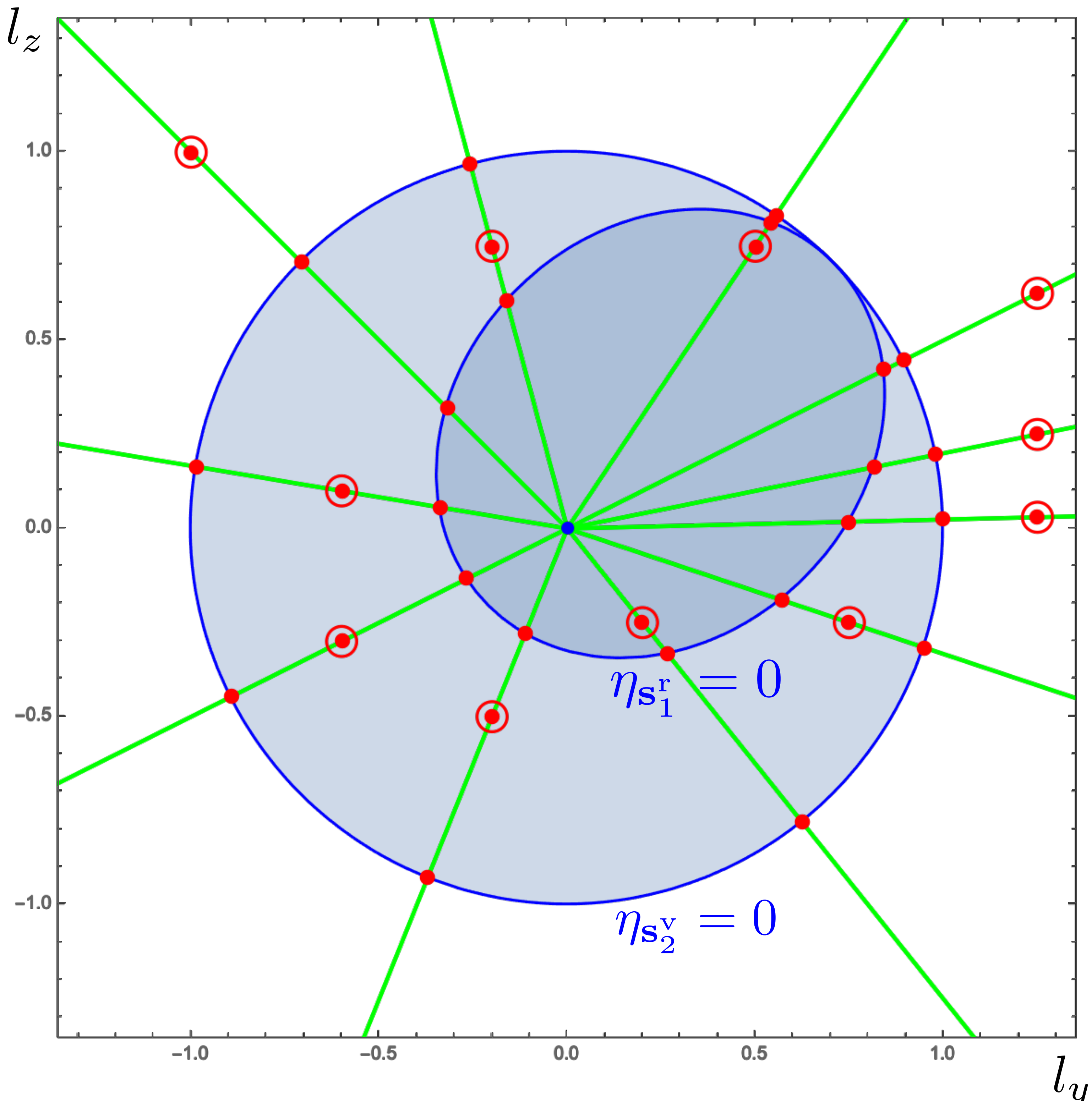

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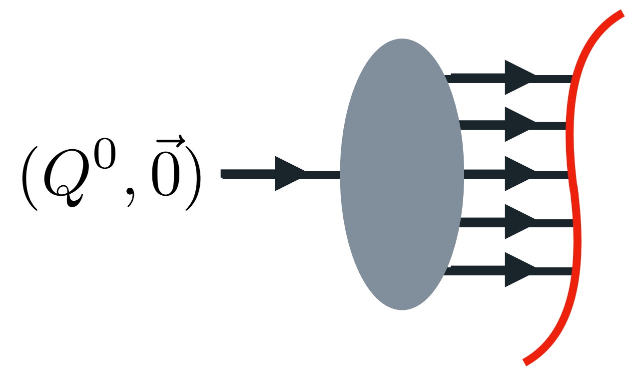

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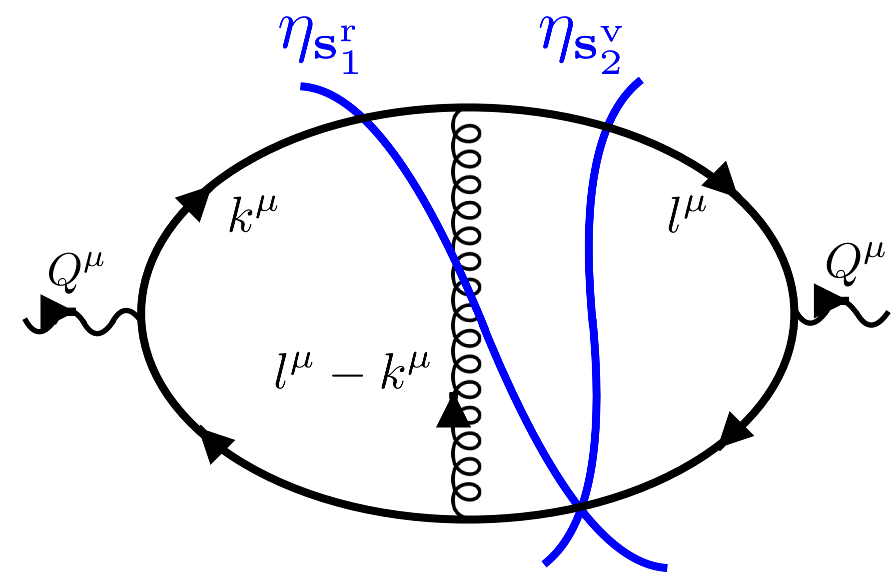

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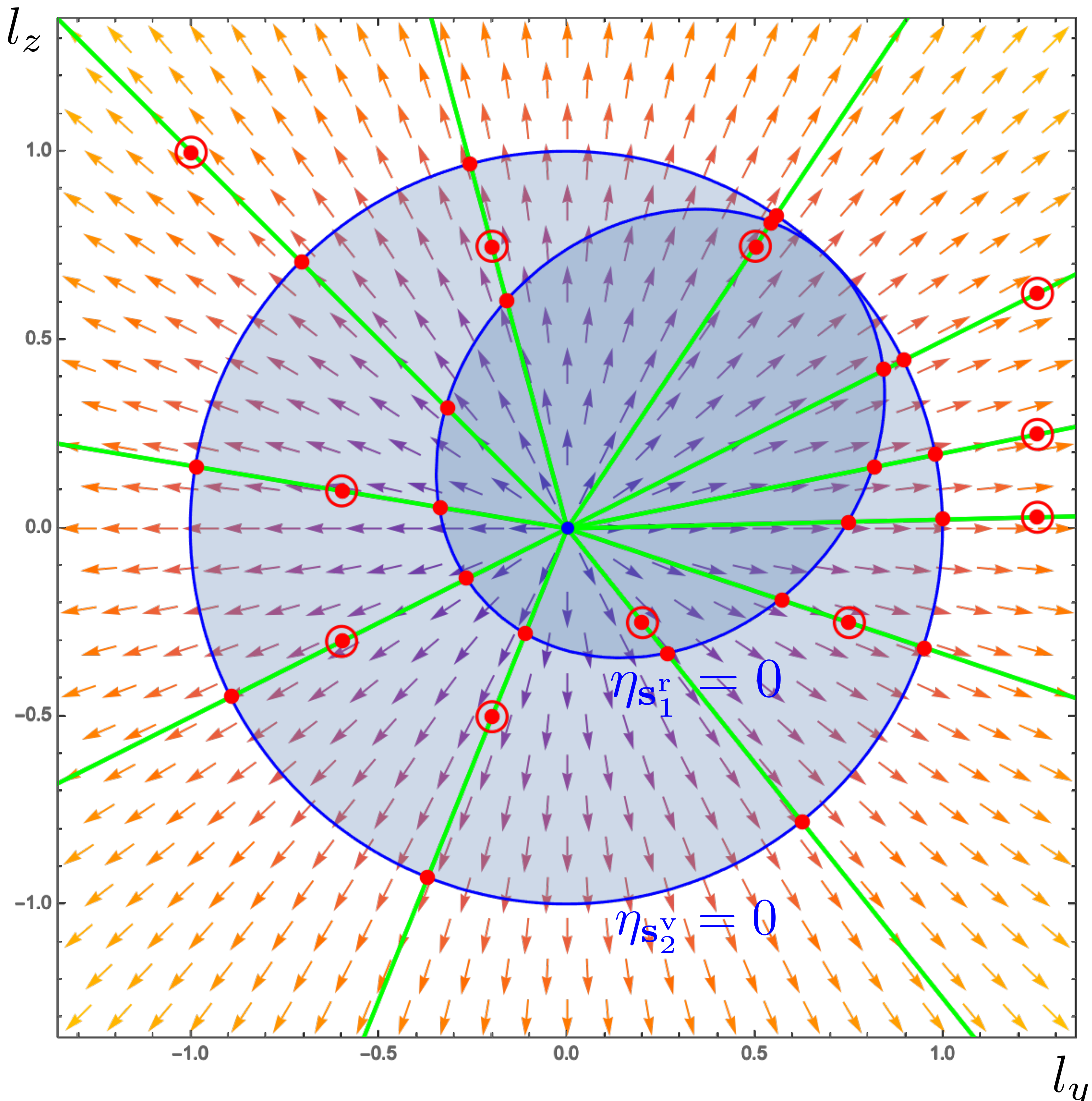

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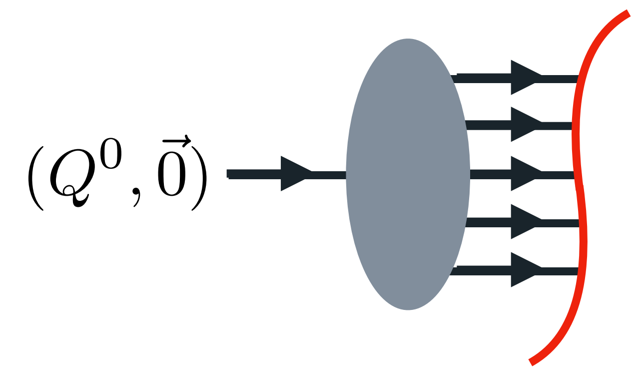

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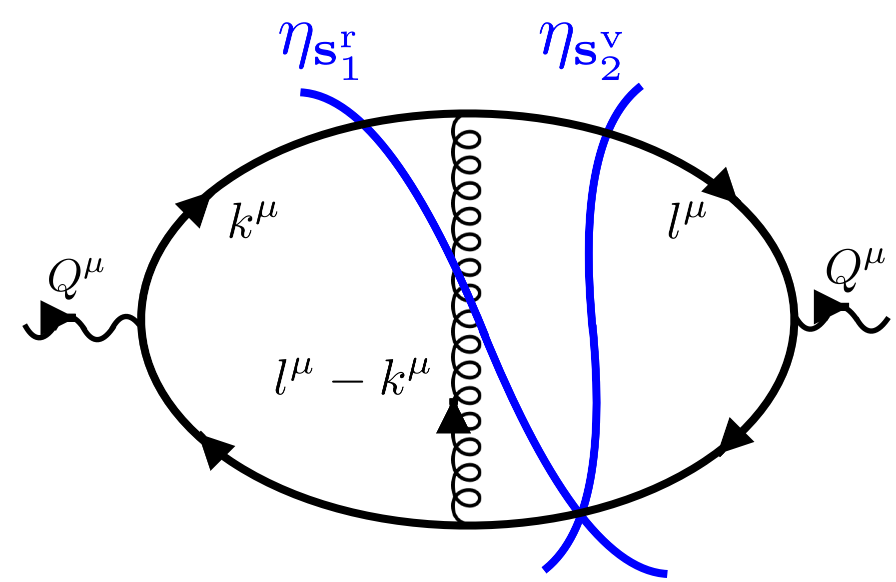

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Locality Unitarity: all-orders proof

hi, B. Ruijl, arXiv : 2010.01068]

[Summary]in proceedings, arXiv : 2110.15662]

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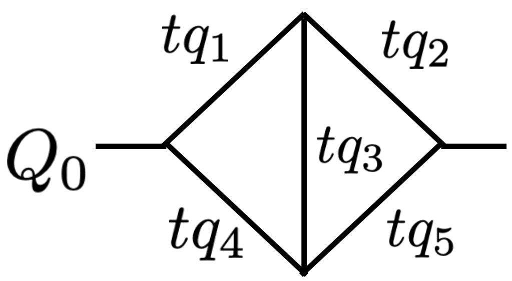
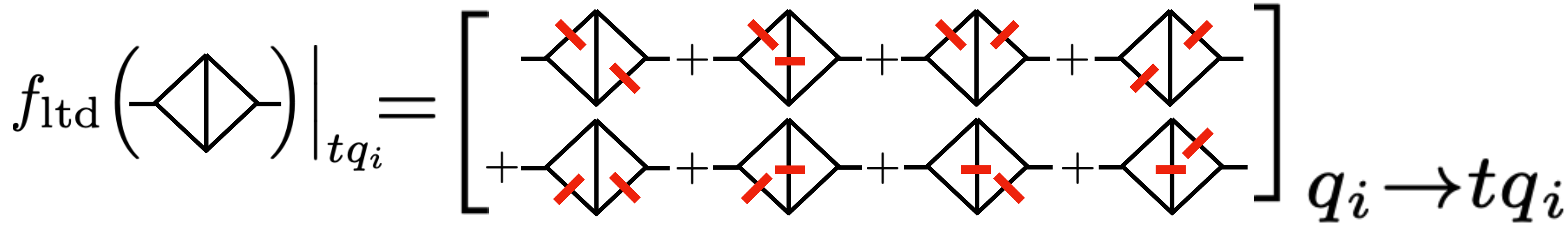
The **LTD representation** of the double triangle with rescaled momenta is





LOCALITY UNITARITY: ALL-ORDERS PROOF

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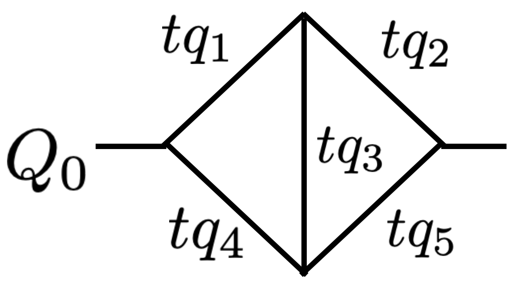
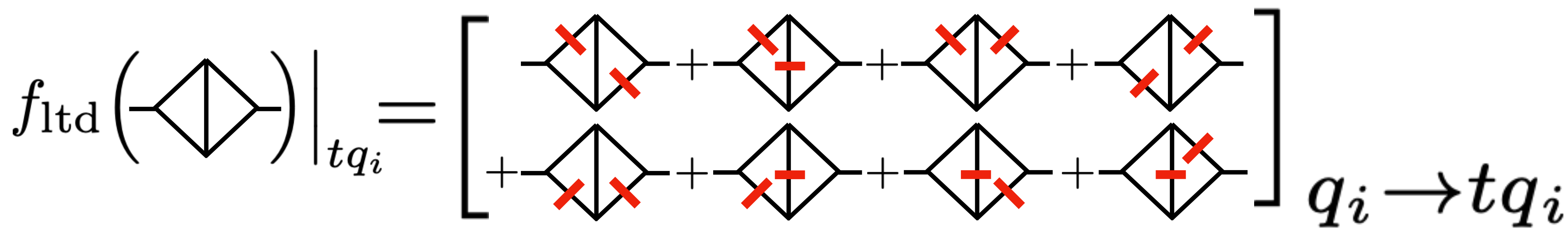
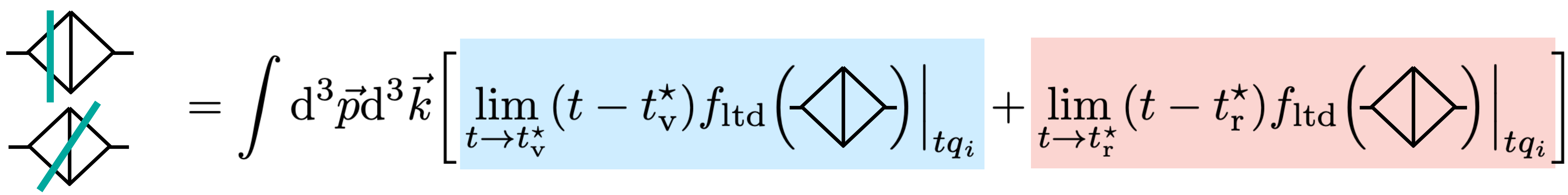




Then one can capture the **thresholds** of this forward-scattering graphs with



$g_{\rm v}$, $g_{\rm r}$ can be written as different limits of the same function!

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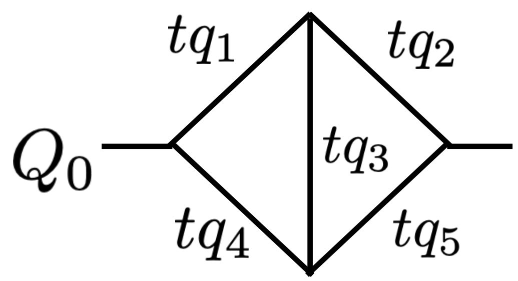

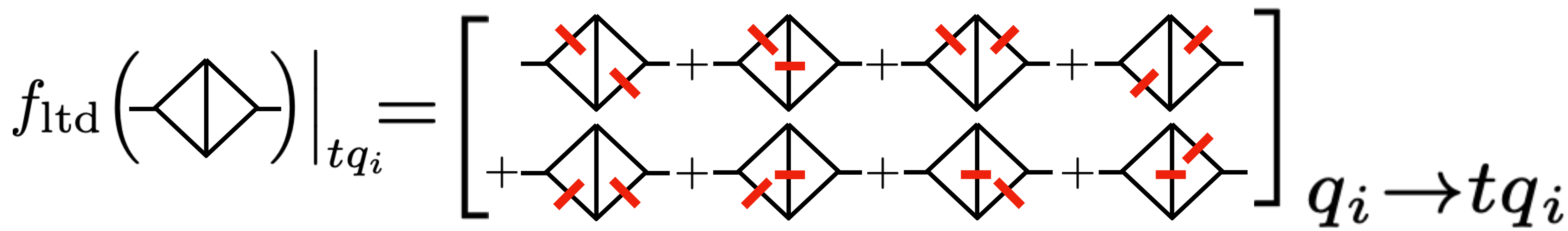

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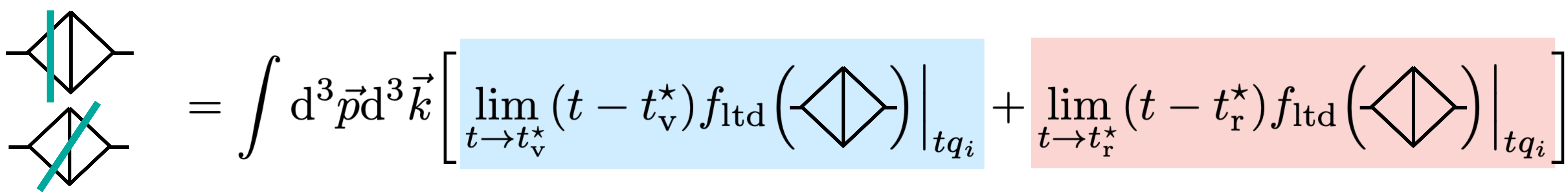

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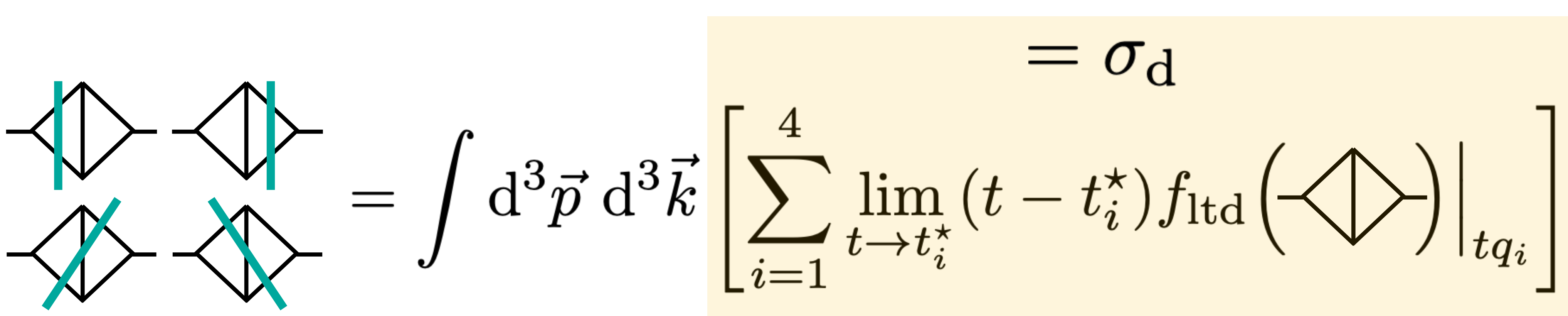
Solving delta in the scaling variable $\Rightarrow$ **1d residue theorem along the line** $\gamma(t) = (t\vec{k}, t\vec{p})$



LU representation

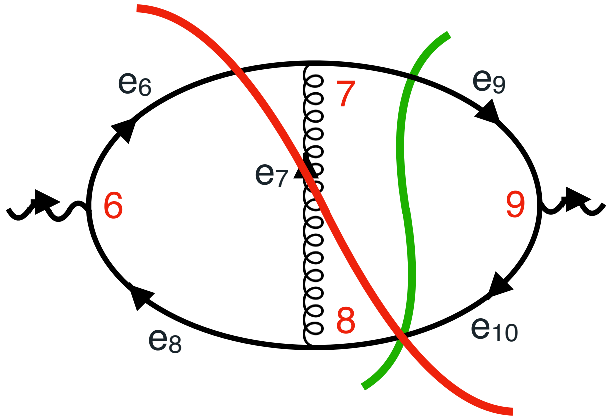
Cutkosky, but at the local level! We prove cancellations by studying the limit $t_r^\star \to t_{\rm v}^\star$

LOCALITY UNITARITY

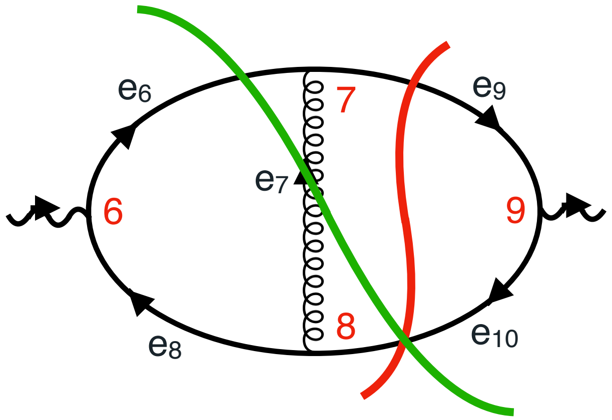
[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :

= Cutkosky cut = threshold singularity



cancels



Locality Unitarity

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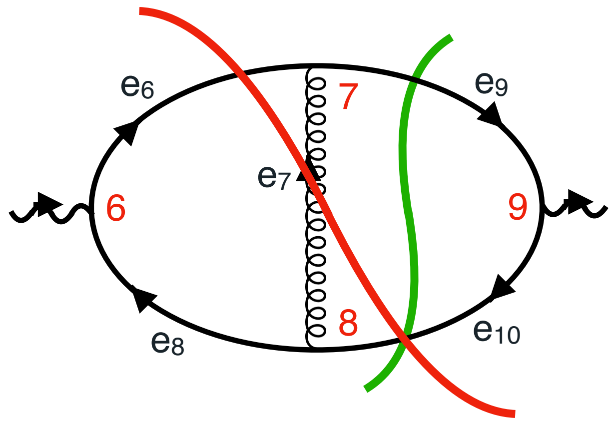

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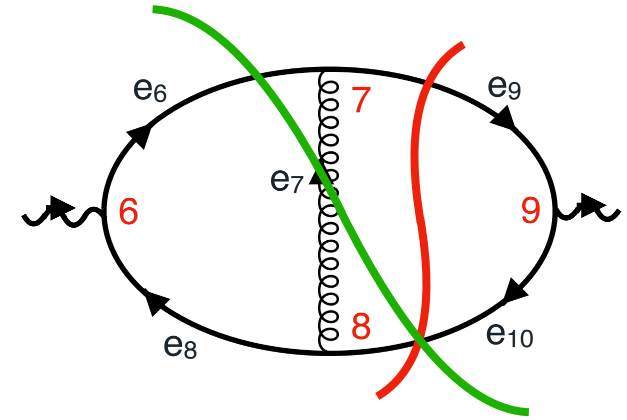

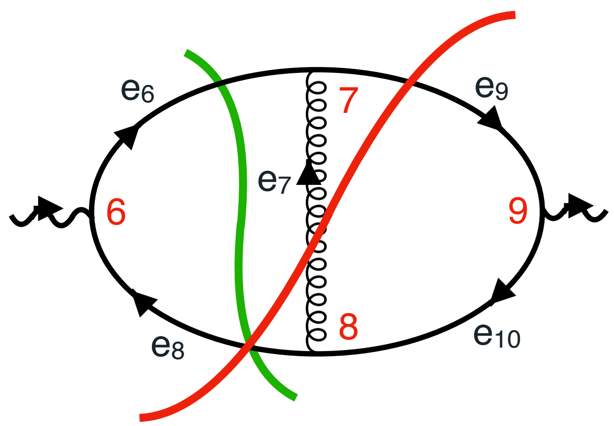

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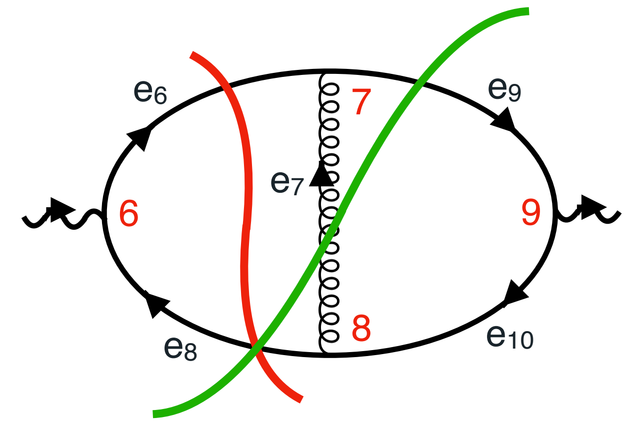

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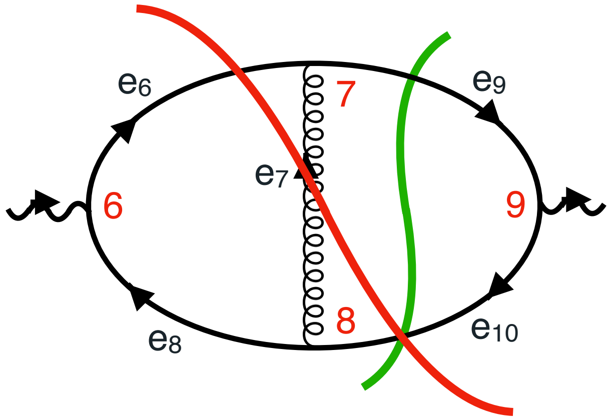

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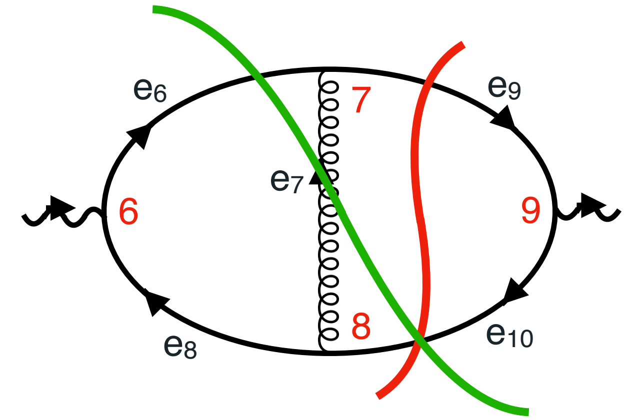

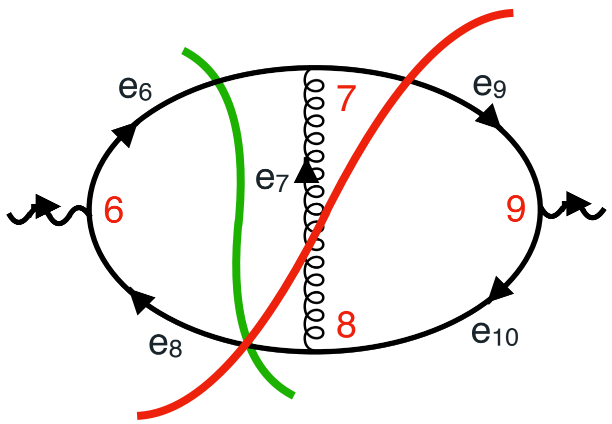

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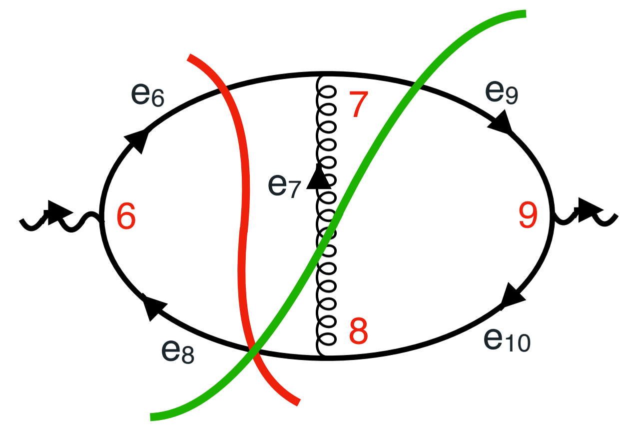
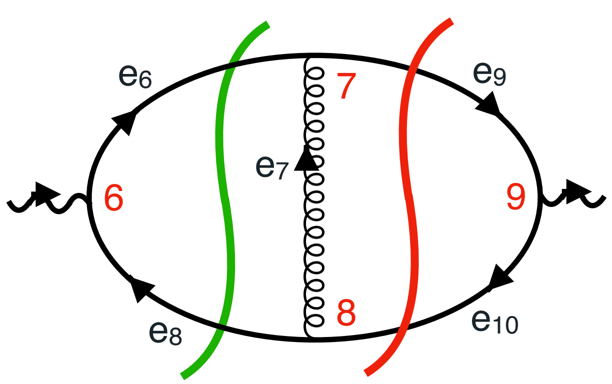
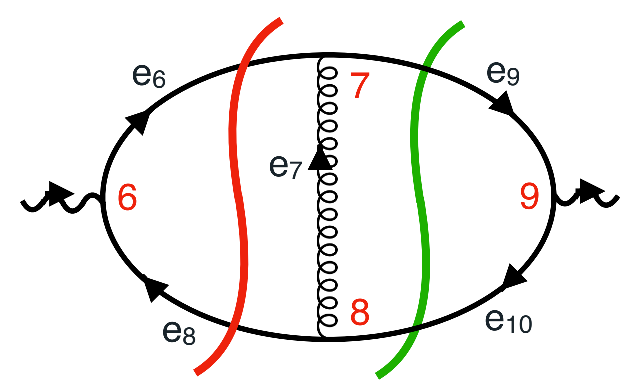


Even for **non-pinched singular threshold** ! (when $\mathcal{O}_\mathbf{s} \equiv 1$) :



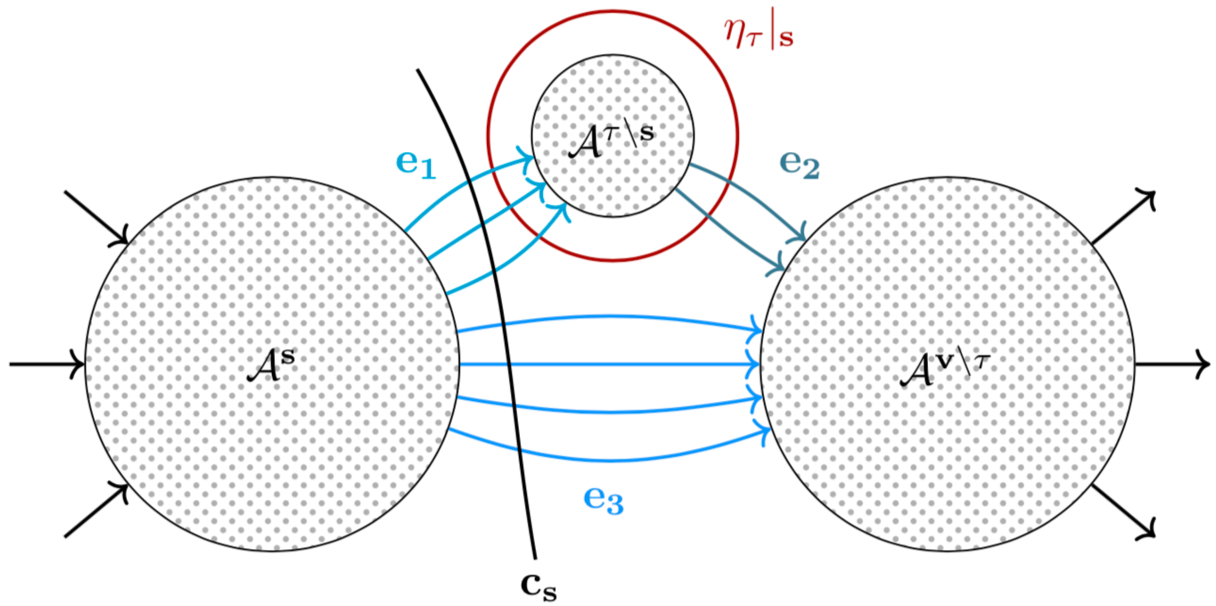
cancels

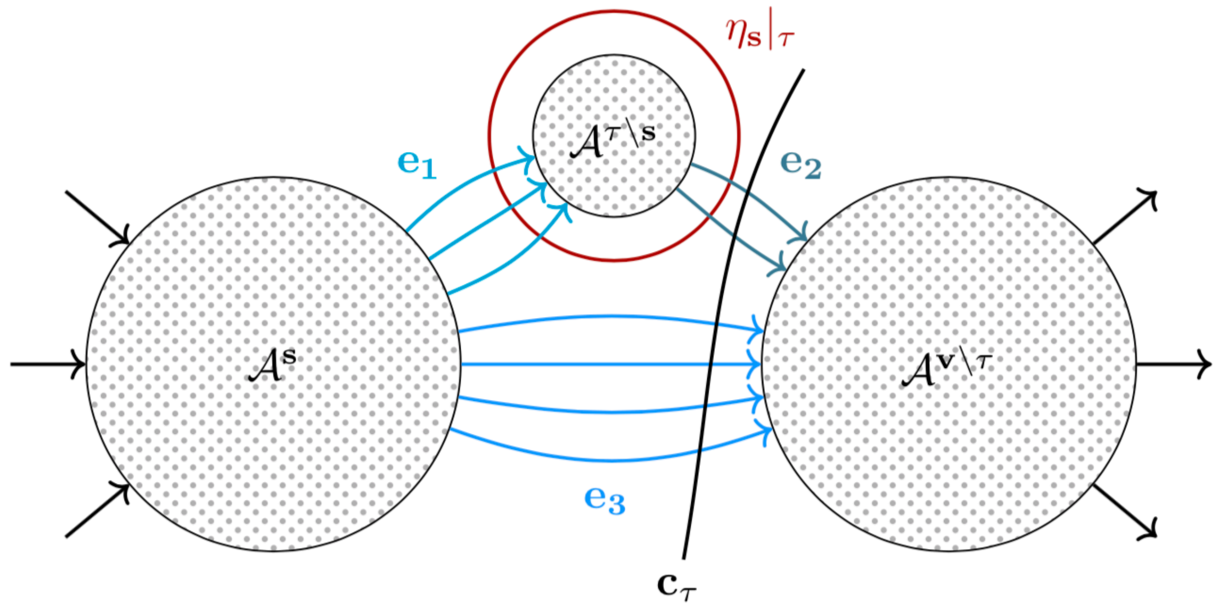


Locality Unitarity

[Capatti, VH, Pelloni, Ruijl, arxiv:2010.01068]

This **pairwise cancellation** pattern holds at **all orders**, and for **all threshold** :





Tropical sampling in momentum space

for taming integrable singularities

$$I^{(\mathrm{eucl.})}[f] = \int d^{DL}\mathbf{k}\, \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}}$$

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$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \quad \underbrace{K(\mathbf{z})}_{\text{bounded}} \quad f(\mathbf{k}(\mathbf{z}))$$

$$\boxed{I^{\text{eucl.}}[f=1]}$$

E	$\ell(G)$	σ_I/I	samples per second	preprocessing time	RAM
6	3	0.9	$1.1 \cdot 10^6$ / s	$3.0 \cdot 10^{-5}$ s	1 KB
8	4	1.1	$7.5 \cdot 10^5$ / s	$1.3 \cdot 10^{-4}$ s	4 KB
10	5	1.3	$5.1 \cdot 10^5$ / s	$6.0 \cdot 10^{-4}$ s	16 KB
12	6	1.6	$4.1 \cdot 10^5$ / s	$2.7 \cdot 10^{-3}$ s	64 KB
14	7	1.8	$3.2 \cdot 10^5$ / s	$1.2 \cdot 10^{-2}$ s	256 KB
16	8	2.1	$2.6 \cdot 10^5$ / s	$5.3 \cdot 10^{-2}$ s	1 MB
18	9	2.5	$2.1 \cdot 10^5$ / s	$2.3 \cdot 10^{-1}$ s	4 MB
20	10	2.8	$1.4 \cdot 10^5$ / s	$1.1 \cdot 10^0$ s	16 MB
22	11	3.2	$1.0 \cdot 10^5$ / s	$4.7 \cdot 10^0$ s	64 MB
24	12	3.7	$8.6 \cdot 10^4$ / s	$2.1 \cdot 10^1$ s	256 MB
26	13	4.2	$6.9 \cdot 10^4$ / s	$9.5 \cdot 10^1$ s	1 GB
28	14	4.8	$5.9 \cdot 10^4$ / s	$4.4 \cdot 10^2$ s	4 GB
30	15	5.3	$5.1 \cdot 10^4$ / s	$1.9 \cdot 10^3$ s	16 GB
32	16	6.3	$4.3 \cdot 10^4$ / s	$8.7 \cdot 10^3$ s	64 GB
34	17	7.2	$3.6 \cdot 10^4$ / s	$3.9 \cdot 10^4$ s	256 GB

Table 1: Benchmark of Feynman integral evaluations with different numbers of edges.

[M. Borinsky, arXiv : 2008.12310]

Tropical sampling of Euclidean Feynman integrals

$$\boxed{I^{\text{eucl.}}[f=1]}$$

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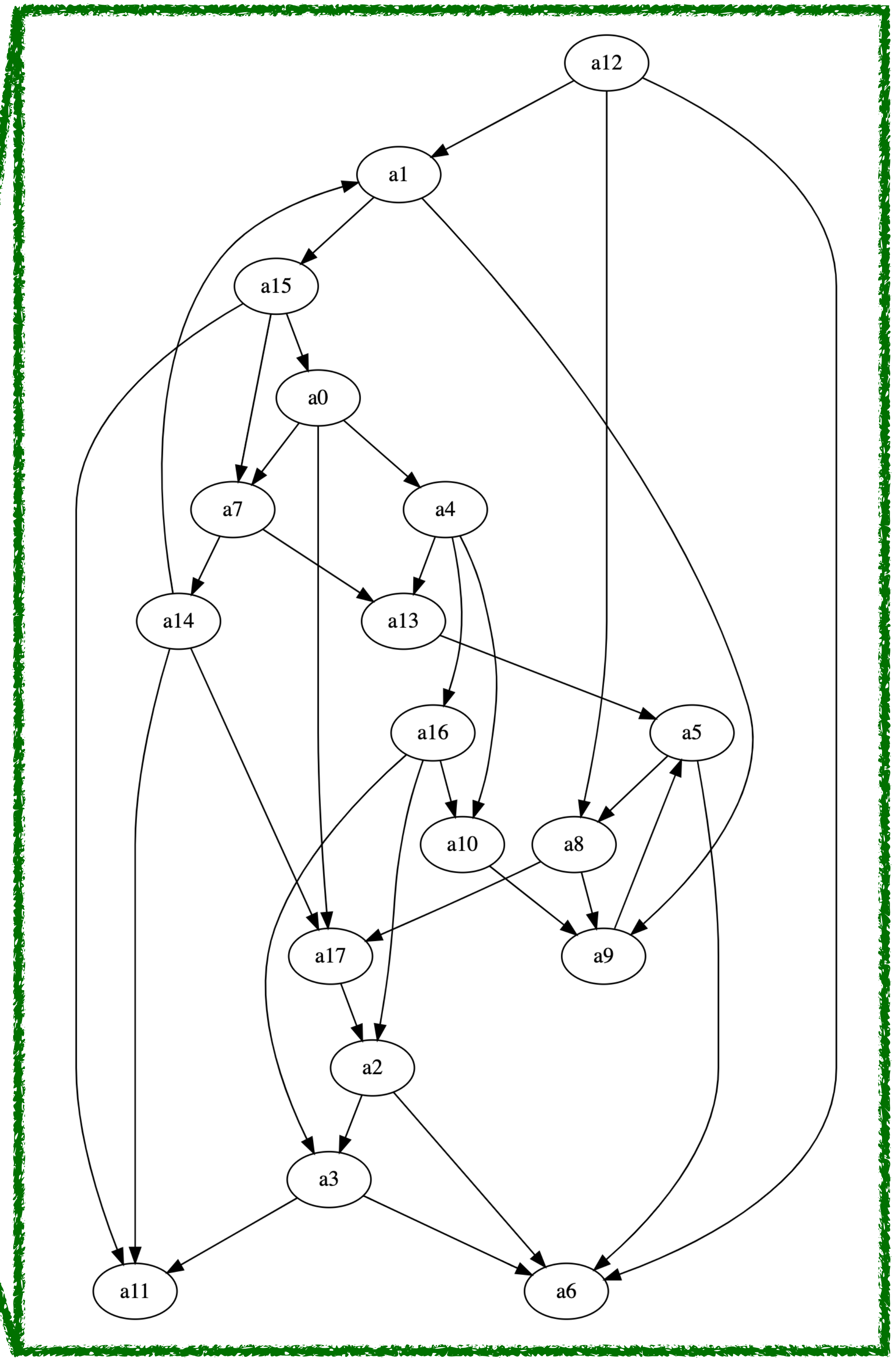
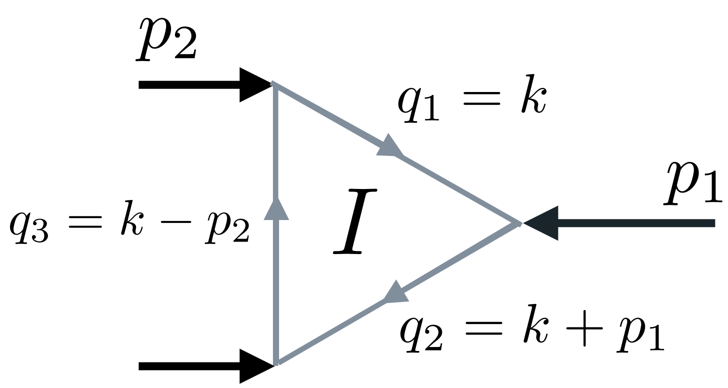


Tropical sampling of Loop-Tree Duality integrands

[Inspired from and in collaboration with M. Borinsky, arXiv : 2008.12310]

[Investigation from **Mathijs Fraije**]

Recall:



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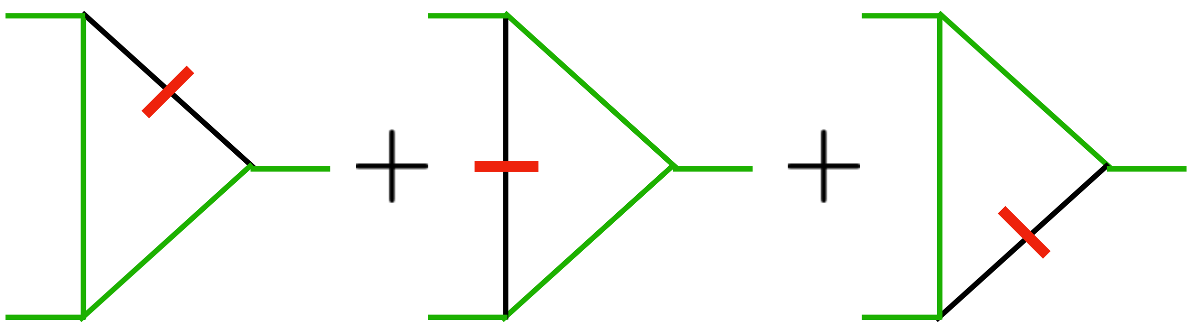

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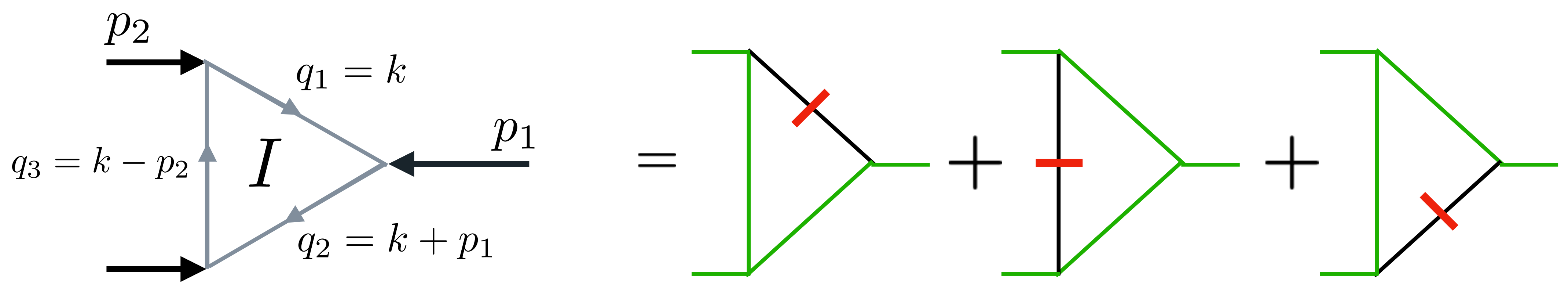

$$= \int \mathrm{d}^4 k \frac{1}{D_1 D_2 D_3} \left(D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3) \right) \supset \int d^3 \vec{k} \frac{1}{E_1 E_2 E_3} \frac{1}{E_1 + E_2 - p_1^0} \frac{1}{E_1 + E_2 + p_1^0}$$

With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \overset{m=0}{=} |\vec{q}_i|$.

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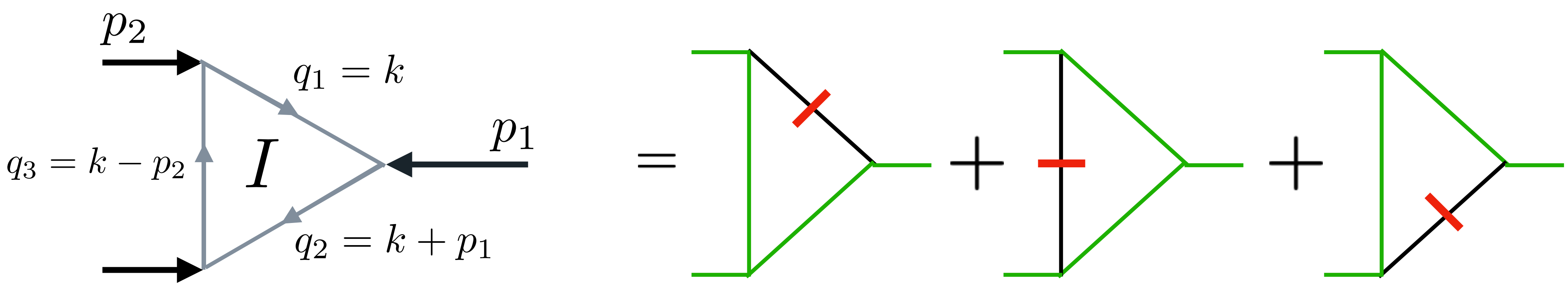

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With $E_i(\vec{k}) \equiv \sqrt{\vec{q}_i \cdot \vec{q}_i + m_i^2} \overset{m=0}{=} |\vec{q}_i|$. The prefactor $\frac{1}{E_1 E_2 E_3}$ contains point-like integrable singularities :

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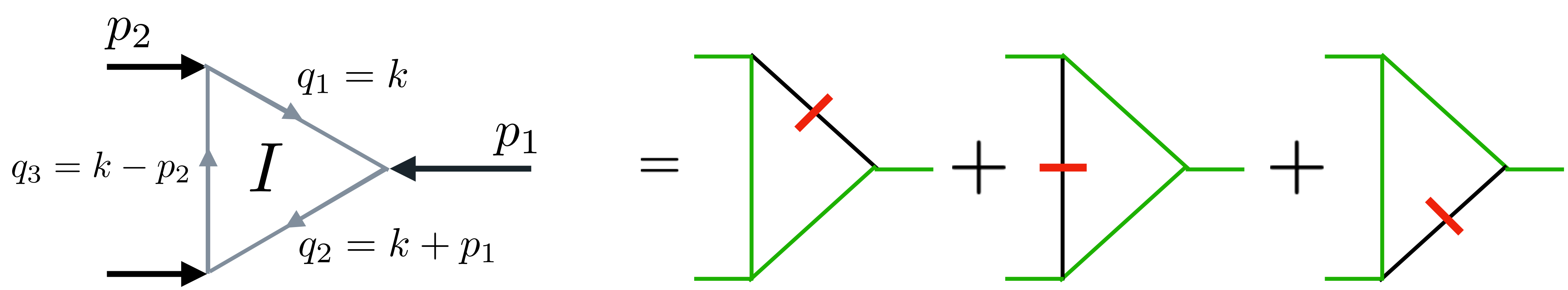

$$= \int \mathrm{d}^4 k \frac{1}{D_1 D_2 D_3} \left(D_1 \delta^{(+)}(D_1) + D_2 \delta^{(+)}(D_2) + D_3 \delta^{(+)}(D_3)\right) \supset \int d^3\vec{k} \frac{1}{E_1 E_2 E_3} \frac{1}{E_1 + E_2 - p_1^0} \frac{1}{E_1 + E_2 + p_1^0}$$

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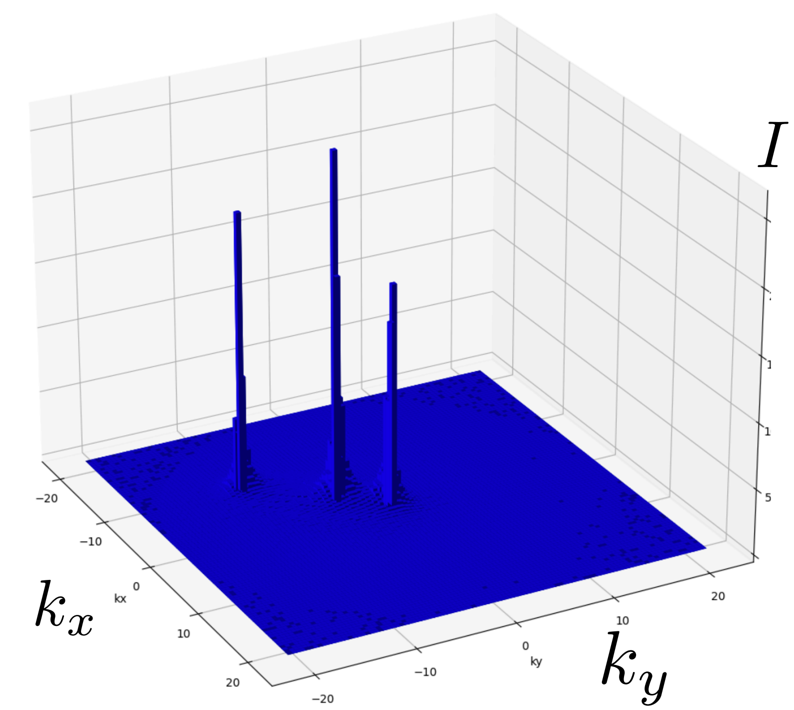
$$\frac{1}{E_1 E_2 E_3} = \frac{1}{|\vec{k}||\vec{k} + \vec{p}_1||\vec{k} - \vec{p}_2|}$$

2D example:



Tropical sampling of Loop-Tree Duality integrands

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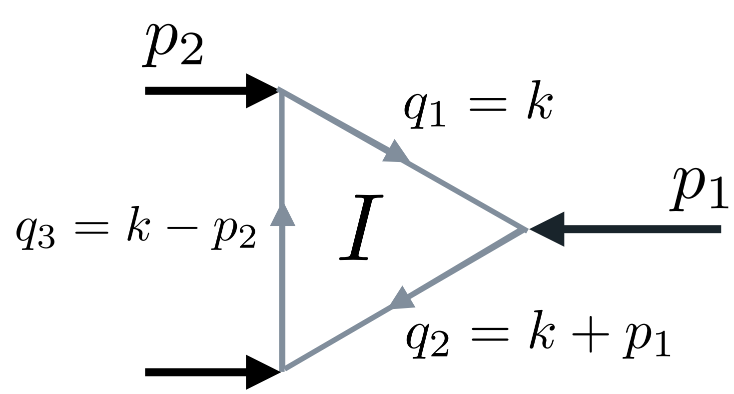

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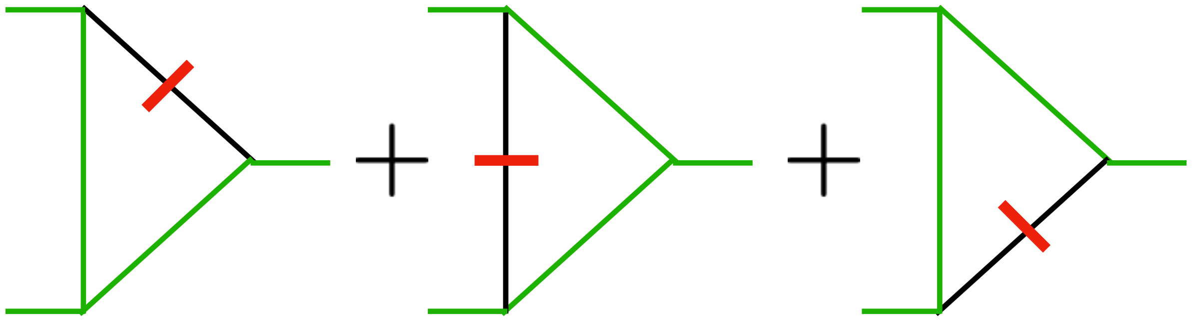

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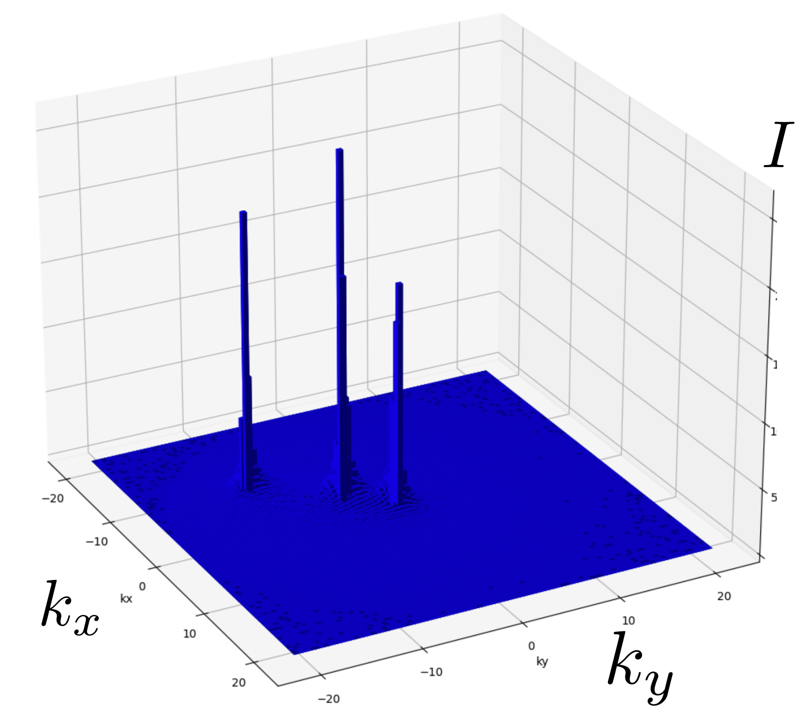

[Originates from $\frac{1}{\left(k^0 - E(\vec{k})\right)\left(k^0 + E(\vec{k})\right)} = \frac{1}{2E(\vec{k})}\left(\frac{1}{E(\vec{k}) - k^0} + \frac{1}{E(\vec{k}) + k^0}\right)$]

Tropical sampling of Loop-Tree Duality integrands

[Inspired from and in collaboration with M. Borinsky, arXiv : 2008.12310]

[Investigation from **Mathijs Fraije**]

$$I^{(\mathrm{LTD})} \supset \int d^3\vec{k}\, \frac{1}{|\vec{k}|\,|\vec{k}+\vec{p_1}|\,|\vec{k}-\vec{p_2}|} \frac{\mathcal{N}(\vec{k})}{\left((E_1+E_2-p_1^0)\,(E_1+E_2+p_1^0)\right)}$$

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Multi-channeling is the canonical approach to flatten these three integrable singularities.

