Local Analytic Sector Subtraction

NNLO subtraction formula

Massless QCD final-state radiation

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(RV)} \right) \delta_{X_n}$$

$$+ \int d\Phi_{n+1} \Big[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \Big]$$

$$+ \int d\Phi_{n+2} \Big[RR\, \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} - K^{(12)} \right) \delta_{X_n} \Big]$$

single-unresolved (yellow: $I^{(1)}$, $K^{(1)}$)

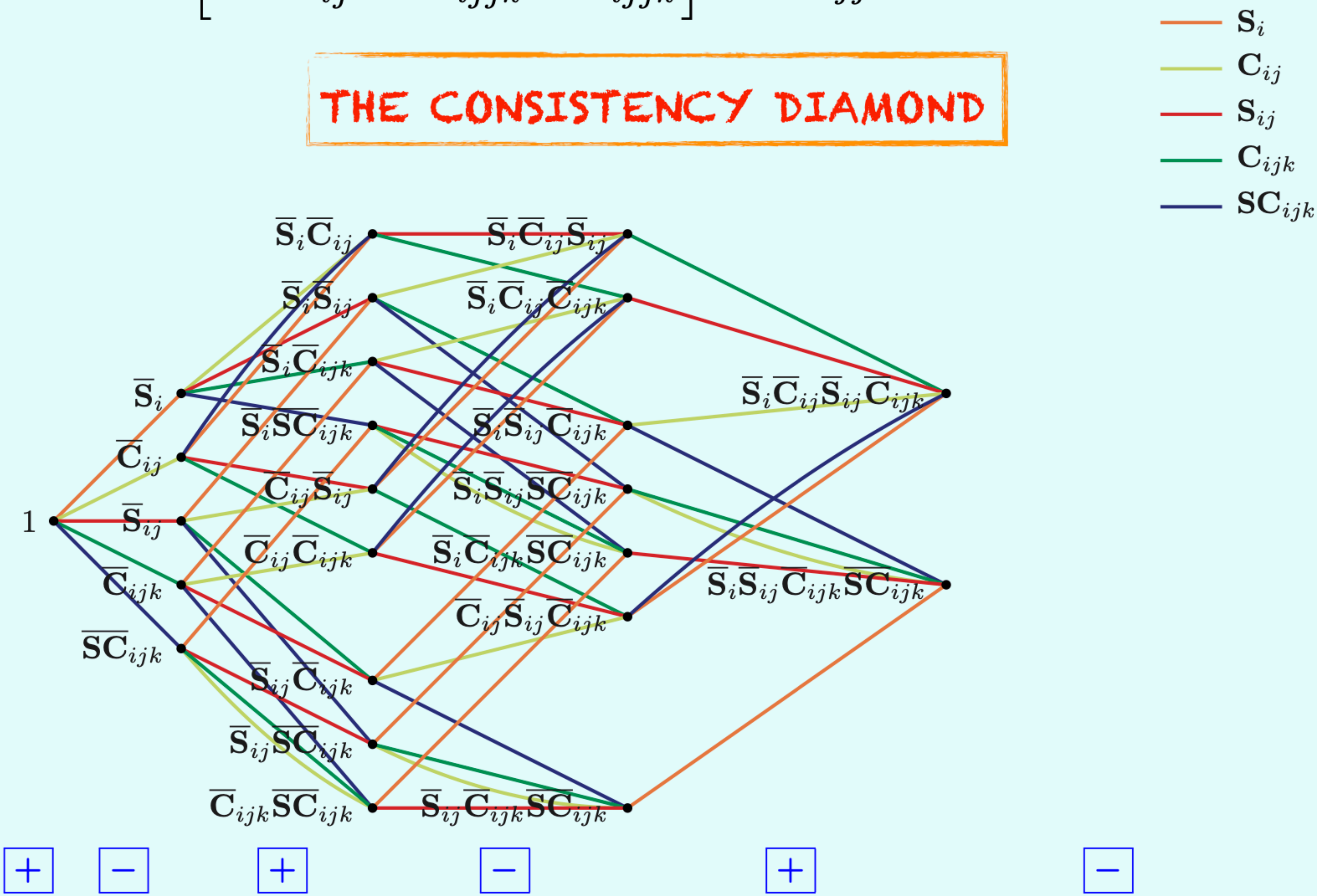
double-unresolved (orange: $I^{(2)}$, $K^{(2)}$)

single-unresolved of RV (blue: $I^{(RV)}$, $K^{(\mathrm{RV})}$)

strongly-ordered double-unresolved (green: $I^{(12)}$, $K^{(12)}$)

$$\left[1-\overline{\mathbf{L}}^{(\mathbf{1})}_{ij}-\overline{\mathbf{L}}^{(\mathbf{2})}_{ijjk}+\overline{\mathbf{L}}^{(\mathbf{12})}_{ijjk}\right] RR\,\mathcal{W}_{ijjk} \quad = \quad \text{finite}$$

THE CONSISTENCY DIAMOND



(Slide by S. Uccirati)

NLO: ISR & Optimisations

- ***Systematic optimisation*** with **damping factors**: multiplicative powers of kinematic invariants smoothly turning off the local counterterms away from the singular regions