But it would be far better to build a single parameterisation whose Jacobian vanishes simultaneously at *all three points* $\vec{k} = \left\{\vec{0}, -\vec{p}_1, \vec{p}_2\right\}$:

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$$= \int d^3\vec{k} \frac{1}{D_1^{\frac{1}{2}} D_2^{\frac{1}{2}} D_3^{\frac{1}{2}}} f(\vec{k})$$

with **euclidean** propagators :

$$D_1 = \vec{k}\cdot\vec{k}$$

$$D_2 = (\vec{k}+\vec{p_1})\cdot(\vec{k}+\vec{p_1})$$

$$D_3 = (\vec{k}-\vec{p_2})\cdot(\vec{k}-\vec{p_2})$$

Ready for being tropical-sampled ! $= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$

Tropical sampling in momentum space

$$I^{\text{eucl.}}[f] = \int d^{DL}\mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}}$$

Tropical sampling in momentum space

$$I^{\mathrm{eucl.}}[f] = \int d^{DL}\mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}} \propto \int d\mathbf{k} \int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{f(\mathbf{k})}{\left(\sum_e x_e D_e(\mathbf{k})\right)^{\sum_e \nu_e}}$$

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$$\propto \int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{1}{\mathcal{U}(\mathbf{x})^{D/2} \mathcal{V}(\mathbf{x})^{\omega}} \int_0^\infty d\lambda \lambda^{\omega-1} e^{-\lambda} \int \prod_l d^D q_l e^{-q_l^2/2} f\left(\mathbf{k}(\mathbf{x}, \mathbf{q}, \lambda)\right)$$

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Feynman params.

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Quite a bit of algebra, leading to ~ Schwinger param.

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Easy to sample momenta from this PDF **exactly**

Tropical sampling in momentum space

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Tropical sampling [M. Borinsky 2008.12310]

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$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

$$K(\mathbf{z}) \propto \left(\frac{\mathcal{U}^{\text{tr}}(\mathbf{x})}{\mathcal{U}(\mathbf{x})}\right)^{D/2} \left(\frac{\mathcal{V}^{\text{tr}}(\mathbf{x})}{\mathcal{V}(\mathbf{x})}\right)^{\omega}$$

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- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\text{tr}}(x,y) = \max\{x^2, y^3, xy\}$

Tropical sampling in momentum space

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- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\mathrm{tr}}(x,y) = \max\left\{x^2, y^3, xy\right\}$
- Moreover, a systematic algorithm exists to sample according to p^{tr} (note:. sampling dimensionality increases)

Tropical sampling in momentum space

$$I^{\text{eucl.}}[f] = \int d^{DL}\mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}} \propto \int d\mathbf{k} \underbrace{\int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e}}_{\text{Feynman params.}} \frac{f(\mathbf{k})}{\left(\sum_e x_e D_e(\mathbf{k})\right)^{\sum_e \nu_e}}$$

$\omega = \sum_e \nu_e - DL/2$ $\qquad \mathcal{V} = \mathcal{F}/\mathcal{U}$ $\qquad$ Quite a bit of algebra, leading to ~ Schwinger param.

$$\propto \underbrace{\int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{1}{\mathcal{U}(\mathbf{x})^{D/2}\mathcal{V}(\mathbf{x})^{\omega}}}_{\text{Tropical sampling [M. Borinsky 2008.12310]}} \underbrace{\int_0^\infty d\lambda \lambda^{\omega-1} e^{-\lambda} \int \prod_l d^D q_l e^{-q_l^2/2}}_{\text{Easy to sample momenta from this PDF exactly}} f\left(\mathbf{k}(\mathbf{x},\mathbf{q},\lambda)\right)$$

$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z})) \qquad K(\mathbf{z}) \propto \left(\frac{\mathcal{U}^{\text{tr}}(\mathbf{x})}{\mathcal{U}(\mathbf{x})}\right)^{D/2} \left(\frac{\mathcal{V}^{\text{tr}}(\mathbf{x})}{\mathcal{V}(\mathbf{x})}\right)^{\omega}$$

- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\text{tr}}(x,y) = \max\left\{x^2, y^3, xy\right\}$
- Moreover, a systematic algorithm exists to sample according to p^{tr} (note:. sampling dimensionality increases)
- One can show that the ratio K = $\frac{p^{\text{tr}}}{p}$ is bounded from below and above when $p \equiv \mathcal{U}$ or $\mathcal{V}$

Tropical sampling in momentum space

$$I^{\text{eucl.}}[f] = \int d^{DL}\mathbf{k} \frac{f(\mathbf{k})}{\prod_e D_e(\mathbf{k})^{\nu_e}} \propto \int d\mathbf{k} \int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{f(\mathbf{k})}{\left(\sum_e x_e D_e(\mathbf{k})\right)^{\sum_e \nu_e}}$$

Feynman params.

$$\omega = \sum_e \nu_e - DL/2 \qquad \mathcal{V} = \mathcal{F}/\mathcal{U}$$

Quite a bit of algebra, leading to ~ Schwinger param.

$$\propto \int_{\mathbb{P}_+^E} \prod_e x_e^{\nu_e} \frac{1}{\mathcal{U}(\mathbf{x})^{D/2}\mathcal{V}(\mathbf{x})^{\omega}} \int_0^\infty d\lambda \lambda^{\omega-1} e^{-\lambda} \int \prod_l d^D q_l e^{-q_l^2/2} f\left(\mathbf{k}(\mathbf{x},\mathbf{q},\lambda)\right)$$

Tropical sampling [M. Borinsky 2008.12310]

Easy to sample momenta from this PDF **exactly**

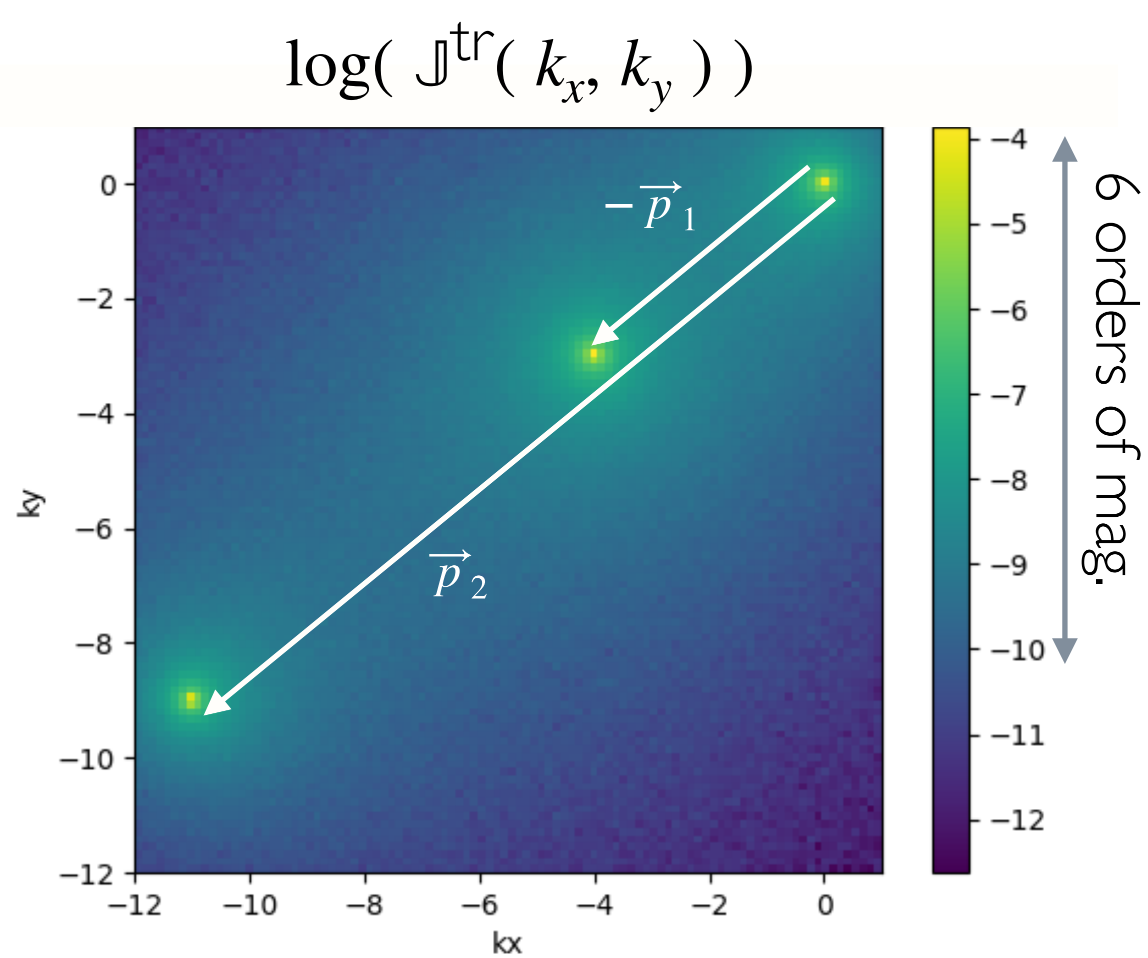
$$= \int_{[0,1]^{(DL+2|E|+1)}} d\mathbf{z} \underbrace{K(\mathbf{z})}_{\text{bounded}} f(\mathbf{k}(\mathbf{z}))$$

$$K(\mathbf{z}) \propto \left(\frac{\mathcal{U}^{\text{tr}}(\mathbf{x})}{\mathcal{U}(\mathbf{x})}\right)^{D/2} \left(\frac{\mathcal{V}^{\text{tr}}(\mathbf{x})}{\mathcal{V}(\mathbf{x})}\right)^{\omega}$$

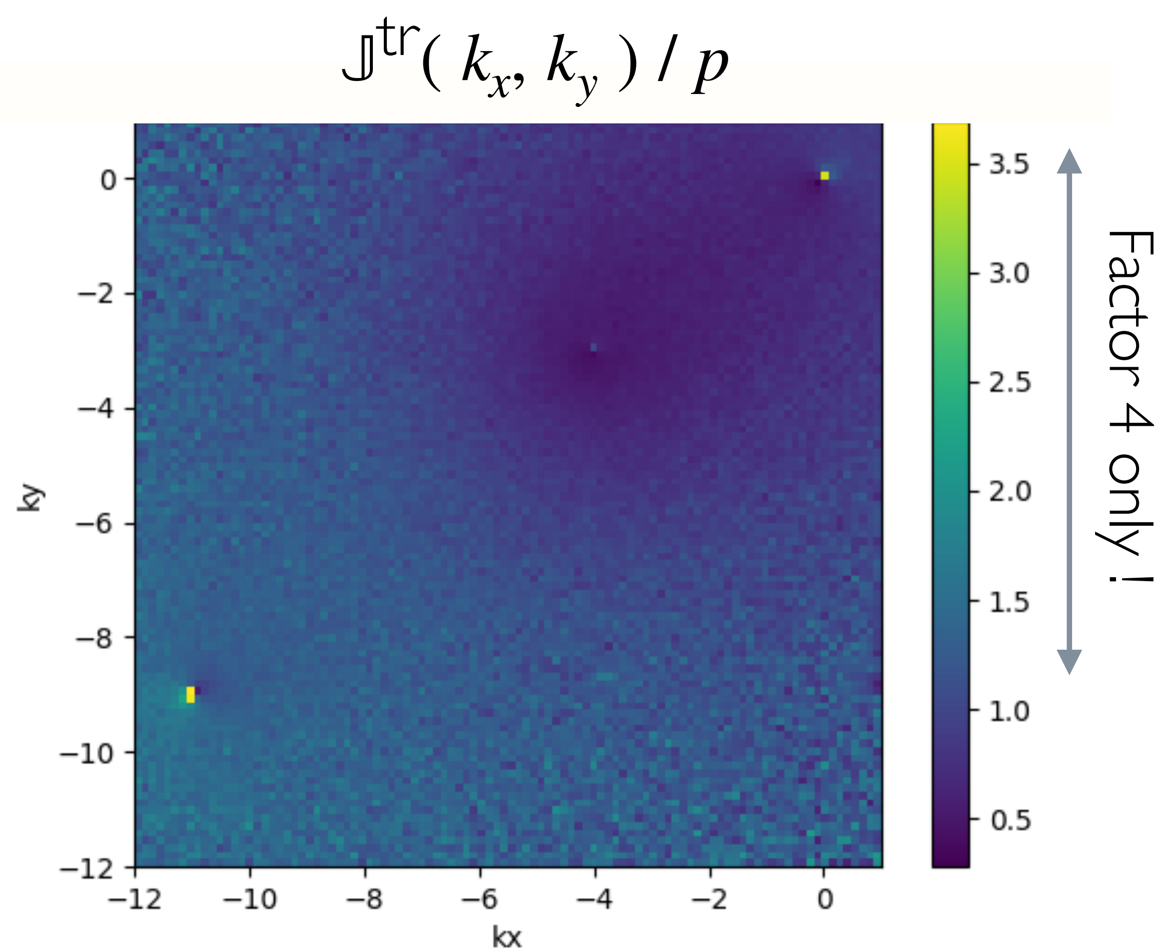
- Tropical sampling idea: let $p(x,y) = 12x^2 - 7y^3 + 5xy$ for which $p^{\text{tr}}(x,y) = \max\left\{x^2, y^3, xy\right\}$
- Moreover, a systematic algorithm exists to sample according to p^{tr} (note:. sampling dimensionality increases)
- One can show that the ratio K = $\frac{p^{\text{tr}}}{p}$ is bounded from below and above when $p \equiv \mathcal{U}$ or $\mathcal{V}$
- M. Fraije and M. Borinsky will soon publish a **Rust** implementation of the map: $\phi^{\text{tr}}[\Gamma](\mathbf{z}) \to (K(\mathbf{z}), \mathbf{k}(\mathbf{z}))$

Let's apply this approach to our example.

In 2D with tropical sampling for removing blow ups from $p := \dfrac{1}{|\vec{k}||\vec{k}+\vec{p}_1||\vec{k}-\vec{p}_2|}$:



Log of tropical ~sampling density



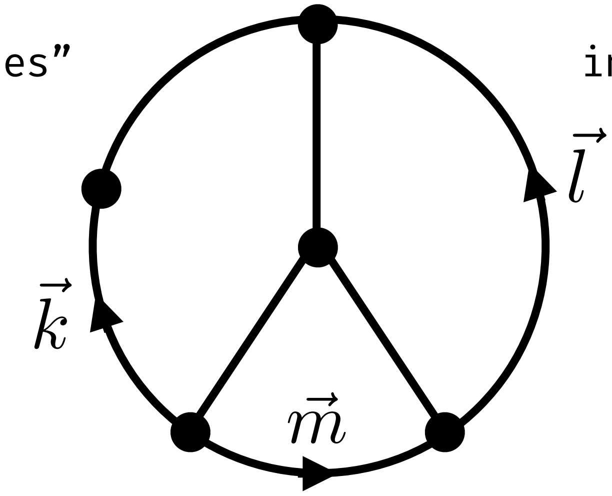
Integrable singularities conquered !

- <u>Many details</u> of this approach <u>omitted</u> here. In our implementation: **arbitrary numerators** supported!

Tropical sampling results

“scalar vacuum mercedes”

internal mass: $m = 1$



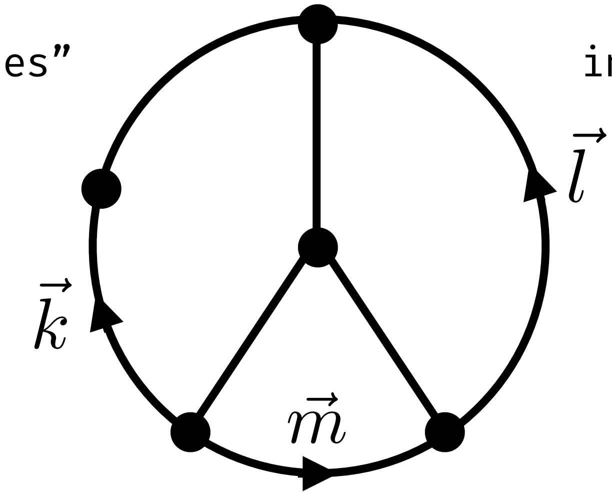
$$\int \left[d\Omega_k |\vec{k}|^2 d|\vec{k}|\right] \times \left[d\Omega_l |\vec{l}|^2 d|\vec{l}|\right] \times \left[d\Omega_m |\vec{m}|^2 d|\vec{m}|\right]$$

Sampling Strategy	# Dim.	Im. Central value	Pull
Analytic result	-	-3.0525714828514e-7	ref.
Prod. of spherical params†	9	-2.990(50)e-7	-1.2σ

†: 5 x 2M samples, 10 μs/sample

Credits: **Mathijs Fraaije**

Tropical sampling results



$$\int \left[d\Omega_k |\vec{k}|^2 d|\vec{k}|\right] \times \left[d\Omega_l |\vec{l}|^2 d|\vec{l}|\right] \times \left[d\Omega_m |\vec{m}|^2 d|\vec{m}|\right]$$

$$1 = \sum_{\text{lmb}} \left[\frac{\prod_{i \in \text{lmb}} E_i^{-\alpha}}{\left(\sum_{\text{lmb}'} \prod_{j \in \text{lmb}'} E_j^{-\alpha}\right)} \right]$$

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Tropical sampling results

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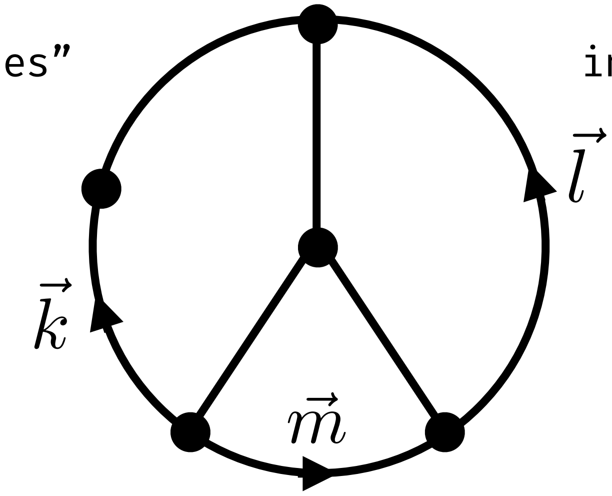

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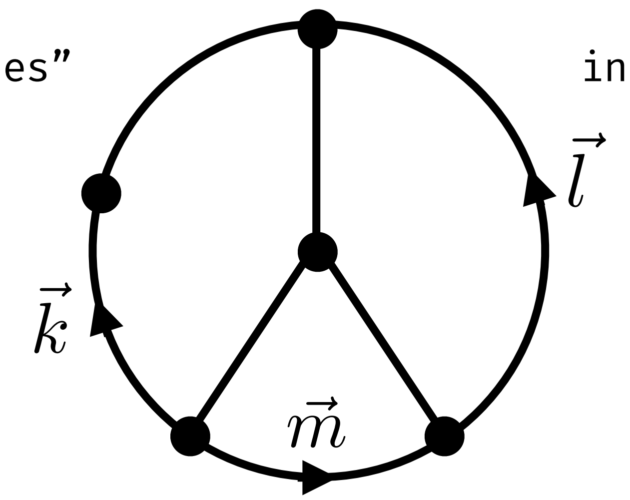
Tropical sampling results

“scalar vacuum mercedes”

internal mass: $m = 1$



$$\int \left[d\Omega_k|\vec{k}|^2 d|\vec{k}|\right] \times \left[d\Omega_l|\vec{l}|^2 d|\vec{l}|\right] \times \left[d\Omega_m|\vec{m}|^2 d|\vec{m}|\right]$$

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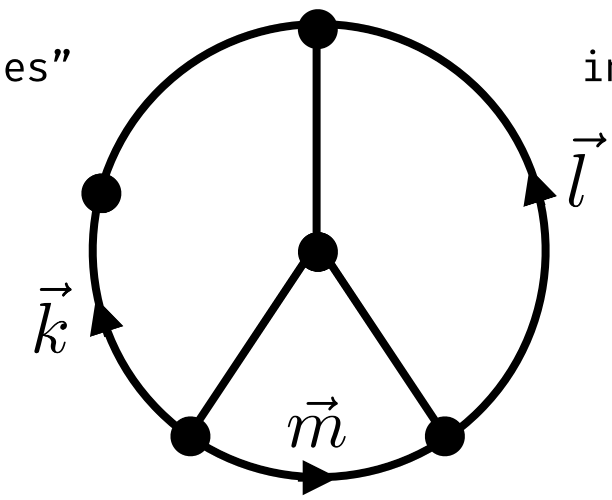
pySecDec (QMC)	7	-3.0525690(21)e-7	+1.2σ

†: 5 x 2M samples, 10 μs/sample

Credits: **Mathijs Fraaije**

Tropical sampling results

“scalar vacuum mercedes”



internal mass: $m = 1$

$$\boxed{\int \left[d\Omega_k |\vec{k}|^2 d|\vec{k}|\right] \times \left[d\Omega_l |\vec{l}|^2 d|\vec{l}|\right] \times \left[d\Omega_m |\vec{m}|^2 d|\vec{m}|\right]}$$

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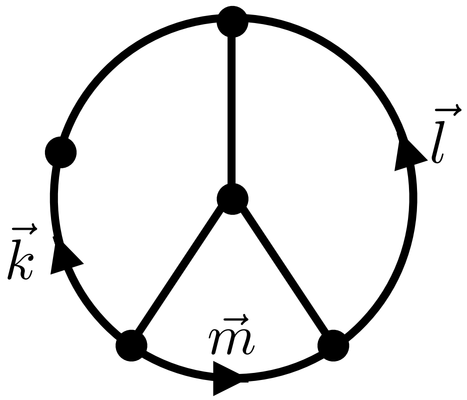
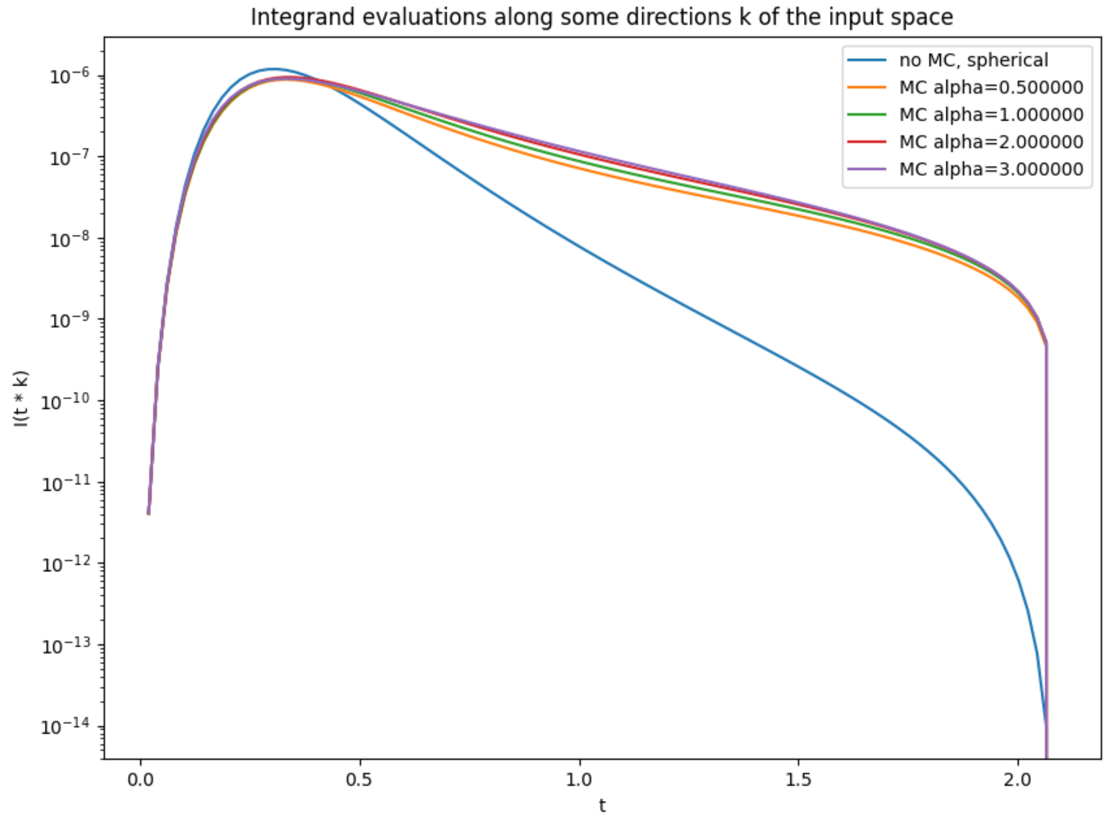
pySecDec (QMC)	7	-3.0525690(21)e-7	**+1.2σ**
Neural Importance Sampling	dep. on sampling	**Theo Heimel** prelim: 2x+ var. reduction !	**\o/**

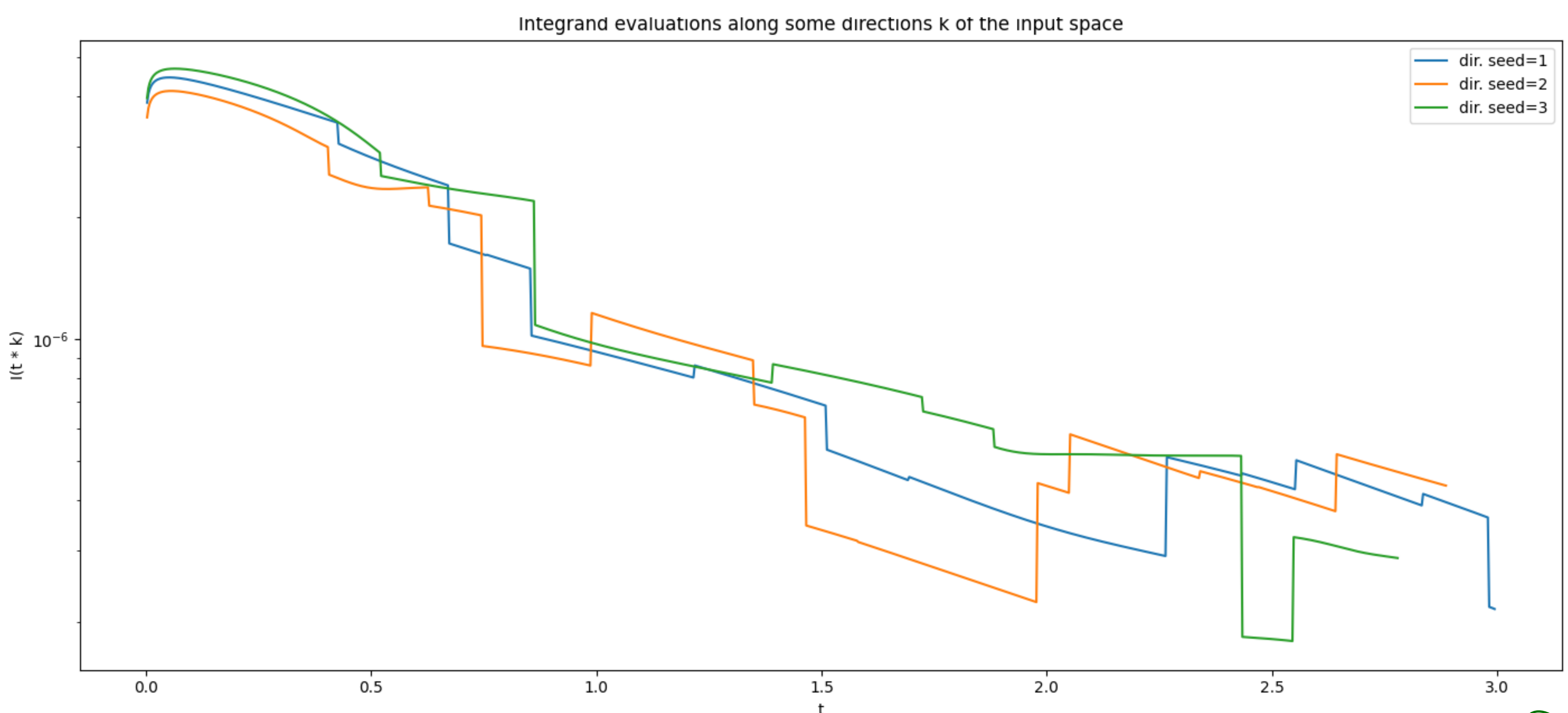
†: 5 x 2M samples, 10 μs/sample

Credits: **Mathijs Fraaije**

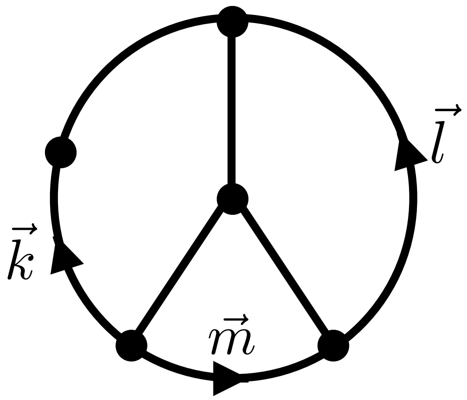
Tropical sampling PDF



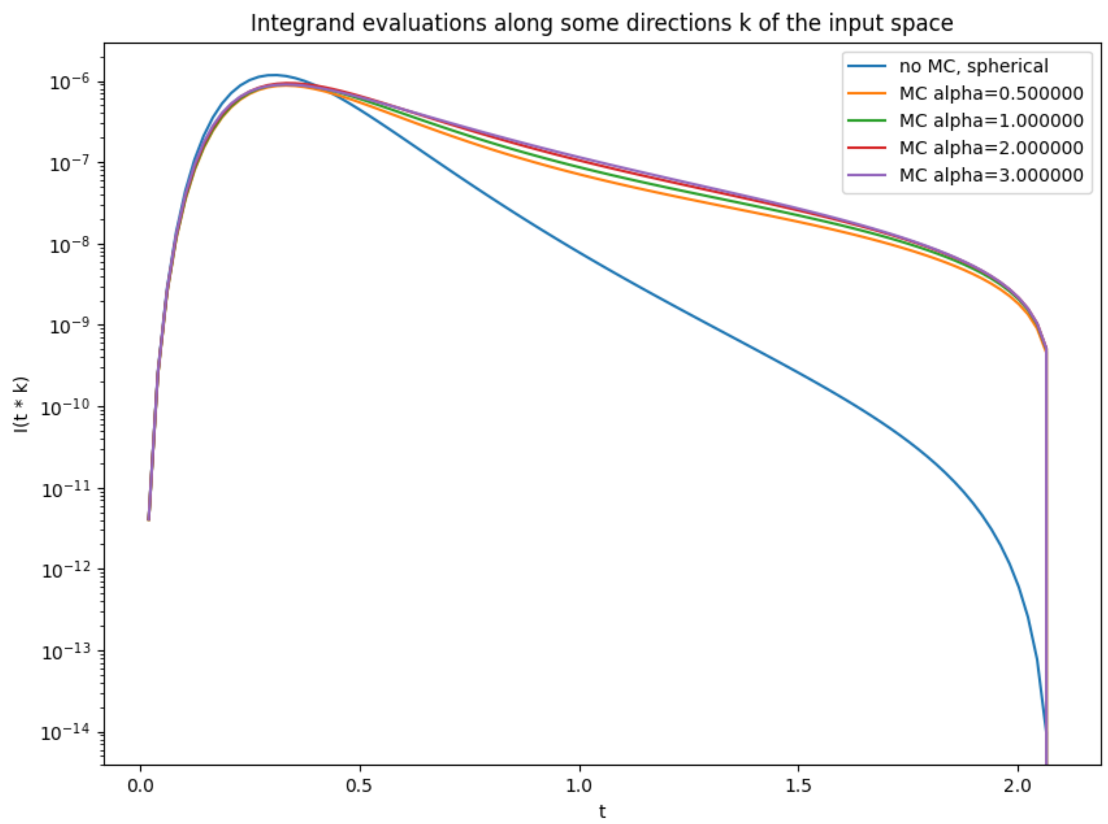


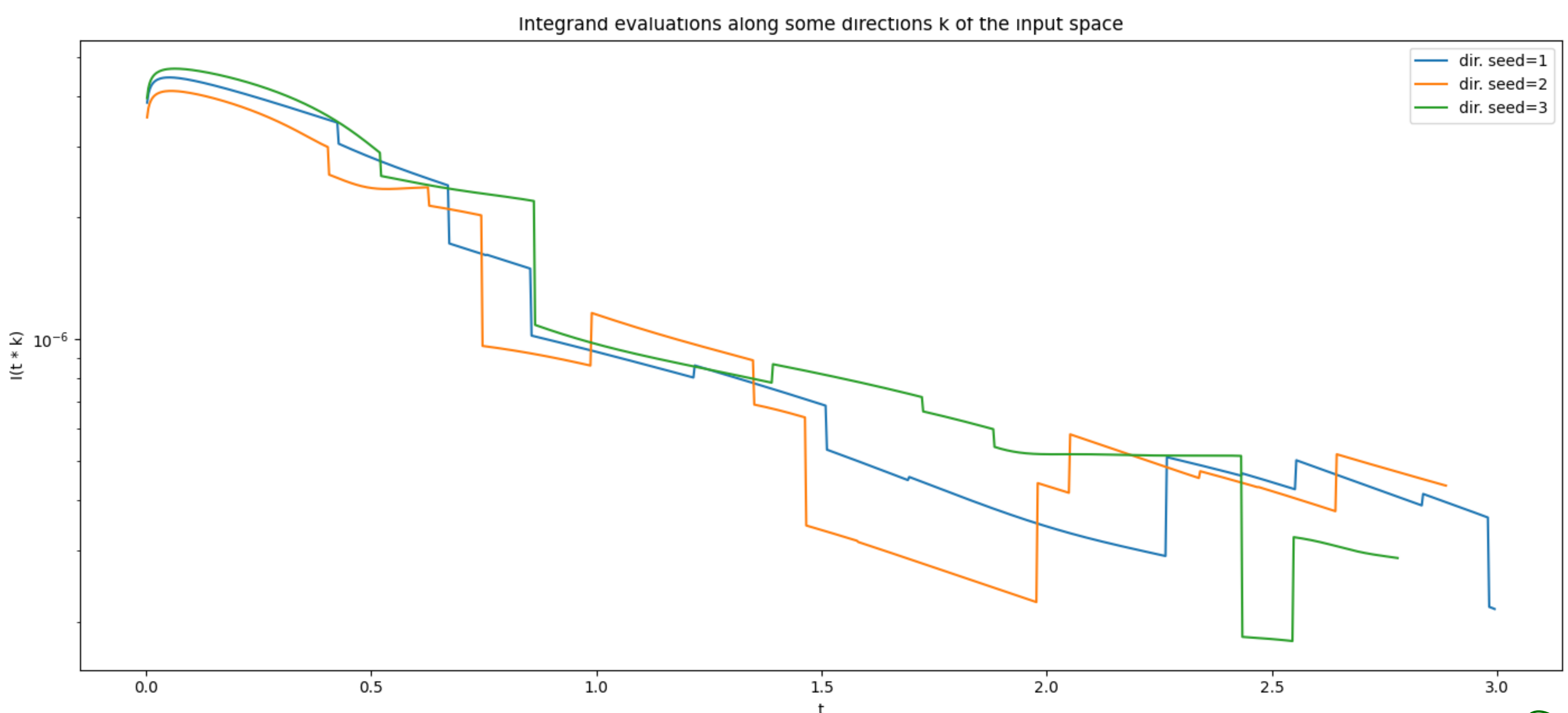


Credits: **Mathijs Fraaije**

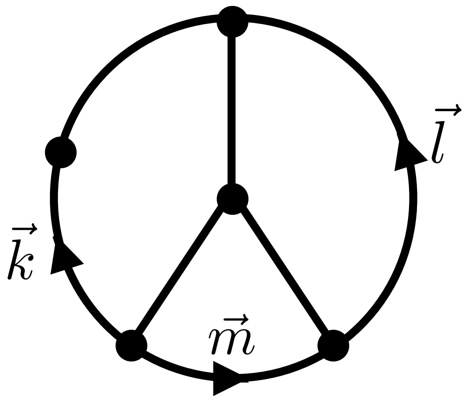


Tropical sampling PDF



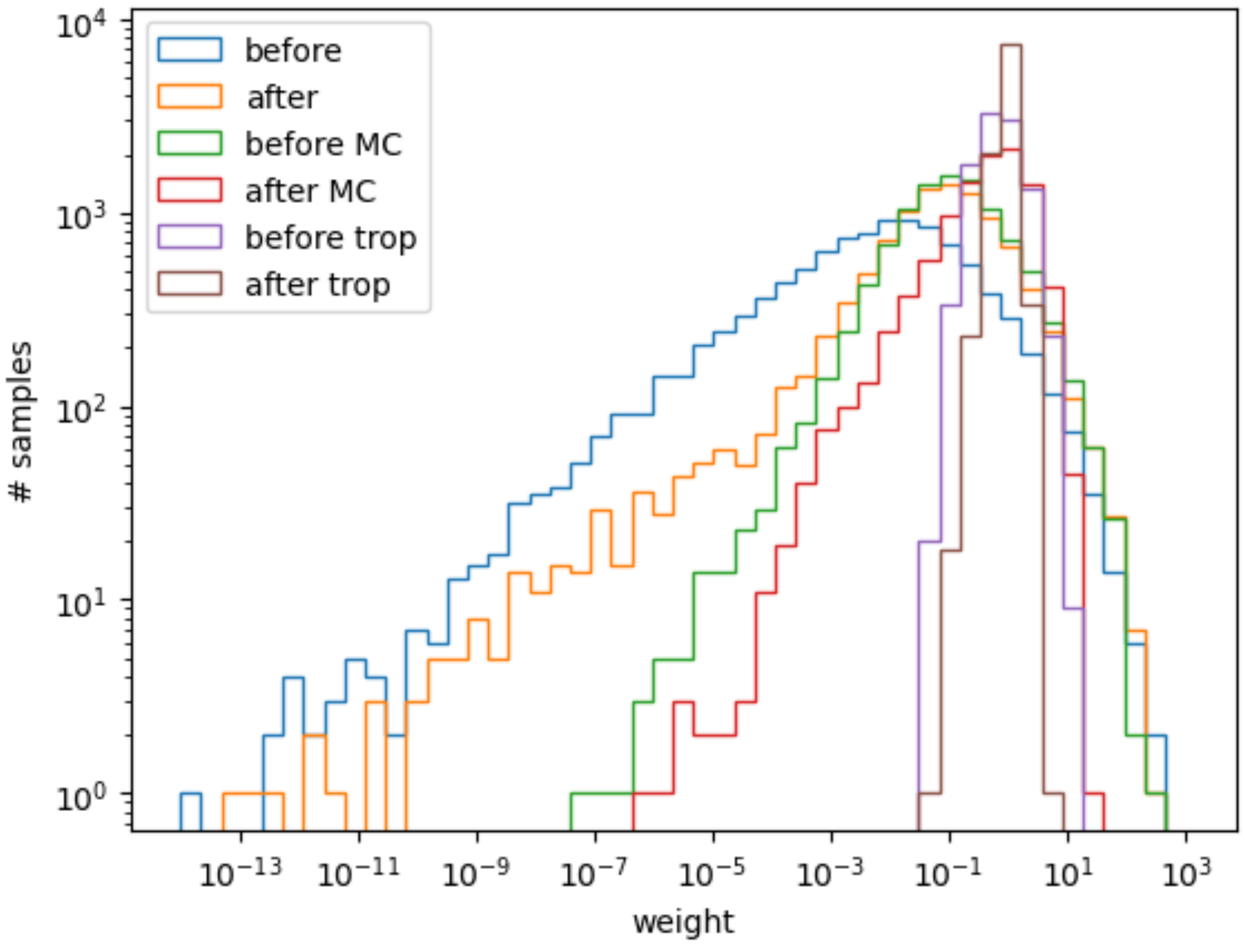


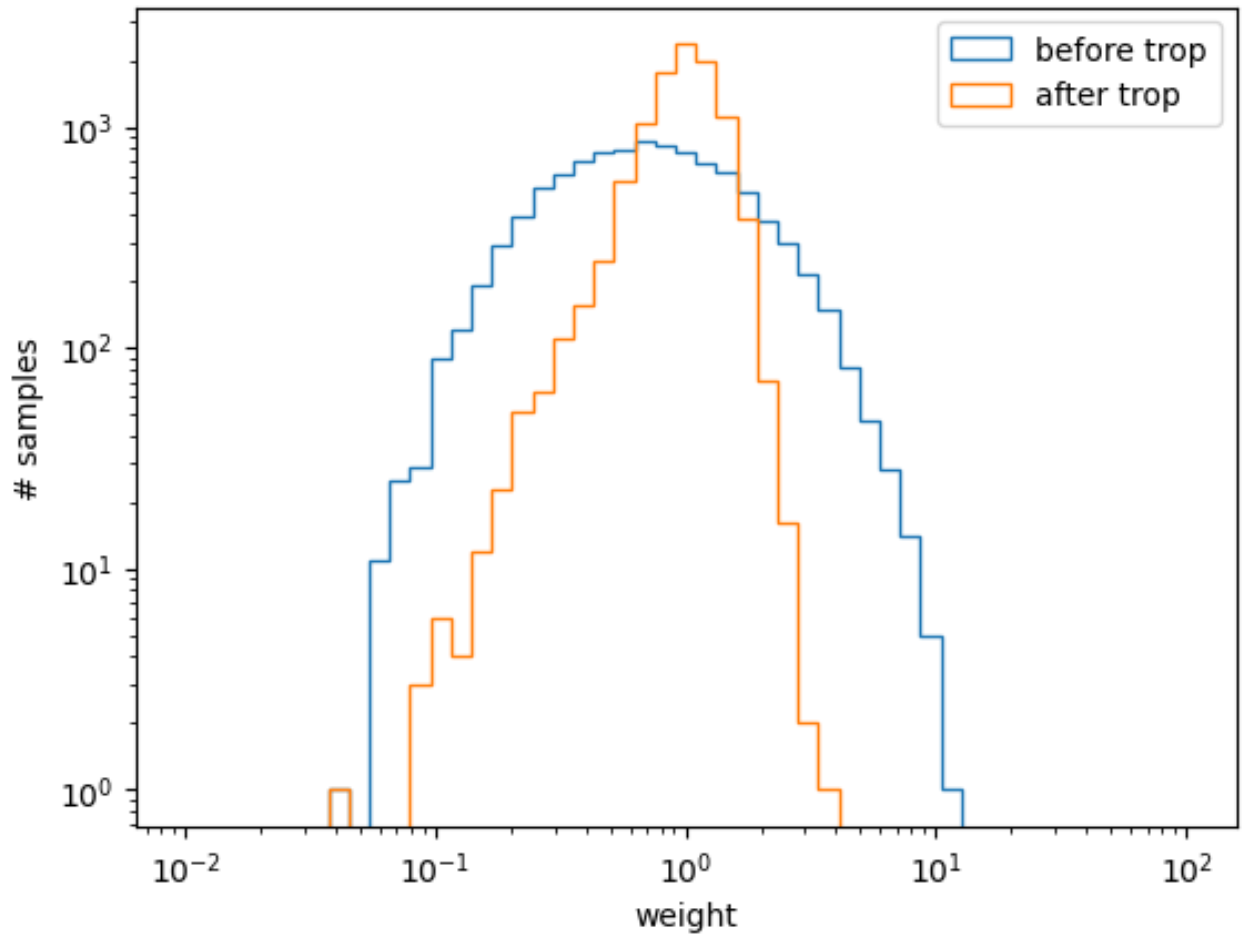
Credits: **Mathijs Fraaije**



Neural Importance Sampling

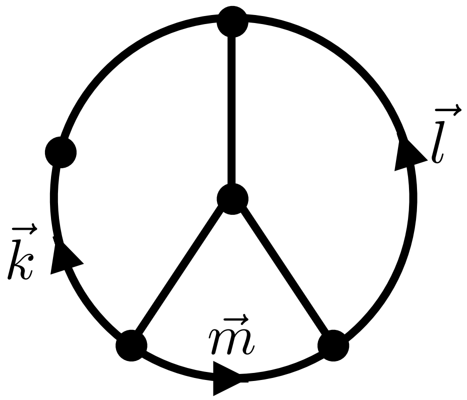
Preliminary





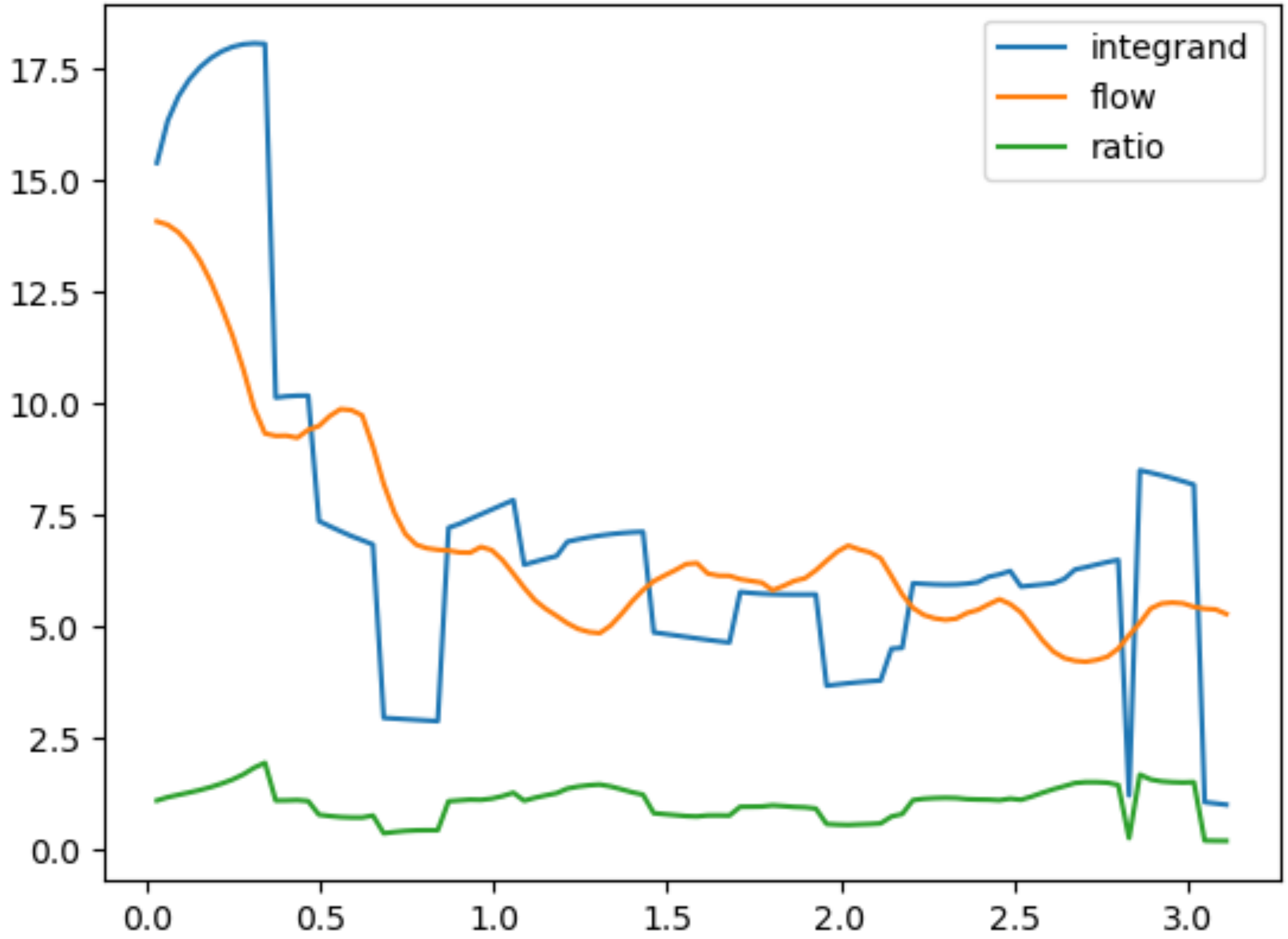
Easy ~2x variance reduction !

Credits: **Theo Heimel**



Neural Importance Sampling

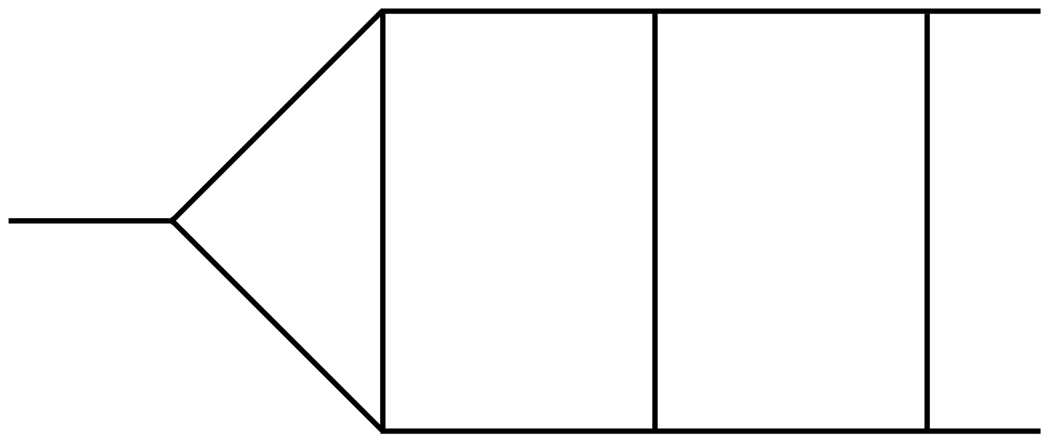
Preliminary



NIS seemingly fine with irregular integrand!

Credits: **Theo Heimel**

Tropical sampling results

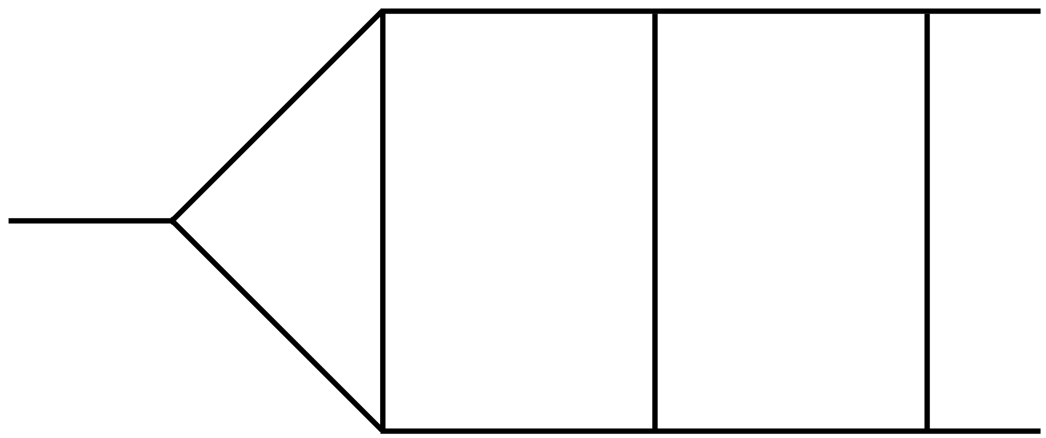


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/18$)

N Samples	Naive	Tropical
0.01M 30 ms	3.51(63)e-8 **18%**	
0.1M 0.3 s	3.78(24)e-8 **6%**	
1M 3 s	3.99(11)e-8 **2.7%**	
10M 30 s	4.045(36)e-8 **0.9%**	

Credits: Mathijs Fraaije

Tropical sampling results

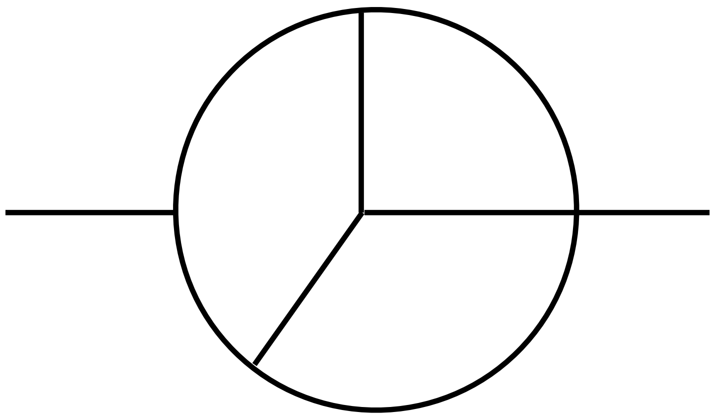


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/18$)

N Samples	Naive	Tropical
0.01M 30 ms	3.51(63)e-8 **18%**	4.050(35)e-8 **0.9%**
0.1M 0.3 s	3.78(24)e-8 **6%**	4.030(11)e-8 **0.3%**
1M 3 s	3.99(11)e-8 **2.7%**	4.0379(35)e-8 **0.09%**
10M 30 s	4.045(36)e-8 **0.9%**	4.0358(11)e-8 **0.03%**

Credits: Mathijs Fraaije

Tropical sampling results

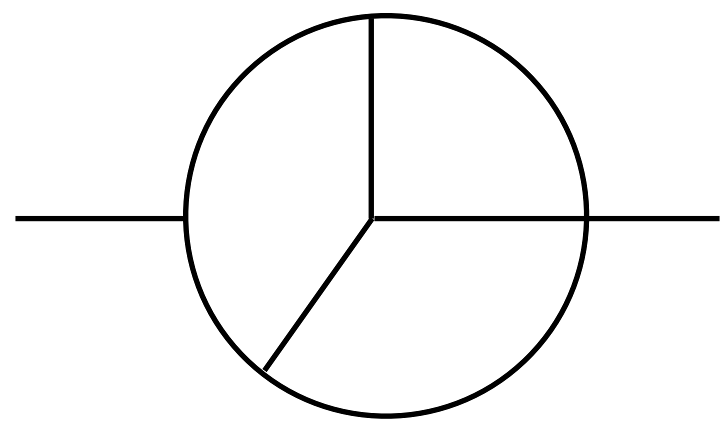


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/14$)

N Samples	Naive	Tropical
0.01M 30 ms	8.9(2.9)e-7 **33%**	
0.1M 0.3 s	3.5(1.0)e-6 **29%**	
1M 3 s	5.6(1.2)e-6 **22%**	
10M 30 s	1.23(41)e-5 **34%**	

Credits: Mathijs Fraaije

Tropical sampling results

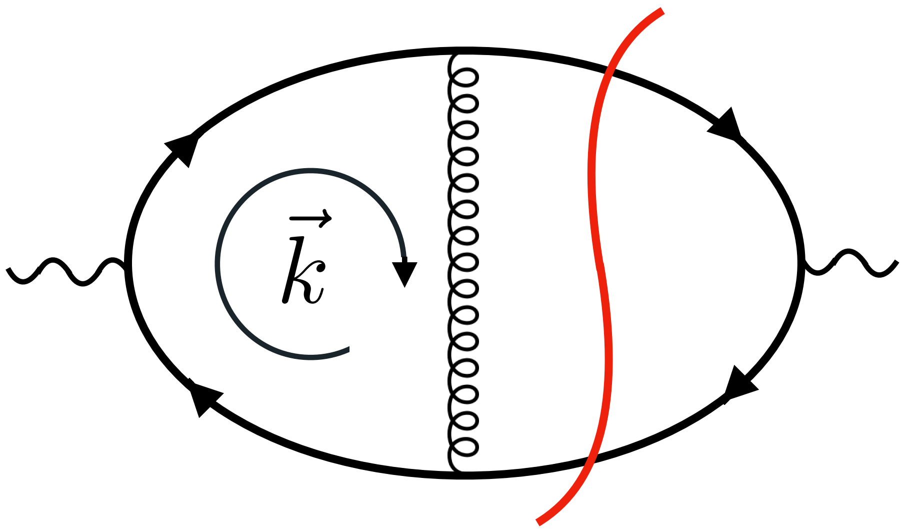


($f(\mathbf{k}) = 1$, denominator powers: $\nu_e = 11/14$)

N Samples	Naive	Tropical
0.01M 30 ms	8.9(2.9)e-7 33%	9.403(60)e-6 0.6%
0.1M 0.3 s	3.5(1.0)e-6 29%	9.518(19)e-6 0.2%
1M 3 s	5.6(1.2)e-6 22%	9.4984(60)e-6 0.06%
10M 30 s	1.23(41)e-5 34%	9.4986(19)e-6 0.02%

Credits: Mathijs Fraaije

Localised Renormalisation: BPHZ



$$\lim_{|\vec{k}|\to\infty} I^{(\text{Local Unitarity})} \to \infty$$

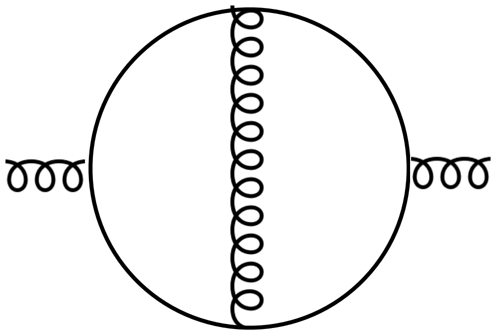
Localised Renormalisation

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

$$R(\Gamma) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$$

Localised Renormalisation

[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

R (Γ = ) $= \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$

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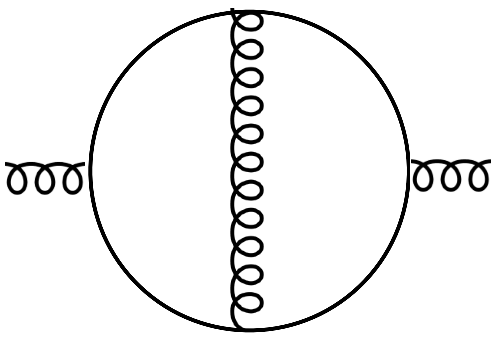
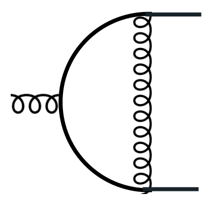
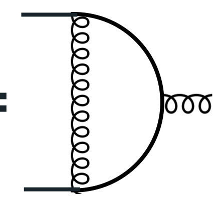
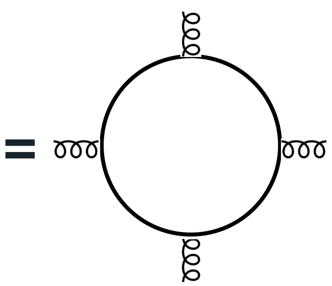
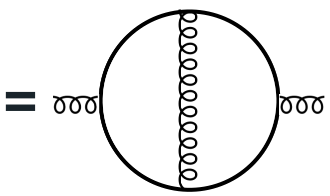
R (Γ = ) $= \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K \left(\sum_{S \in W(\gamma) \setminus \gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$

UV subgraphs :

$\mathrm{dod}\left(\gamma_{\{1,2,3\}}\right) = 0$

$\mathrm{dod}\left(\gamma_4\right) = 2$

$\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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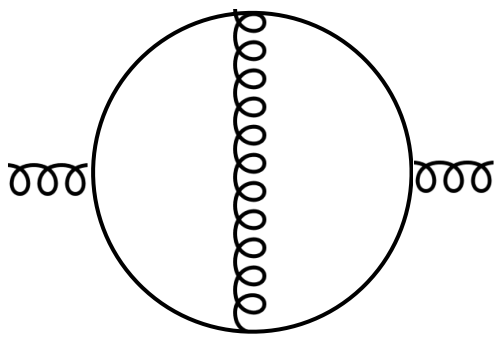
$R\Big(\Gamma =$  $\Big) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S \in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma')\right)$

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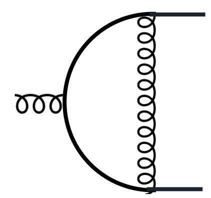
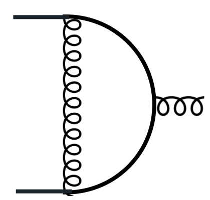
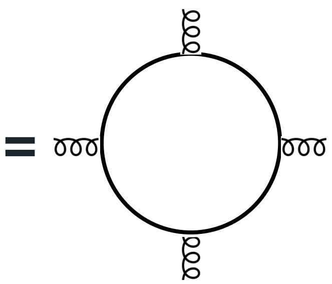
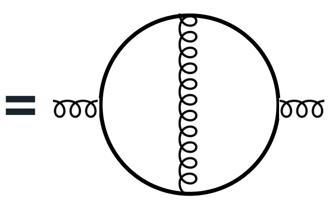
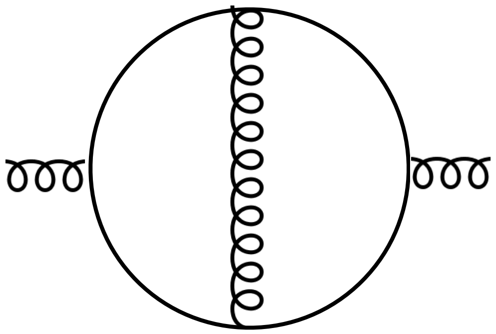
$\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$R(\Gamma) = \Gamma - K(\gamma_1) * \Gamma\setminus\gamma_1 - K(\gamma_2) * \Gamma\setminus\gamma_2 - K(\gamma_3) * \Gamma\setminus\gamma_3 - K(\gamma_4) * \Gamma\setminus\gamma_4$$

$$+ K\big(K(\gamma_1) * \Gamma\setminus\gamma_1\big) + K\big(K(\gamma_2) * \Gamma\setminus\gamma_2\big) + K\big(K(\gamma_3) * \Gamma\setminus\gamma_3\big)$$

Localised Renormalisation

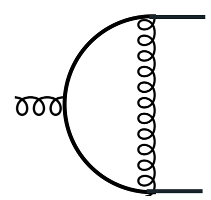
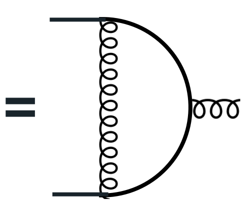
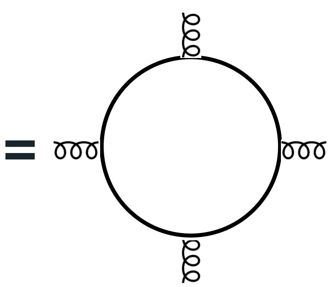
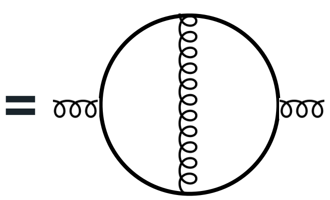
[Capatti, VH, Ruijl, arxiv : 2203.11038] [BPHZ refs.]

$R\Big(\Gamma =$  $\Big) = \sum_{S \in W(\Gamma)} \Gamma \setminus S * \prod_{\gamma \in S} Z(\gamma), \quad Z(\gamma) = -K\left(\sum_{S \in W(\gamma)\setminus\gamma} \gamma \setminus S * \prod_{\gamma' \in S} Z(\gamma') \right)$

UV subgraphs :

$\mathrm{dod}\left(\gamma_{\{1,2,3\}}\right) = 0$

$\mathrm{dod}\left(\gamma_4\right) = 2$

$\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$R(\Gamma) = \Gamma - K(\gamma_1)*\Gamma\setminus\gamma_1 - K(\gamma_2)*\Gamma\setminus\gamma_2 - K(\gamma_3)*\Gamma\setminus\gamma_3 - K(\gamma_4)*\Gamma\setminus\gamma_4$$
$$+ K(K(\gamma_1)*\Gamma\setminus\gamma_1) + K(K(\gamma_2)*\Gamma\setminus\gamma_2) + K(K(\gamma_3)*\Gamma\setminus\gamma_3)$$

What is the operator $K(\gamma)$? Anything we want ! so long as it:

- Locally cancels UV divergences of γ, even in the presence of nestings
- Yields results immediately renormalised in the chosen scheme $(\overline{\mathrm{MS}} + \mathrm{OS})$
- Minimal analytics: at most single-scale all-massive vacuum integrals

Local renormalisation operator K

Our solution: $\mathsf{K}(\gamma) := \mathsf{T}(\gamma)$

$\mathsf{T}(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

LOCAL RENORMALISATION OPERATOR K

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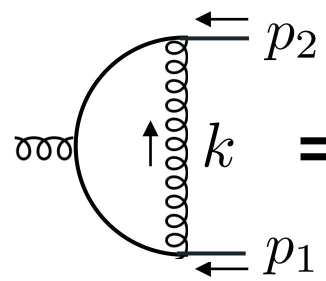
$\gamma_1 =$ 

$= \dfrac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$

Local renormalisation operator K

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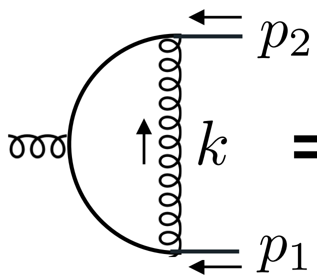
γ_1 =



$$= \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\ (k-p_1)^2 - m^2\ )(\ k^2\ )(\ (k+p_2)^2 - m^2\ )}$$

$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k, \lambda p_1, \lambda p_2, \lambda m)}{(k-\lambda p_1)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2))(k^2 - m_{\mathrm{UV}}^2)((k+\lambda p_2)^2 - m_{\mathrm{UV}}^2 - \lambda^2(m^2 - m_{\mathrm{UV}}^2)\ )}$$

Local renormalisation operator K

Our solution: $\mathrm{K}(\gamma) := \mathrm{T}(\gamma)$

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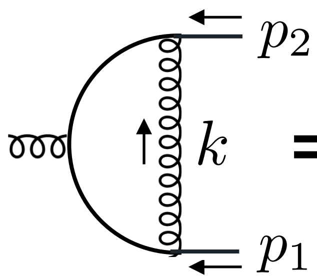
$\gamma_1 =$ 

$= \dfrac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\,(k-p_1)^2 - m^2\,)(\,k^2\,)(\,(k+p_2)^2 - m^2\,)}$

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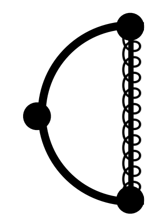
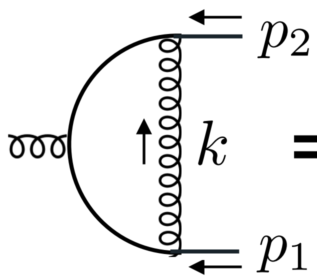
$$\mathrm{T}(\gamma) = \mathrm{T}_{\mathrm{dod}(\gamma)}(\gamma^{\lambda}) = \sum_{j=0}^{\mathrm{dod}(\gamma)} \frac{1}{j!} \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} \gamma^{\lambda} \bigg|_{\lambda=0}, \qquad \mathrm{T}_0(\gamma_1) = \frac{\mathcal{N}(k,0,0,0)}{(k^2 - m_{\mathrm{UV}}^2)^3} \sim$$



Local renormalisation operator K

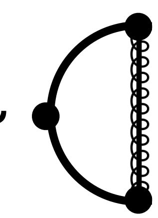
Our solution: $K(\gamma) := T(\gamma) - [T](\gamma)$

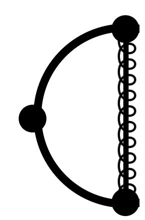
$T(\gamma)$:= **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

$\gamma_1 =$ 

$= \dfrac{\mathcal{N}_{\gamma_1}(k,p_1,p_2,m)}{(\ (k-p_1)^2-m^2\ )(\ k^2\ )(\ (k+p_2)^2-m^2\ )}$

$$\gamma_1^{\lambda} := \frac{\mathcal{N}_{\gamma_1}(k,\lambda p_1,\lambda p_2,\lambda m)}{(k-\lambda p_1)^2-m_{\mathrm{UV}}^2-\lambda^2(m^2-m_{\mathrm{UV}}^2))(k^2-m_{\mathrm{UV}}^2)((k+\lambda p_2)^2-m_{\mathrm{UV}}^2-\lambda^2(m^2-m_{\mathrm{UV}}^2)\ )}$$

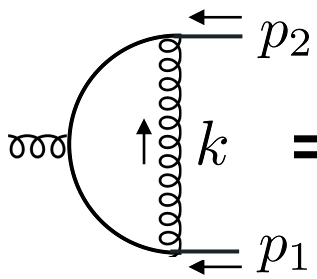
$T(\gamma) = T_{\mathrm{dod}(\gamma)}(\gamma^{\lambda}) = \displaystyle\sum_{j=0}^{\mathrm{dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma^{\lambda}\Big|_{\lambda=0}$, $\quad T_0(\gamma_1) = \dfrac{\mathcal{N}(k,0,0,0)}{(k^2-m_{\mathrm{UV}}^2)^3} \sim$ 

$[T](\gamma)$:= **Integrated CT** , $[T](\gamma_1) = \left(\dfrac{\mu_r^2}{4\pi e^{-\gamma_{\mathrm{E}}}}\right)^{\epsilon} \displaystyle\int d^{4-2\epsilon}k$  $= \displaystyle\sum_{k=-\infty}^{+\infty} \alpha_k \epsilon^k$

Local renormalisation operator K

Our solution: $\mathrm{K}(\gamma) := \mathrm{T}(\gamma) - [\mathrm{T}](\gamma) + \delta^{\overline{\mathrm{MS}}+\mathrm{OS}}(\gamma)$

$\mathrm{T}(\gamma) :=$ **Local CT** : Taylor expansion around the "UV point" up to $\mathrm{dod}(\gamma)$

$\gamma_1 =$ 

$$= \frac{\mathcal{N}_{\gamma_1}(k, p_1, p_2, m)}{(\,(k-p_1)^2 - m^2\,)(\,k^2\,)(\,(k+p_2)^2 - m^2\,)}$$

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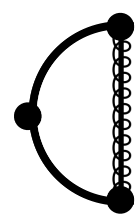
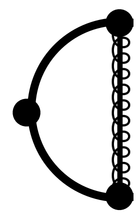
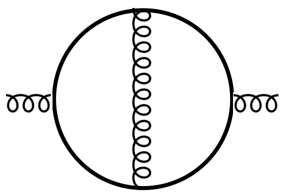
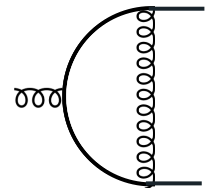
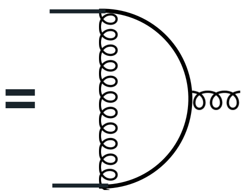
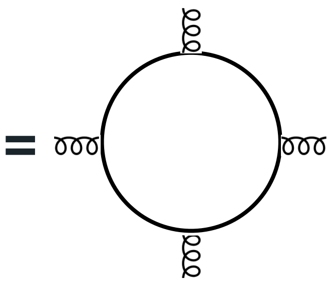
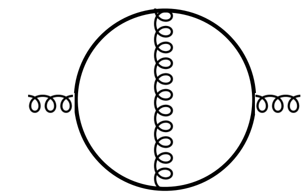
$$\mathrm{T}(\gamma) = \mathrm{T}_{\mathrm{dod}(\gamma)}(\gamma^{\lambda}) = \sum_{j=0}^{\mathrm{dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma^{\lambda}\Big|_{\lambda=0}, \qquad \mathrm{T}_0(\gamma_1) = \frac{\mathcal{N}(k,0,0,0)}{(k^2 - m_{\mathrm{UV}}^2)^3} \sim$$ 

$[\mathrm{T}](\gamma) :=$ **Integrated CT** , $[\mathrm{T}](\gamma_1) = \left(\frac{\mu_r^2}{4\pi e^{-\gamma_{\mathrm{E}}}}\right)^{\epsilon} \int d^{4-2\epsilon}k$  $= \sum_{k=-\infty}^{+\infty} \alpha_k \epsilon^k$

$\delta^{\mathrm{X}}(\gamma) :=$ **Renormalisation CT** in scheme X , $(-[\mathrm{T}] + \delta^{\overline{\mathrm{MS}}}) := \bar{\mathrm{K}}$, $\bar{\mathrm{K}}(\gamma_1) = \sum_{k=0}^{+\infty} \alpha_k \epsilon^k$

R-operator unfolding

$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$R(\Gamma) = \Gamma - T_0(\gamma_1) * \Gamma\backslash\gamma_1 - T_0(\gamma_2) * \Gamma\backslash\gamma_2 - T_0(\gamma_3) * \Gamma\backslash\gamma_3 - T_2(\gamma_4) * \Gamma\backslash\gamma_4$$

$$+ T_2\big(T_0(\gamma_1) * \Gamma\backslash\gamma_1\big) + T_2\big(T_0(\gamma_2) * \Gamma\backslash\gamma_2\big) + T_2\big(T_0(\gamma_3) * \Gamma\backslash\gamma_3\big) + \bar{K} \text{ terms}$$

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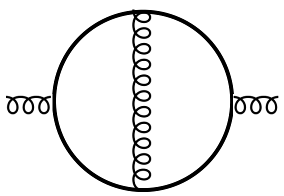
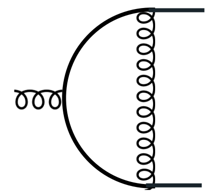
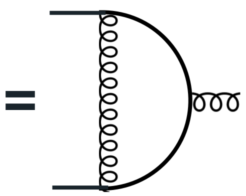
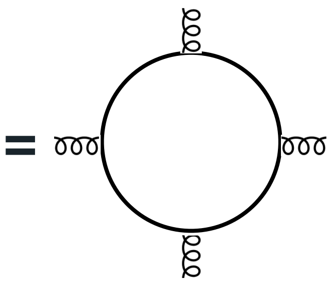
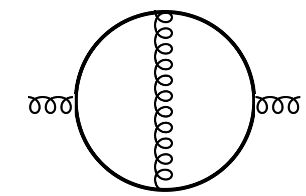
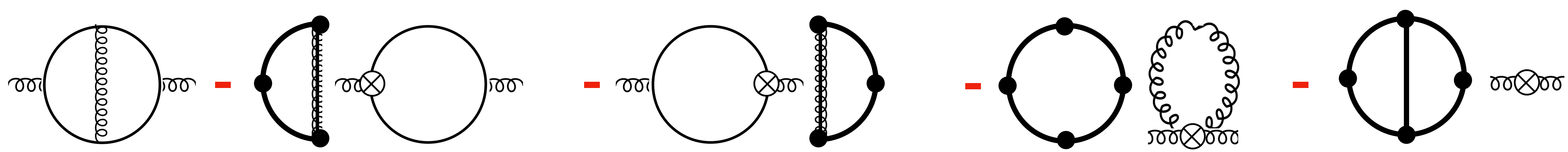
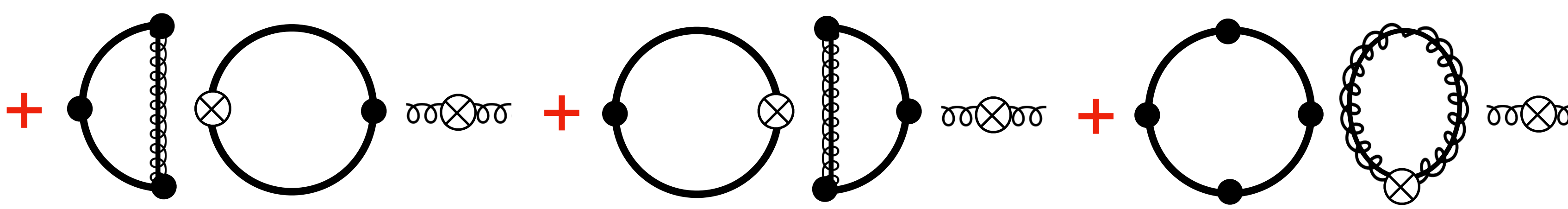
$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

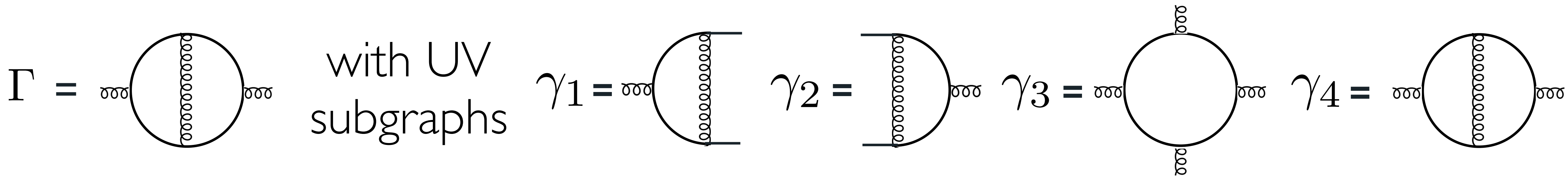
$$R(\Gamma) = \Gamma - T_0(\gamma_1) * \Gamma\backslash\gamma_1 - T_0(\gamma_2) * \Gamma\backslash\gamma_2 - T_0(\gamma_3) * \Gamma\backslash\gamma_3 - T_2(\gamma_4) * \Gamma\backslash\gamma_4$$

$$+ T_2\left(T_0(\gamma_1) * \Gamma\backslash\gamma_1\right) + T_2\left(T_0(\gamma_2) * \Gamma\backslash\gamma_2\right) + T_2\left(T_0(\gamma_3) * \Gamma\backslash\gamma_3\right) + \bar{K} \text{ terms}$$

= 

 + $\bar{K}$ terms

R-OPERATOR UNFOLDING



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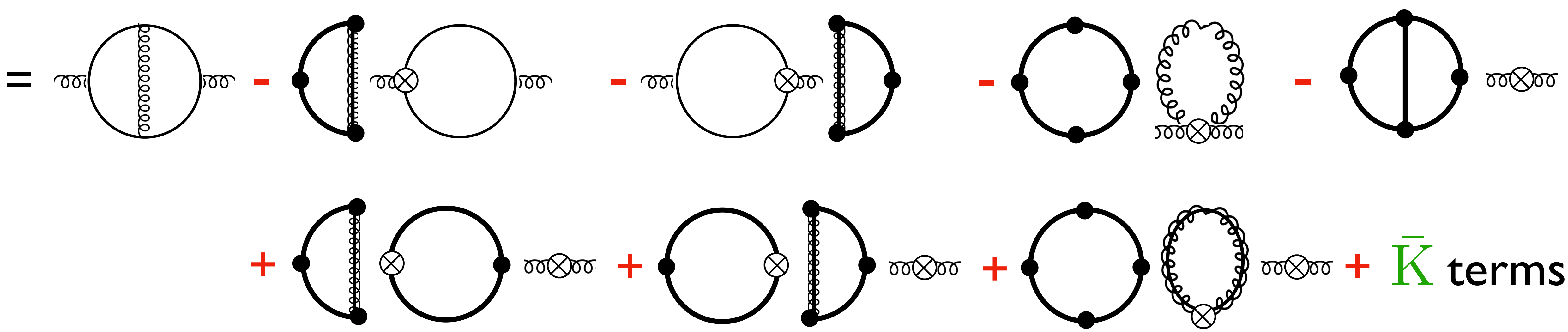
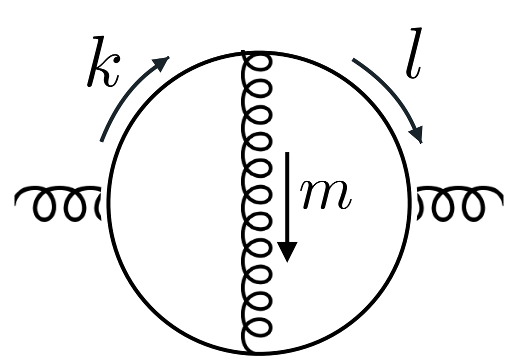
$$+T_2\left(T_0(\gamma_1)*\Gamma\backslash\gamma_1\right) + T_2\left(T_0(\gamma_2)*\Gamma\backslash\gamma_2\right) + T_2\left(T_0(\gamma_3)*\Gamma\backslash\gamma_3\right) + \bar{K} \text{ terms}$$



The four different types of UV limits are now **finite** !



$k, m \to \infty$, l finite

$l, m \to \infty$, k finite

$k, l \to \infty$, m finite

$k, l, m \to \infty$

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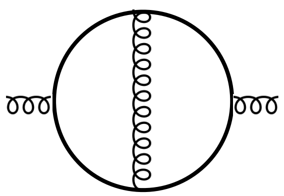
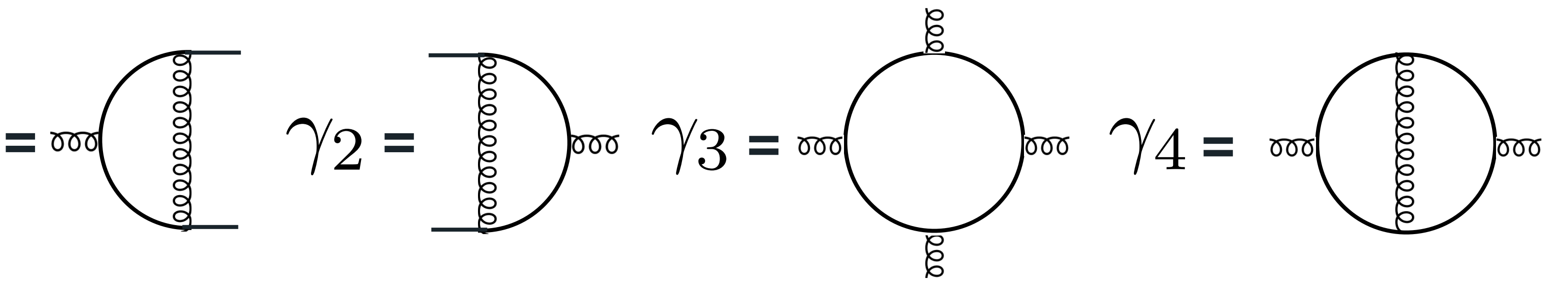
$\Gamma =$  with UV subgraphs $\gamma_1 =$ $\gamma_2 =$ $\gamma_3 =$ $\gamma_4 =$



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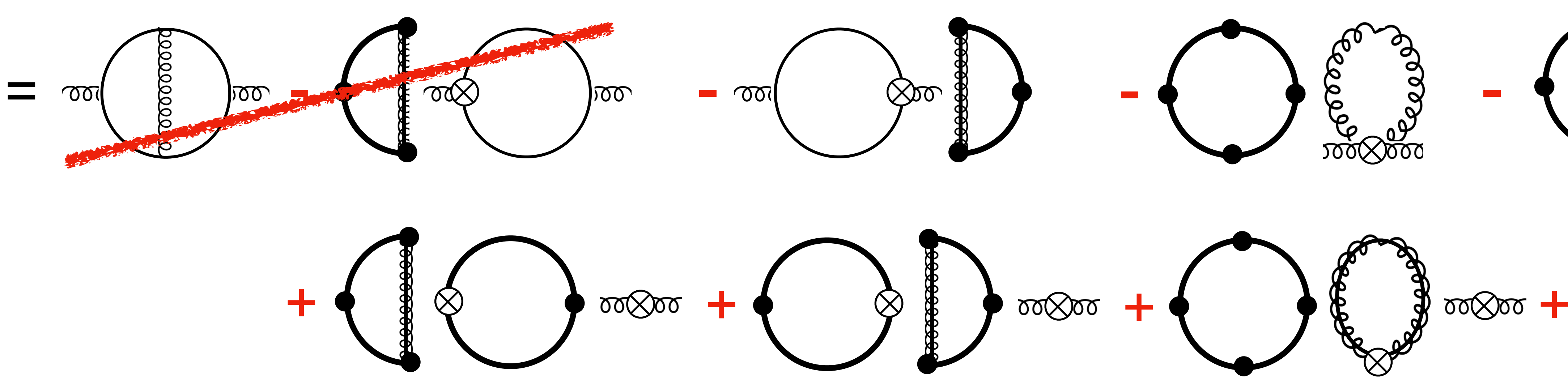
$$+ T_2\big(T_0(\gamma_1) * \Gamma\backslash\gamma_1\big) + T_2\big(T_0(\gamma_2) * \Gamma\backslash\gamma_2\big) + T_2\big(T_0(\gamma_3) * \Gamma\backslash\gamma_3\big) + \bar{K} \text{ terms}$$



$+ \bar{K}$ terms

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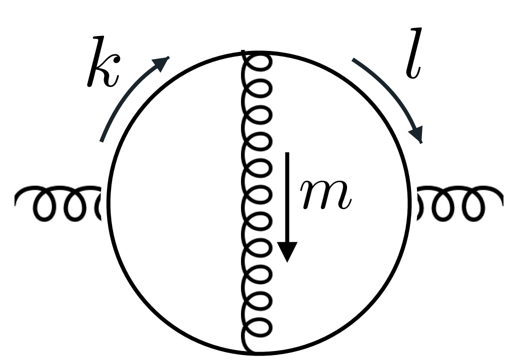

$$\boxed{k, m \rightarrow \infty \;, l \text{ finite}}$$
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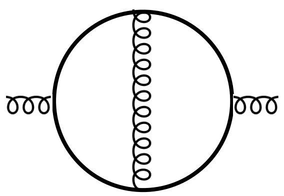
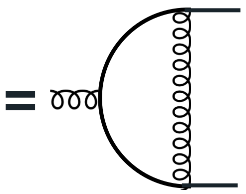
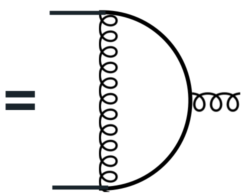
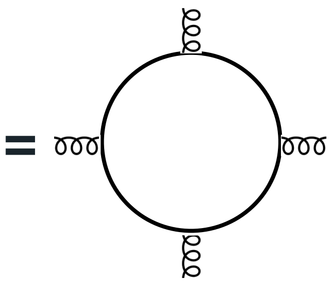
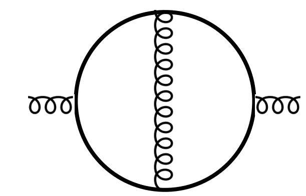
R-operator unfolding

$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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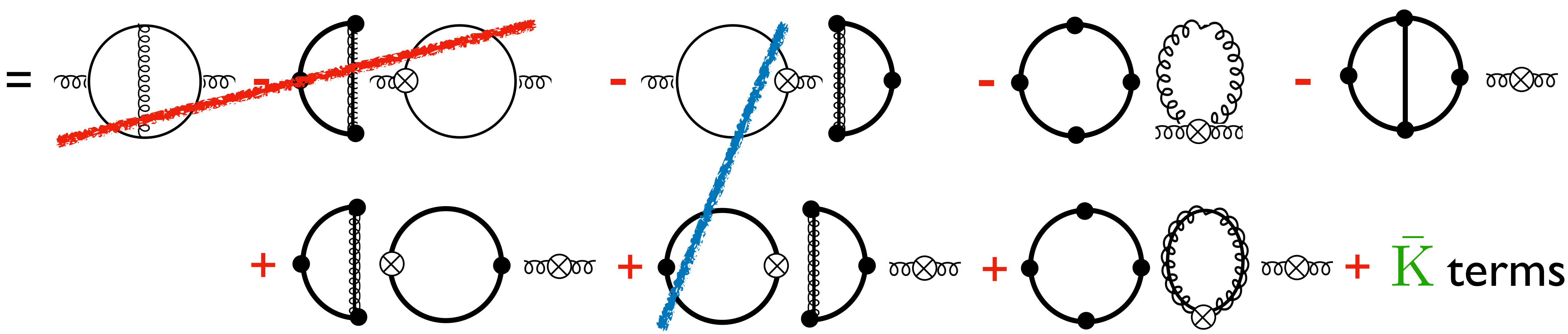
$$+T_2(T_0(\gamma_1)*\Gamma\backslash\gamma_1) + T_2(T_0(\gamma_2)*\Gamma\backslash\gamma_2) + T_2(T_0(\gamma_3)*\Gamma\backslash\gamma_3) + \bar{K}\ \text{terms}$$



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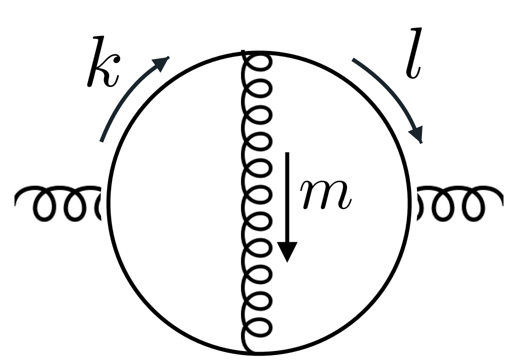

$$\boxed{k, m \to \infty\ , l\ \text{finite}}$$
$$l, m \to \infty\ , k\ \text{finite}$$
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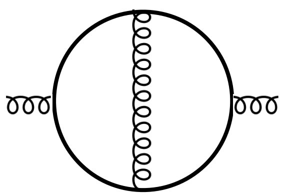
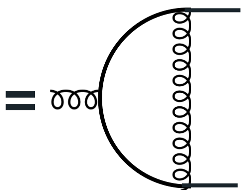
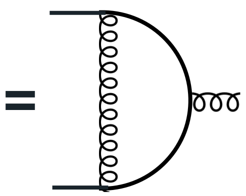
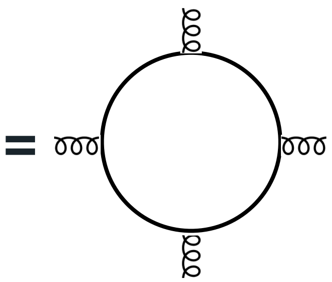
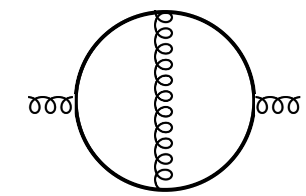
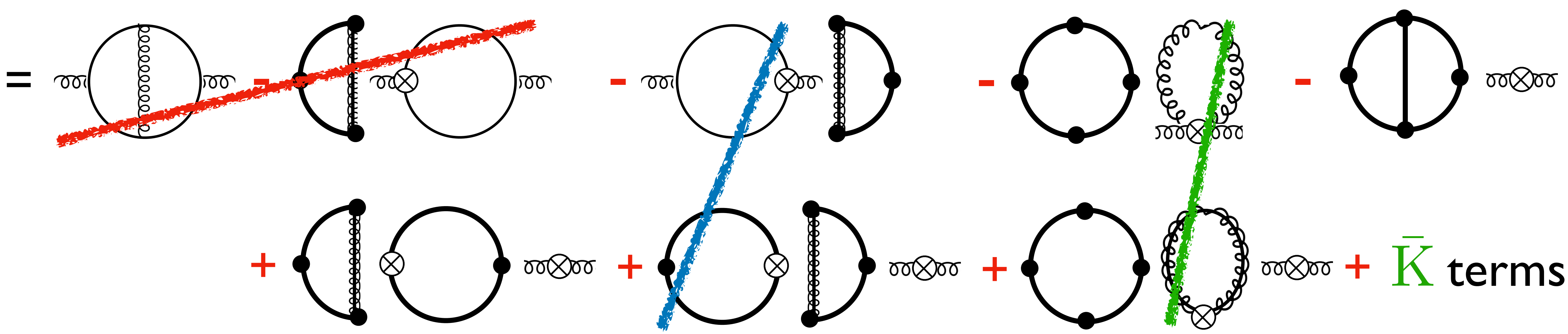
$$k, l, m \to \infty$$

R-OPERATOR UNFOLDING

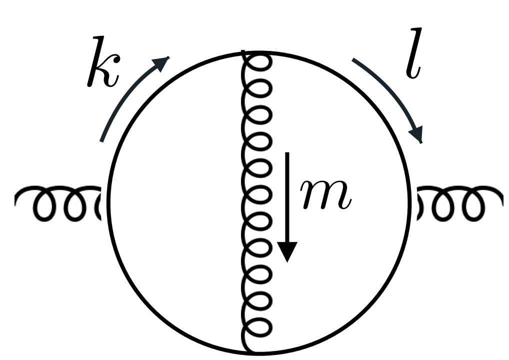
$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

$$R(\Gamma) = \Gamma - T_0(\gamma_1) * \Gamma \backslash \gamma_1 - T_0(\gamma_2) * \Gamma \backslash \gamma_2 - T_0(\gamma_3) * \Gamma \backslash \gamma_3 - T_2(\gamma_4) * \Gamma \backslash \gamma_4$$

$$+ T_2\left(T_0(\gamma_1) * \Gamma \backslash \gamma_1\right) + T_2\left(T_0(\gamma_2) * \Gamma \backslash \gamma_2\right) + T_2\left(T_0(\gamma_3) * \Gamma \backslash \gamma_3\right) + \bar{K} \textbf{ terms}$$



The four different types of UV limits are now **finite** !



$\boxed{k, m \to \infty \ , l \ \text{finite}}$

$l, m \to \infty \ , k \ \text{finite}$

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$k, l, m \to \infty$

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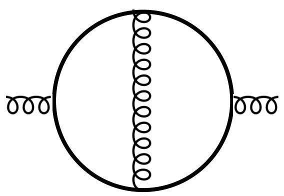
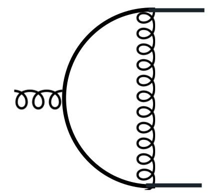
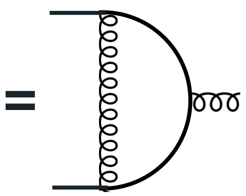
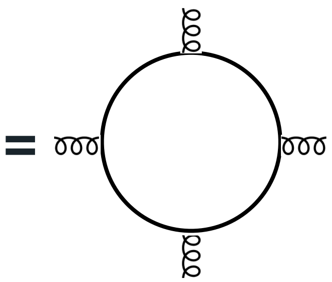
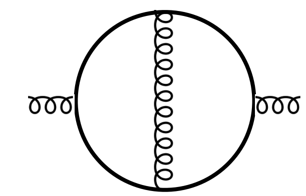
$\Gamma =$  with UV subgraphs $\gamma_1 =$  $\gamma_2 =$  $\gamma_3 =$  $\gamma_4 =$ 

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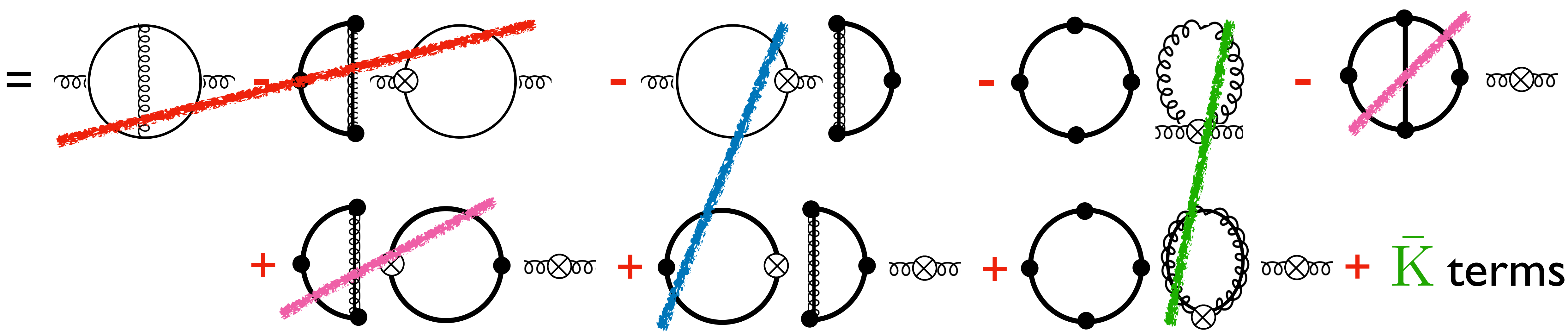

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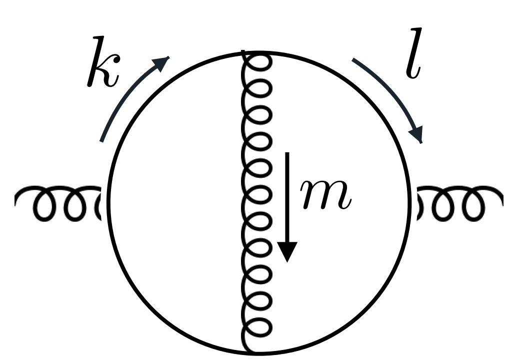

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$$l, m \to \infty \ , k \ \text{finite}$$
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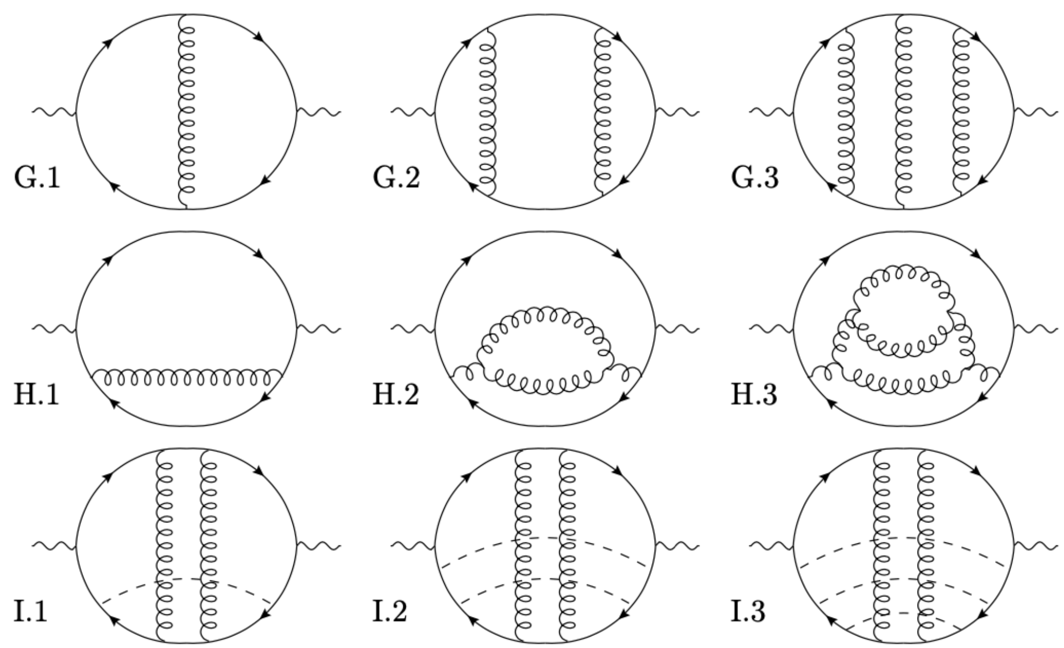
Efficiency in LU implementation

αLoop → γLoop



SG	proc.	order	t_{gen} [s]	M_{disk} [MB]	N_{sg} [-]	N_{cuts} [-]	t_{eval} [ms]	$t_{\text{eval}}^{(\text{f128})}$ [ms]
G.1	$1 \to 2$	NLO	0.1	0.13	2	4	0.004	0.13
G.2	$1 \to 2$	NNLO	4.7	3.0	17	9	0.04	2.1
G.3	$1 \to 2$	N3LO	36K	509	220	16	17.6	281
H.1	$1 \to 2$	NLO	0.07	0.12	2	2	0.006	0.14
H.2	$1 \to 2$	NNLO	1.5	1.3	17	3	0.056	1.9
H.3	$1 \to 2$	N3LO	255	43	220	4	2.35	56
I.1	$1 \to 3$	NNLO	126	22	266	9	0.32	12.4
I.2	$1 \to 4$	NNLO	1.9K	120	4492	9	4.4	67
I.3	$1 \to 5$	NNLO	36K	20K	$\mathcal{O}$(100K)	9	3.6K	17.3K

NB: most recent version of αLoop does better, but scaling is similar

Numerical gamma chains with spenso.rs

spenso.rs

[Lucien Huber : crates.io/crates/spenso]

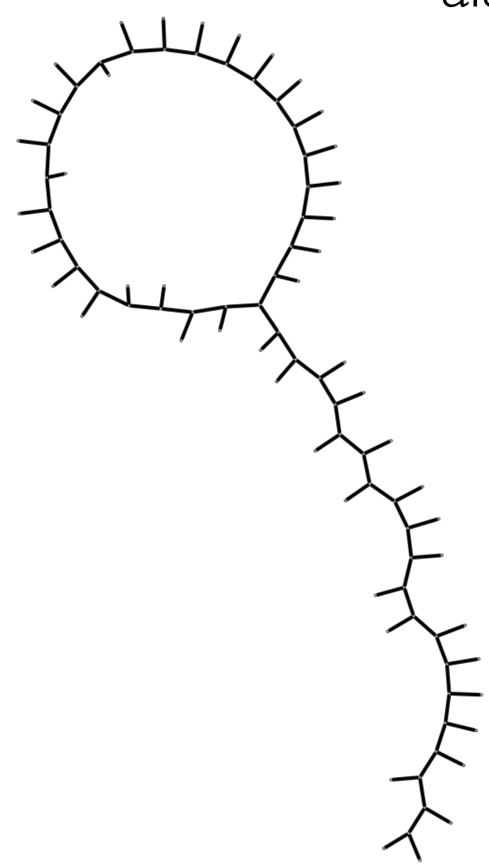
$$p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} \mathbf{Tr}(\gamma_\mu \gamma_{\mu_1} \gamma_\nu \gamma_{\mu_2} \gamma_\rho \gamma_{\mu_3})$$

γ -algebra

$$\begin{aligned}
&-4(p_1 \cdot p_2) p_{3,\mu} \eta_{\nu\rho} \\
&+ 4(p_1 \cdot p_2) p_{3,\nu} \eta_{\mu\rho} \\
&- 4(p_1 \cdot p_2) p_{3,\rho} \eta_{\mu\nu} \\
&+ 4(p_1 \cdot p_3) p_{2,\mu} \eta_{\nu\rho} \\
&- 4(p_1 \cdot p_3) p_{2,\nu} \eta_{\mu\rho} \\
&- 4(p_1 \cdot p_3) p_{2,\rho} \eta_{\mu\nu} \\
&- 4p_{1,\mu} (p_2 \cdot p_3) \eta_{\nu\rho} \\
&\quad + 4p_{1,\mu} p_{2,\nu} p_{3,\rho} \\
&\quad + 4p_{1,\mu} p_{2,\rho} p_{3,\nu}
\end{aligned}$$

necessary for d-dimensions
but scales badly for $d = 4$

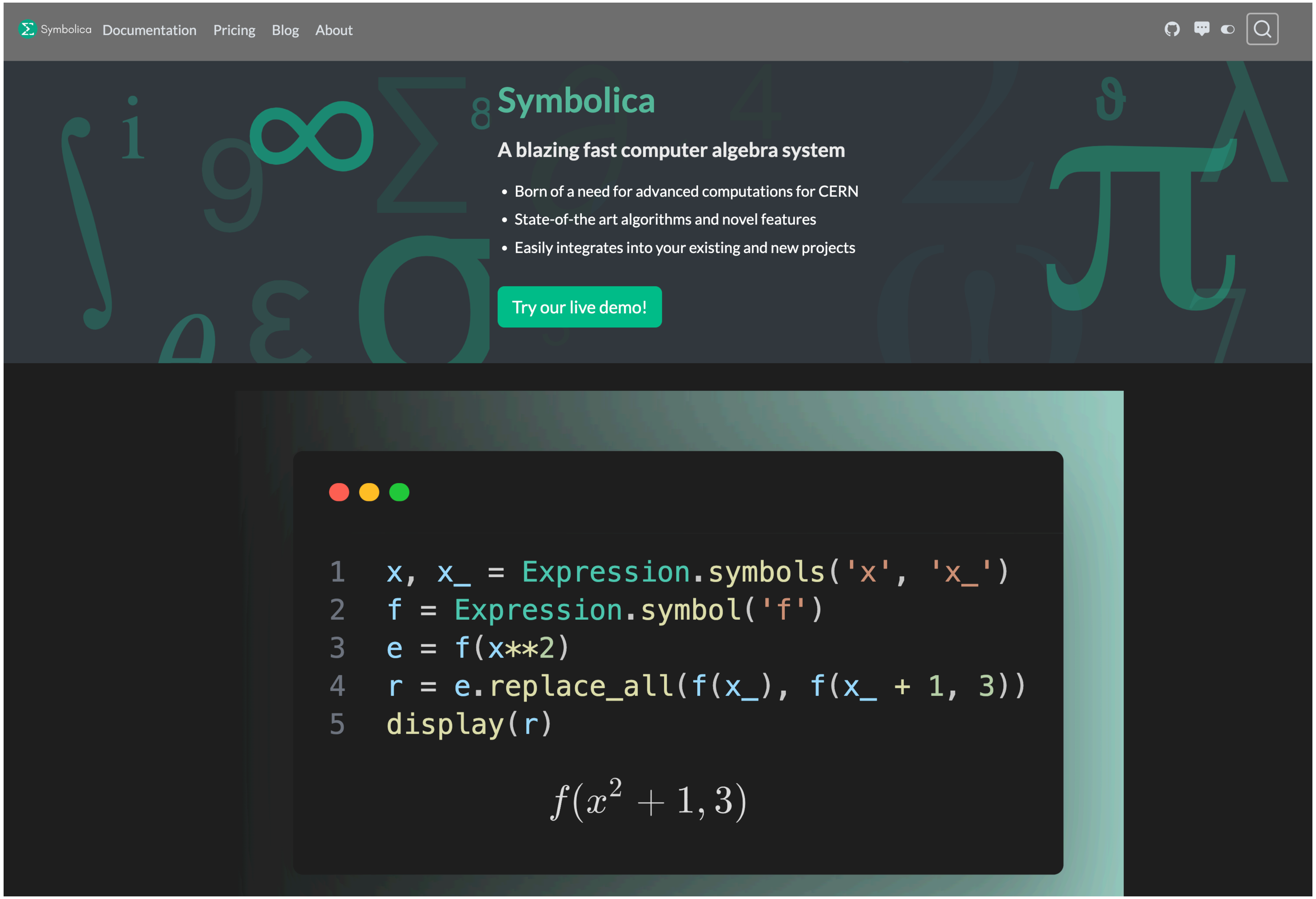
numerical tensors
"ala MadGraph"



Spenso	Spenso Compiled	Hardcode Fortran
104 μs	16 μs	31μs

Symbolica

Ben Ruijl's successor to FORM



Try it for yourself in this colab notebook !

Summary - Possible overlap with LSS

Formalism :

- **Local** cancellation of all **final-state IR singularities**
- Completely **generic** (masses, kinematics, topologies, observables…)
- New **theoretical perspectives** on perturbative expansions
- **Automated renormalisation** with minimal analytical computations
- **Falsehood :** "Analytical = solved && Numerical = black box"

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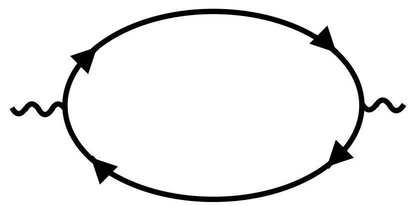
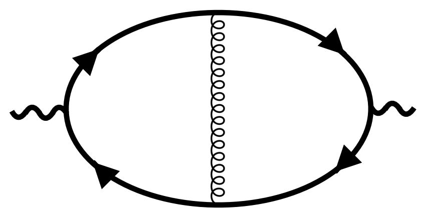
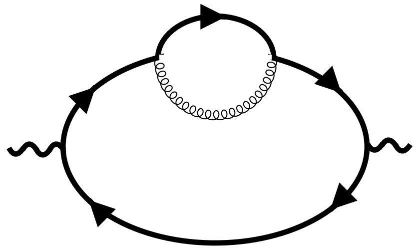
Outlook :

- Rewrite **LU** impl. **cleaner and more efficient**: $\alpha\text{Loop} \rightarrow \gamma\text{Loop}$
- Apply **LU** to **new unknown corrections** : e.g. $b\bar{b}$ N^3LO AFB or decays
- Applications in finite **T** and finite μ pQFT : see [Navarrete & al.: 2403.02180]
- Generalise **LU** so as to apply of **initial-state singularities** as well
- Match **LU** to some form of **numerical resummation** (PSMC).
- Method likely well-suited for deployment on GPU.

Back-up slides

Local Unitarity: x-sec results

Numerical results for $e^+e^- \to \gamma^\star \to d\bar{d}$ @ NLO

$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+\, 2\times$  $\Big]$

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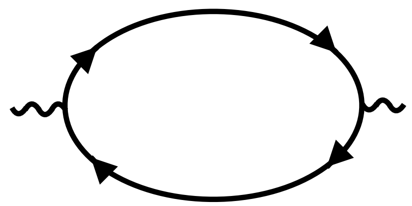
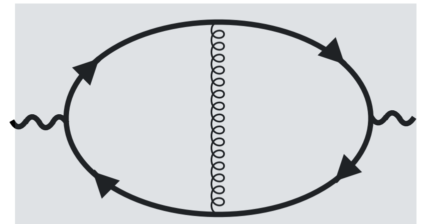
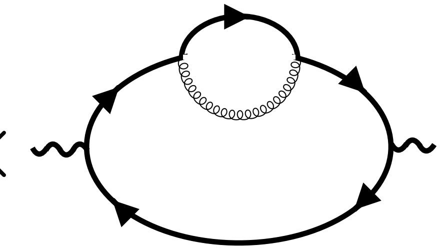
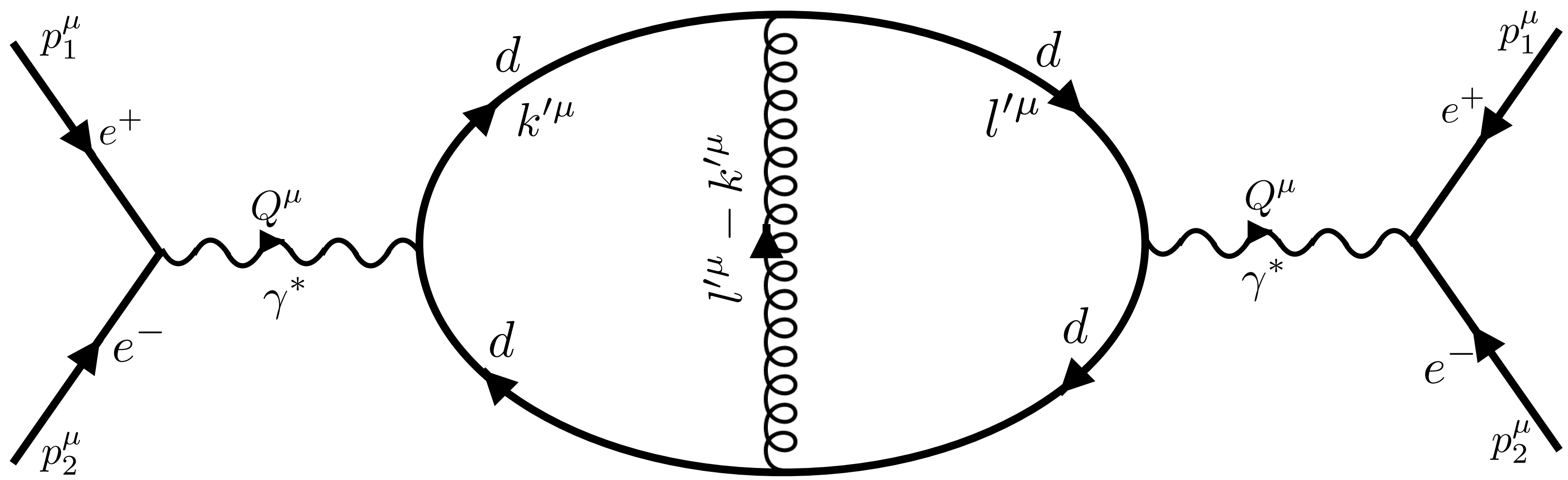
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+ 2\times$  $\Big]$

Visualisation of the **LU integrand** for the **Double-Triangle** supergraph and :



$$p_1^\mu = (1,0,0,1) \qquad 0.4 < p_{t,j_1} < 0.8$$

$$p_2^\mu = (1,0,0,-1) \qquad (\,\vec{k},\vec{l}\,) = (\,(0,k_y,\frac{1}{\sqrt{2}})\,,(0,\frac{1}{\sqrt{2}},l_z)\,)$$

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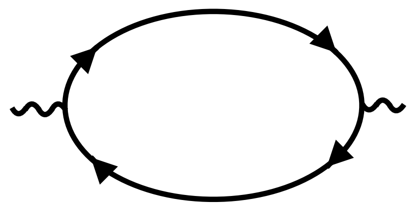
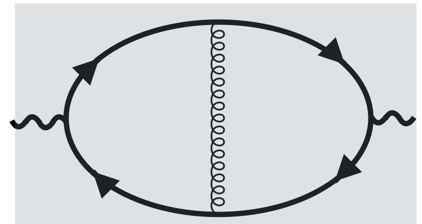
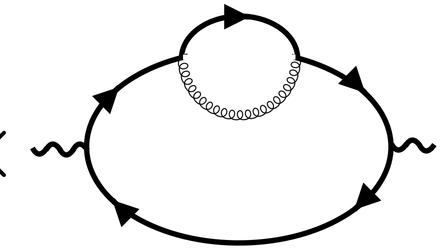
$$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$$  $+$  $+\, 2\times$  $\Big]$

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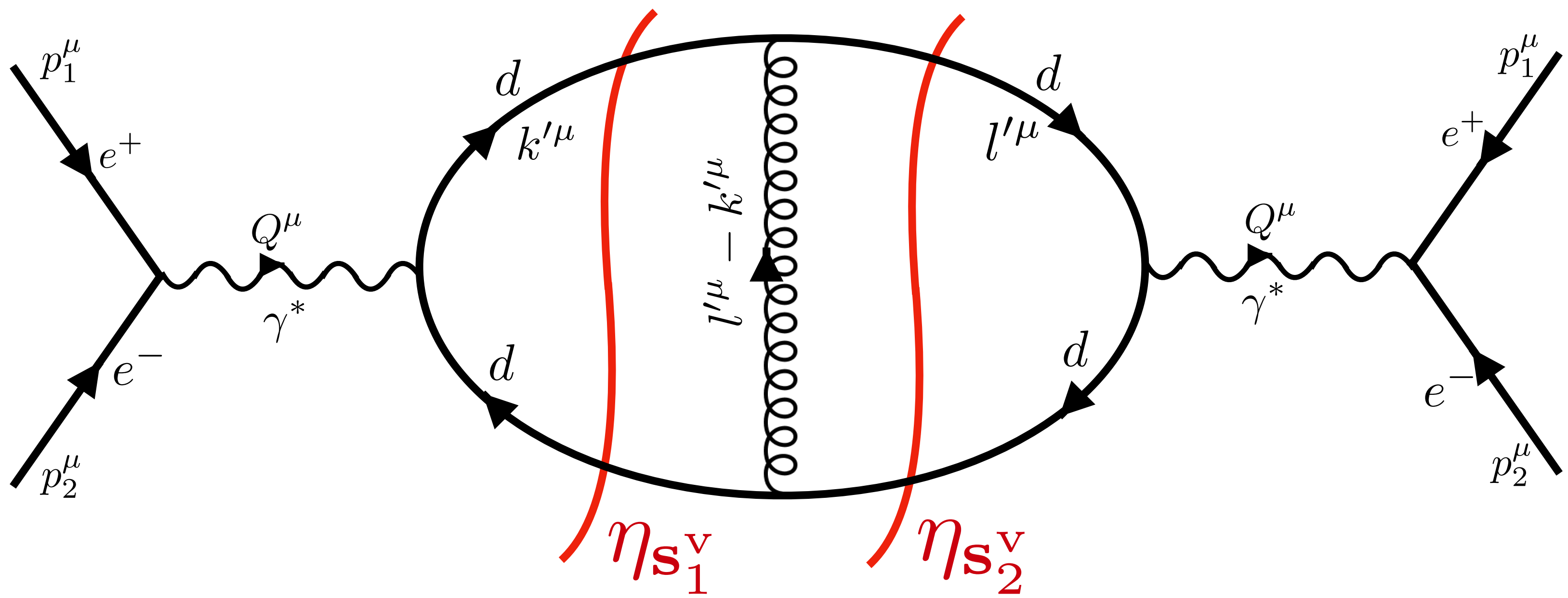

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Cancellation of **non-pinched E-surfaces** for : $\eta_{\mathbf{s}_1^{\mathrm{v}}} = \eta_{\mathbf{s}_2^{\mathrm{v}}} \;\to\; k'_y = l'_z$

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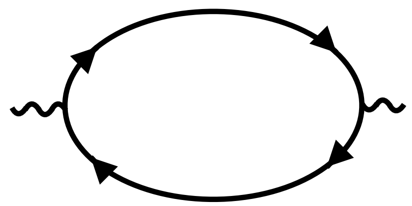
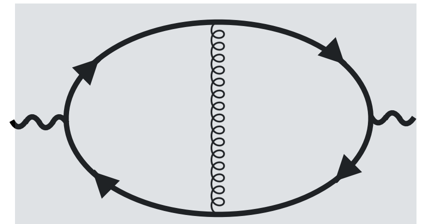
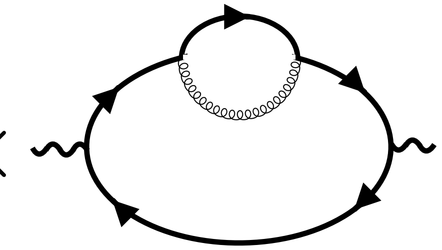
$\sigma^{(\mathrm{LU})}_{\gamma^\star \to d\bar{d}} = \mathrm{LU}\Big[$  $+$  $+\, 2\times$  $\Big]$

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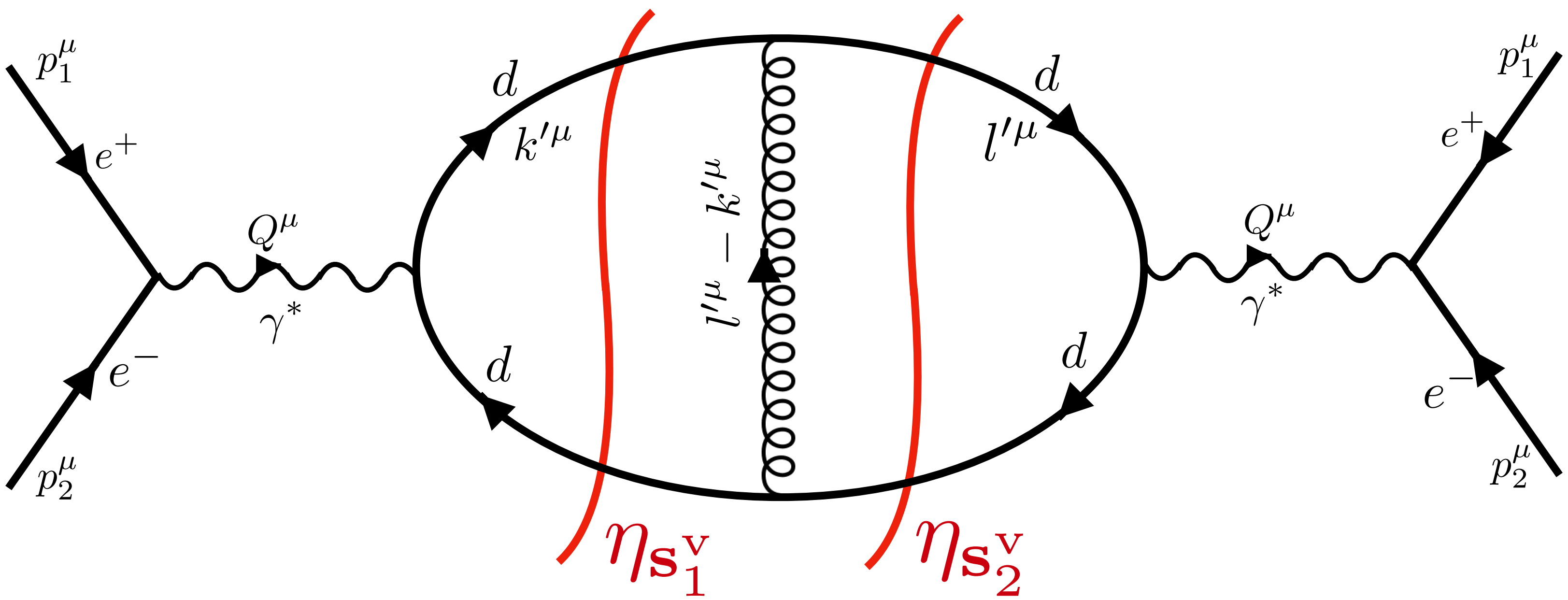

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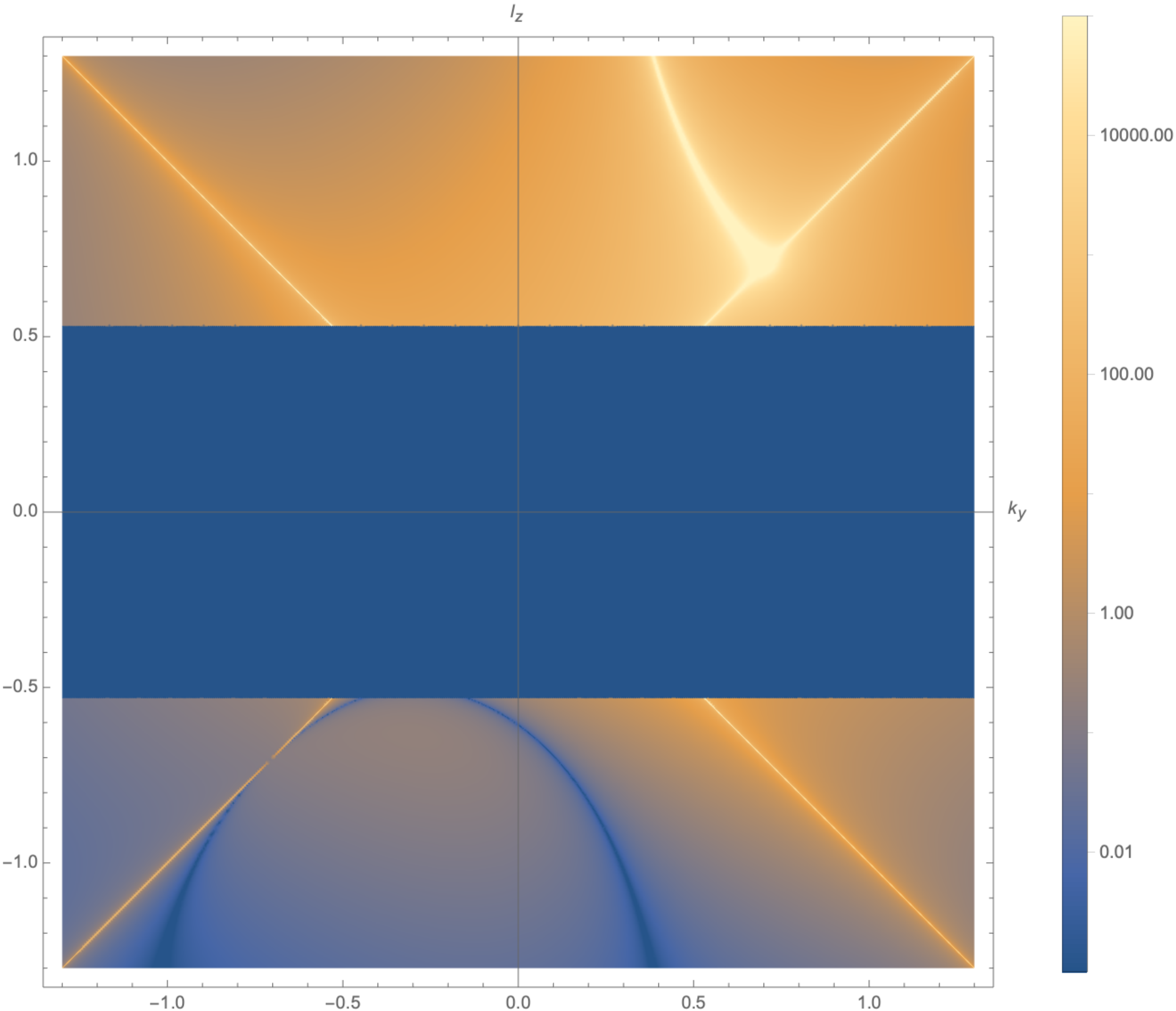
Cancellation of **non-pinched E-surfaces** for : $\eta_{\mathbf{s}_1^{\mathrm{v}}} = \eta_{\mathbf{s}_2^{\mathrm{v}}} \;\to\; k'_y = l'_z$

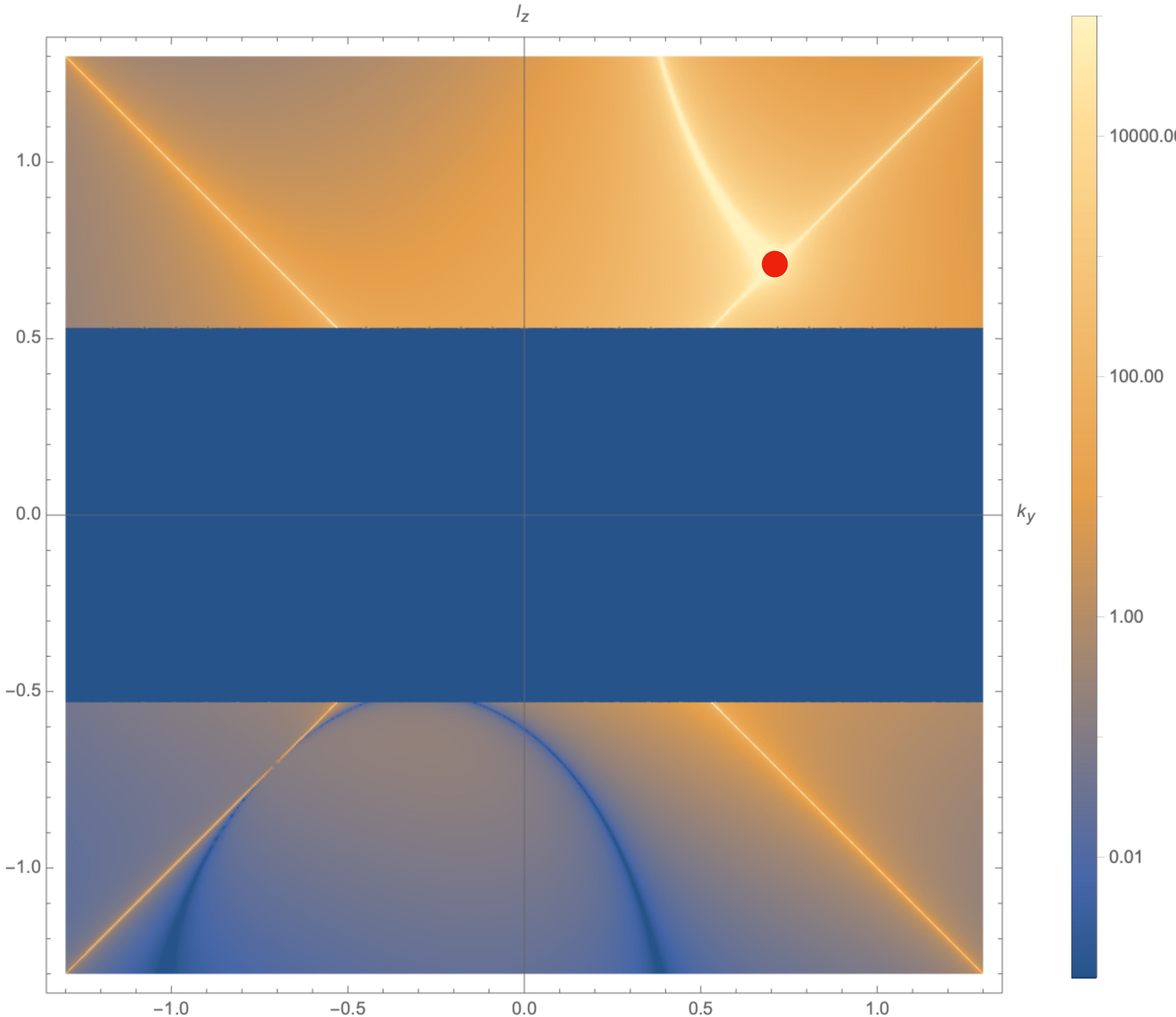
Soft configuration for : $|\vec{l}\,' - \vec{k}| = 0 \;\to\; k'_y = l'_z = \frac{1}{\sqrt{2}}$

$I_{\mathbf{s}_1^{\mathrm{v}}}$



$I_{\mathbf{s}_1^{\mathrm{v}}}$

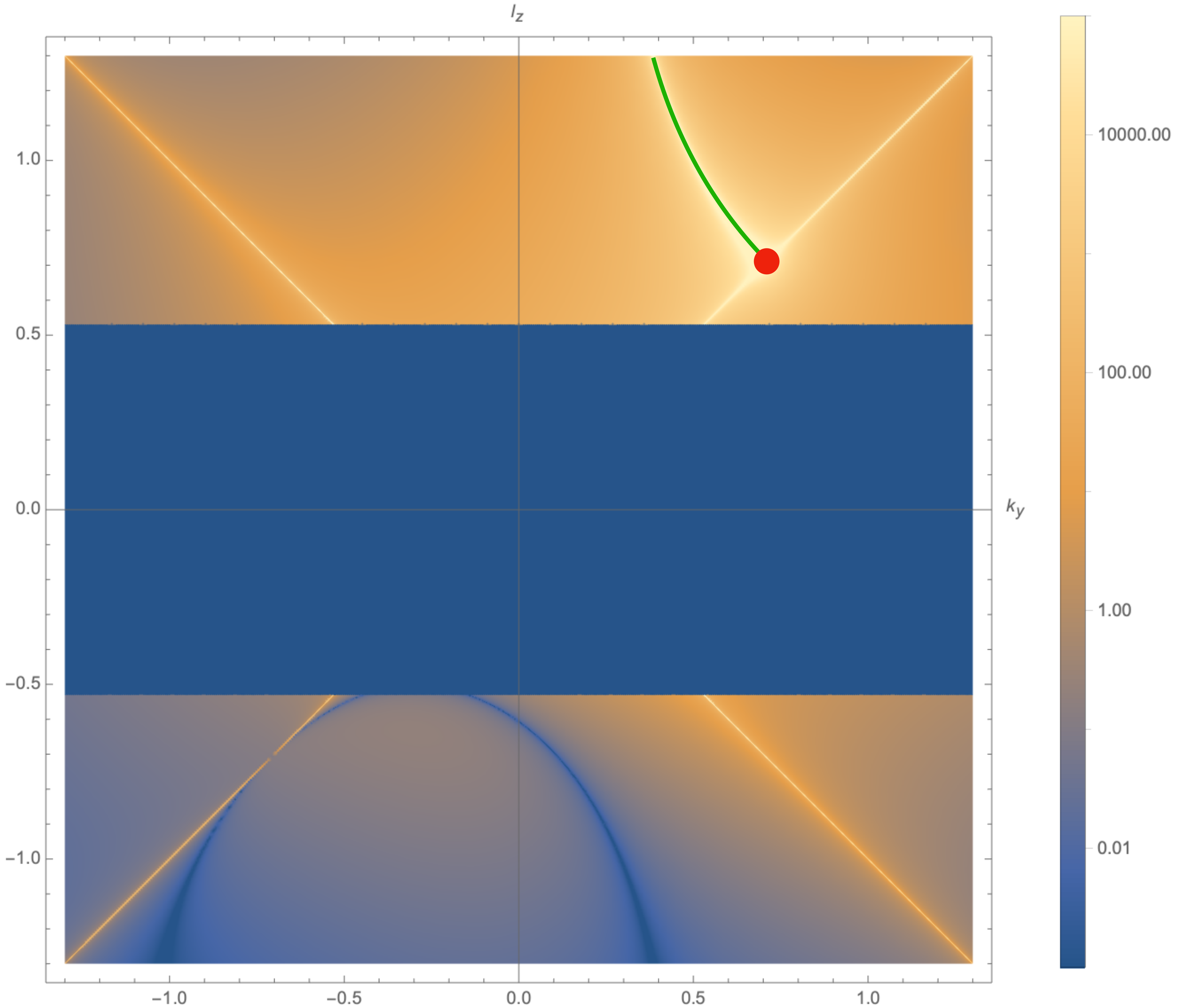
soft



$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

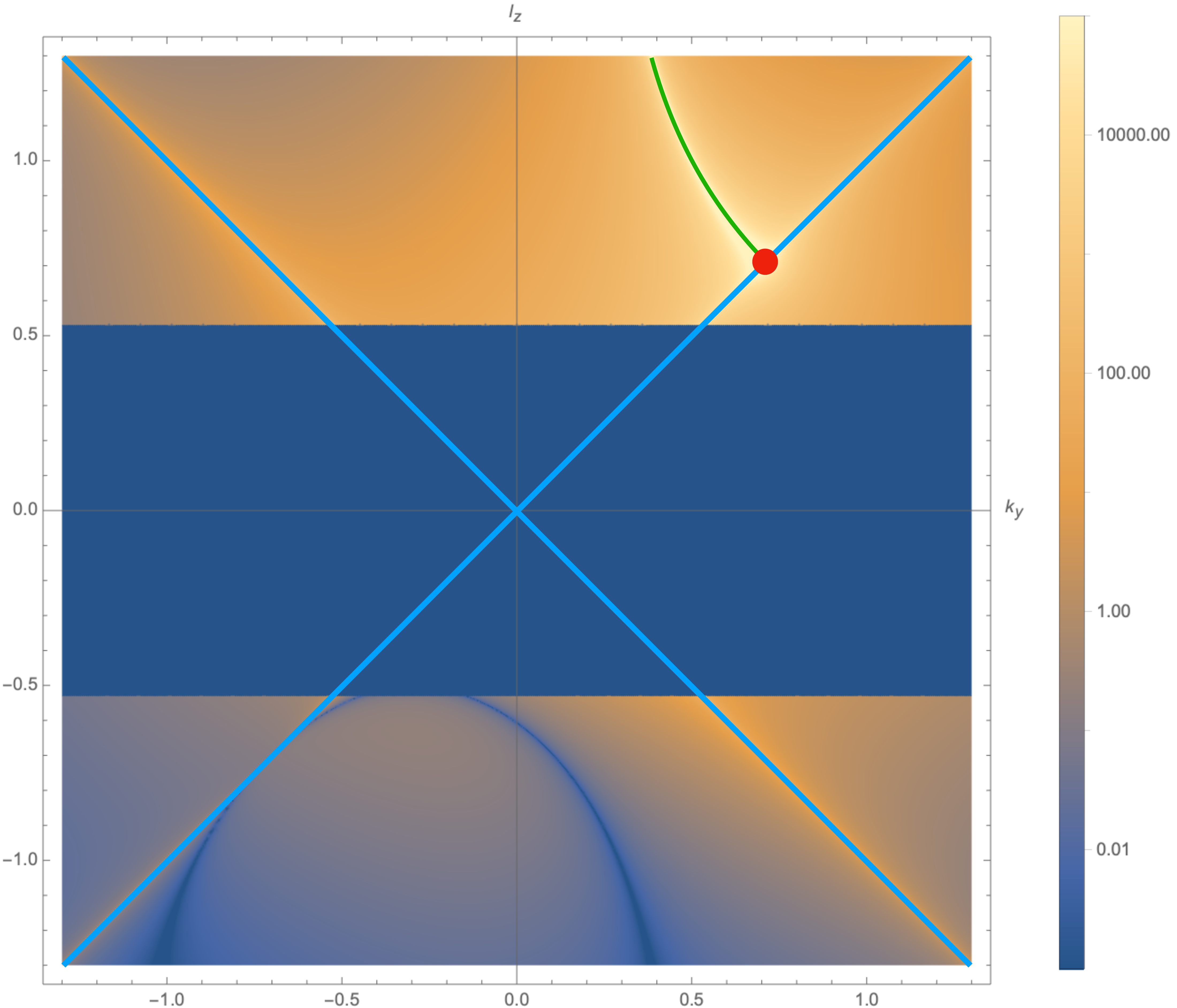


$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

collinear

non-pinch cancellation



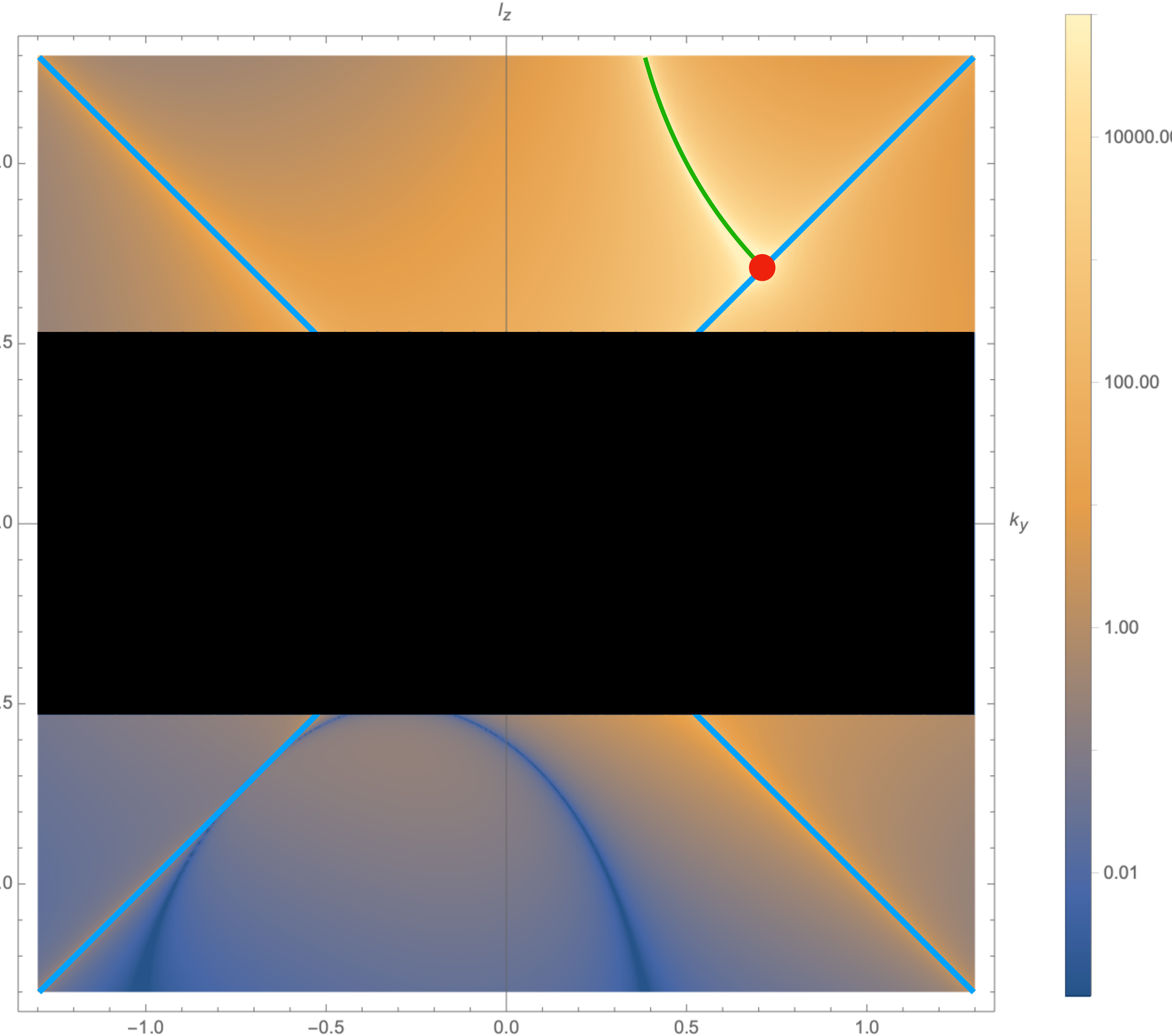
$I_{\mathbf{s}_1^{\mathrm{v}}}$

soft

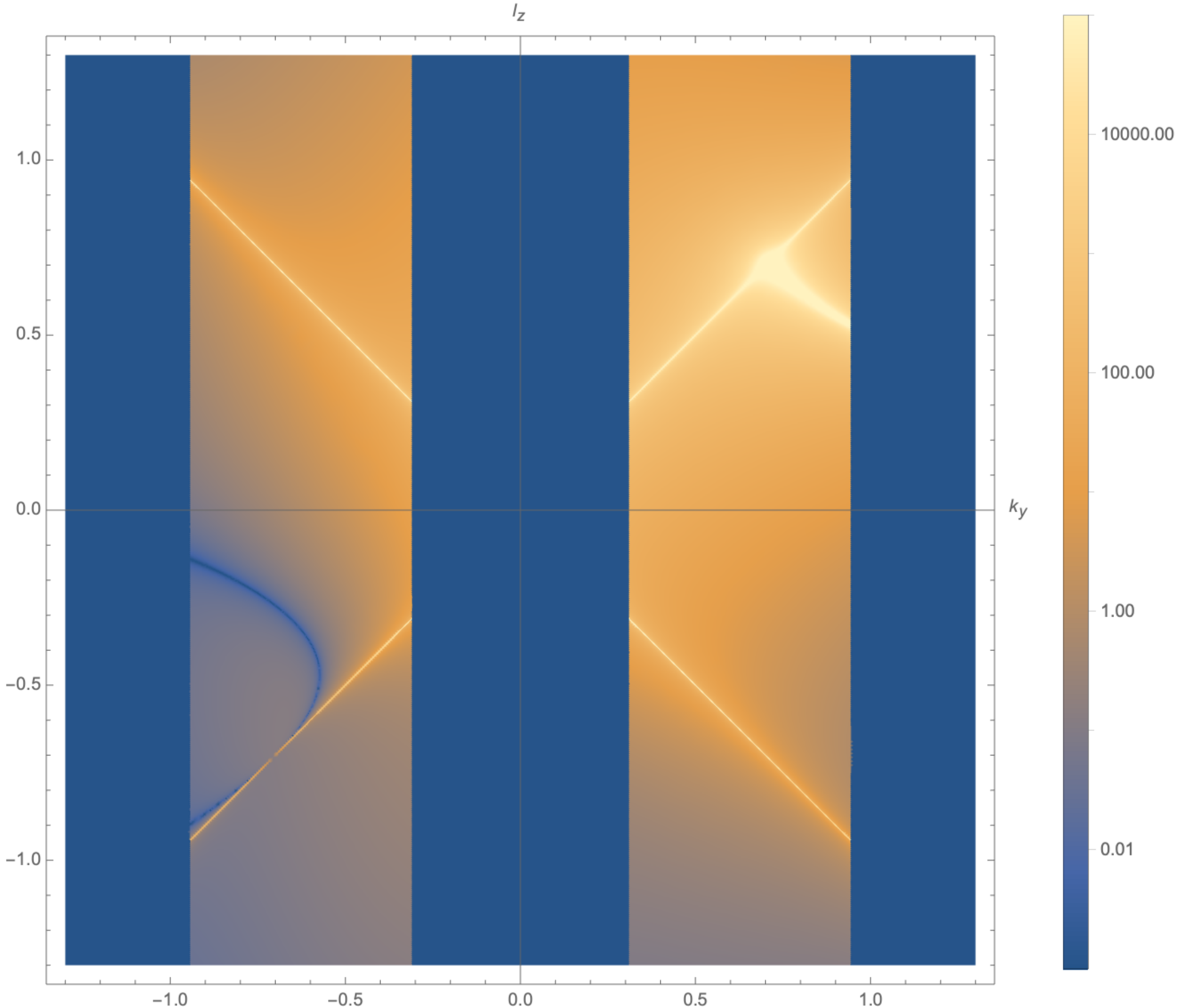
collinear

non-pinch cancellation

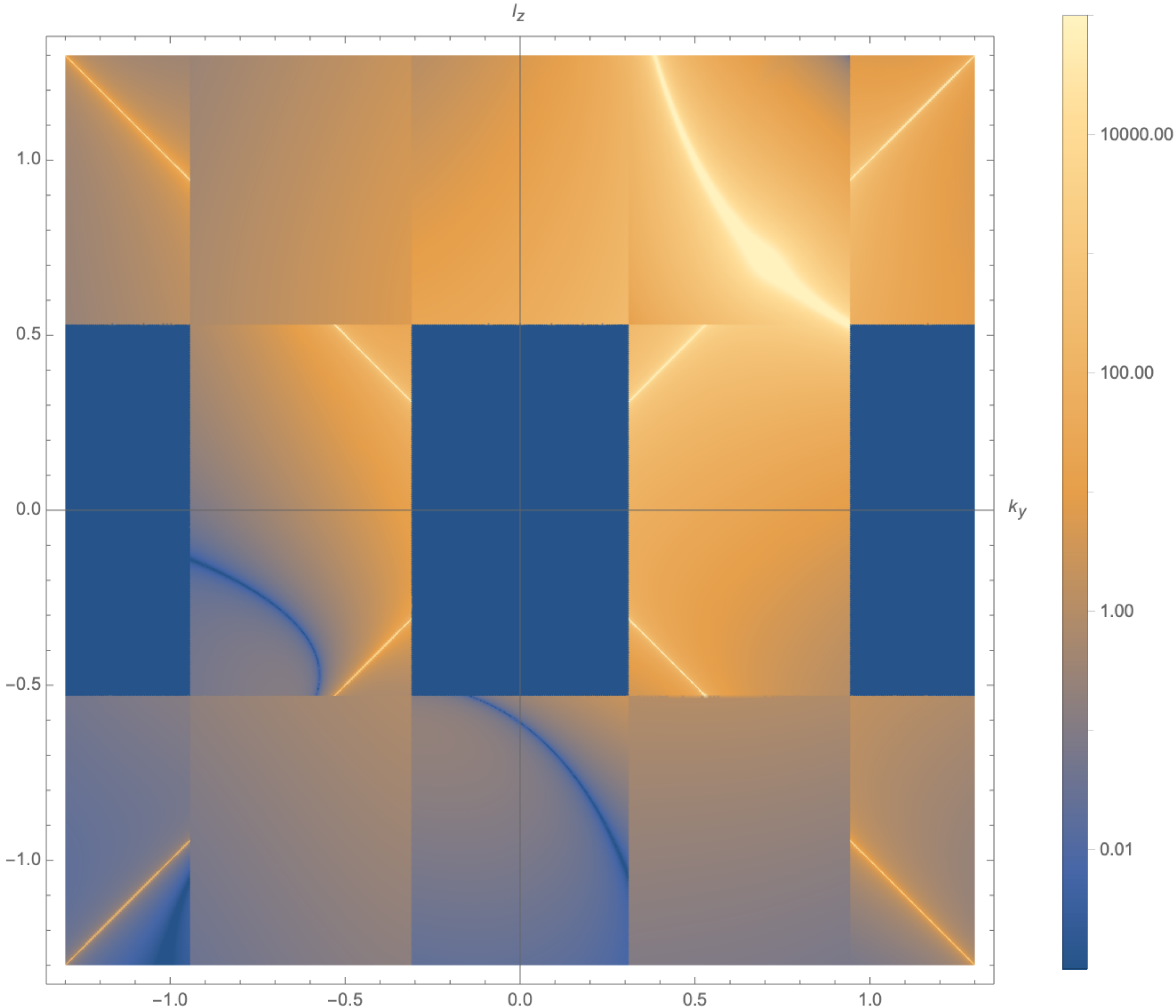
observable cuts



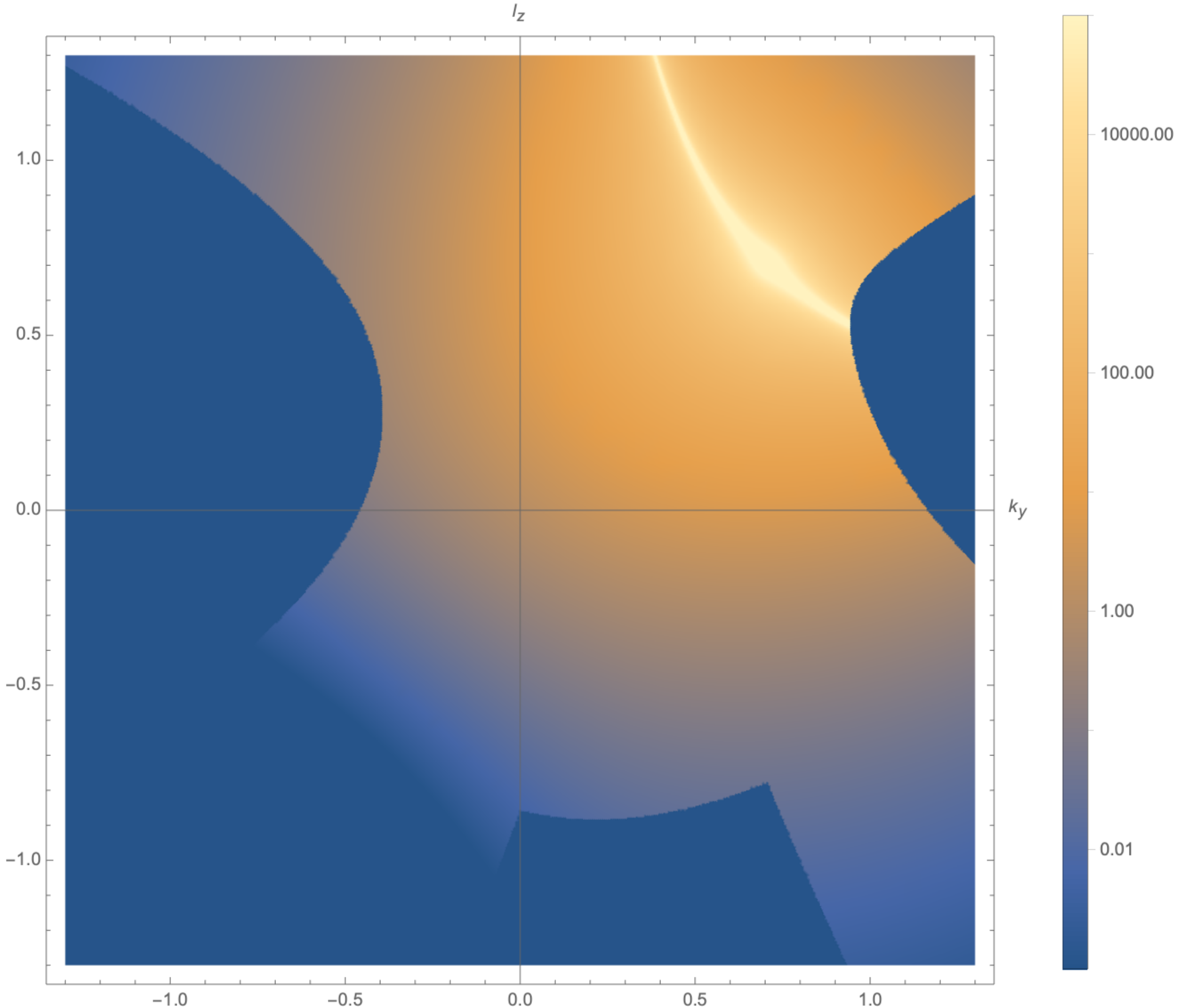
$I_{\mathbf{s}_2^{\mathrm{v}}}$



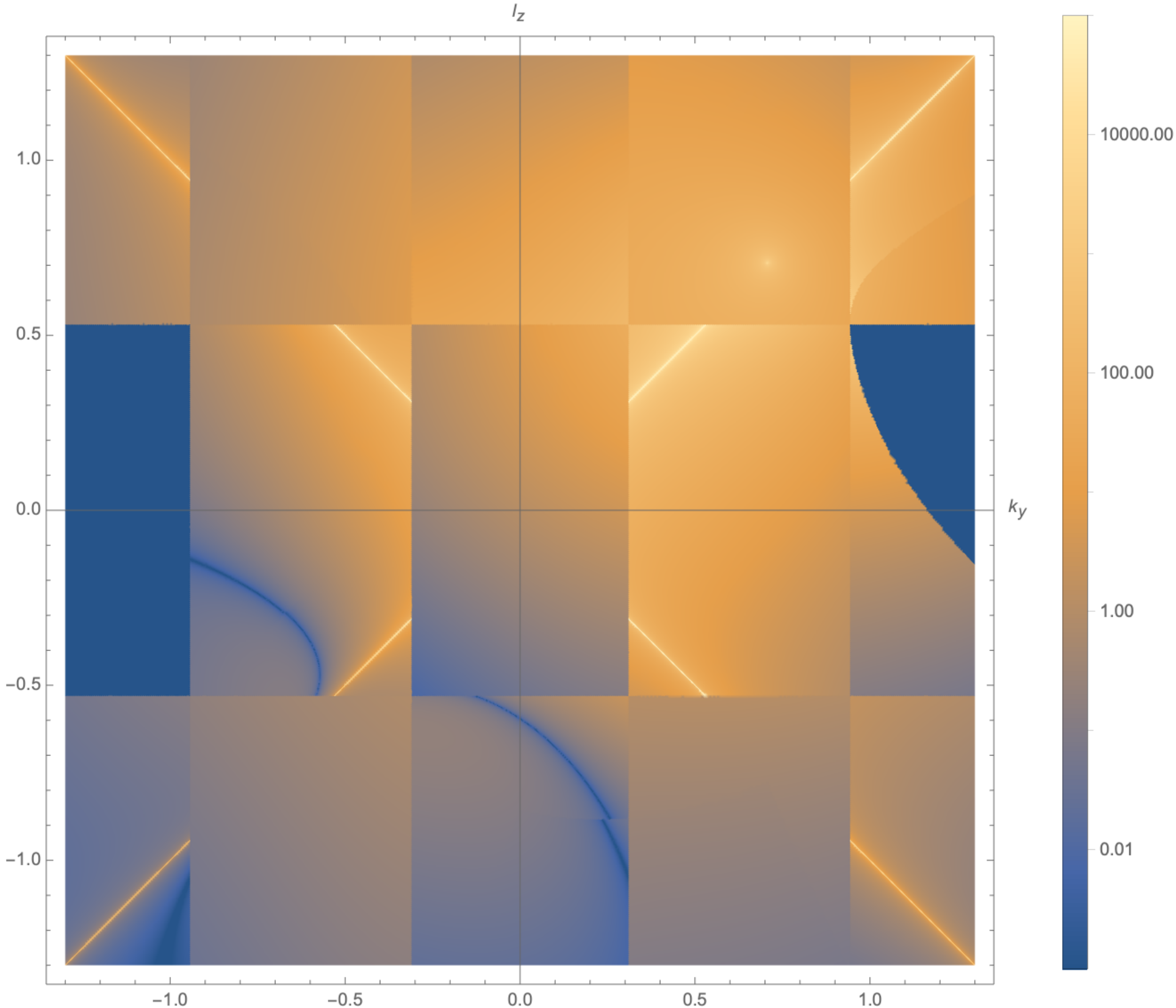
$I_{\mathbf{s}_1^{\mathrm{v}}} + I_{\mathbf{s}_2^{\mathrm{v}}}$



$$I_{\mathbf{s}_1^{\mathrm{r}}} + I_{\mathbf{s}_2^{\mathrm{r}}}$$

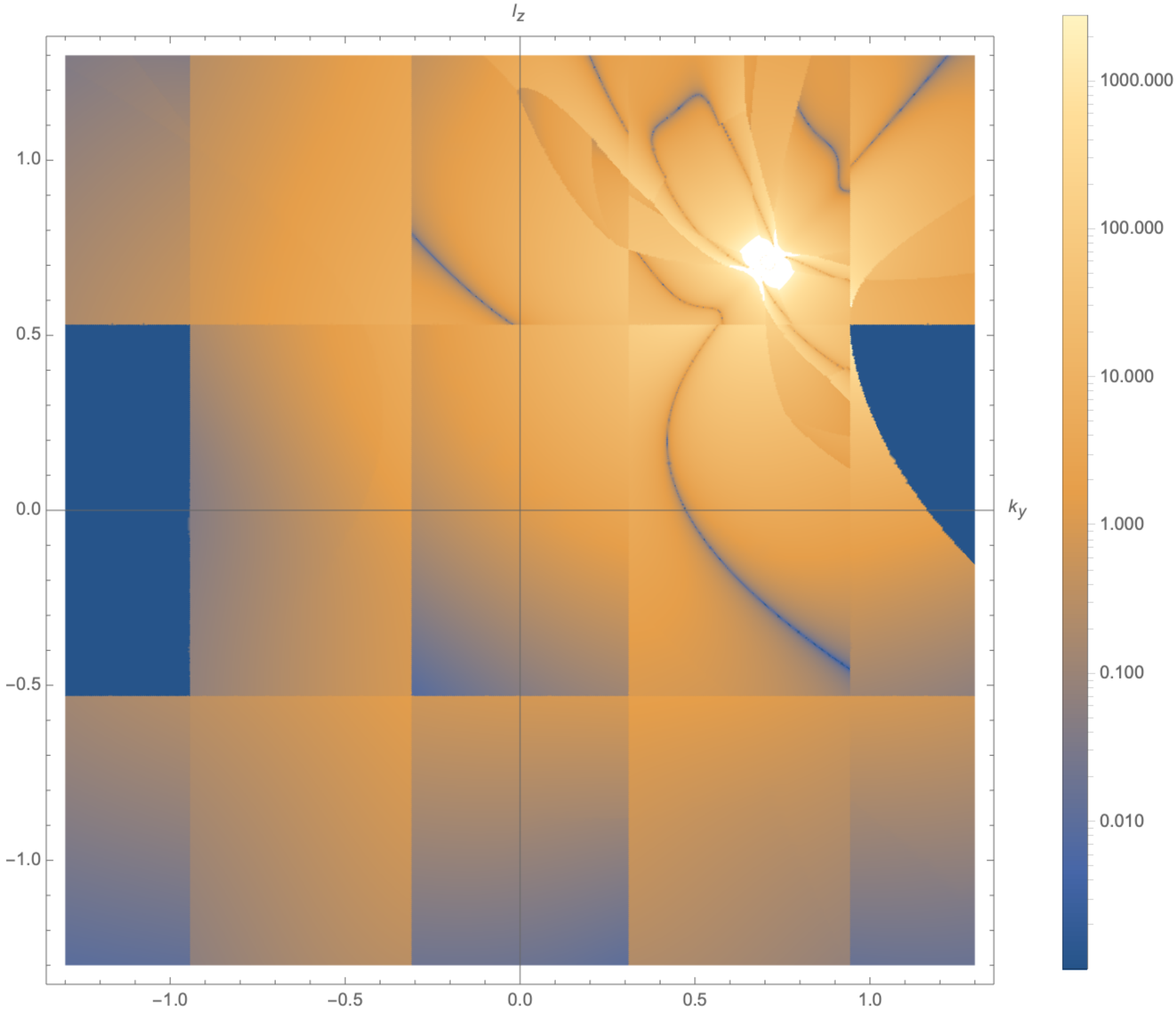


I_{Σ}



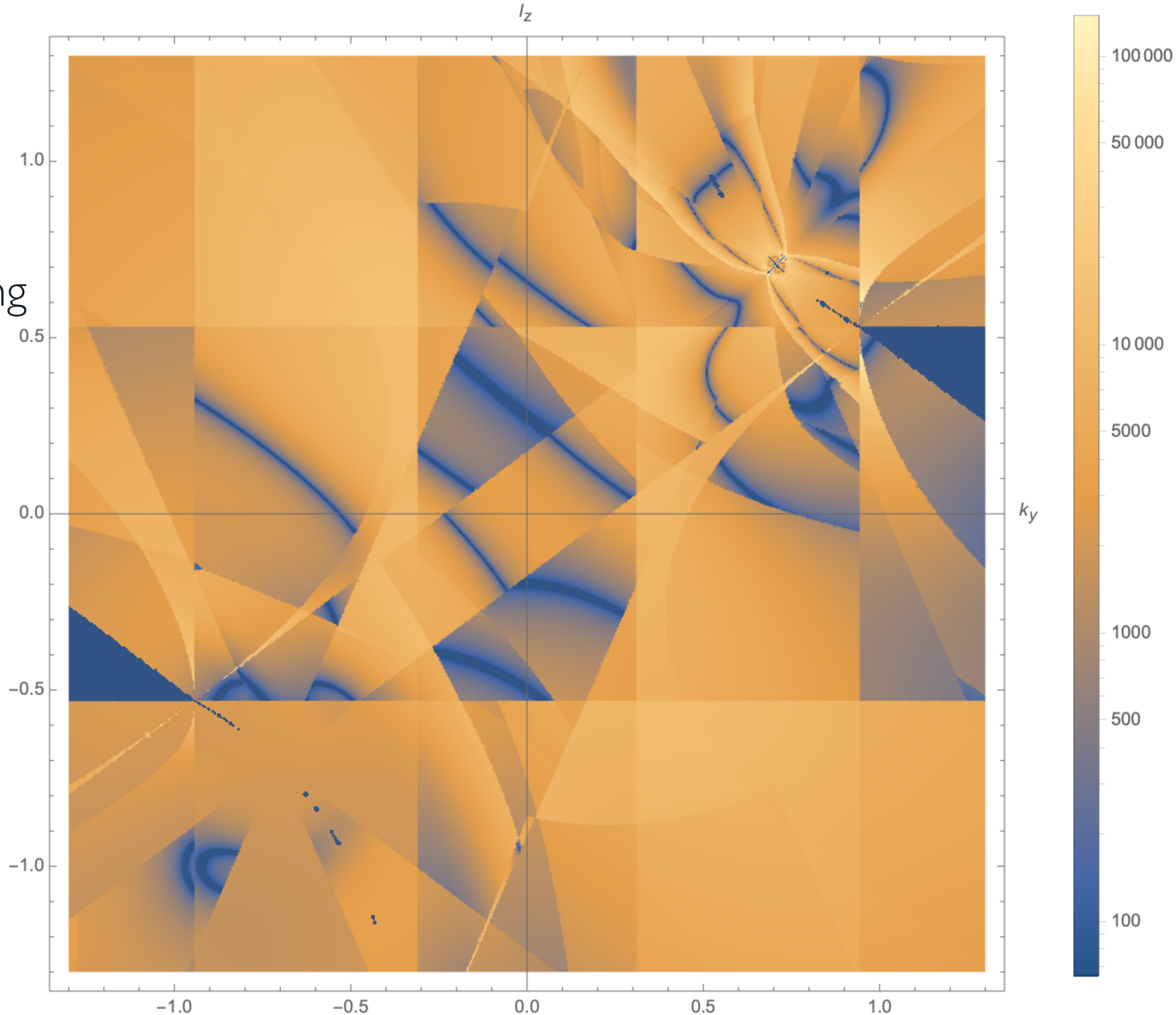
$\mathrm{Re}\left[I_{\sum}\right]$

with deformation



$\mathrm{Re}\left[I_{\sum}\right]$

with deformation and multichanneling



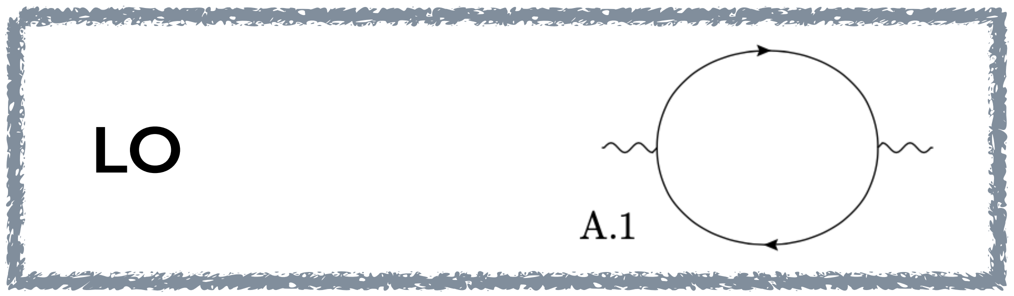
Numerical results

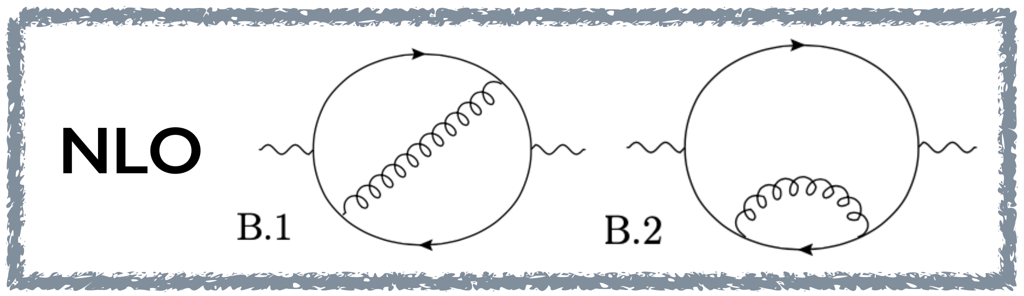
We have computed the following fixed-order processes with **Local Unitarity**:

NLO
- $e^+e^- \to \gamma \to jj$ $p_t(j_1)$ distribution
- $e^+e^- \to \gamma \to jjj$ semi-inclusive
- $e^+e^- \to \gamma \to t\bar{t}h$ (semi-)inclusive

NNLO
- $\gamma^\star \to jj$ inclusive
- $\gamma^\star \to t\bar{t}$ inclusive

First NNLO cross-sections computed **fully numerically in momentum space**.





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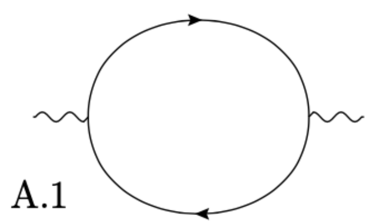
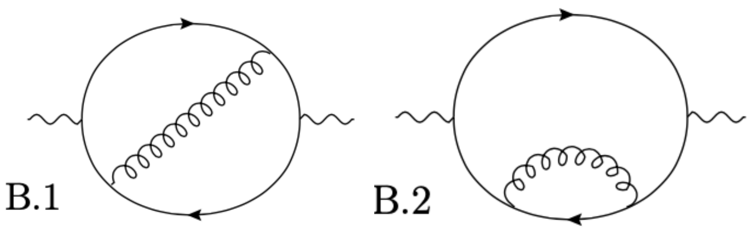
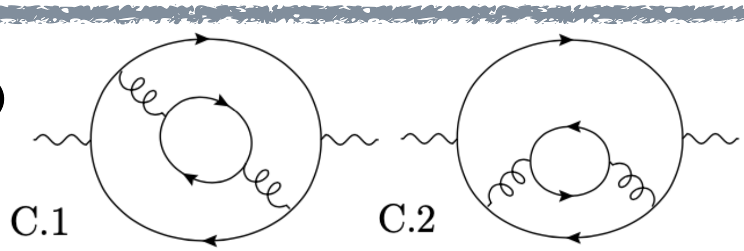


NLO



NNLO
$\mathcal{O}(n_f)$



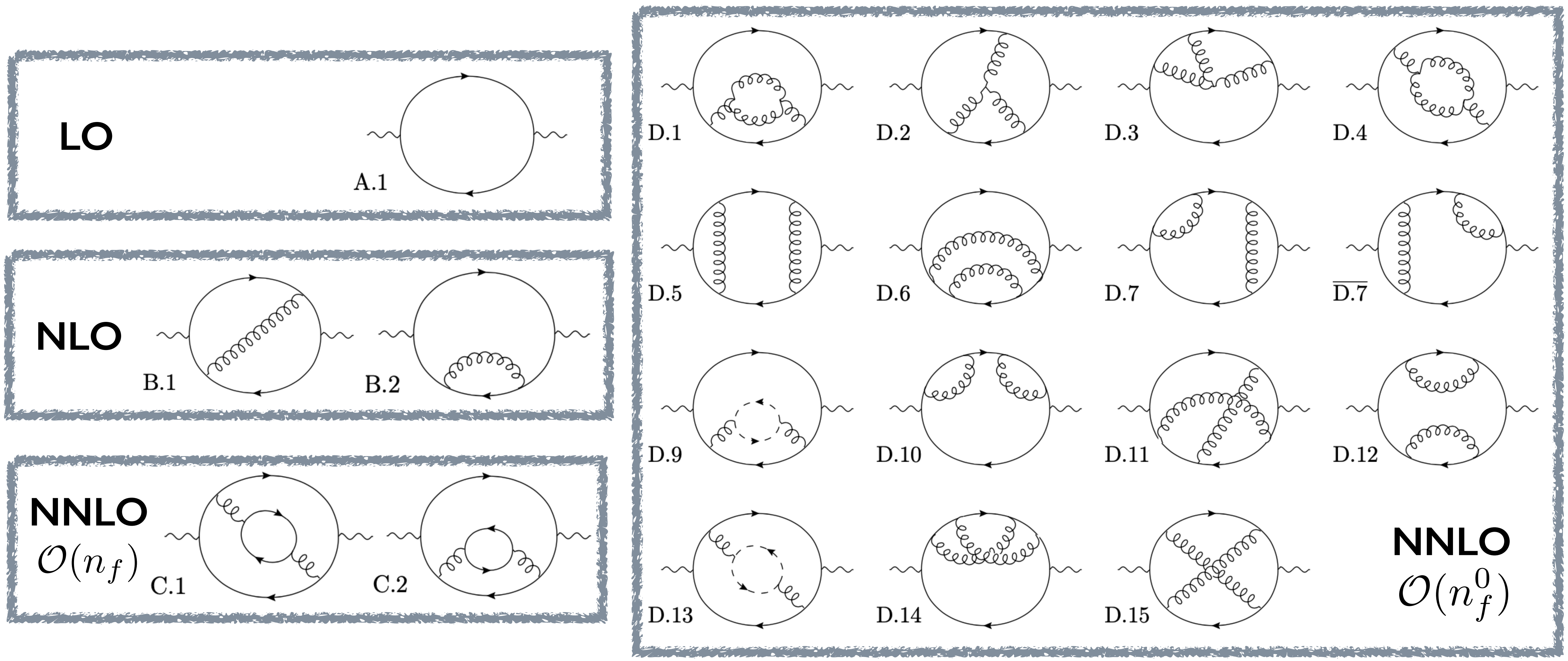
Numerical results

We have computed the following fixed-order processes with **Local Unitarity**:

NLO
- $e^+e^- \to \gamma \to jj$ $p_t(j_1)$ distribution
- $e^+e^- \to \gamma \to jjj$ semi-inclusive
- $e^+e^- \to \gamma \to t\bar{t}h$ (semi-)inclusive

NNLO
- $\gamma^\star \to jj$ inclusive
- $\gamma^\star \to t\bar{t}$ inclusive

First NNLO cross-sections computed **fully numerically in momentum space**.



Numerical results

SG id	Ξ	$\sigma^{(\overline{\text{MS}})}_{\gamma^\star\to jj}$ [GeV^{-2}] $p^2_{\gamma^\star}=\mu_r^2=(400\text{ GeV})^2$	Δ [%]	$\sigma^{\left[\alpha_s^{(\overline{\text{MS}})},m_t^{(\text{OS})}\right]}_{\gamma^\star\to t\bar{t}}$ [GeV^{-2}] $\mu_r^2=m_t^2,\ p^2_{\gamma^\star}=(400\text{ GeV})^2$	Δ [%]
LO		$\mathcal{O}(\alpha_s^0)$			
A.1	1	$5.031049\cdot10^{-01}$	0.0018	$1.387586\cdot10^{+00}$	0.0011
Total		$5.031049\cdot10^{-01}$	0.0018	$1.387586\cdot10^{+00}$	0.0011
NLO		$\mathcal{O}(\alpha_s^1)$			
B.1	1	$5.03926\cdot10^{-02}$	0.0075	$2.52705\cdot10^{-01}$	0.034
B.2	2	$-3.14956\cdot10^{-02}$	0.018	$1.80050\cdot10^{-01}$	0.049
Total		$1.88970\cdot10^{-02}$	0.036	$4.3276\cdot10^{-01}$	0.028
Benchmark		$1.889690\cdot10^{-02}$	0.00053	$4.32831\cdot10^{-01}$	-0.018
NNLO		$\mathcal{O}(\alpha_s^2 n_f)$			
C.1	1	$-4.66342\cdot10^{-04}$	0.019	$-1.0022\cdot10^{-03}$	0.17
C.2	2	$3.8448\cdot10^{-04}$	0.036	$-4.6982\cdot10^{-03}$	0.081
Total		$-8.186\cdot10^{-05}$	0.20	$-5.7004\cdot10^{-03}$	0.073
Benchmark		$-8.1834\cdot10^{-05}$	0.036	$-5.6982\cdot10^{-03}$	0.038
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D.1	2	$-2.30886\cdot10^{-03}$	0.017	$3.8886\cdot10^{-02}$	0.031
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[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

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[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

$$\gamma^\star\to t\bar{t}$$

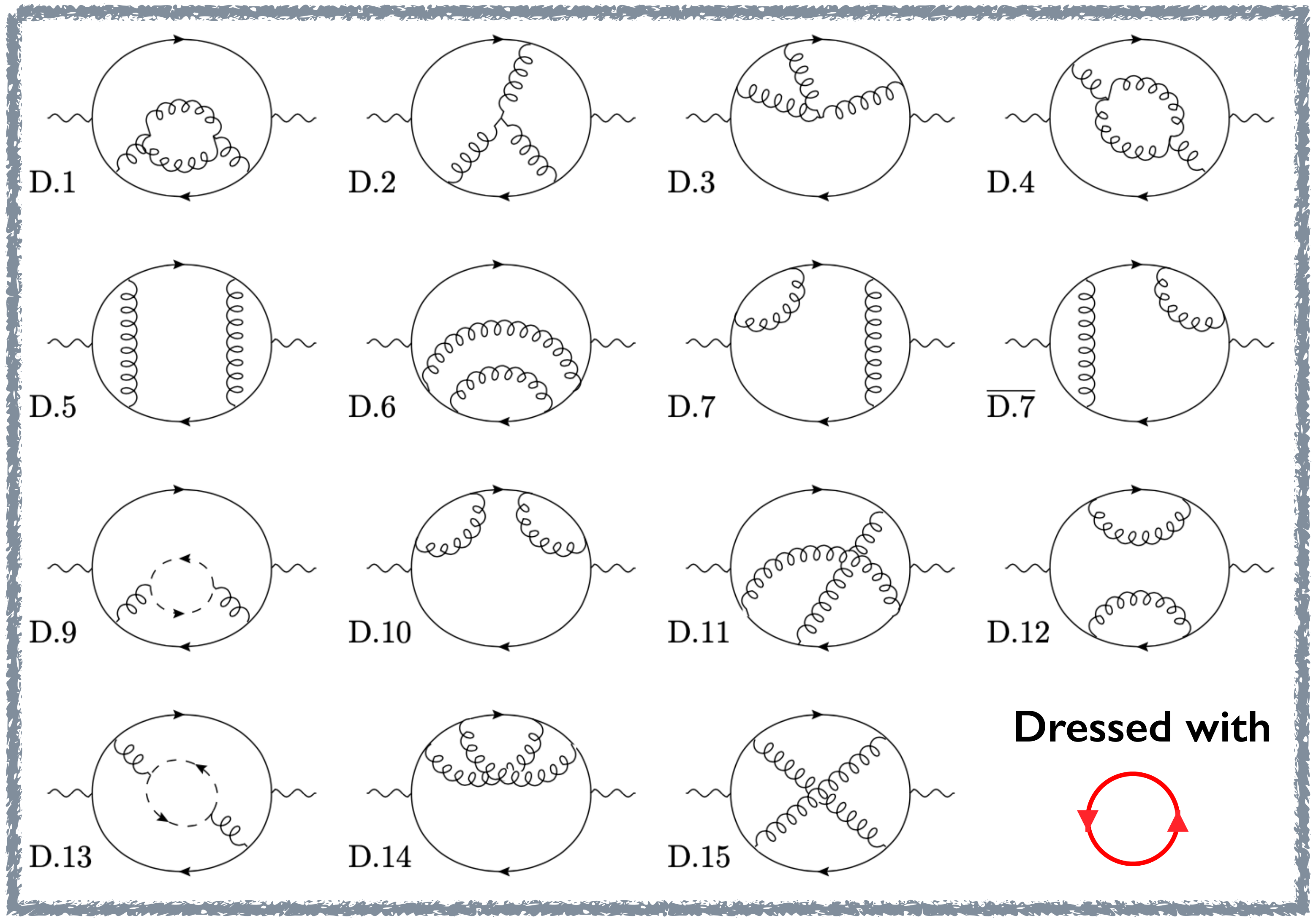
$$\boxed{K_{t\bar{t}}}=$$

$$\begin{aligned}\delta^{(2)} = &-\frac{(3-v^2)\,(1+v^2)}{6}\times\\ &\Big\{\mathrm{Li}_3(p)-2\,\mathrm{Li}_3(1-p)-3\,\mathrm{Li}_3(p^2)-4\,\mathrm{Li}_3\Big(\frac{p}{1+p}\Big)-5\,\mathrm{Li}_3(1-p^2)+\frac{11}{2}\zeta(3)\\ &+\mathrm{Li}_2(p)\,\ln\Big(\frac{4\,(1-v^2)}{v^4}\Big)+2\,\mathrm{Li}_2(p^2)\,\ln\Big(\frac{1-v^2}{2\,v^2}\Big)+2\,\zeta(2)\Big[\ln p-\ln\Big(\frac{1-v^2}{4\,v}\Big)\Big]\\ &-\frac{1}{6}\ln\Big(\frac{1+v}{2}\Big)\Big[36\,\ln 2\,\ln p-44\,\ln^2 p+49\,\ln p\,\ln\Big(\frac{1-v^2}{4}\Big)+\ln^2\Big(\frac{1-v^2}{4}\Big)\Big]\\ &-\frac{1}{2}\ln p\,\ln v\Big[36\,\ln 2+21\,\ln p+16\,\ln v-22\,\ln(1-v^2)\Big]\Big\}\\ &+\frac{1}{24}\Big\{(15-6\,v^2-v^4)\Big(\mathrm{Li}_2(p)+\mathrm{Li}_2(p^2)\Big)+3\,(7-22\,v^2+7\,v^4)\,\mathrm{Li}_2(p)\\ &-(1-v)\,(51-45\,v-27\,v^2+5\,v^3)\,\zeta(2)\\ &+\frac{(1+v)\,(-9+33\,v-9\,v^2-15\,v^3+4\,v^4)}{v}\ln^2 p\\ &+\Big[(33+22\,v^2-7\,v^4)\,\ln 2-10\,(3-v^2)\,(1+v^2)\,\ln v\\ &\qquad-(15-22\,v^2+3\,v^4)\,\ln\Big(\frac{1-v^2}{4\,v^2}\Big)\Big]\ln p\\ &+2\,v\,(3-v^2)\,\ln\Big(\frac{4\,(1-v^2)}{v^4}\Big)\Big[\ln v-3\,\ln\Big(\frac{1-v^2}{4\,v}\Big)\Big]\\ &+\frac{237-96\,v+62\,v^2+32\,v^3-59\,v^4}{4}\ln p-16\,v\,(3-v^2)\,\ln\Big(\frac{1+v}{4}\Big)\\ &-2\,v\,(39-17\,v^2)\,\ln\Big(\frac{1-v^2}{2\,v^2}\Big)-\frac{v\,(75-29\,v^2)}{2}\Big\}\ldots\end{aligned}\qquad\text{(B.3)}$$

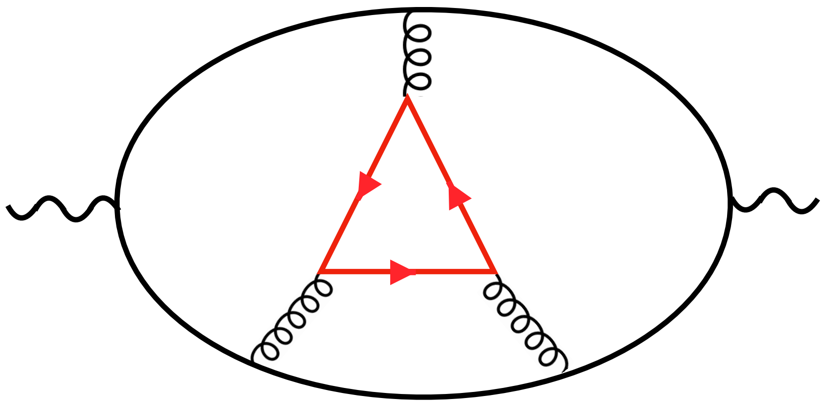
[Chetyrkin, Kuehn, Steinhauser, arxiv : 9606230]

Preliminary N3LO results

n_f contributions :



+ new topologies, such as:

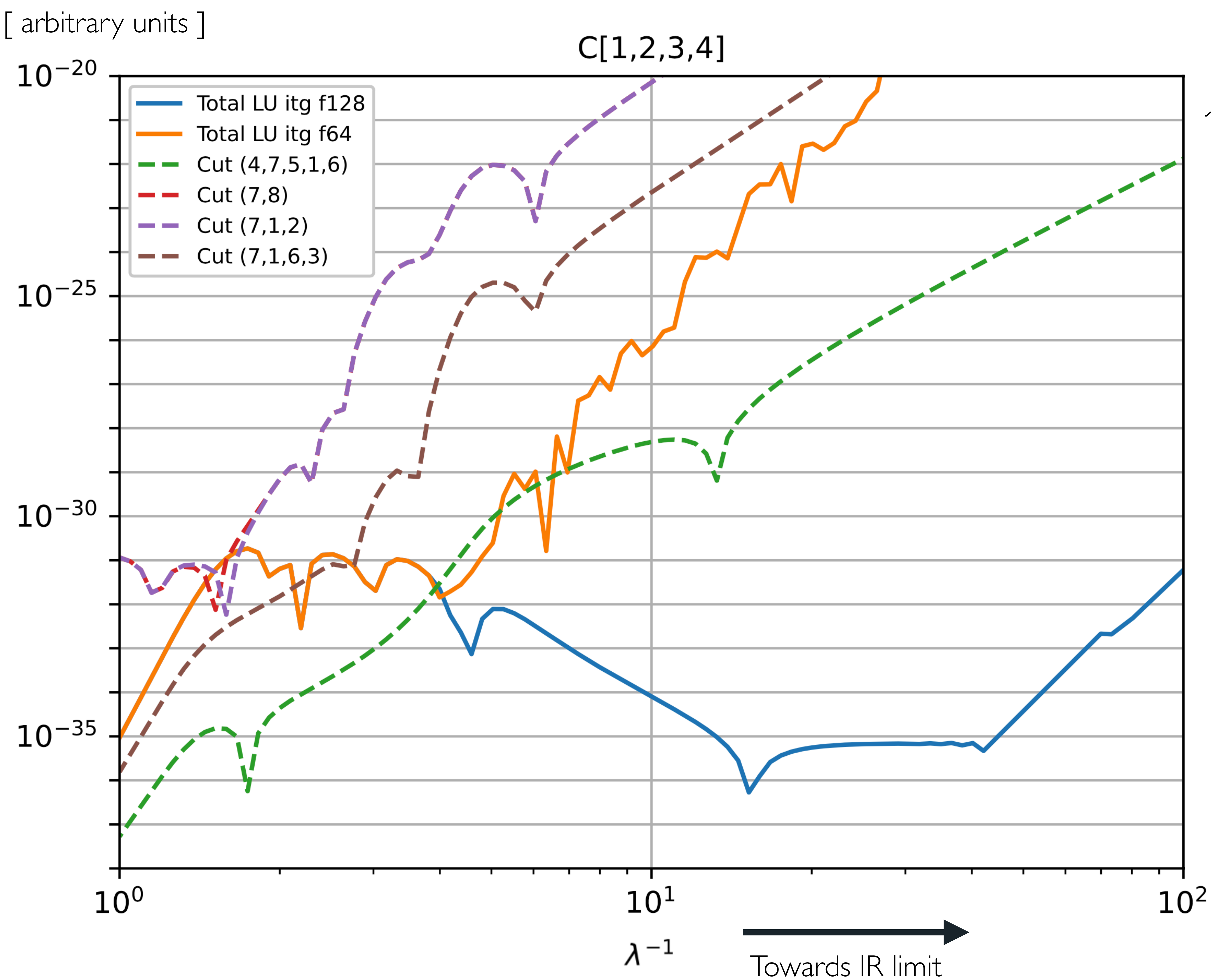


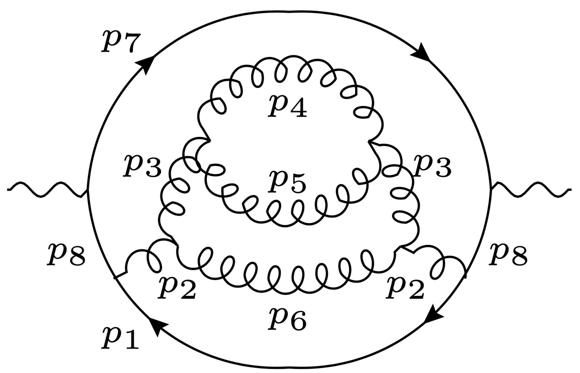
$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f),(\mathrm{MC\ LU})} = -77.1(1.7)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f)} = -C_F^2\left(\frac{29}{2} - 152\zeta_3 + 160\zeta_5\right) - C_F C_A\left(\frac{15520}{27} - \frac{88}{3}\zeta_2 - \frac{3584}{9}\zeta_3 - \frac{80}{3}\zeta_5\right) = -76.8086$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

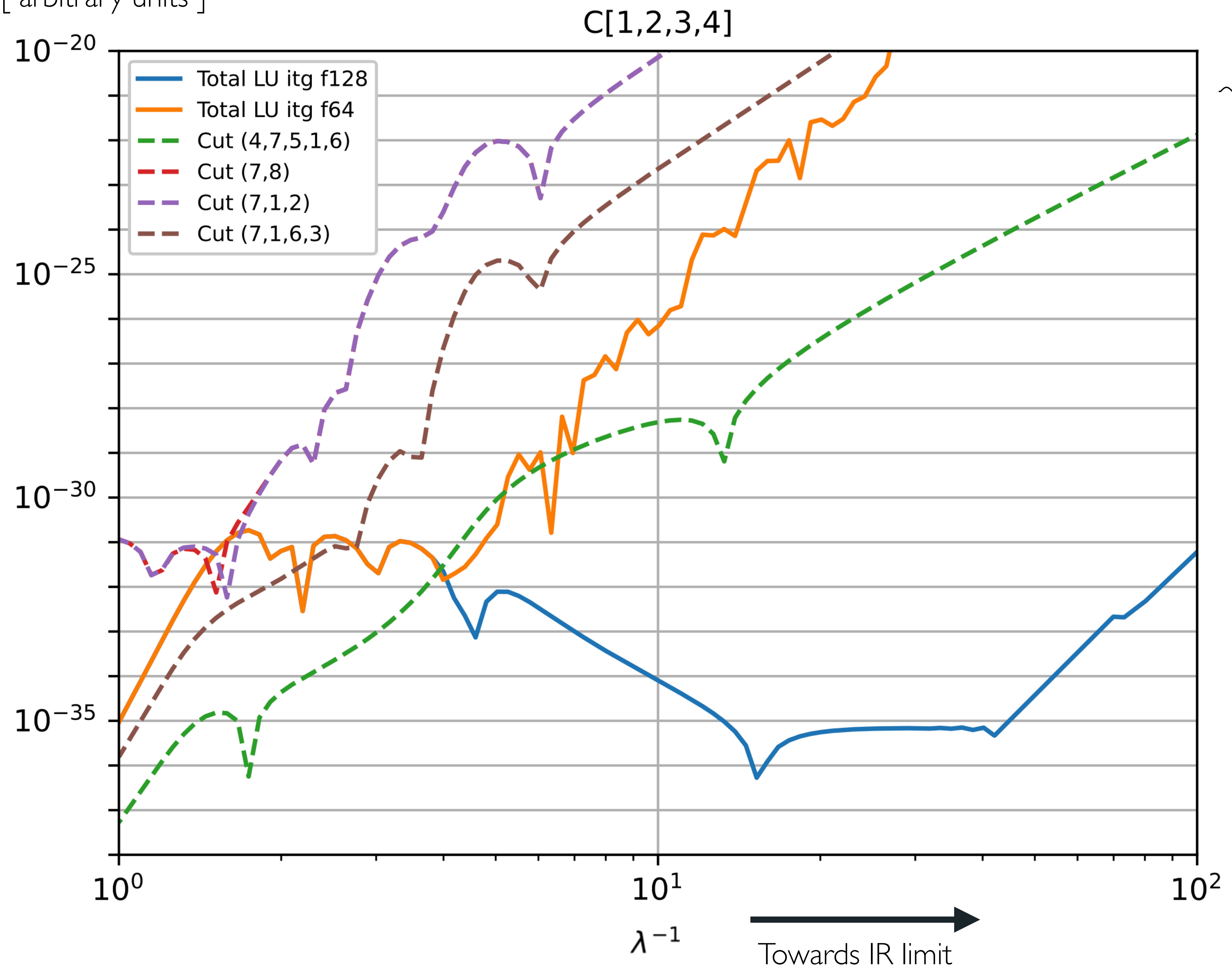
Testing IR quadruple collinear Limits

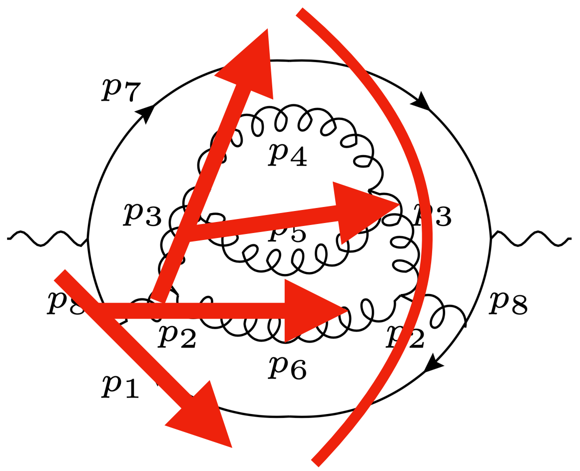




Testing IR quadruple collinear Limits

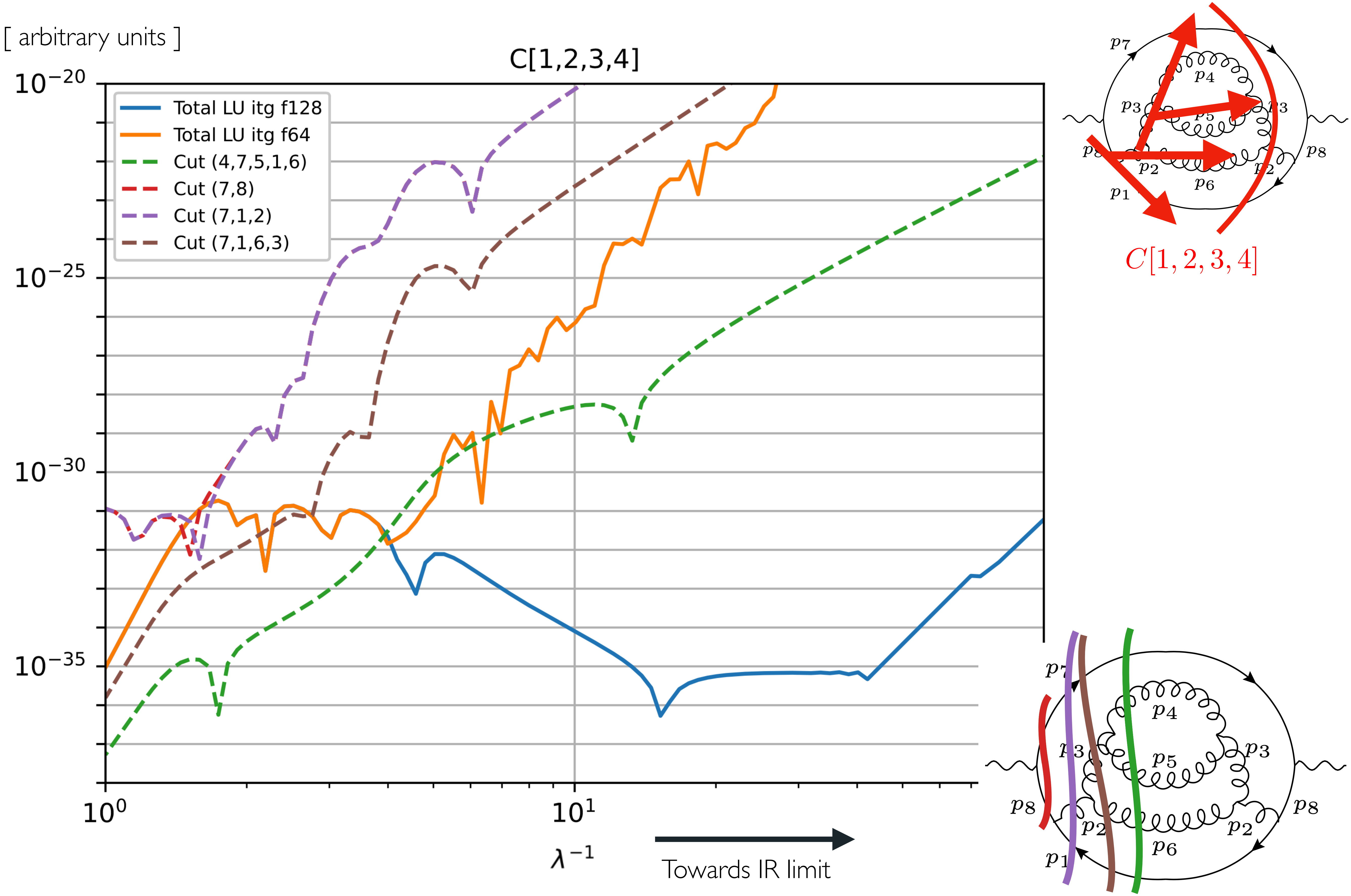
[arbitrary units]





$C[1,2,3,4]$

Testing IR quadruple collinear Limits



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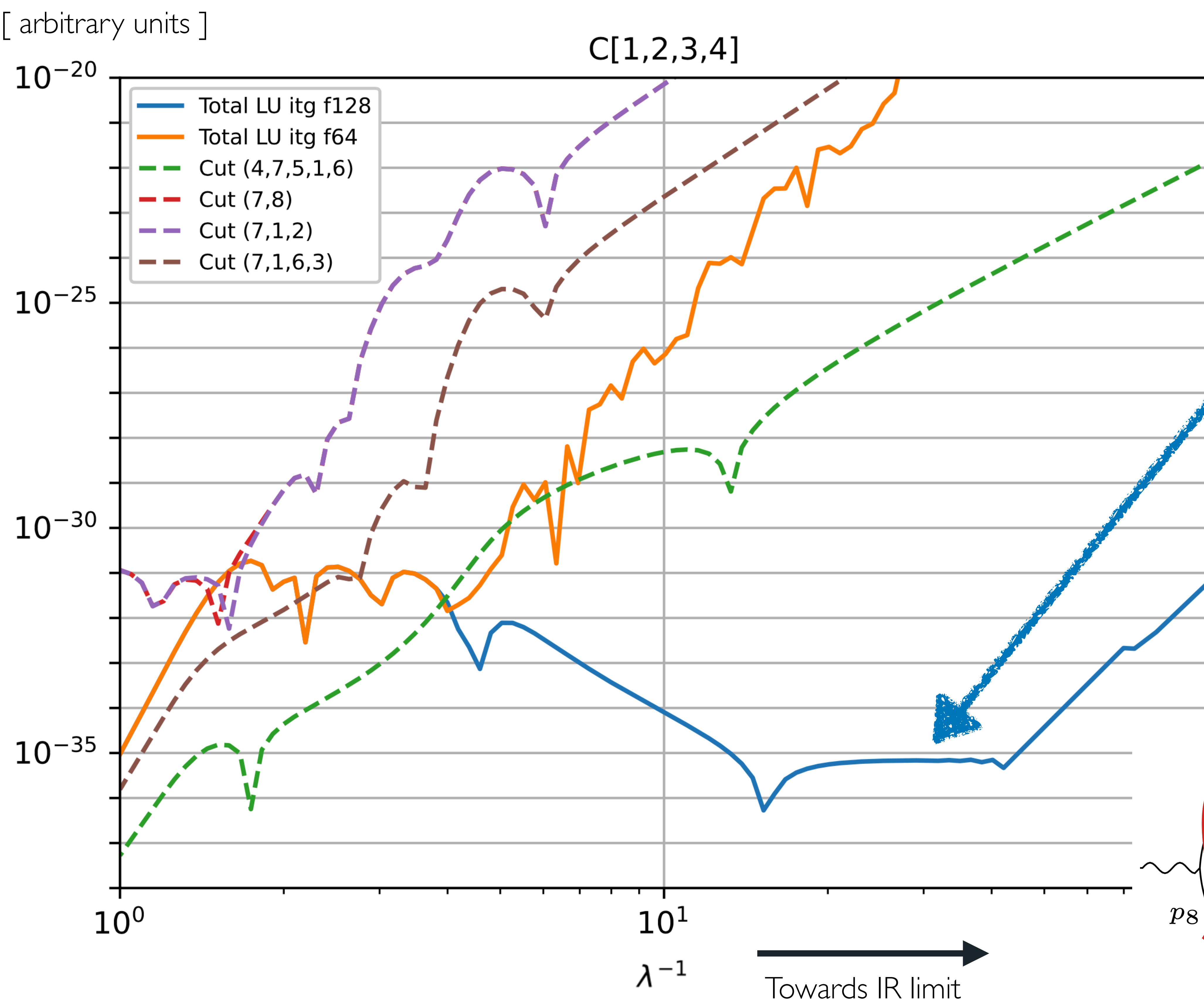
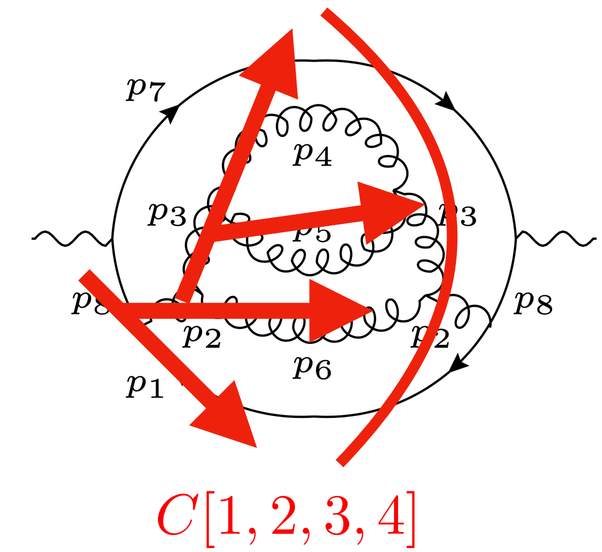
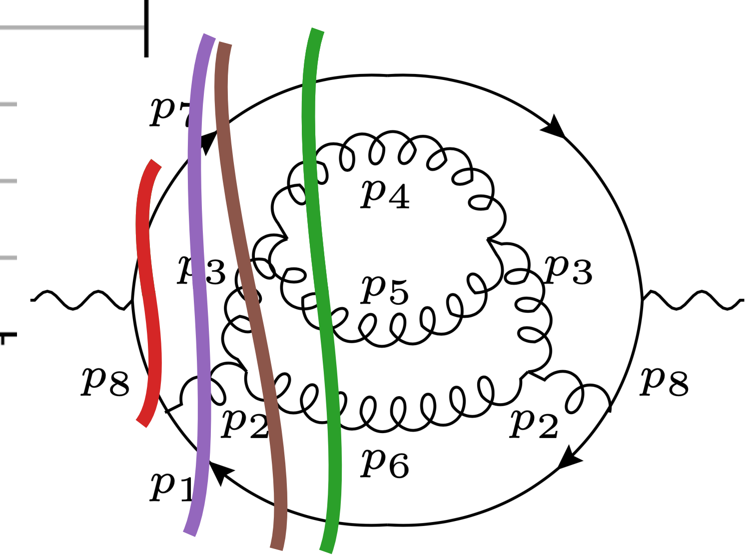




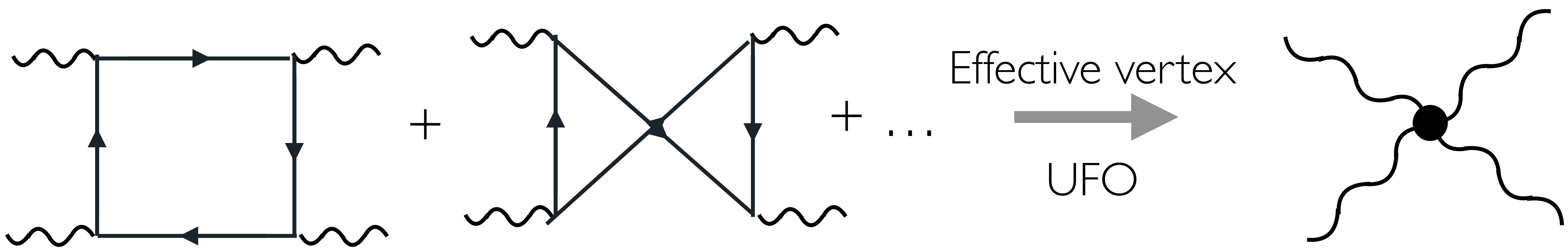
LU integrand always **goes to a constant** on **collinear limits** without incl. *any* scaling of the measure !
No residual integrable singularity.



Photon-Photon scattering in HI UPC

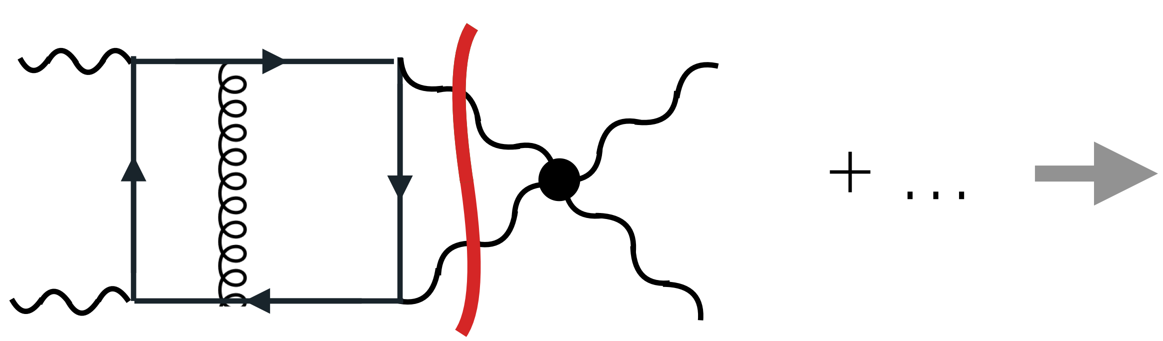
laboration with H.S. -Shao, M. Fraaije, E. Chaubet]

- Too early to present / show much, but alphaLoop delivered a first NLO unknown x-sec!



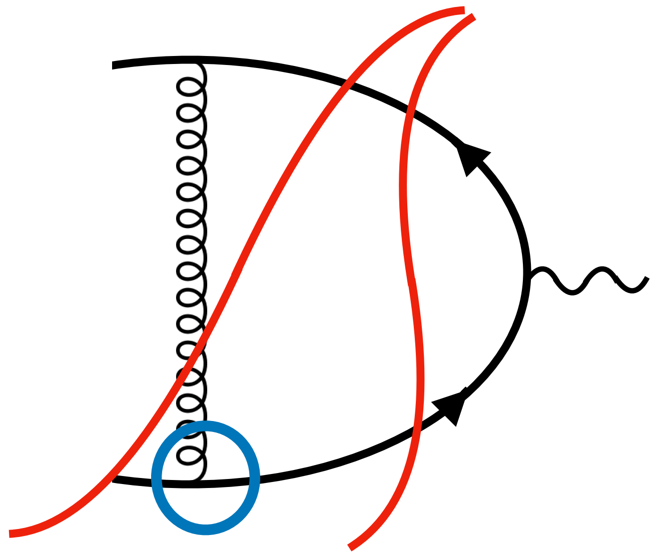
$$\sim \sum_{i=1}^{3} C_i(s,t,u,m_f^2)\, T_i^{\mu_1\mu_2\mu_3\mu_4}(\{p_i\})$$

- In alphaloop we then do



Yielding our first "prediction" for a piece of a yet unknown cross-section: NLO photon scattering. Successful validation vs analytics!

Initial-state Singularities



KLN CAN WORK FOR INITIAL-STATE !

Initial-state singularities: Idea

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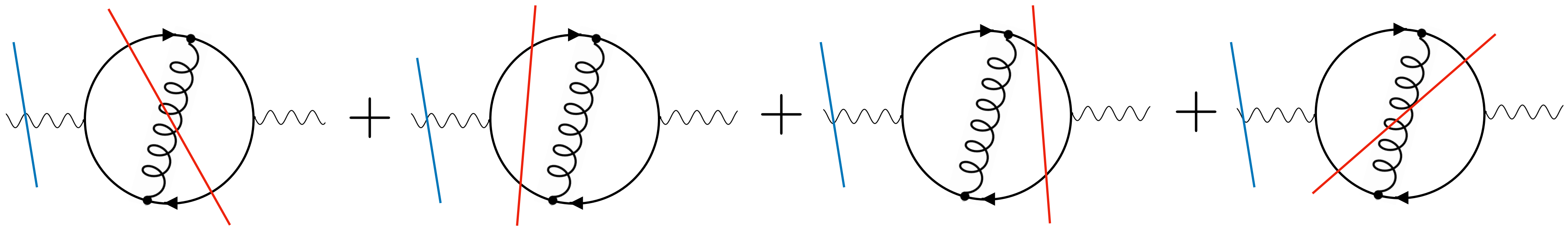
For $e^+e^- \to 2j$ @ NLO we used



(Including all degenerate configurations, higher final-state multiplicities)

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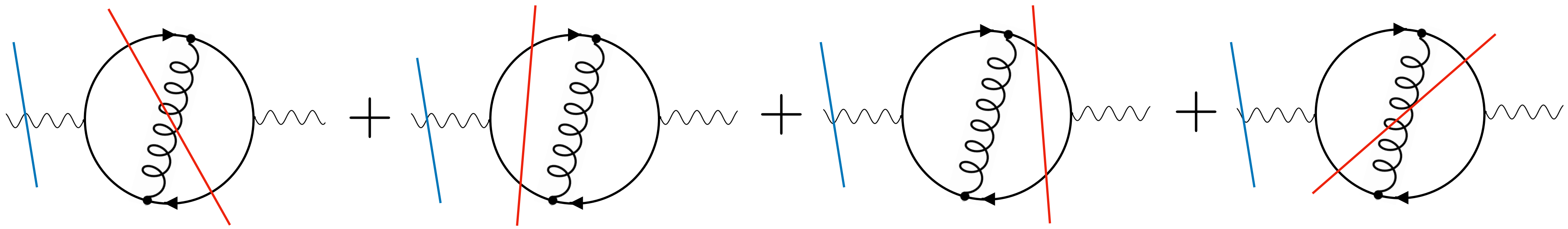

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Flip it, and obtain the answer for Drell-Yan, $2\,\mathrm{j} \rightarrow e^+e^-$

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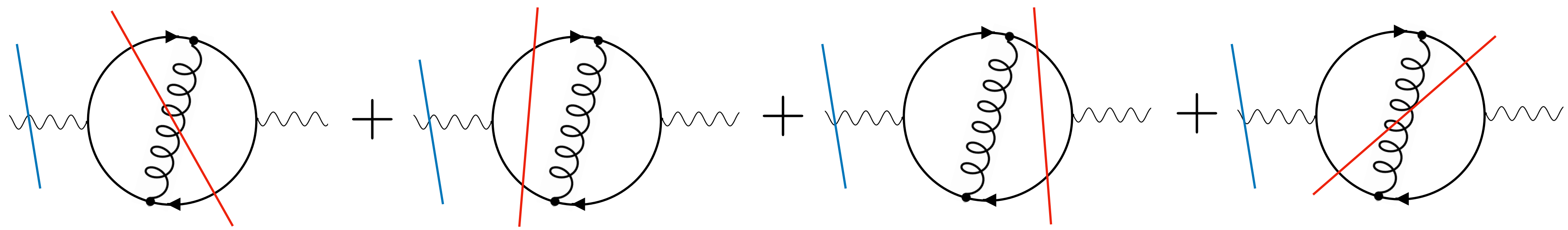

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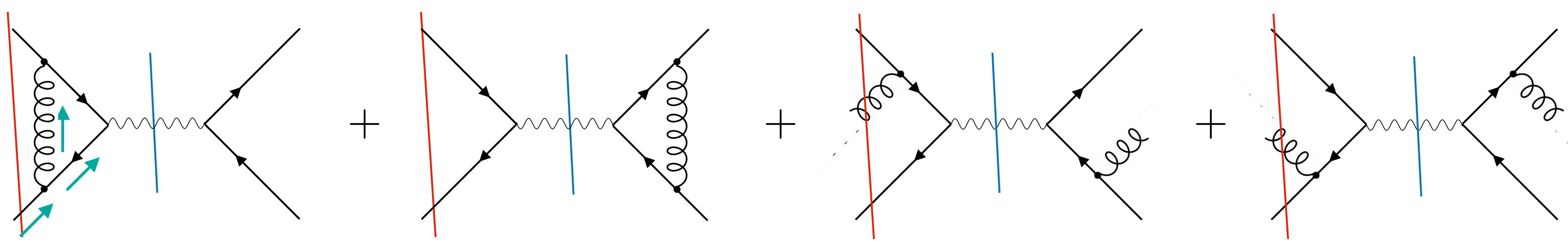

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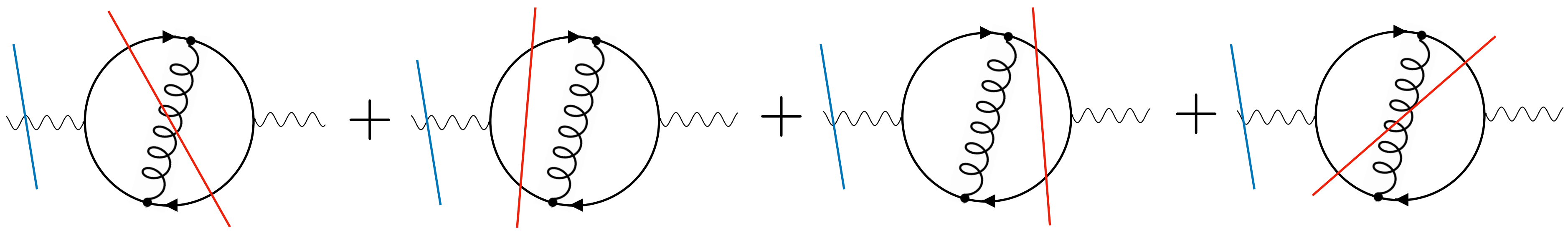

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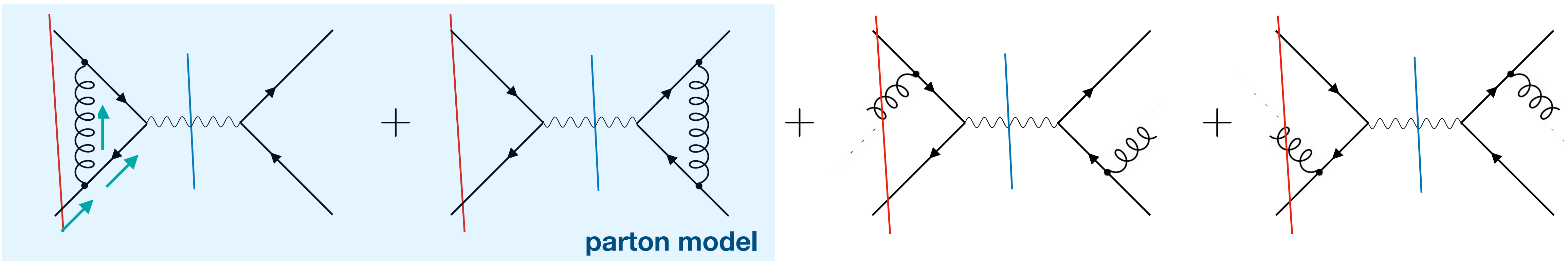

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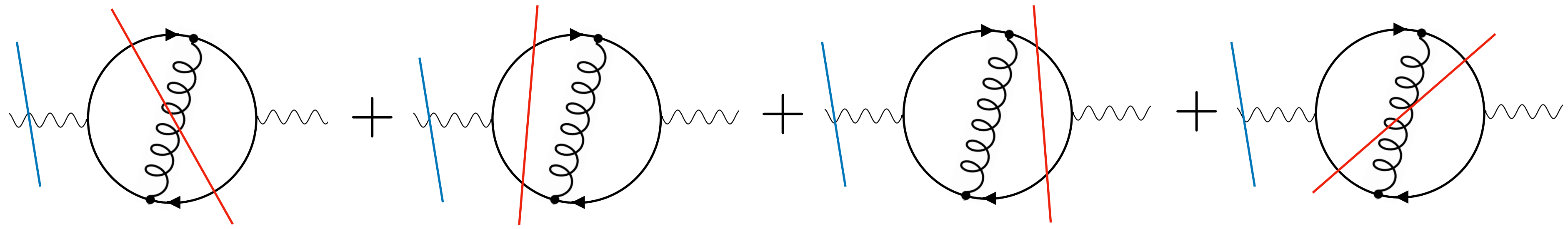

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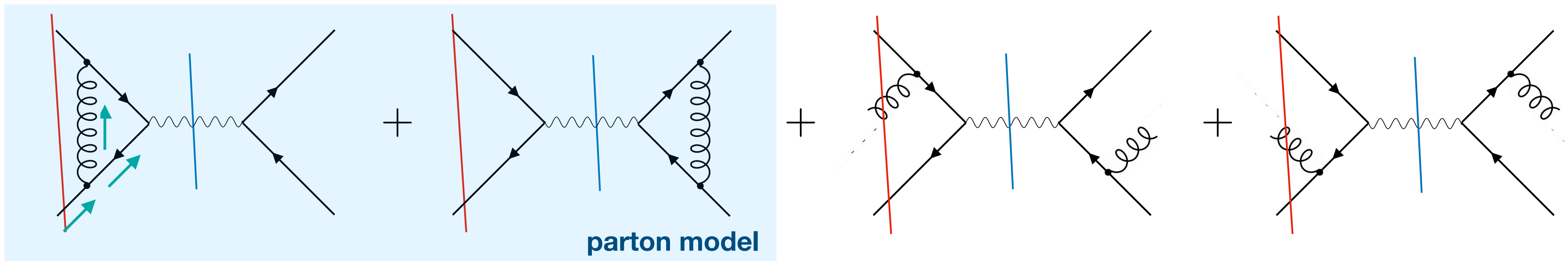
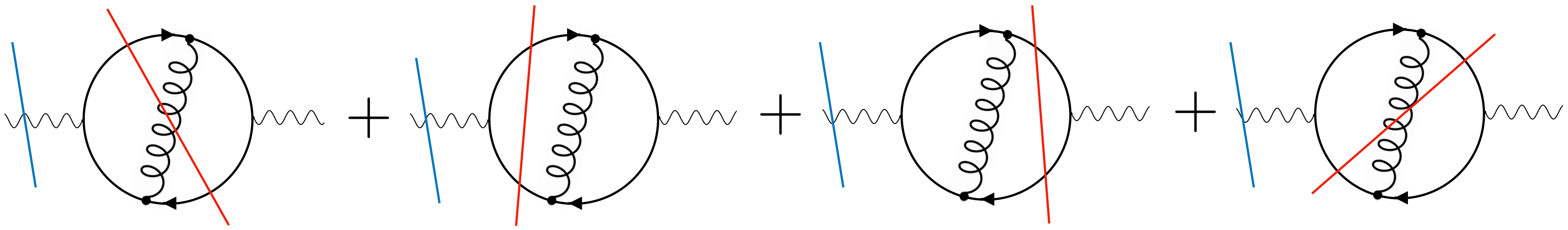


The initial state singularity is now absent!

Initial-state singularities: Idea

For $e^+e^- \to 2j$ @ NLO we used



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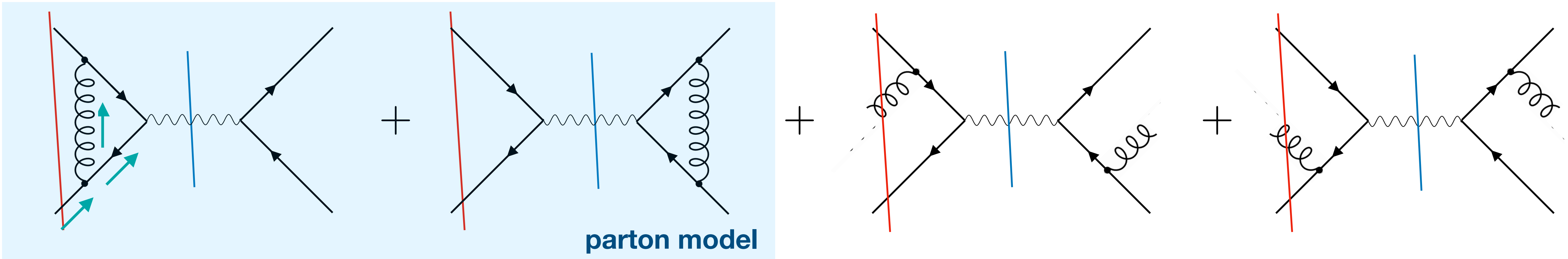

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Include degenerate initial states $\longrightarrow$ **Higher multiplicity initial states**

Initial-state singularities: Idea

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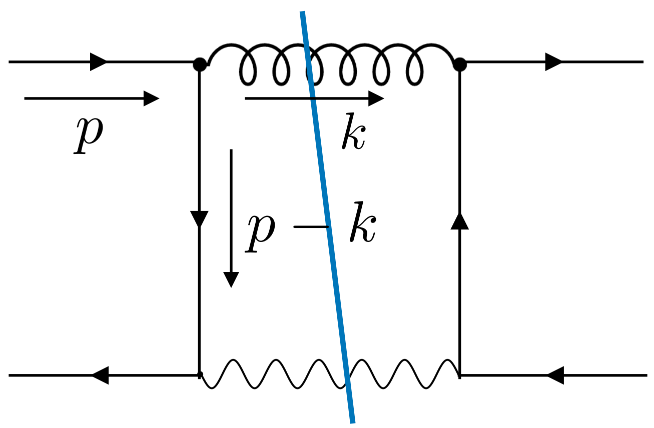
What about this diagram?



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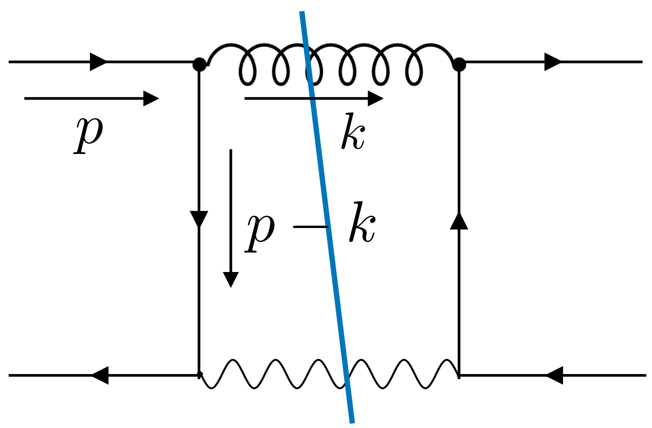

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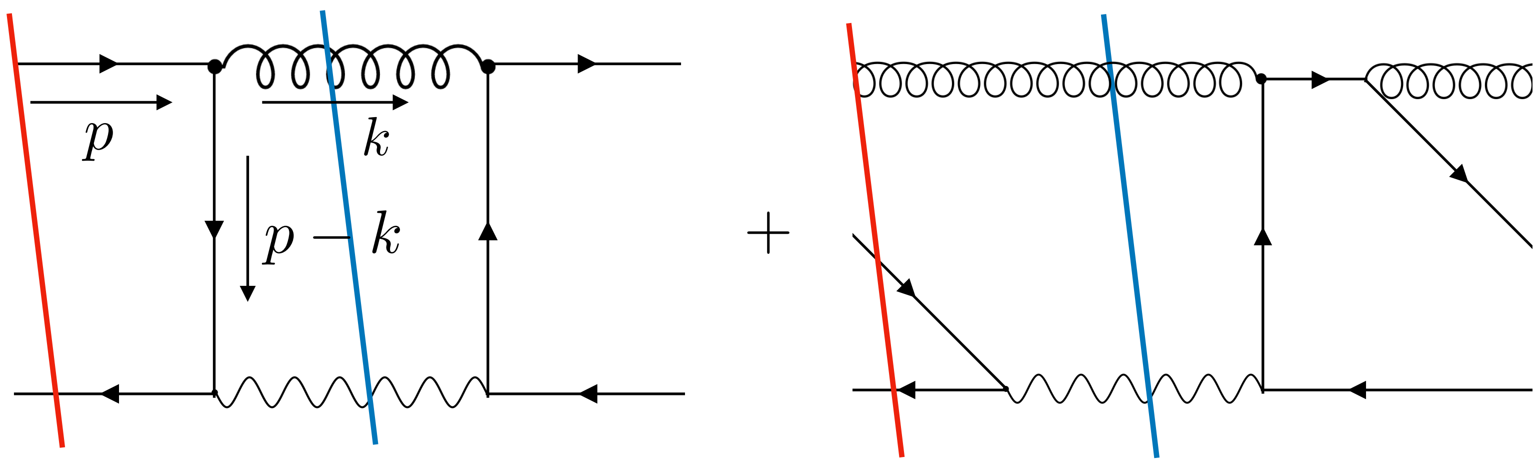

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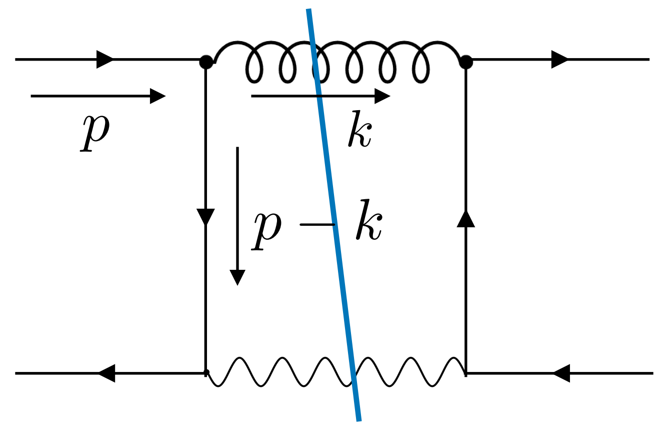

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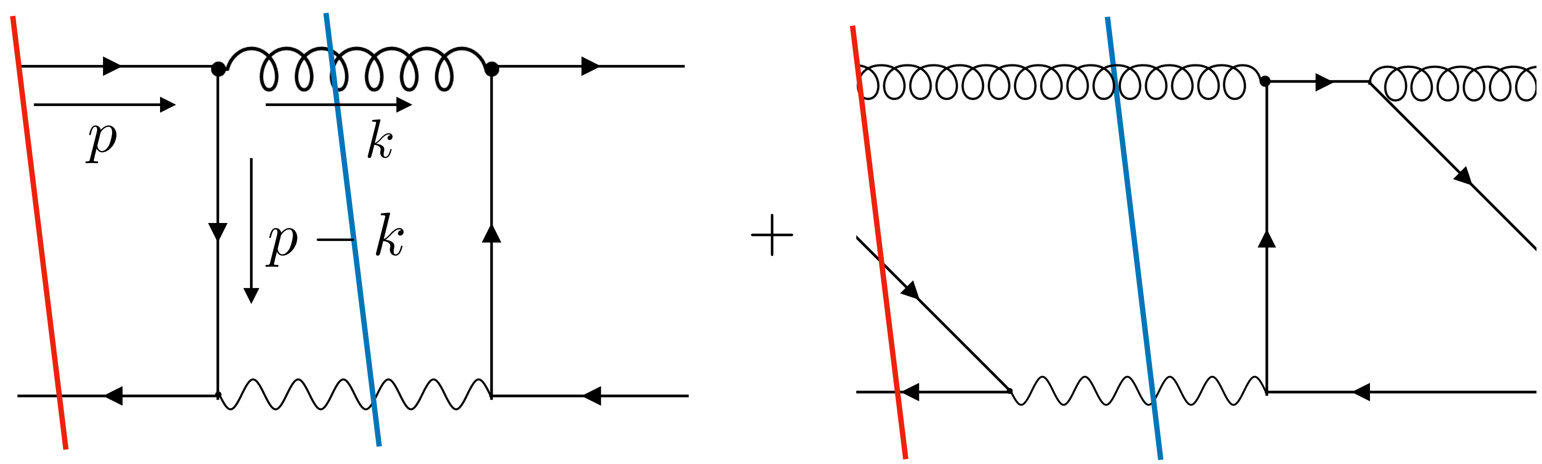

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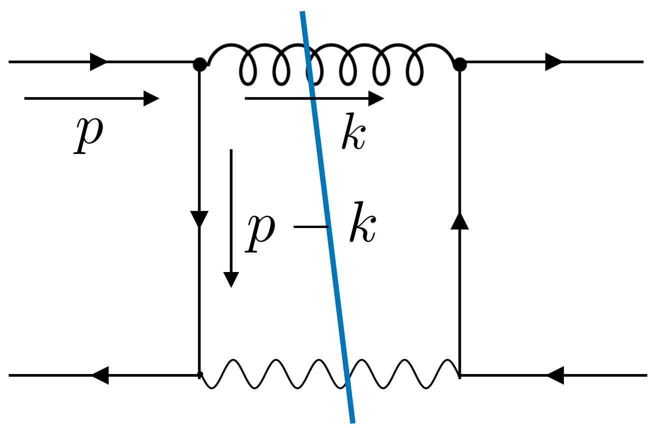

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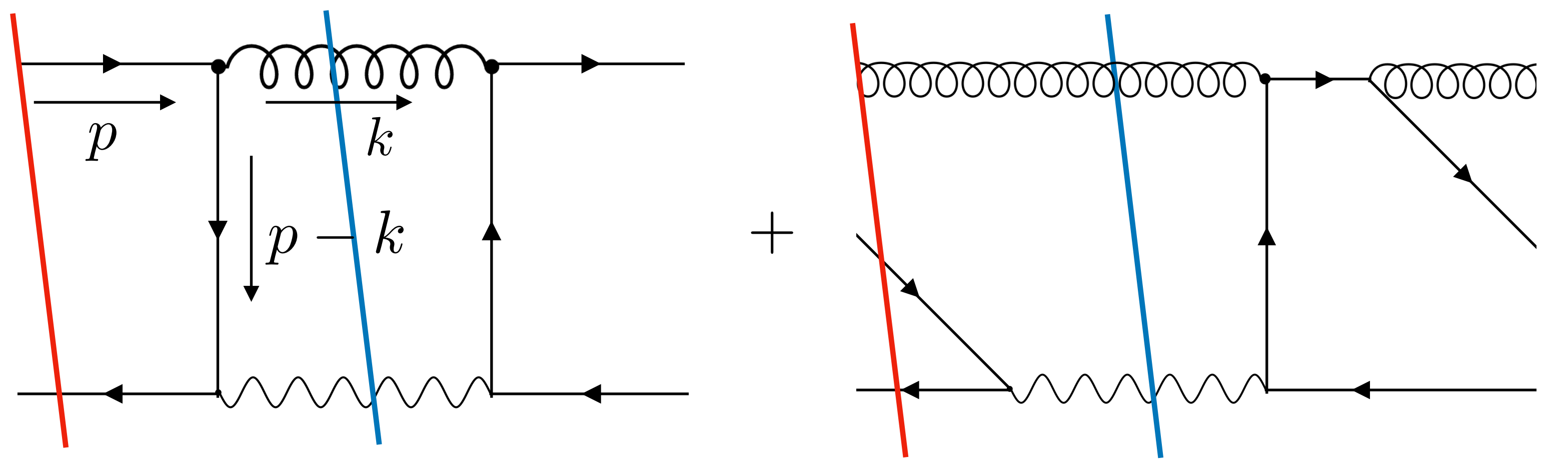

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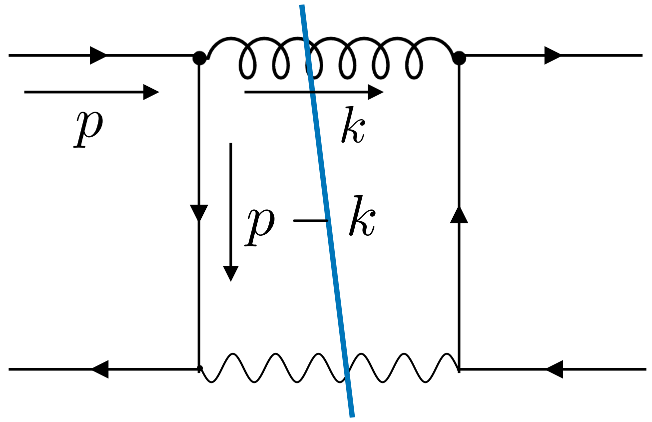

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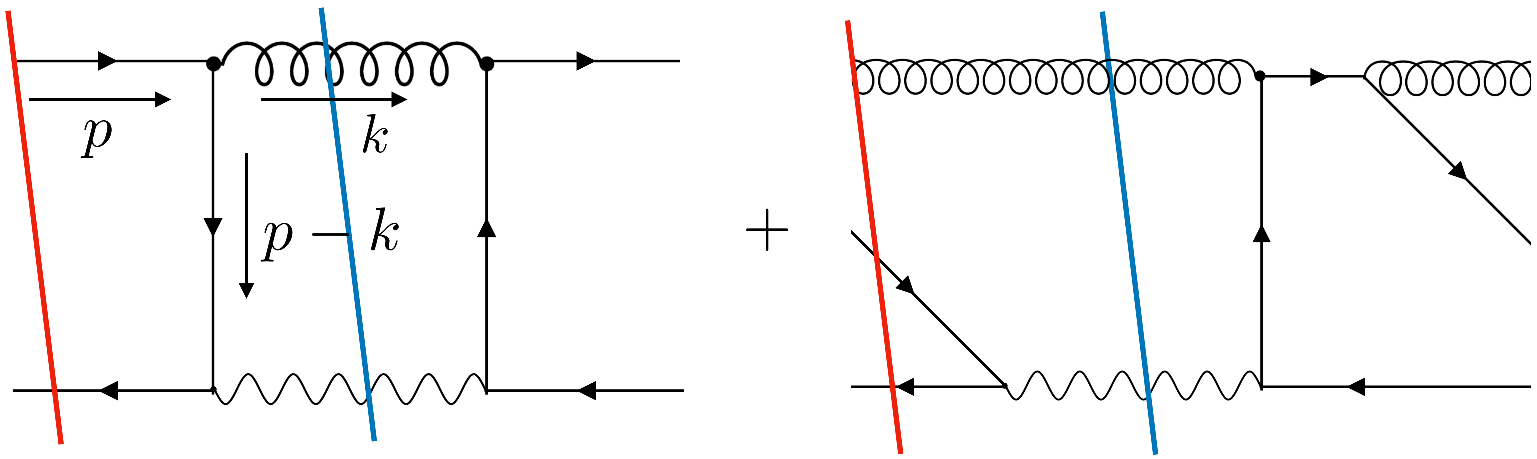

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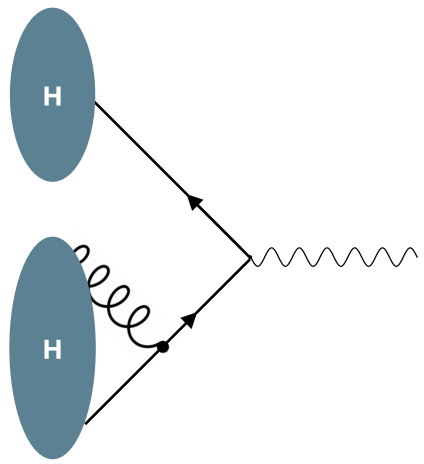
But also more recently, they were studied in:

Frye, Hannesdottir, Paul, Schwartz, Yan
arXiv:1810.10022 (2019)

Initial-state singularities: preliminary tests

This argument suggests that, in order to maintain IR-finiteness, one requires more than two initial state partons



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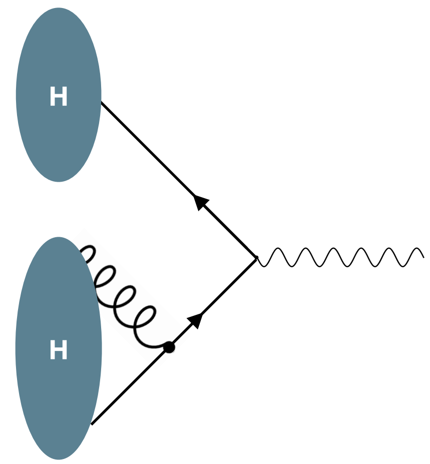
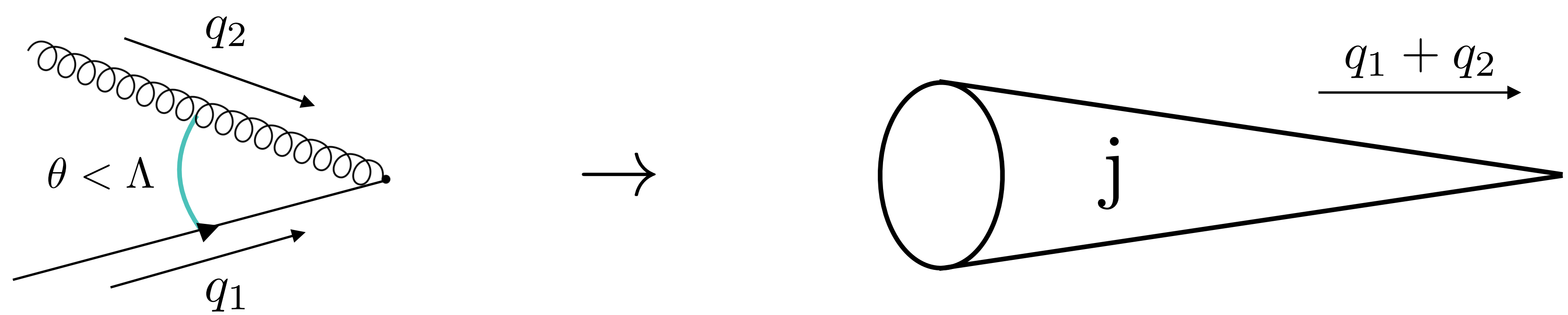


and that the multiple partons should be clustered into **two jet-like objects** that resemble boosted hadrons



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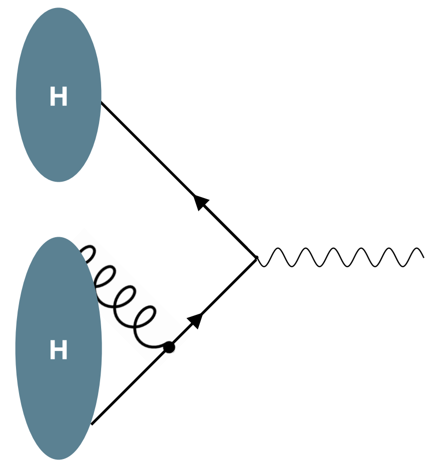

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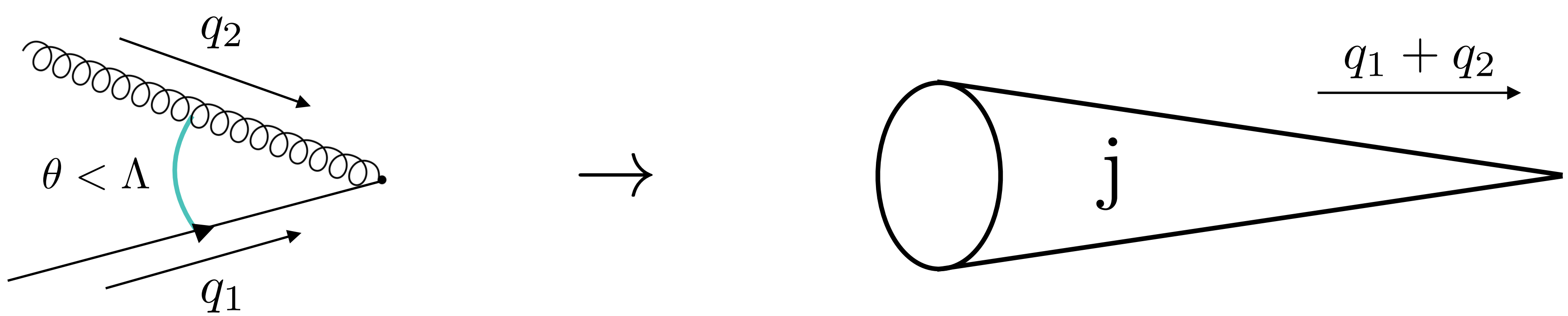

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Cluster initial states analogously to final states: symmetry initial-final state

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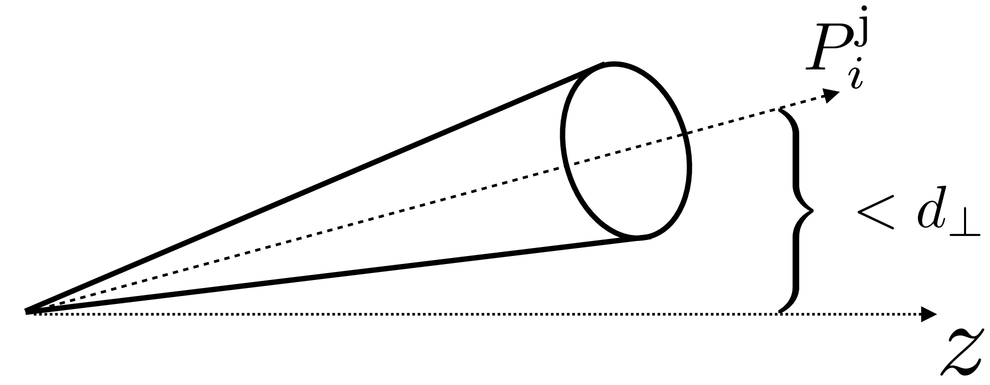

If the scale is zero, the jet lies **exactly** on the z axis

If $d_\perp = 0$ the two jets are exactly back-to-back. This is equivalent to the parton's model

$$p_1 = (x_1\sqrt{s}, 0, 0, x_1\sqrt{s}), \quad p_2 = (x_2\sqrt{s}, 0, 0, -x_2\sqrt{s})$$

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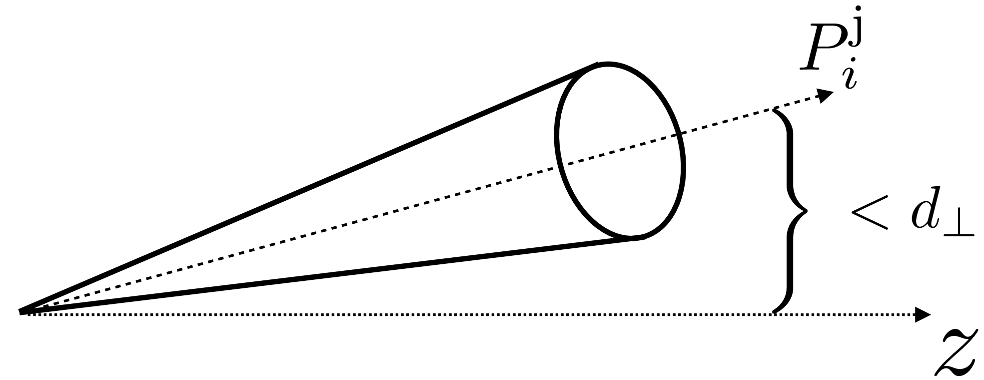

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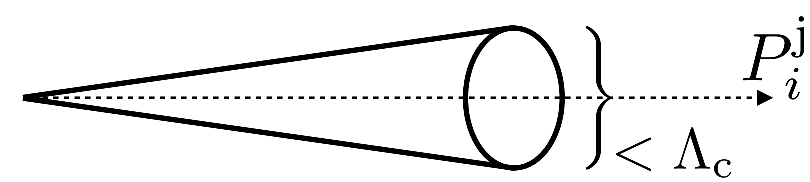

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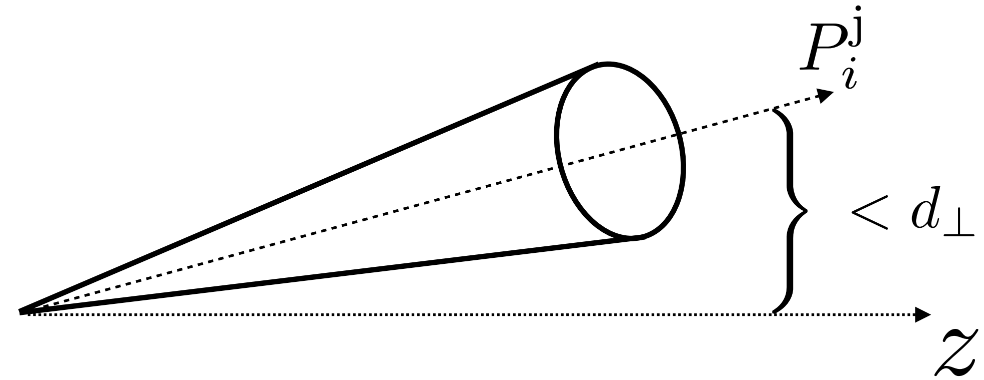

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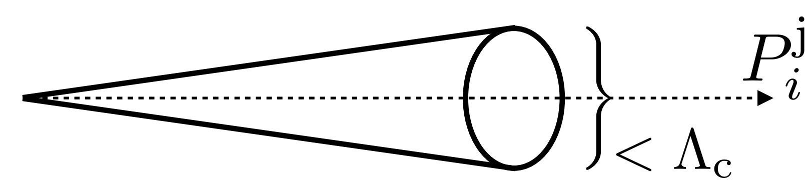

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Λ_{c} **is the equivalent of the factorisation scale!** $\approx \log(\Lambda_{\mathrm{c}})$

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- Vary the factorisation scale Λ_c and interpolate the dependence on the factorisation scale **Numerical resummation?**

Banfi, Salam, Zanderighi, arXiv:0407286 (2004)

Initial-state singularities: “PDFs”

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We started with a very generic formalism for scattering

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And we “forced” the initial-state observable to reproduce the usual factorised structure:

$$\sigma(HH \to X + n\,\mathrm{j}) = \int \mathrm{d}x_1 \mathrm{d}x_2 \; f(x_1, \Lambda_\mathrm{c}) \; f(x_2, \Lambda_\mathrm{c}) \, \frac{\mathrm{d}^2 \sigma_p}{\mathrm{d}x_1 \mathrm{d}x_2}(2\,\mathrm{j} \to X + n\,\mathrm{j}, \Lambda_\mathrm{c})$$

But we did not need to start from this factorised ansatz!

Initial-state singularities: preliminary tests

Numerical example result for this finite sum of two interference diagrams:

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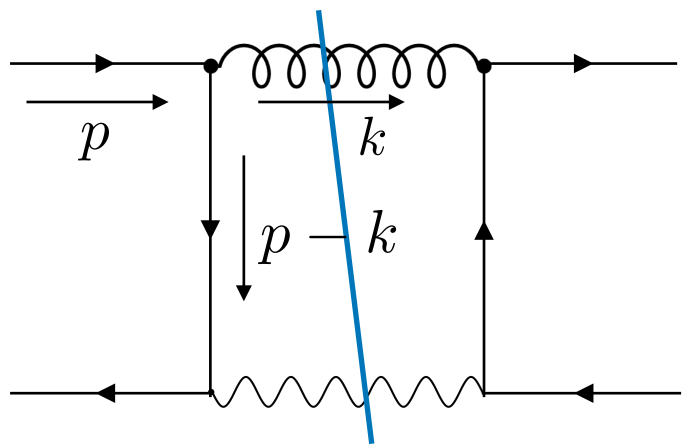

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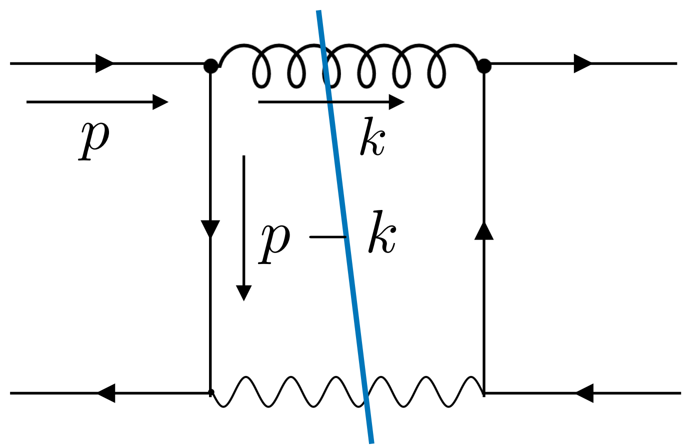

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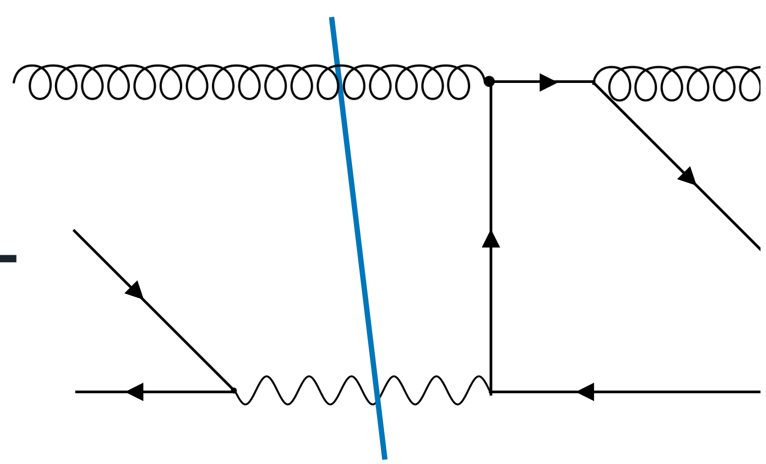

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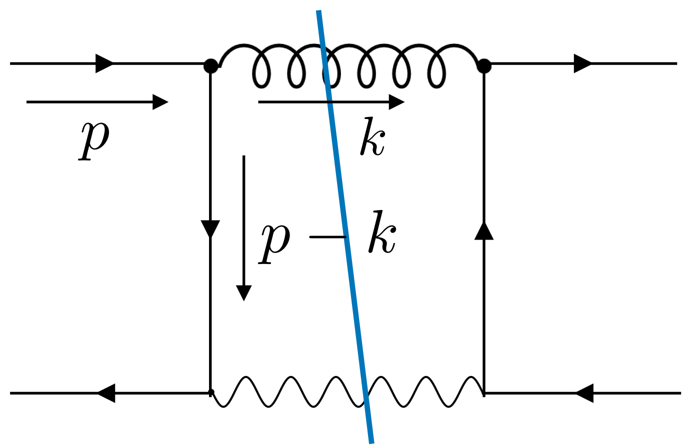

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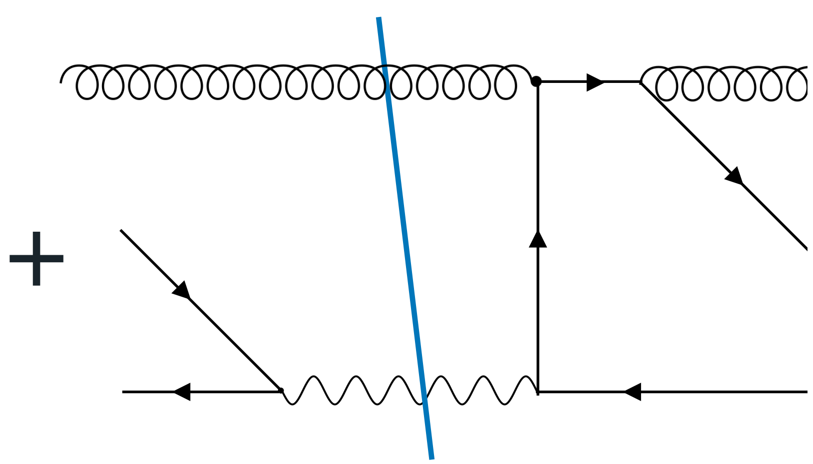

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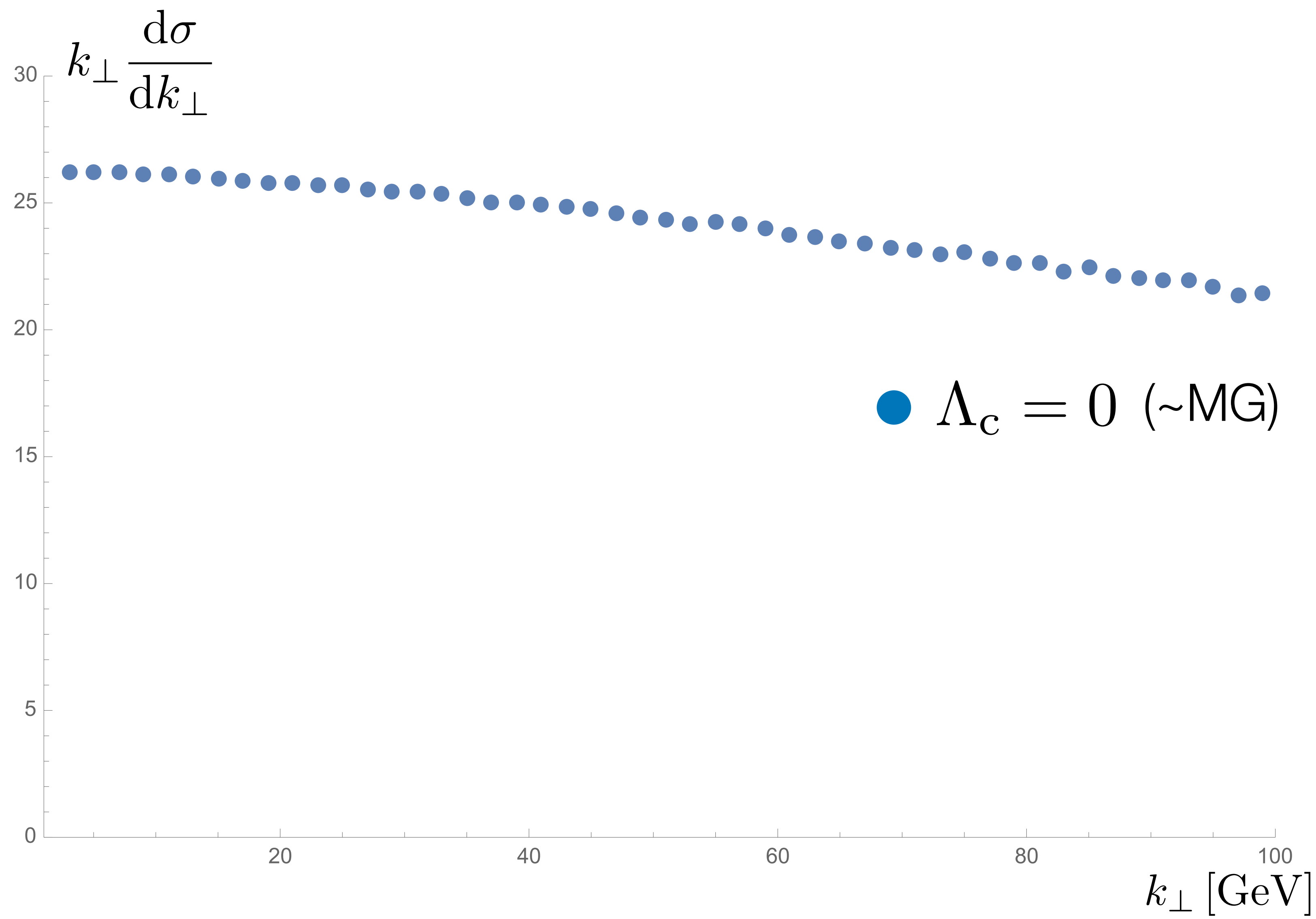

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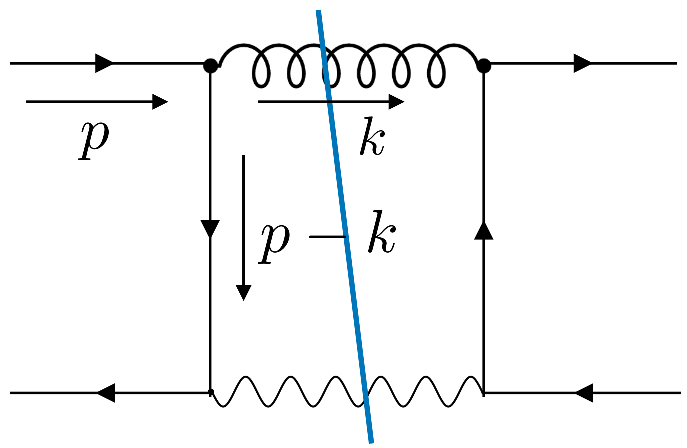

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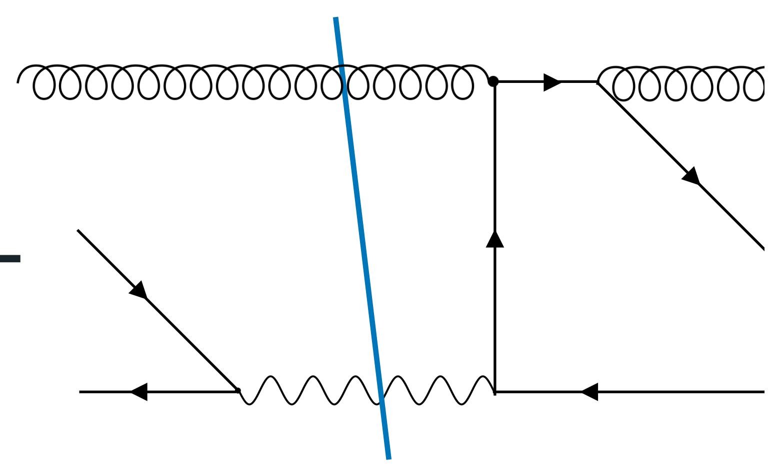

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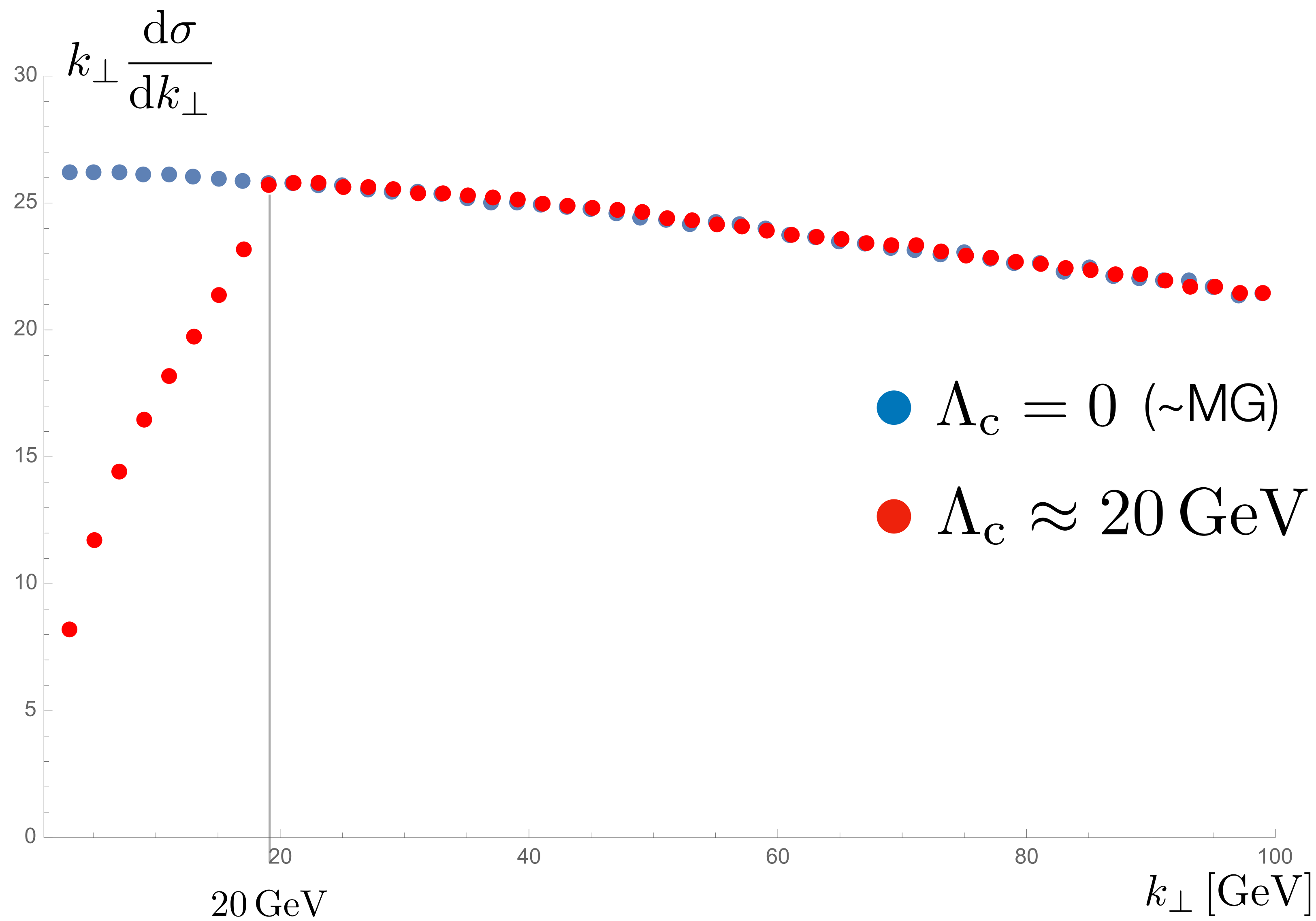

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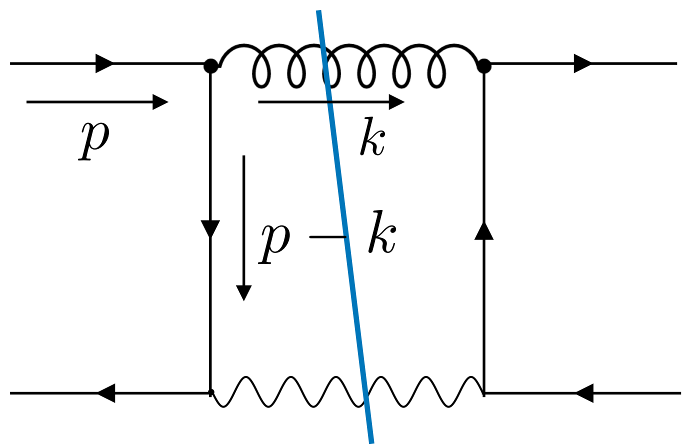

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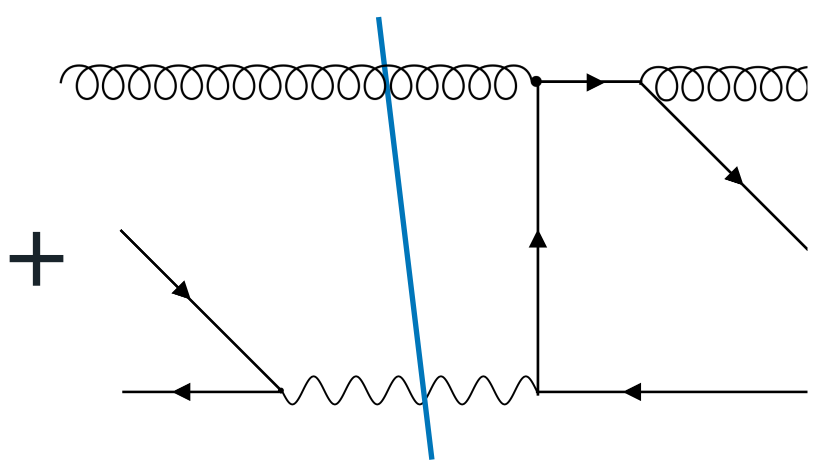

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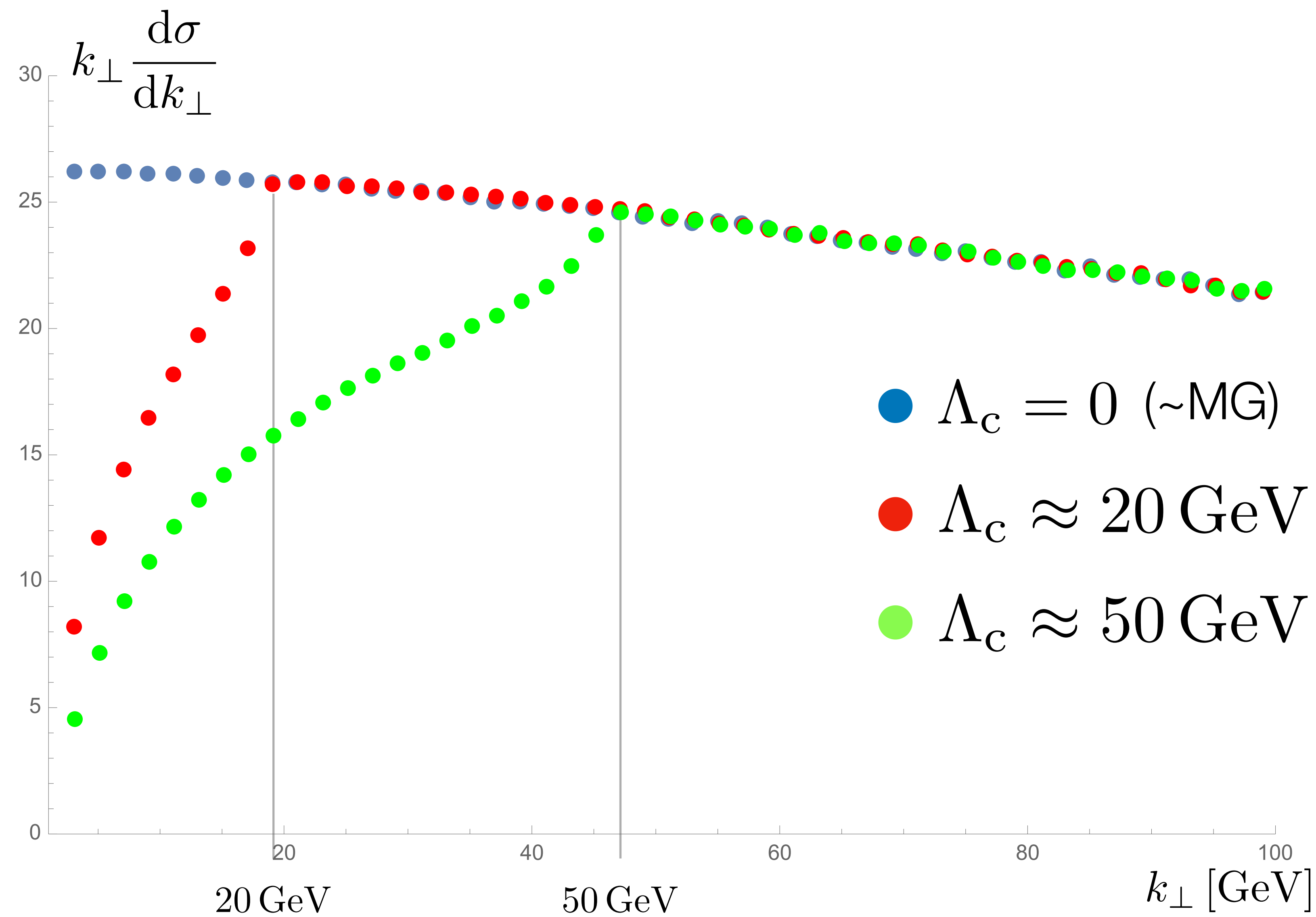

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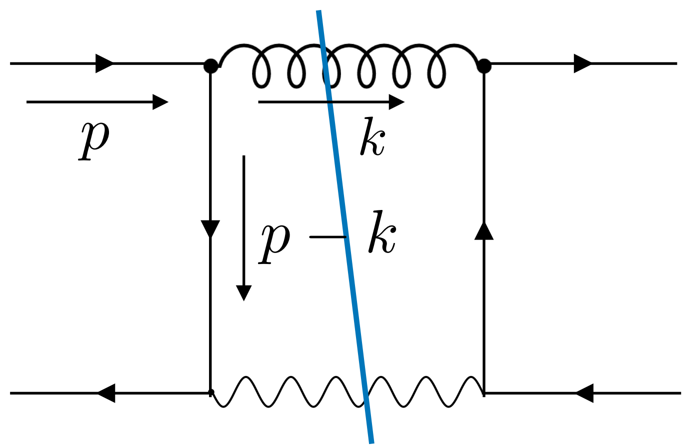

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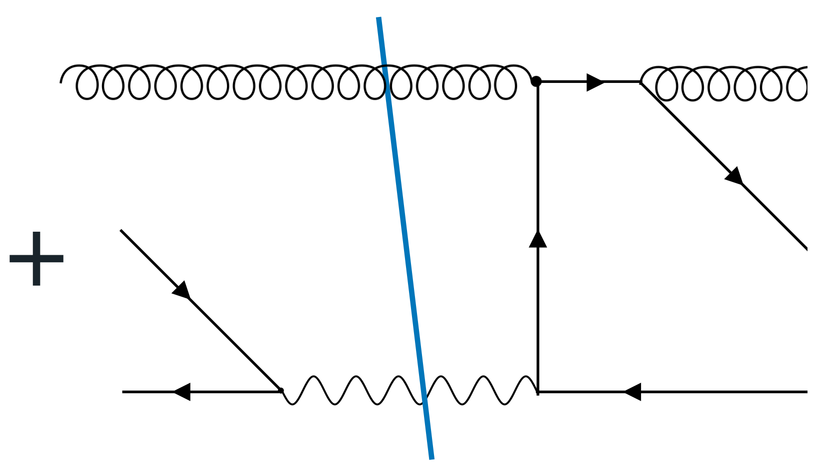

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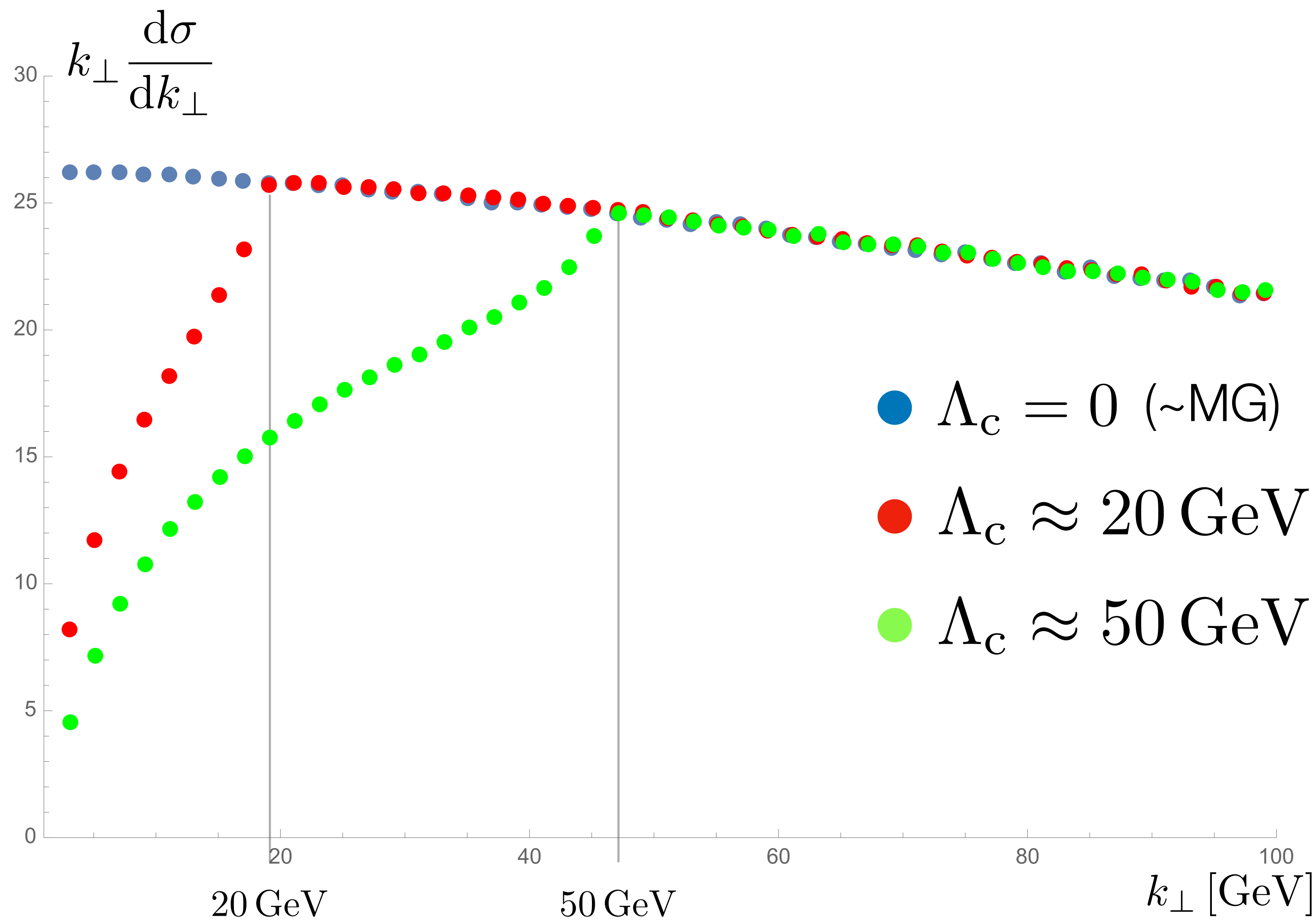

Note that : for $\Lambda_c > 50\,\mathrm{GeV}$
the distribution does not change anymore
because highest separation of two partons in a jet is of **order of Z mass**

Initial-state singularities: scale dependence

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$$I(\Lambda_c) = \int_0^{100} \mathrm{d}k_\perp \frac{\mathrm{d}\sigma}{\mathrm{d}k_\perp}(\Lambda_c)$$

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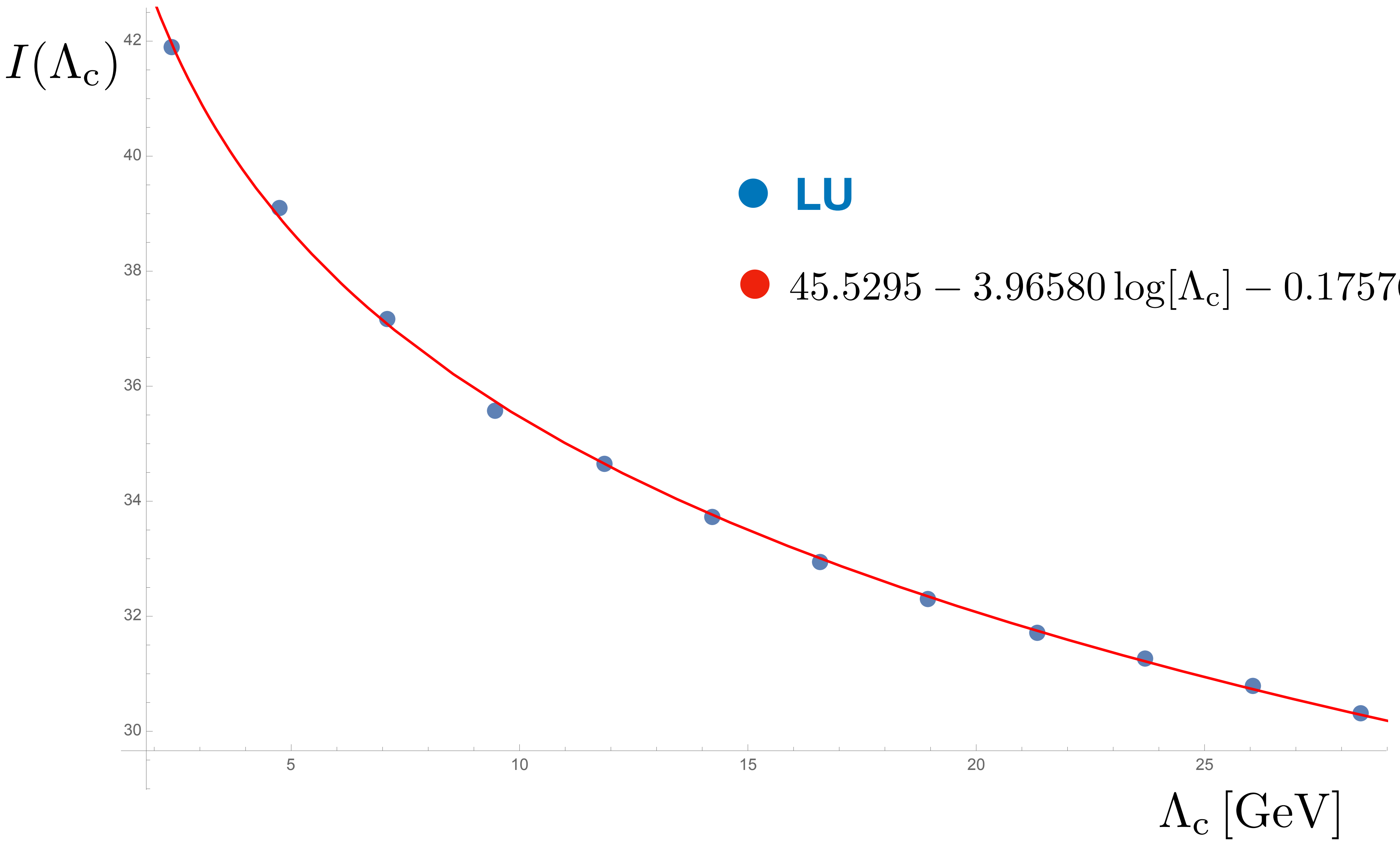

Subtraction vs Local Unitarity

- One can introduce the following local (in x) **counterterm**:

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left(\left[\int_0^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

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- To finally arrive at a **finite** result, differential in $x \in [0, 10]$

$$= \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \lim_{\epsilon\to 0}\left(\cancel{\log(\epsilon)} + \gamma - \mathrm{Ei}(-1) + \log(10) - \cancel{\log(\epsilon)}\right)\mathcal{J}(0)$$

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$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) - \frac{1}{x}\mathcal{J}(0)\right] + \left(\left[\int_0^1 dy \frac{-e^{-y}}{y}\right] + \left[\int_0^{10} dx \frac{1}{x}\right]\right)\mathcal{J}(0)$$

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- To finally arrive at a **finite** result, differential in $x \in [0, 10]$

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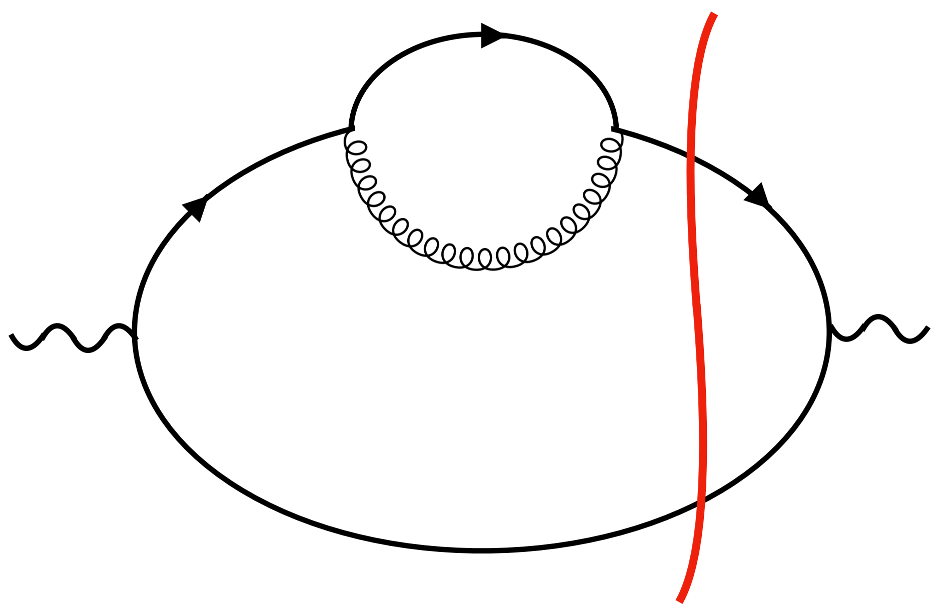
- **Local Unitarity** aligns the measure and combines "real and virtual":

$$\sigma^{(R+V)}(\mathcal{J}) = \int_0^{10} dx \left[\frac{\cos(x)}{x}\mathcal{J}(x) + \frac{-e^{-x}}{x}\mathcal{J}(0)\Theta(1-x)\right]$$

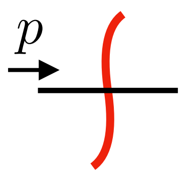
Locality Unitarity: raised propagators

[Capatti, VH, Ruijl, arxiv : 2203.11038]

In **LU**, we cannot consider ***truncated*** amplitudes only :



Traditional Cutkosky rule



$$= -2\pi \mathrm{i} \frac{\delta(p^0 - E(\vec{p}))}{2E(\vec{p})}$$

$$E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$$

would not apply here !

LOCALITY UNITARITY: RAISED PROPAGATORS

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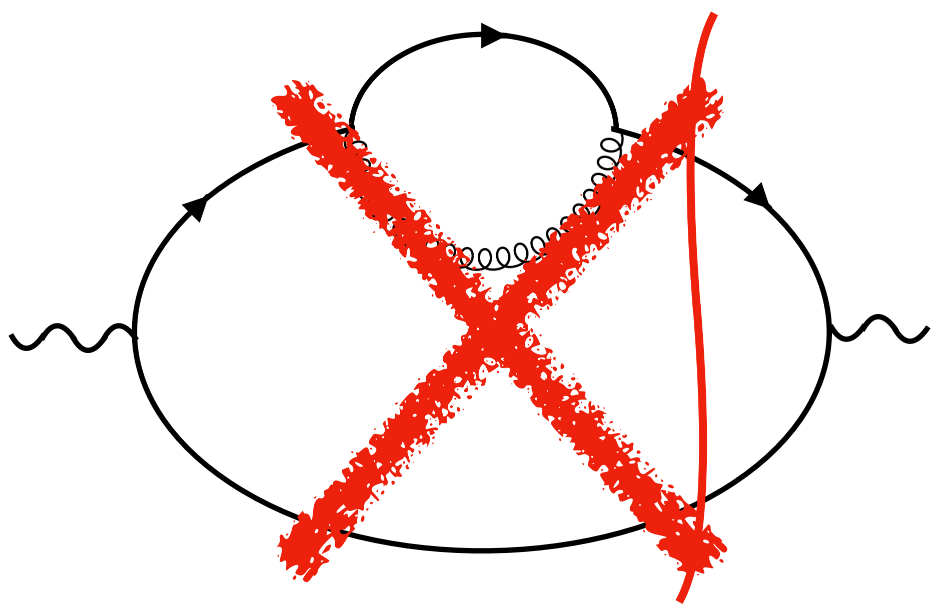

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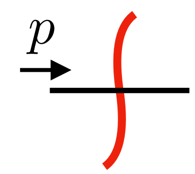

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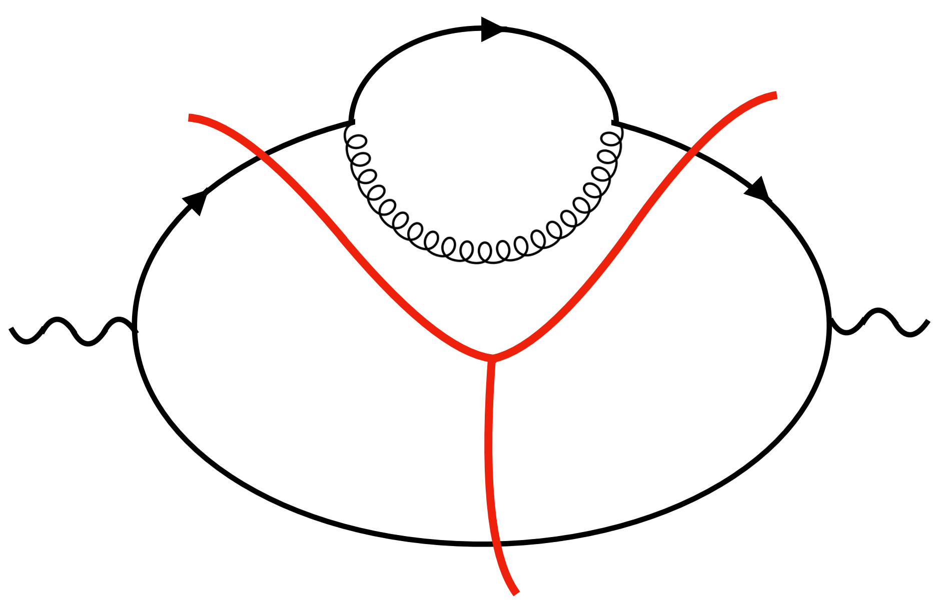
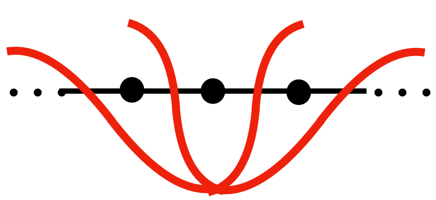
So consider this Cutkosky cut as a **higher-order residue** ➔ **Generalised cutting rule**





$$= -2\pi \mathrm{i} \frac{\delta^{(n)}\left[p^0 - E(\vec{p})\right]}{(p^0 + E(\vec{p}))^2}$$

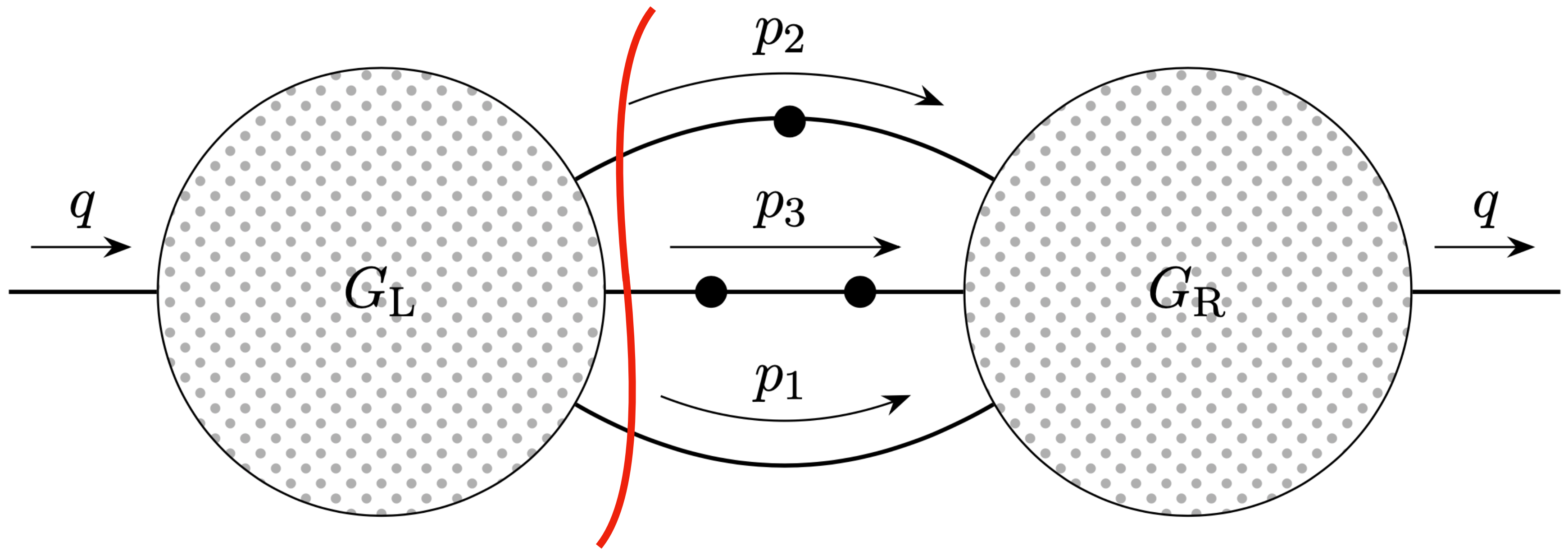
$n -$ times

$$\int \mathrm{d}x\, \delta^{(n+1)}[x] f(x) = \frac{1}{n!} \frac{\mathrm{d}^n f}{\mathrm{d}x^n}\bigg|_{x=0}$$

Locality Unitarity: raised propagators

[Capatti, VH, Ruijl, arxiv : 2203.11038]

This is well understood for **raised loop propagators**,
but for **raised external propagators** of supergraphs, there are subtleties :



$$\propto \delta^{(1)}[p_1^0 - E(\vec{p_1})]\delta^{(2)}[p_2^0 - E(\vec{p_2})]\delta^{(3)}[p_3^0 - E(\vec{p_3})]$$

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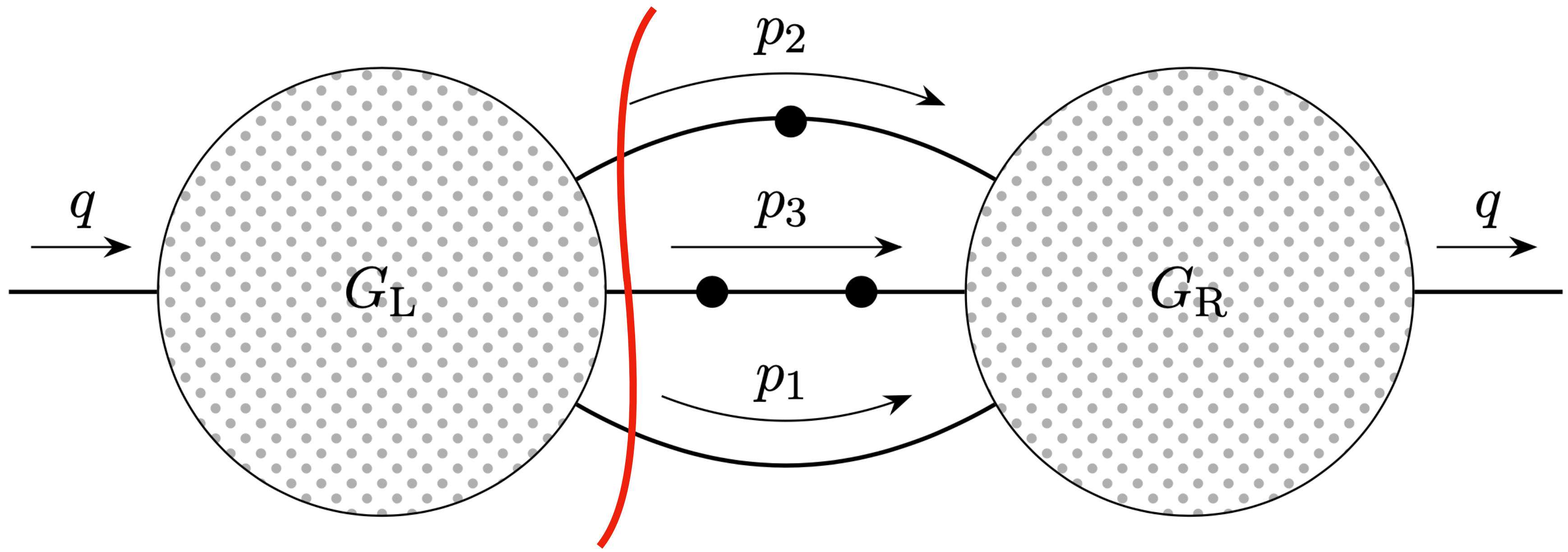

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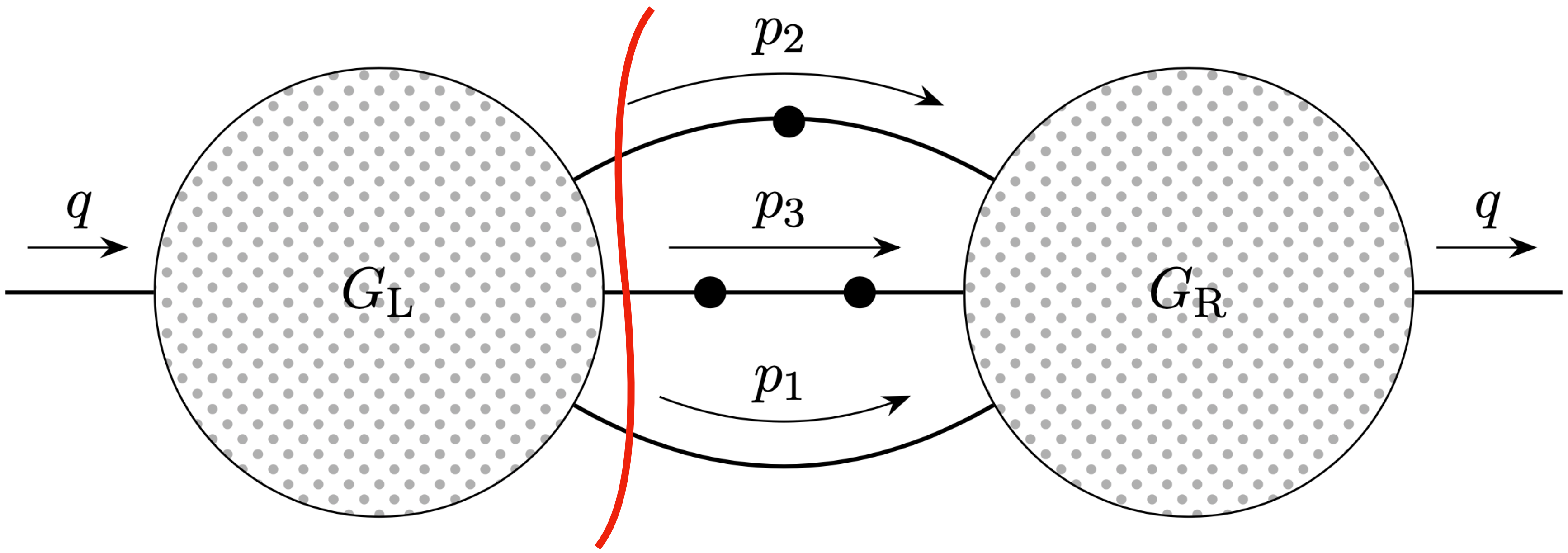

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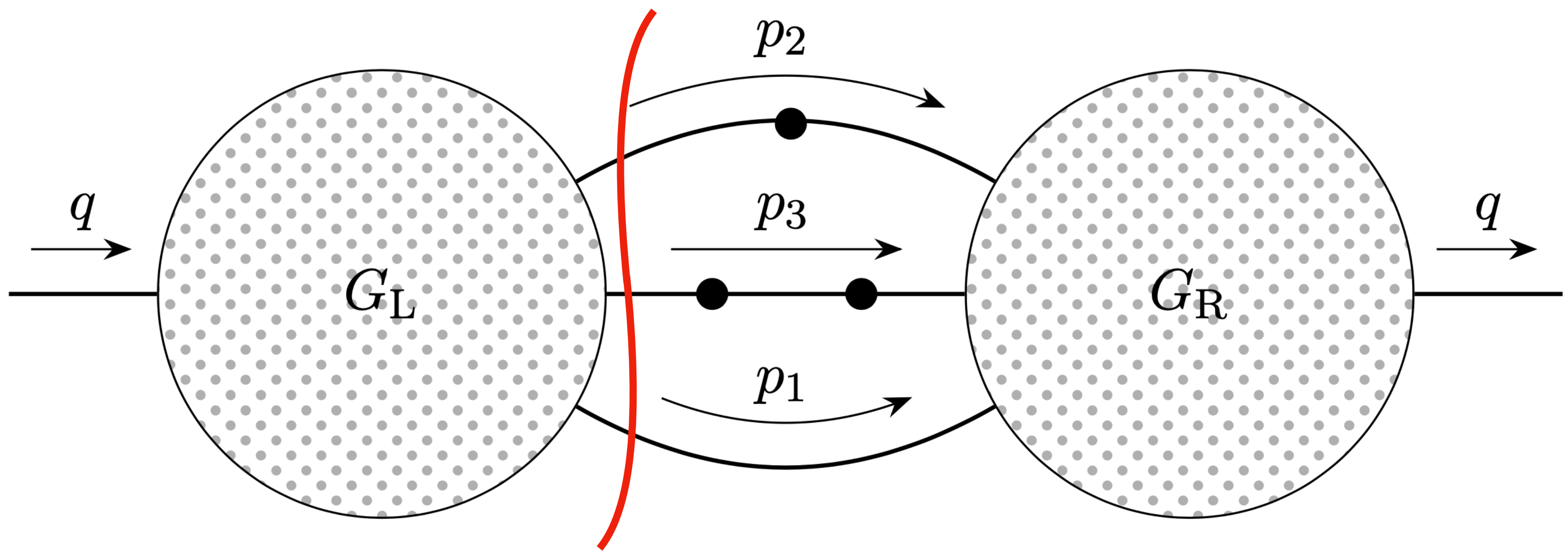

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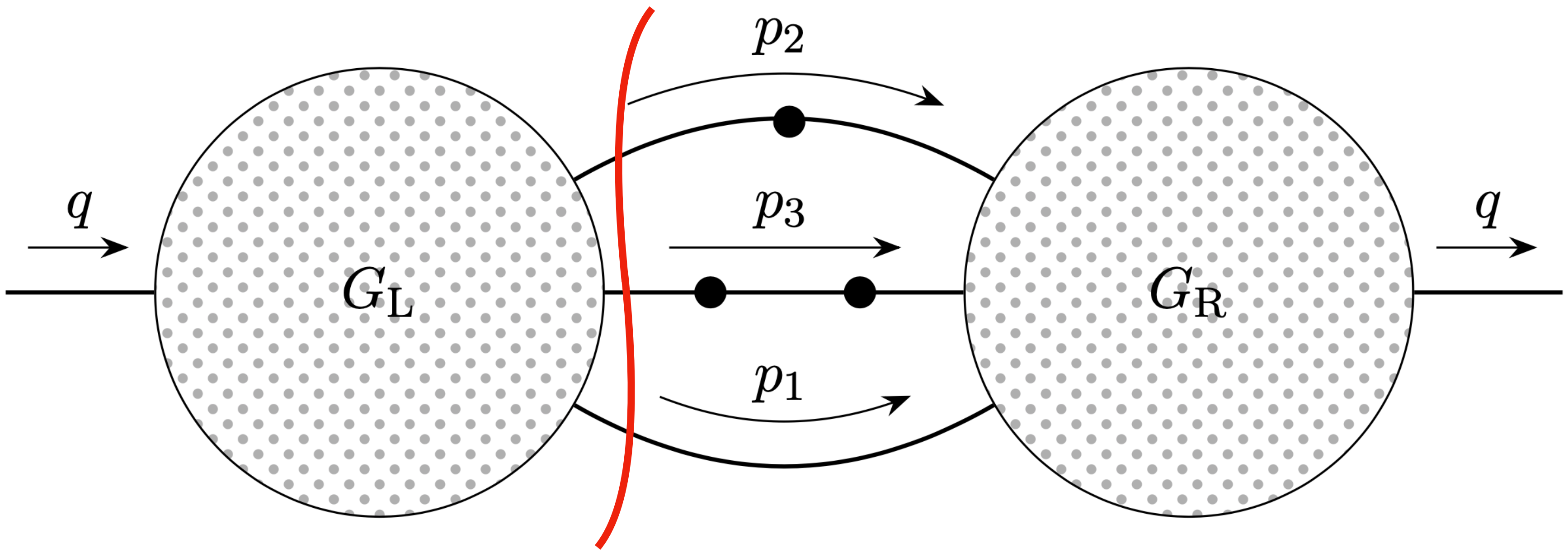

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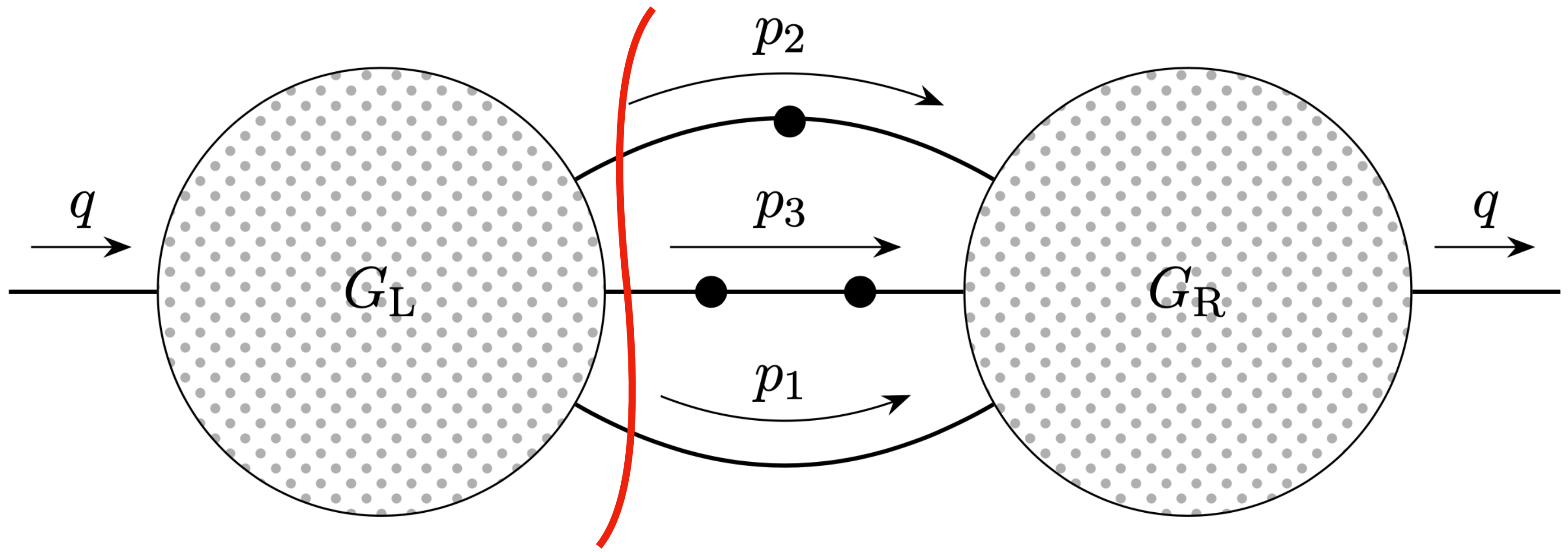

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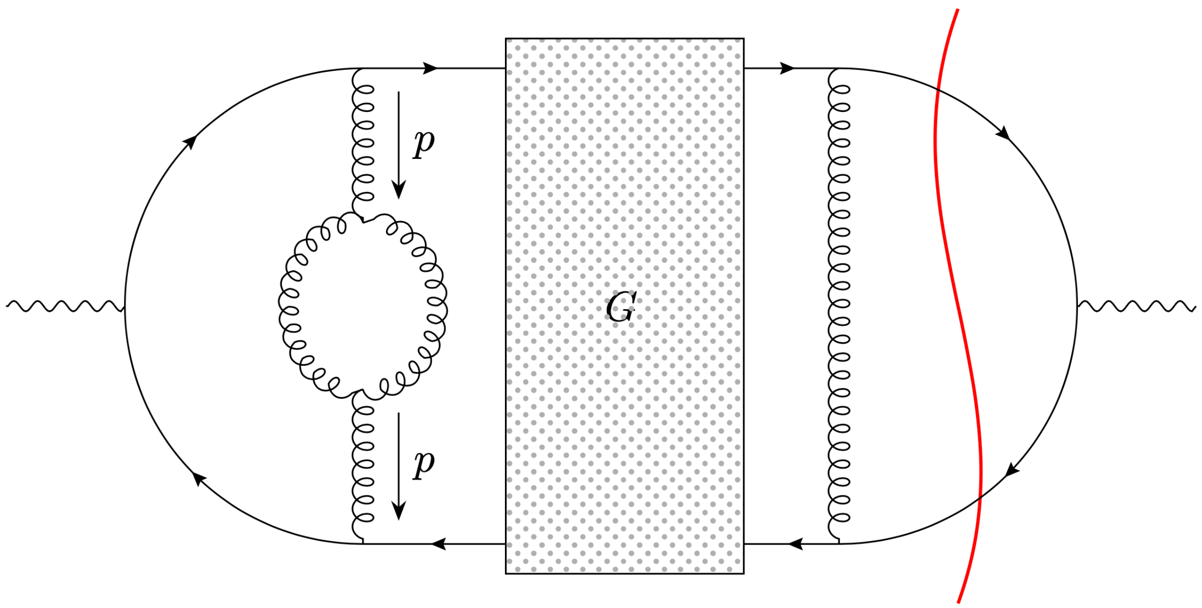
Use **multivariate dual numbers** (auto-differentiation) in order to **efficiently compute amplitude derivatives** of G_L and G_R in p_2^0 and t (in this example)

Spurious soft singularities

[Capatti, VH, Ruijl, arxiv : 2203.11038]

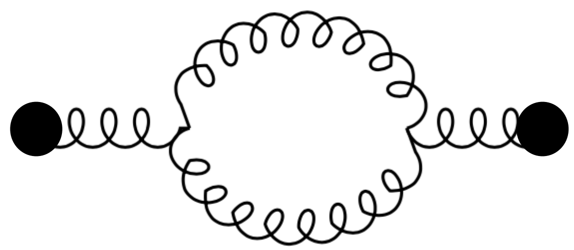


$\propto \dfrac{1}{(p^2)^2}$

For $p = 0$ this induces a spurious soft divergence whose cancellation has nothing to do with KLN!

Only beyond NLO (needs a soft propagator dressed with a self-energy correction)

At the integrated level, we have



$$\propto \frac{1}{p^2}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\frac{1}{p^2}$$

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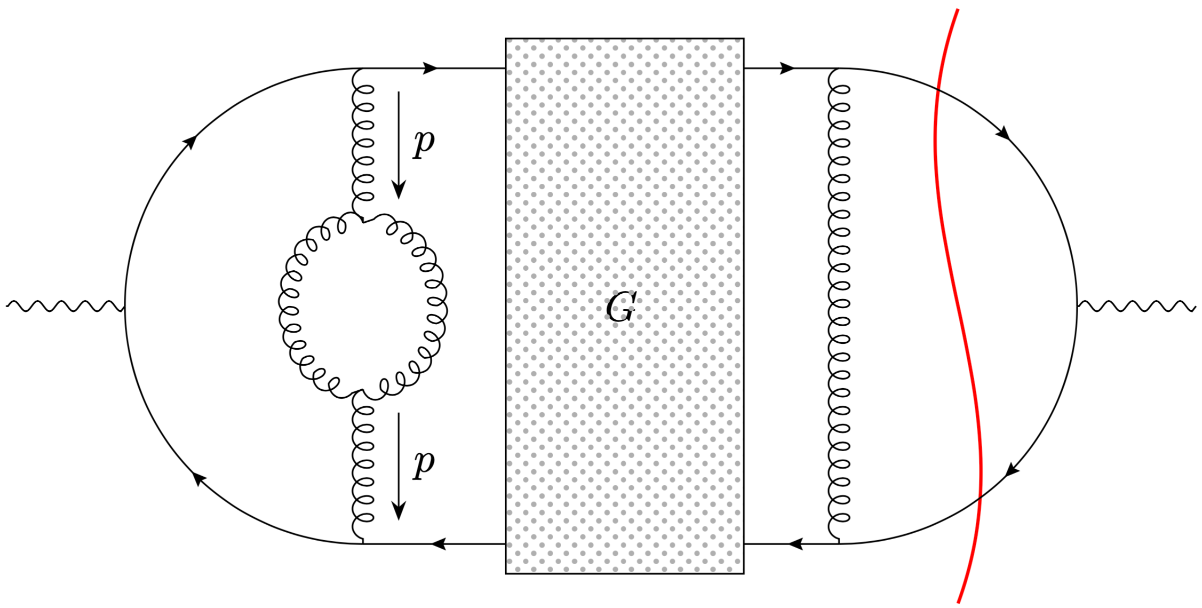

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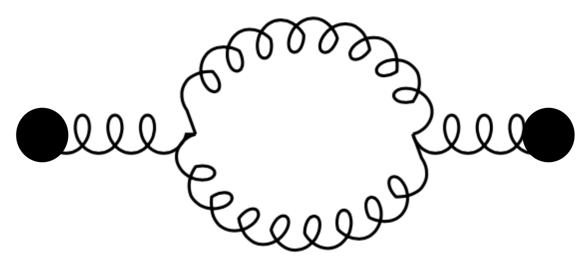
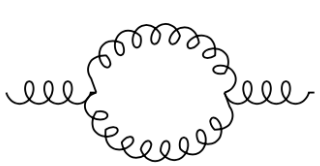
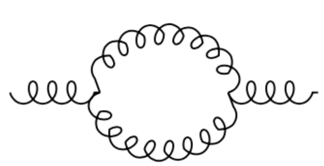
At the integrated level, we have  $\propto \dfrac{1}{p^2}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)\dfrac{1}{p^2}$

but not at the local level; we must introduce **spurious soft counterterms** :

 $-\tilde{T}_1\Big($  $\Big)$, $\tilde{T}_{\text{soft\_dod}}(\gamma) = \displaystyle\sum_{j=0}^{\text{soft\_dod}(\gamma)} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}\lambda^j}\gamma(\lambda\, p)\Big|_{\lambda=0}$, $[\tilde{T}] = 0$

Combined UV and Spurious IR forest

Eurekâ moment:

- Remarkably, we always have : $\text{soft\_dod} = \text{UV\_dod} - 1$
- Spurious soft expansion also valid as UV counter term.
- Spurious soft IR forest similar to the one produced by the R-operation

so that we can combine the UV and spurious soft subtraction as one !

$$\boxed{\hat{T}_{\text{dod}} = T_{\text{dod}} + \tilde{T}_{\text{dod}-1} - T_{\text{dod}}\tilde{T}_{\text{dod}-1}}$$

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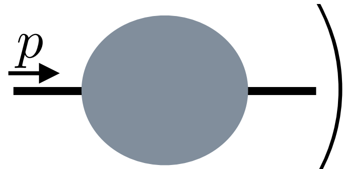
Novelty though: **automatic renormalisation** of fermion **masses in the OS scheme:**

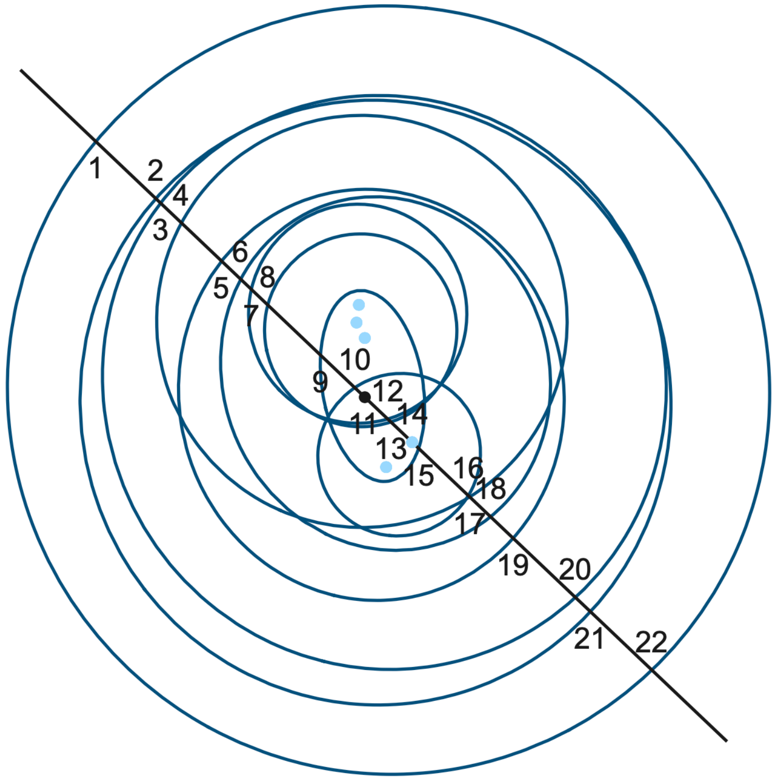
$T^{\text{os}\pm}\Big(\Sigma =$  $\Big) = (1 \pm \gamma^0)\Sigma(p = \pm p^{\text{os}})\,, \quad p^{\text{os}} = (m, 0, 0, 0)$

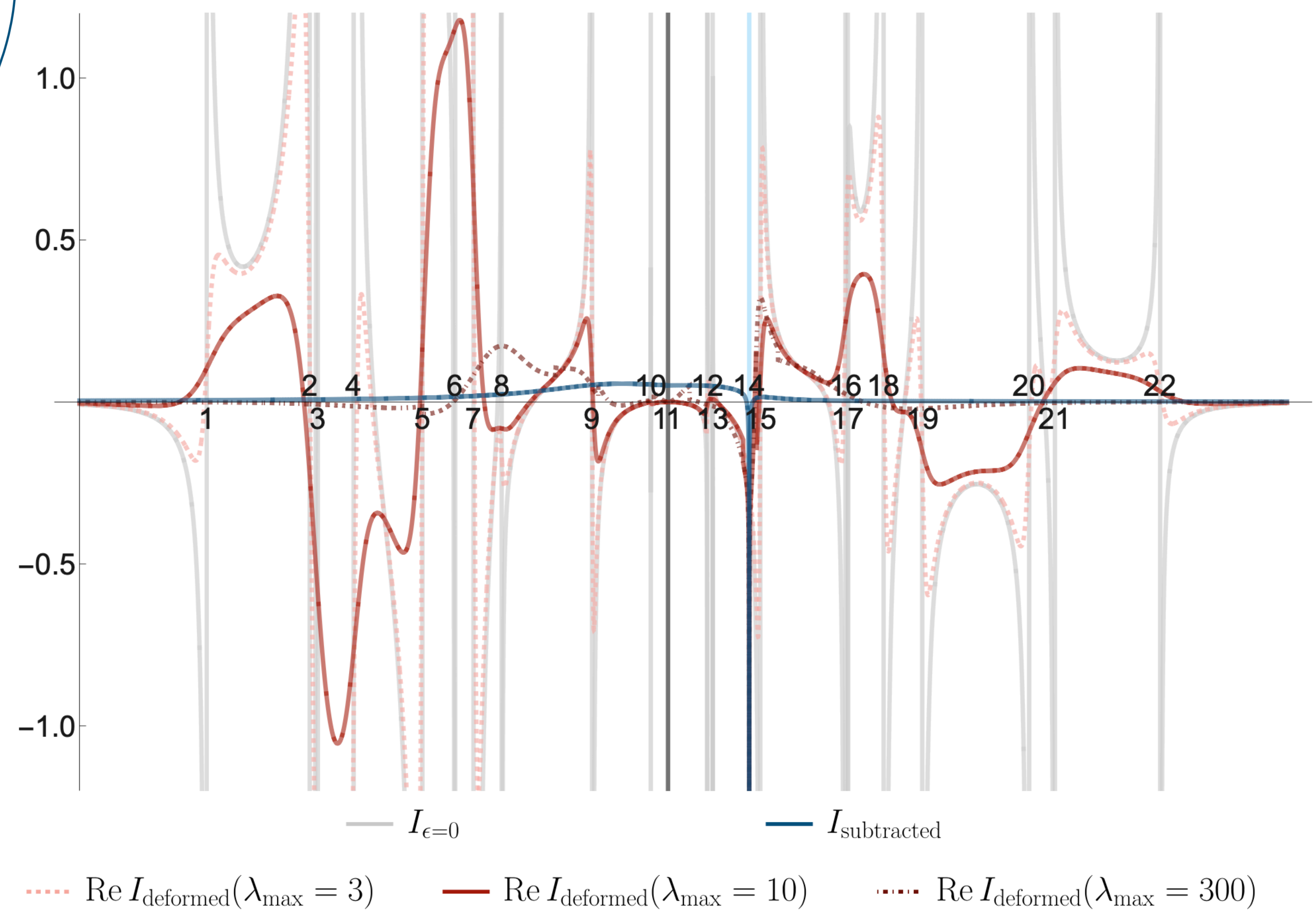
$$\frac{1}{2}\left([T^{\text{os}+}(\Sigma)] + [T^{\text{os}-}(\Sigma)]\right) = \delta m^{\text{os}}$$

Implying that our local UV counterterm T^{os} automatically generates the OS mass renormalisation counterterm !

THRESHOLD SUBTRACTION INSTEAD OF DEFORMATION

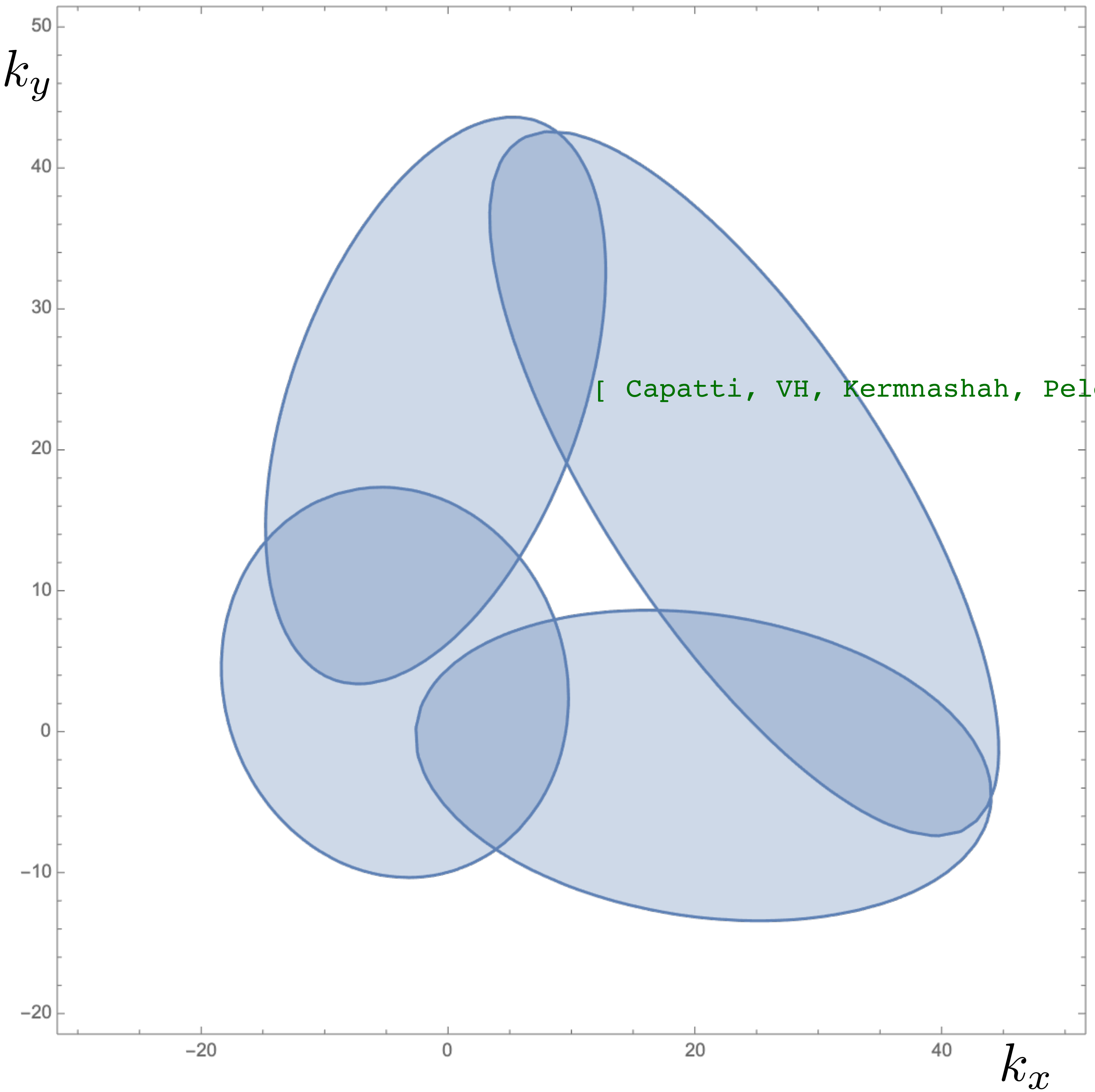
[D. Kermanschah, arXiv : 2110.06869]

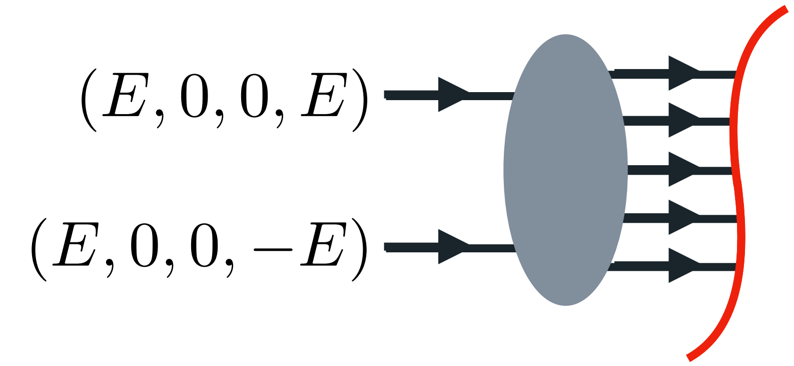




Locality Unitarity: Causal Flow

The **rescaling** change of variables is however **not general** : (<u>**but always sufficient in practice**</u>!)





Ex: Box_4E from sect. 3.1 of

[Capatti, VH, Kermnashah, Pelonni, Ruijl, arxiv:1912.09291]

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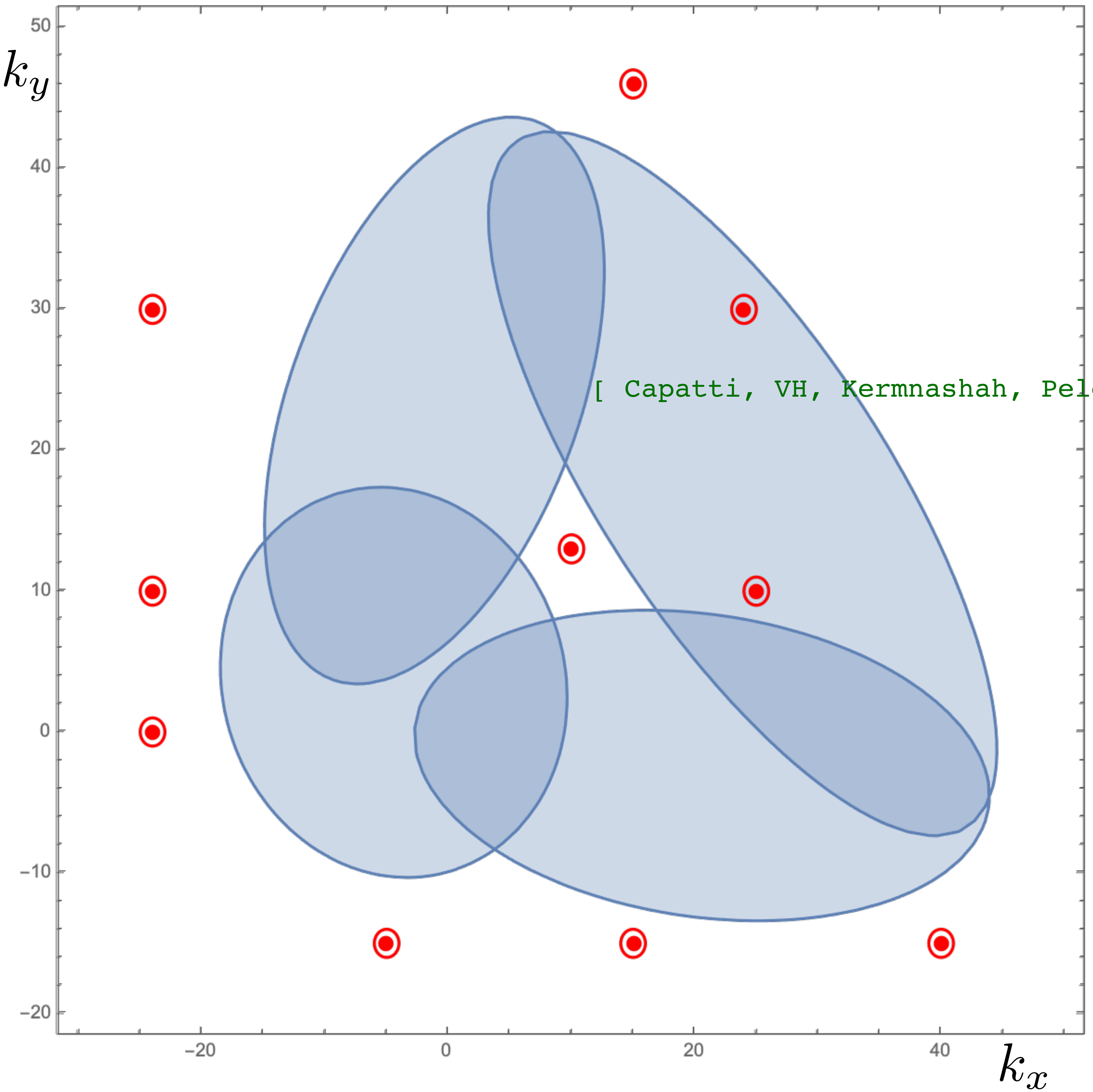

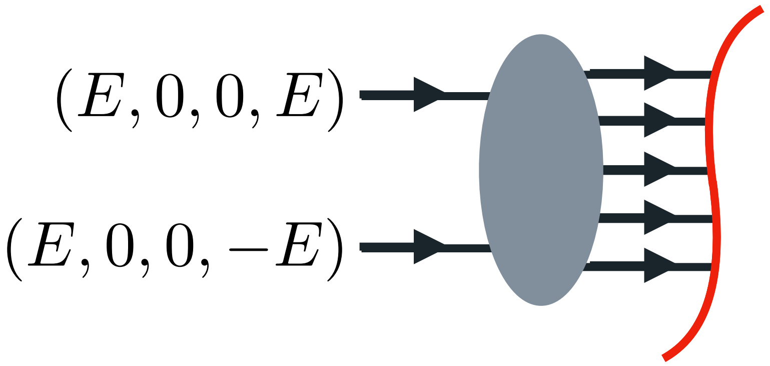

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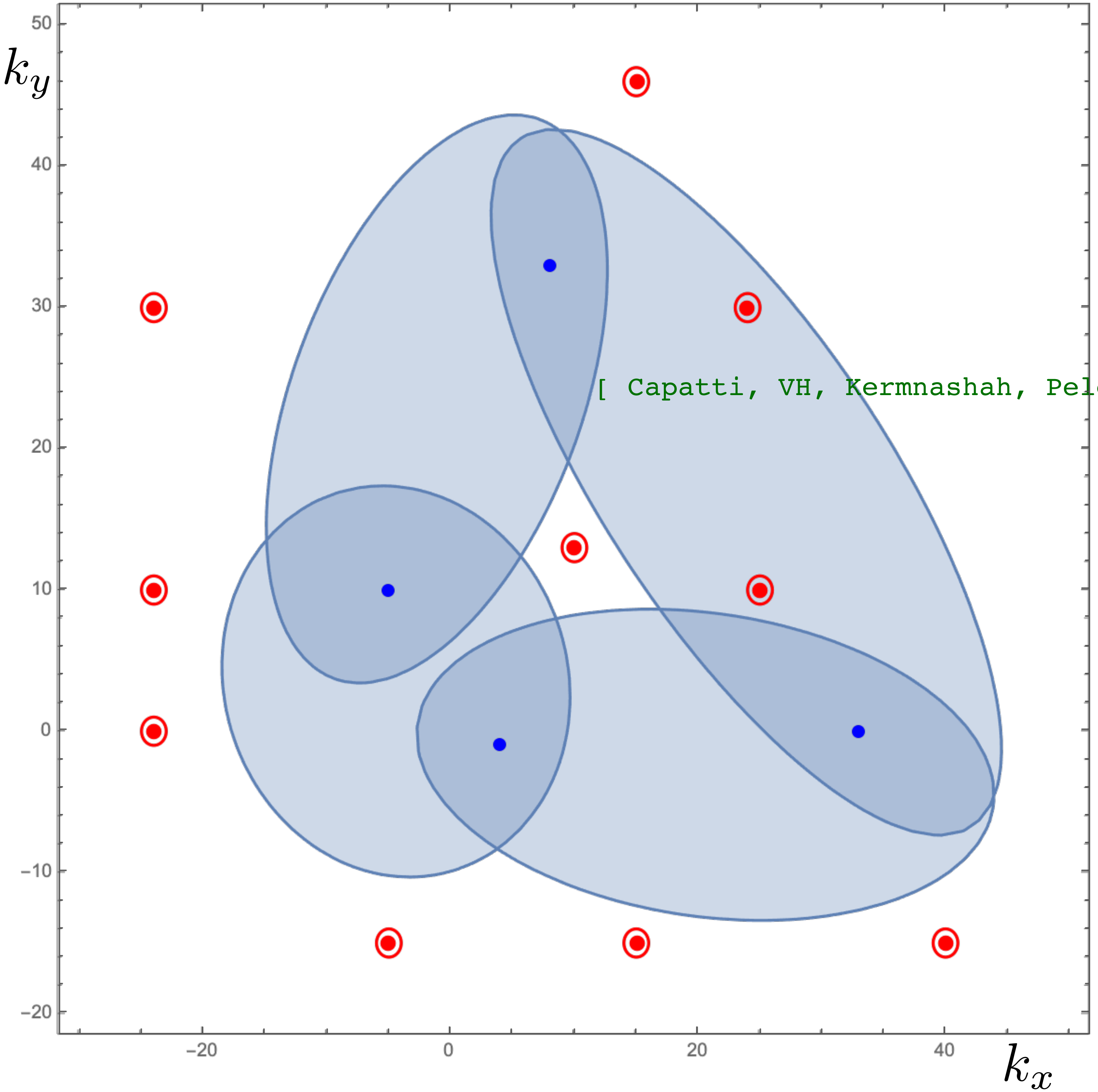

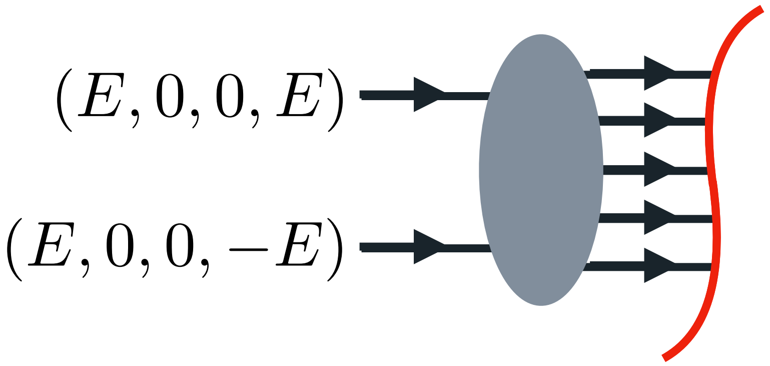

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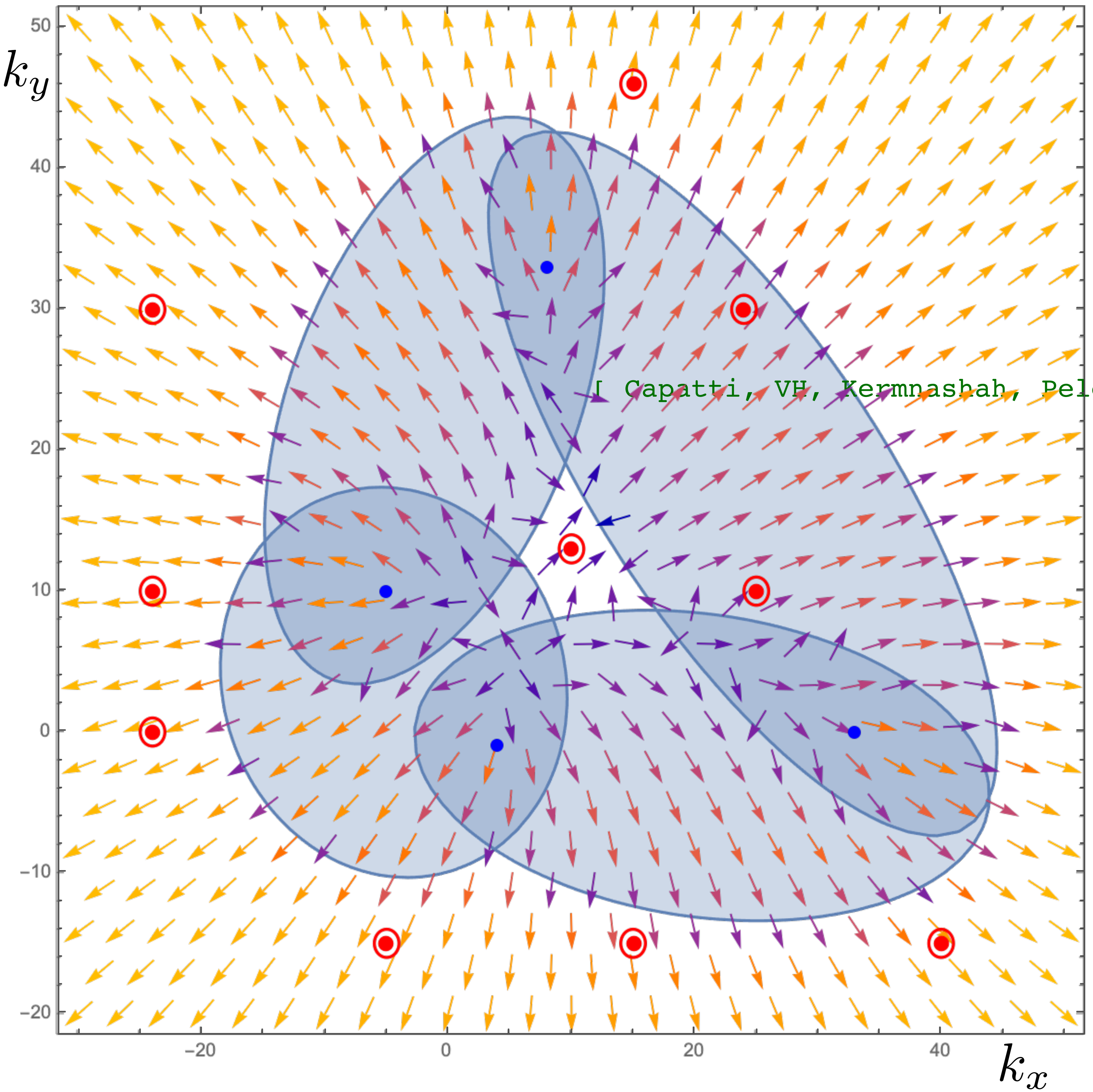

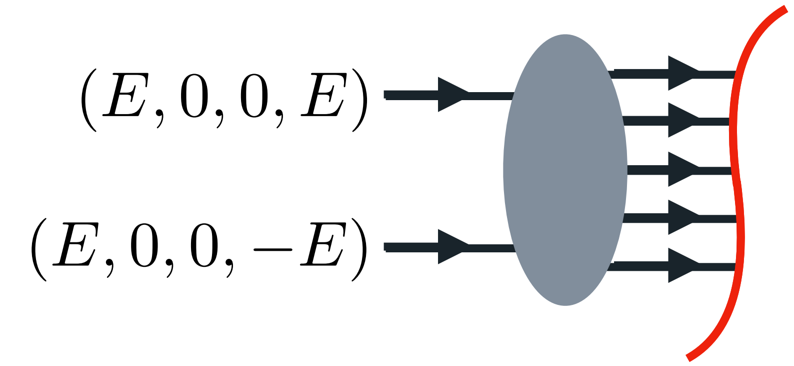

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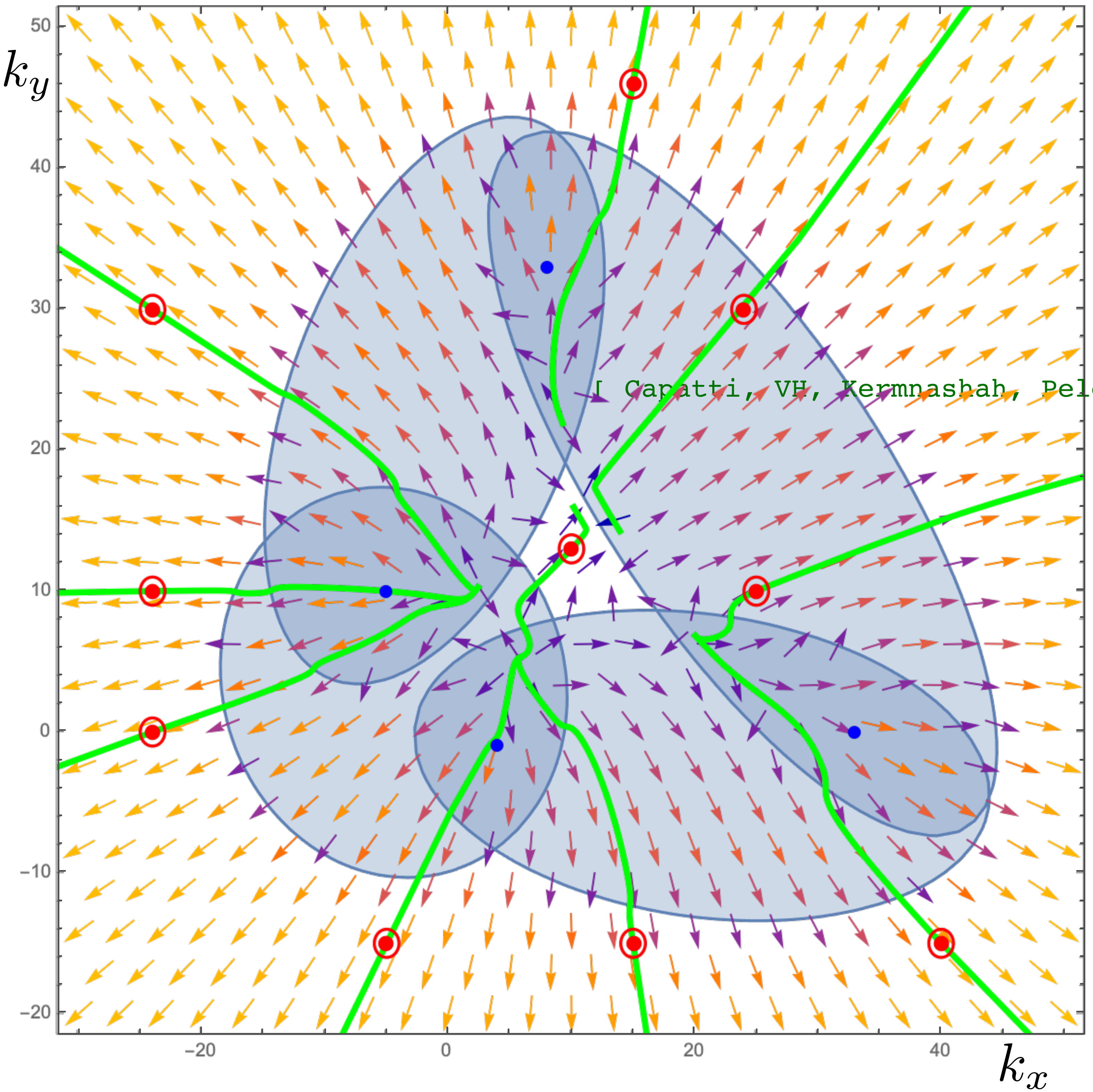

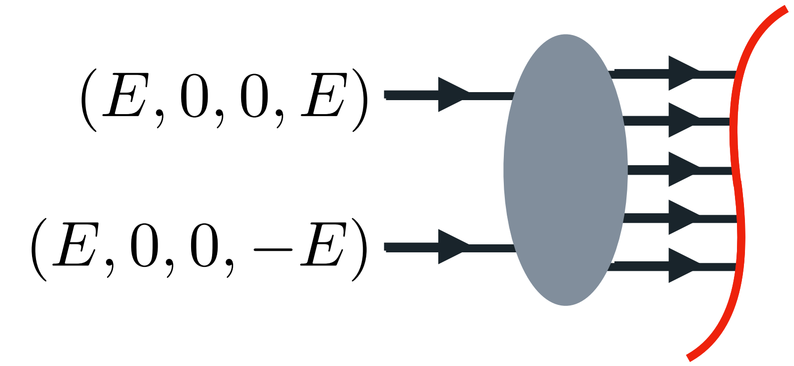

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Compute a **causal flow** $\vec{\phi}$ from our existing construction of a **deformation field** $\vec{\kappa}$:

$$\partial_t \vec{\phi}(t, \vec{k}) = \vec{\kappa}(\vec{\phi}(t, \vec{k}))$$

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In general, this **ODE** can be **solved numerically**.

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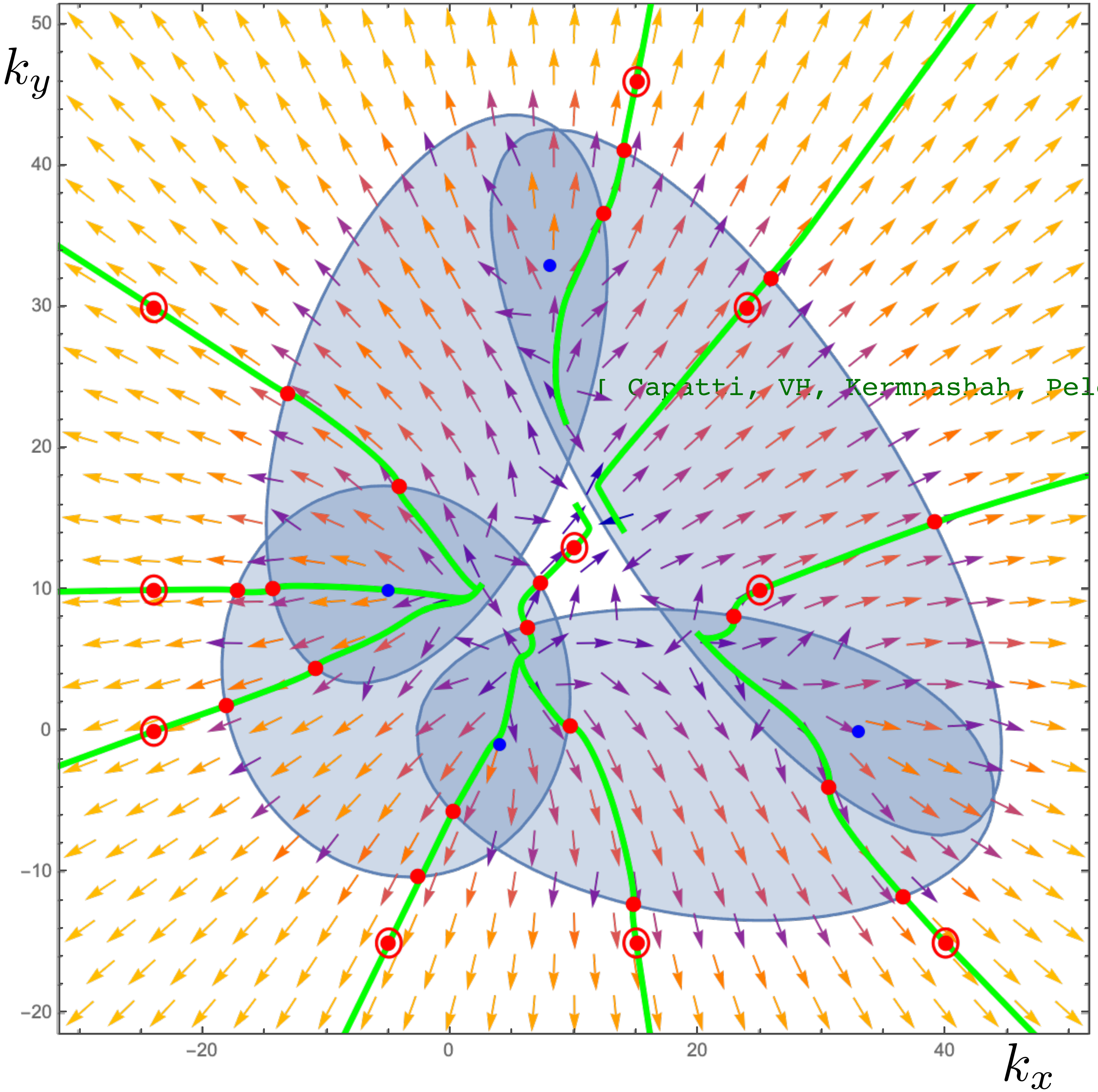

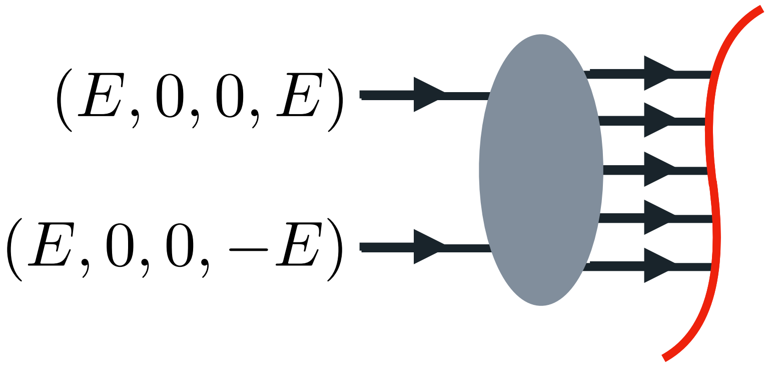

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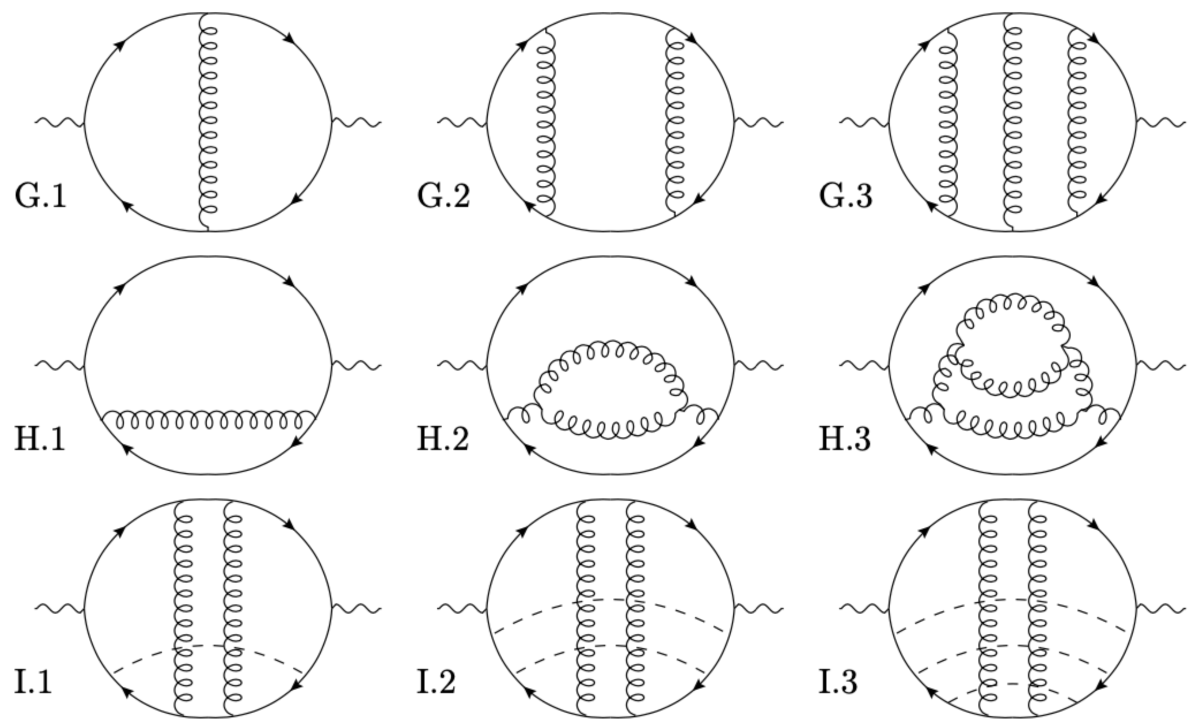
$$\partial_t \vec{\phi}(t,\vec{k}) = \vec{\kappa}(\vec{\phi}(t,\vec{k}))$$

$$\vec{\phi}(0,\vec{k}) = \vec{k}$$

In general, this **ODE** can be **solved numerically**.

Implementation run-time performance



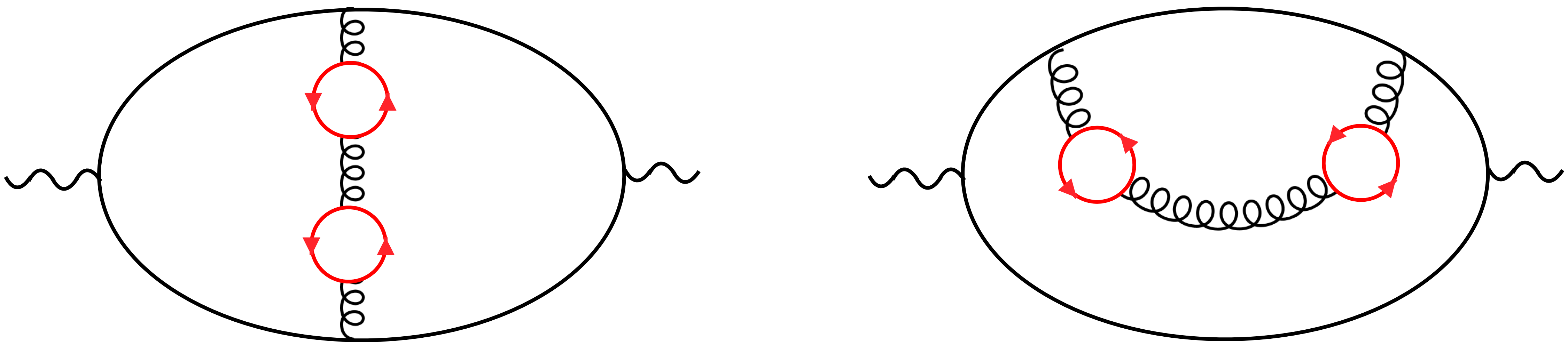
SG	proc.	order	t_{gen} [s]	M_{disk} [MB]	N_{sg} [-]	N_{cuts} [-]	t_{eval} [ms]	$t_{\mathrm{eval}}^{(\mathrm{f128})}$ [ms]
G.1	$1 \to 2$	NLO	0.1	0.13	2	4	0.004	0.13
G.2	$1 \to 2$	NNLO	4.7	3.0	17	9	0.04	2.1
G.3	$1 \to 2$	N3LO	36K	509	220	16	17.6	281
H.1	$1 \to 2$	NLO	0.07	0.12	2	2	0.006	0.14
H.2	$1 \to 2$	NNLO	1.5	1.3	17	3	0.056	1.9
H.3	$1 \to 2$	N3LO	255	43	220	4	2.35	56
I.1	$1 \to 3$	NNLO	126	22	266	9	0.32	12.4
I.2	$1 \to 4$	NNLO	1.9K	120	4492	9	4.4	67
I.3	$1 \to 5$	NNLO	36K	20K	$\mathcal{O}$(100K)	9	3.6K	17.3K

NB: these are **integrand** performance.
(Note: we recently found massive speedup w.r.t the above)

Integration (sampling) not optimised yet.
so we do not report quantitatively on it yet.

Partial N3LO results

n_f^2 contributions :



$$K_{jj}^{(\mathrm{MC\ LU})\ \mathrm{I}} = 24.45(10) \qquad K_{jj}^{(\mathrm{MC\ LU})\ \mathrm{II}} = -24.80(22)$$

(Large accidental cancellation between the two graphs, but validation otherwise successful)

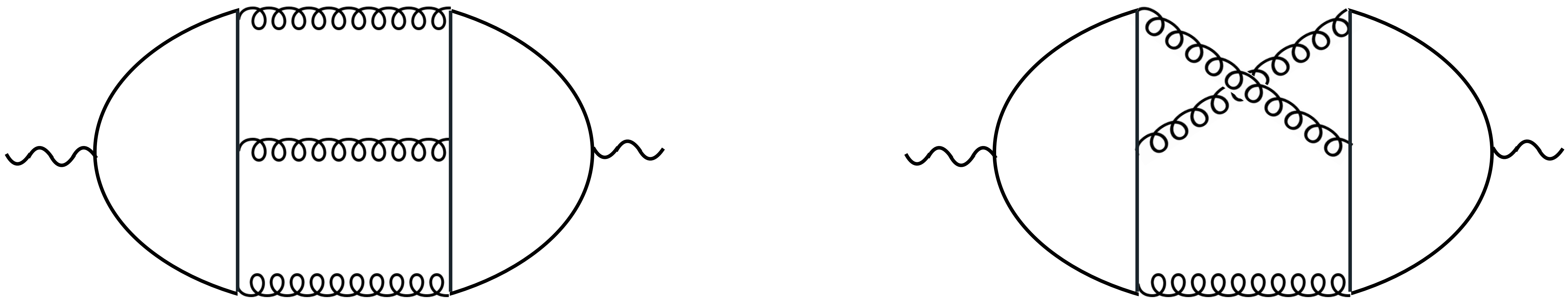
$$K_{jj}^{(\mathrm{MC\ LU})\ \mathrm{I+II}} = -0.35(24)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3 n_f^2)} = C_F\left(\frac{1208}{27} - \frac{8}{3}\zeta_2 - \frac{304}{9}\zeta_3\right) = -0.331415$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

Partial N3LO results

Singlet contributions : (Results for low Monte-Carlo statistics here)



$$K_{jj}^{(\mathrm{MC\ LU})\mathrm{I}} = 48.4(1.0) \qquad K_{jj}^{(\mathrm{MC\ LU})\mathrm{II}} = -74.0(1.1)$$

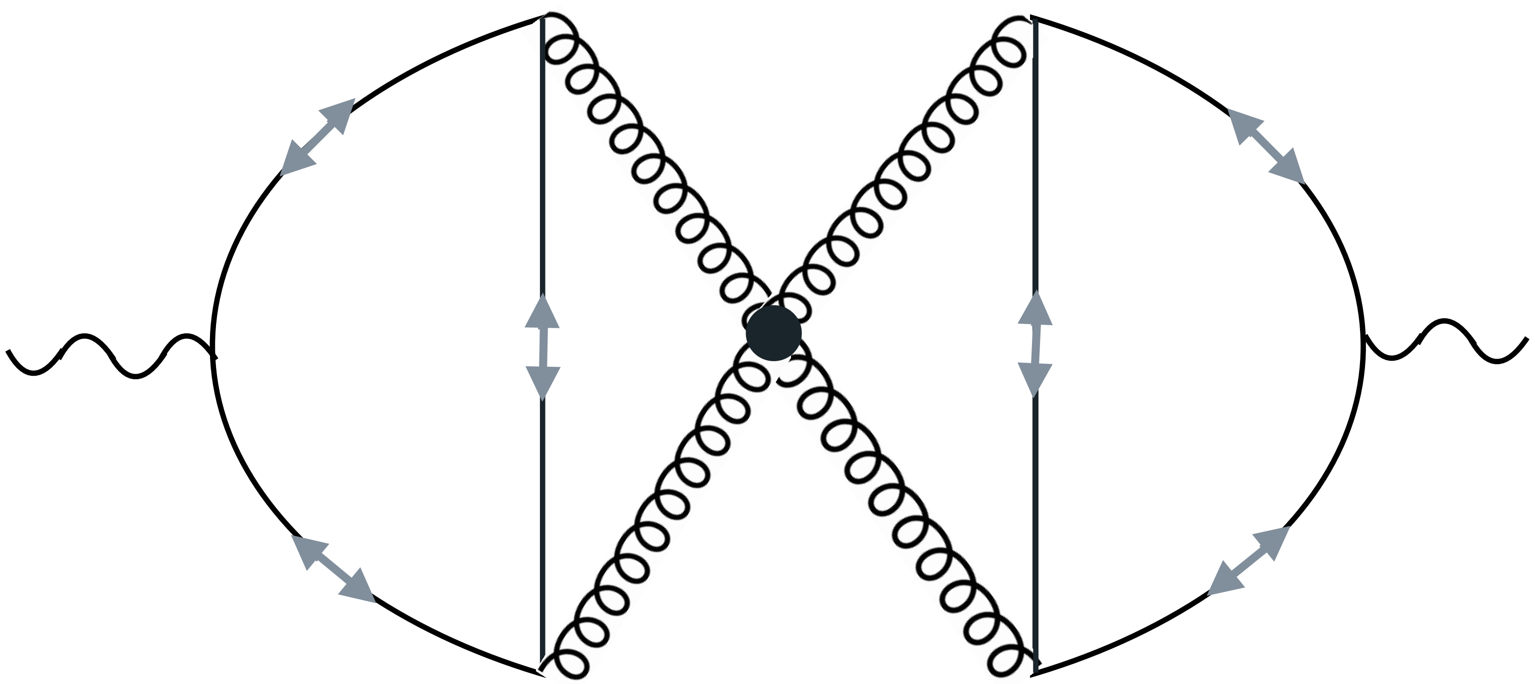
$$K_{jj}^{(\mathrm{MC\ LU})\mathrm{I+II}} = -25.6(1.5)$$

$$K_{jj}^{\mathcal{O}(\alpha_s^3),\,\mathrm{singlet}} = \frac{d_F^{abc} d_F^{abc}}{N_R}\left(\frac{176}{3} - 128\zeta_3\right) = -26.4435$$

[e.g. Herzog, Ruijl, Ueda, Vermaseren, Vogt : 1707.01044]

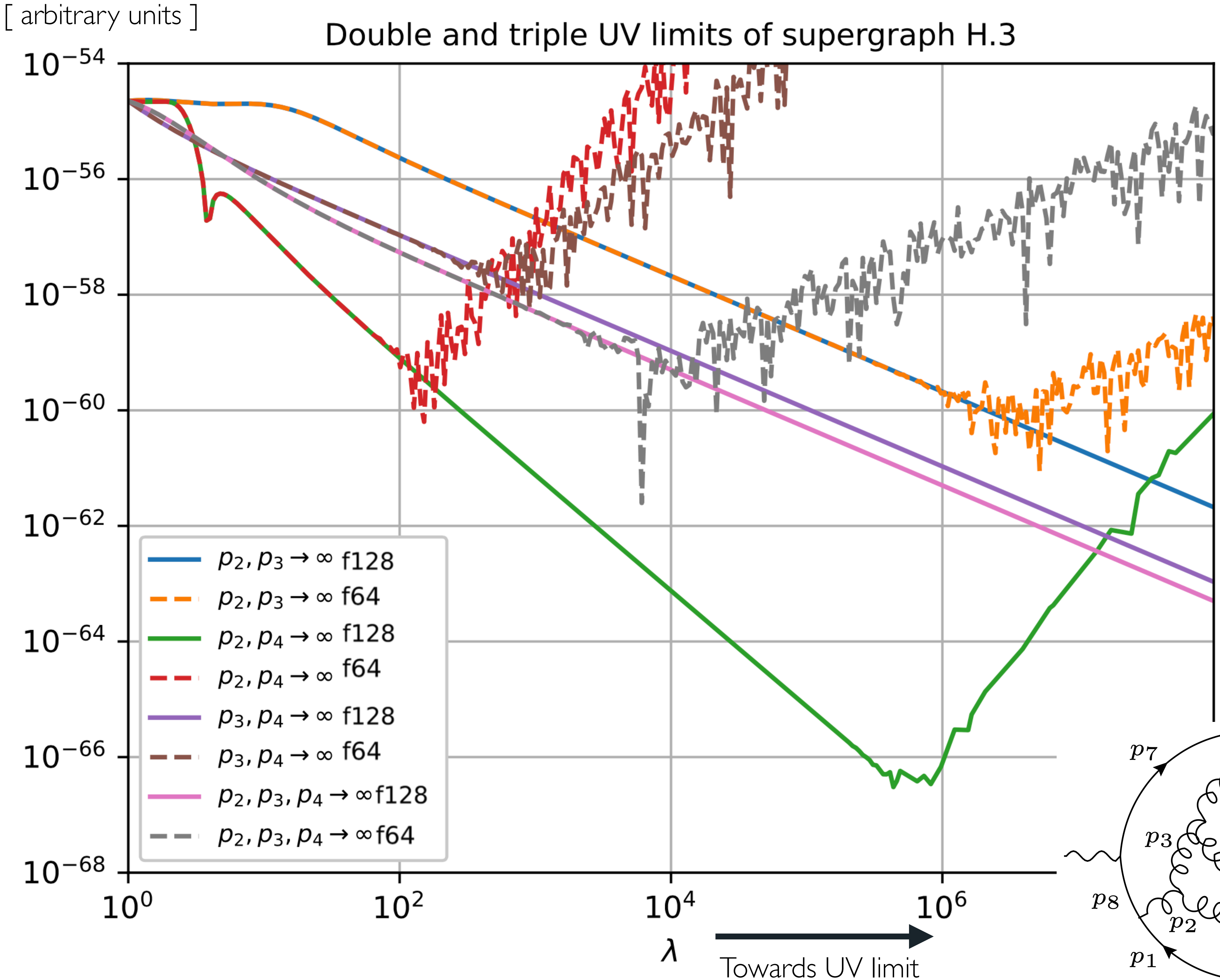
Curiosity

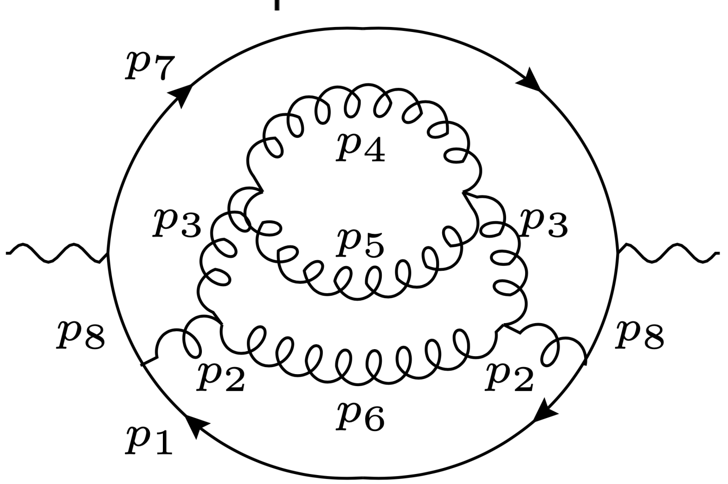
Singlet contributions : A non-obvious zero contribution…



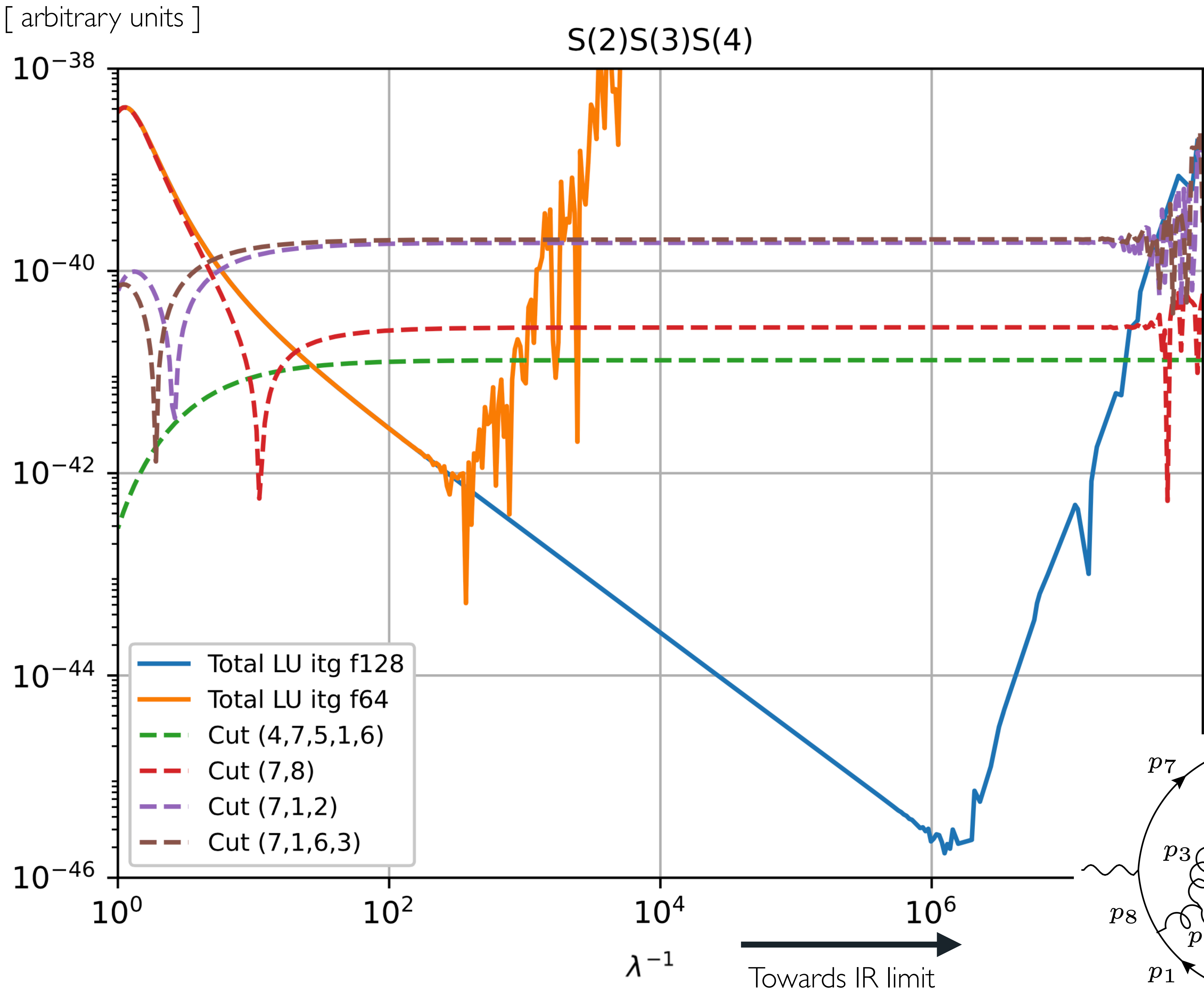
This graph gives no contribution inclusively.
I have not looked into it with any depth, but I don't see
an obvious really as to why it should be zero…

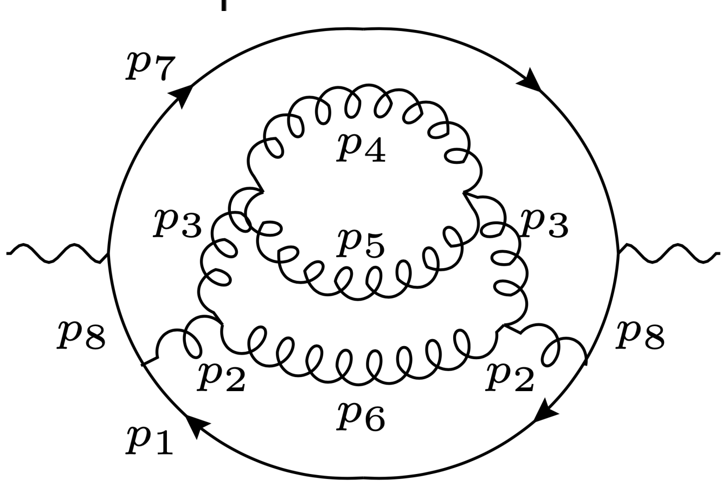
Testing N3LO UV Limits



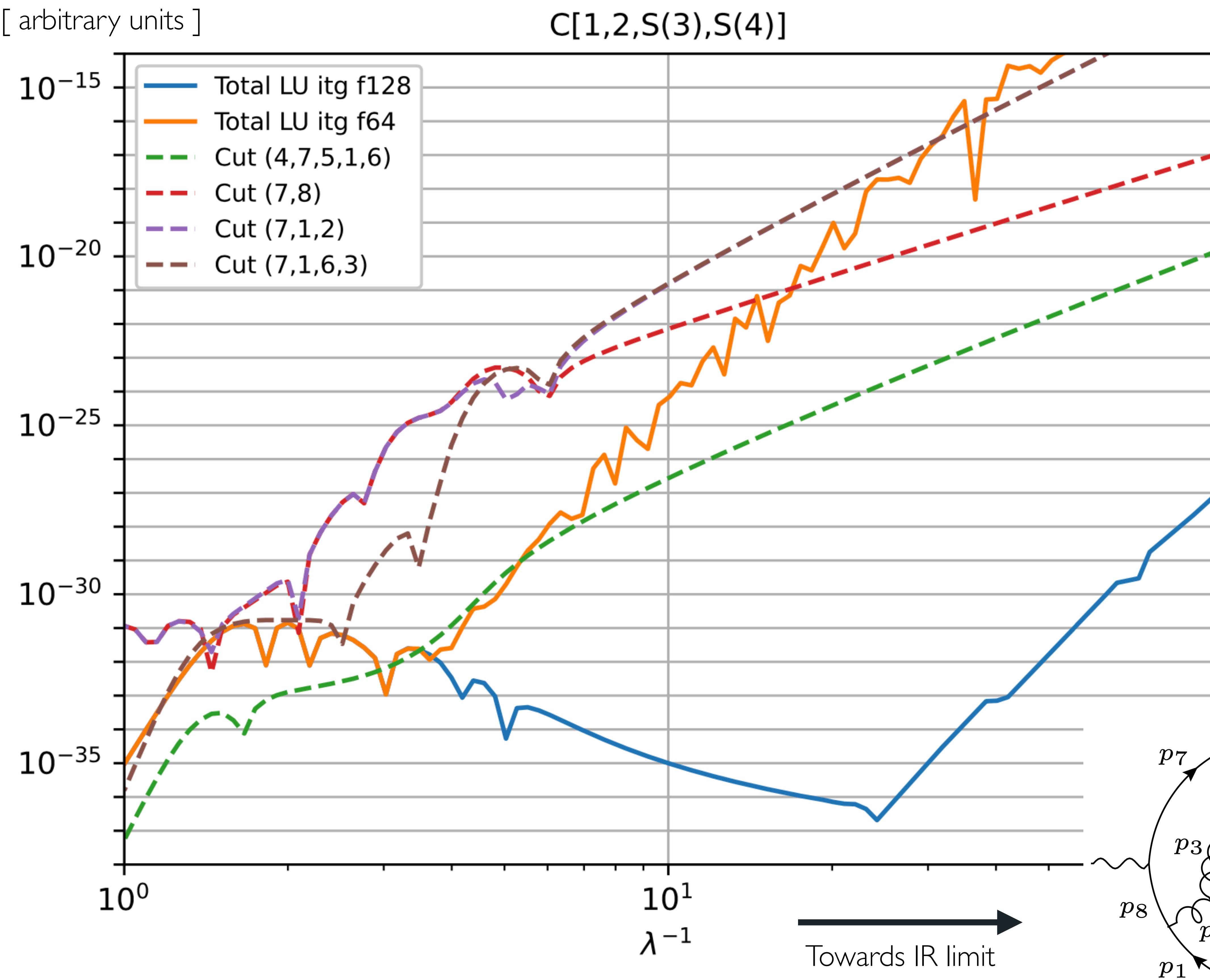


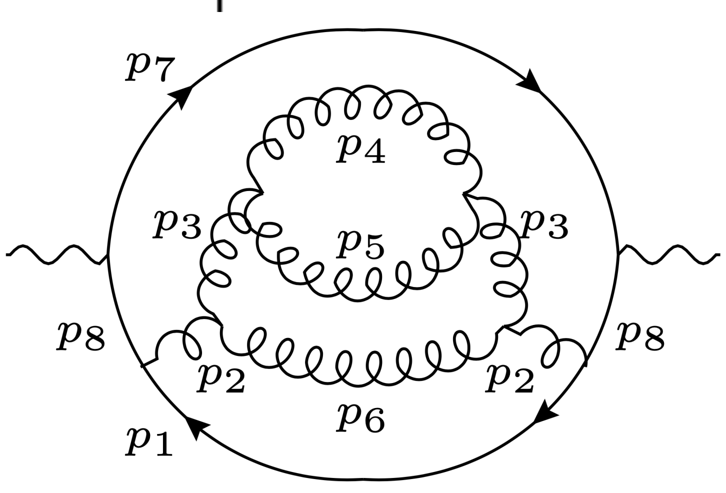
Testing IR soft Limits



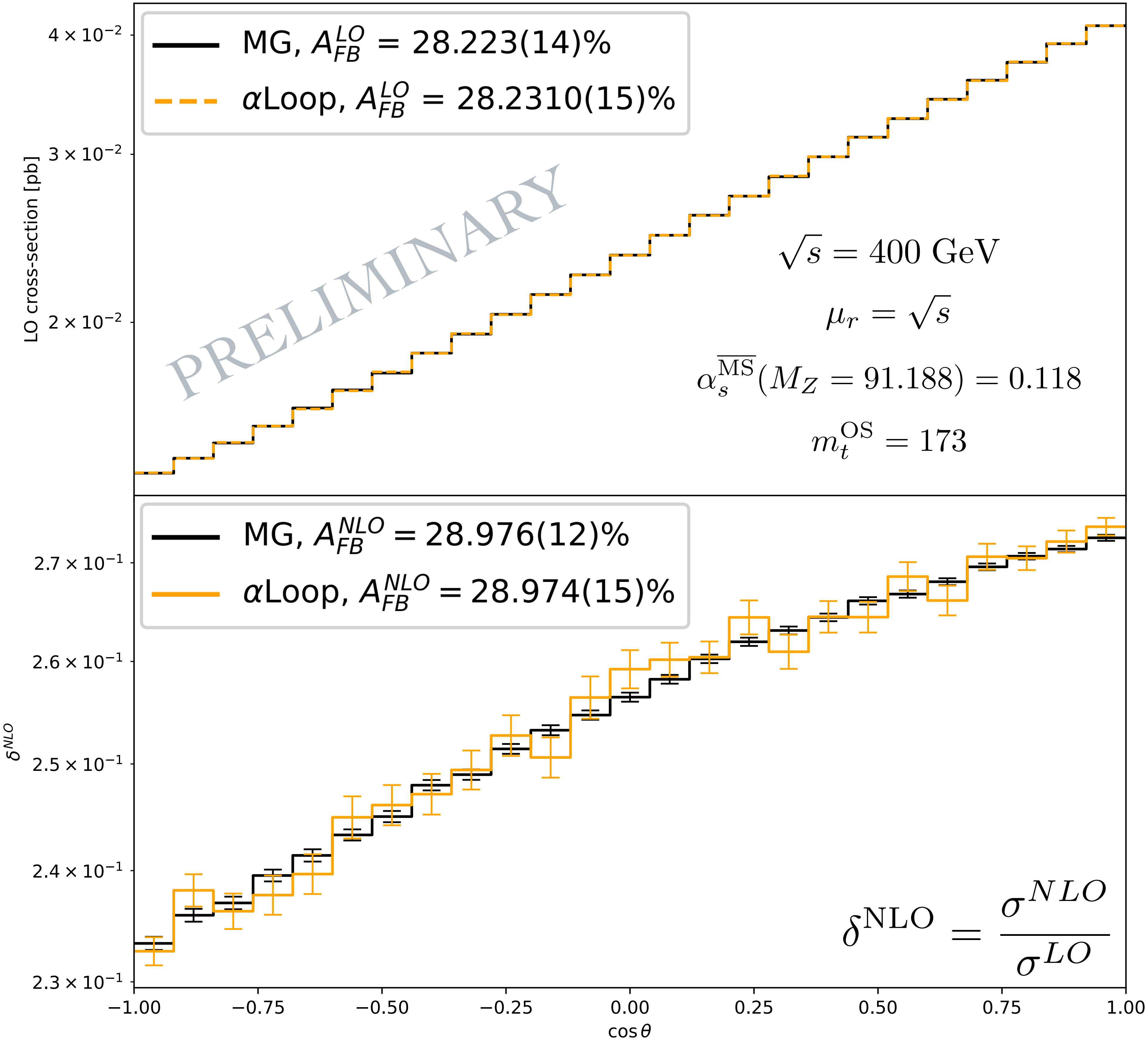


Testing IR soft-collinear Limits





Example II : NLO AFB for $e^+e^- \to \gamma^\star/Z \to t\bar{t}$



First result in LU with γ^5 and EW-boson

Contour deformation well-behaved in this case

Credits to ETHZ student

Max Hofer