✱ Final-state splitting case

$$\bar{\mathbf{S}}_i\, R \;=\; -2\,\mathcal{N}_1 \sum_{k\neq i}\sum_{\substack{l\neq i\\ l<k}} \delta_{f_i g}\,\frac{s_{kl}}{s_{ik}s_{il}}\,(1-z)^{\alpha}(1-y)^{\alpha}\,\bar{B}^{(ikl)}_{kl}$$

$$\bar{\mathbf{C}}_{ij}\, R \;=\; \mathcal{N}_1\,\frac{P^{\mu\nu}_{ij}}{s_{ij}}\,(1-y)^{\beta}\,\bar{B}^{(ijr)}_{\mu\nu}$$

$$\bar{\mathbf{S}}_i\,\bar{\mathbf{C}}_{ij}\, R \;\propto\; \mathcal{N}_1\,\delta_{f_i g}\,\frac{s_{jr}}{s_{ij}s_{ir}}\,(1-z)^{\alpha}(1-y)^{\beta}\,\bar{B}^{(ijr)}$$

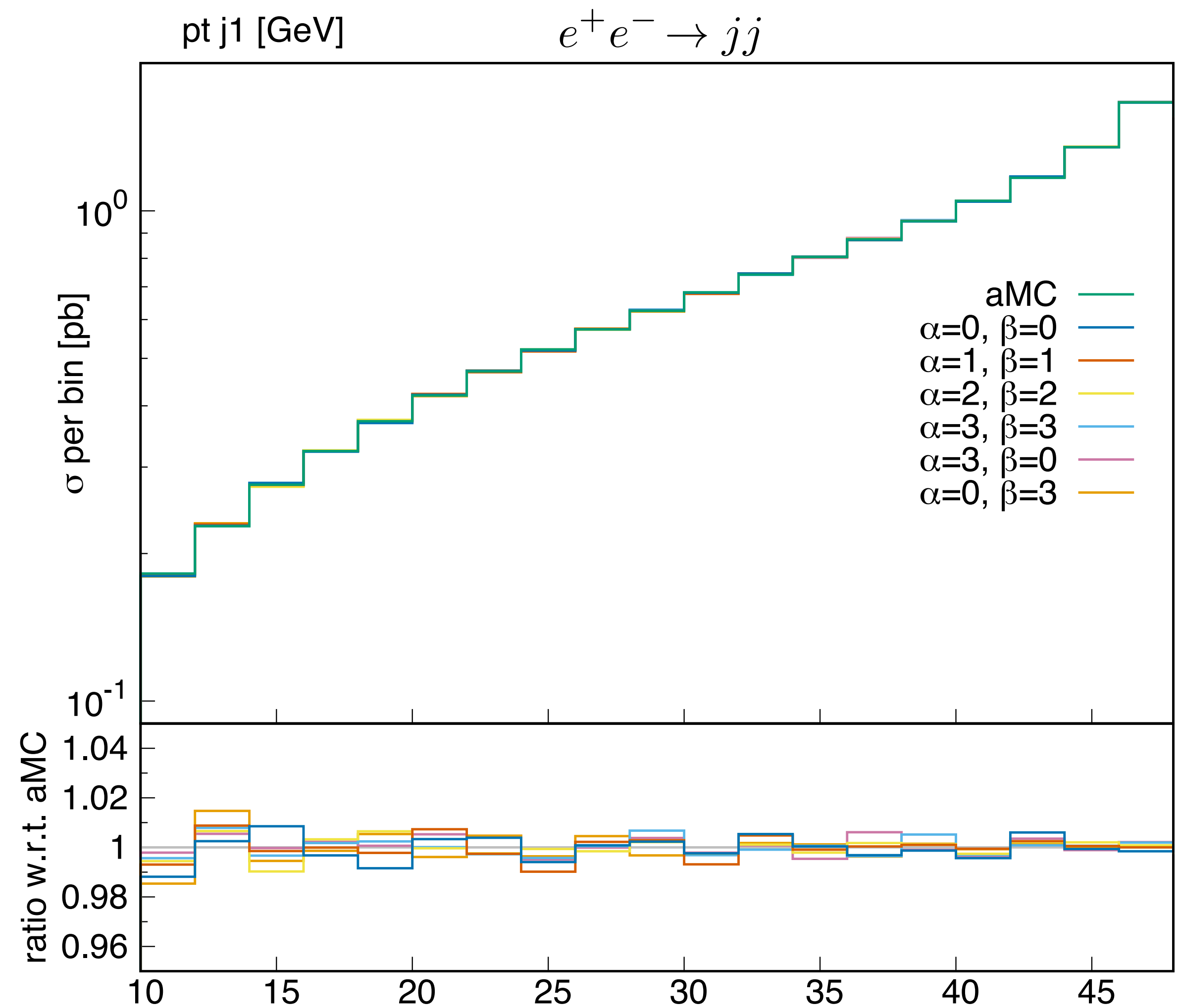
✱ Final results **independent** of the damping exponents

Process	MadNkLO $\alpha=\beta=\gamma=0$	MadNkLO $\alpha=\beta=\gamma=1$	MadNkLO $\alpha=\beta=\gamma=2$
$e^+e^-\to jj$			
V+I	0.02664732(9)	0.01998531(7)	0.00666183(2)
R-K	-0.00666(1)	0.000004(6)	0.013329(6)
NLO corr.	0.019991(10)	0.019985(6)	0.019991(6)
$pp\to Z$			
V+I+C+J	3981.5(4)	-3472.7(4)	-9163.2(5)
R-K	2829.3(2)	10284.3(4)	15974.1(6)
NLO corr.	6810.8(4)	6811.6(6)	6810.9(8)
$pp\to Zj$			
V+I+C+J	7172(2)	5246(2)	3624(2)
R-K	-3395(17)	-1469(25)	156(22)
NLO corr.	3776(17)	3777(25)	3780(22)

[pb]

[**GB**, et al. 2209.09123]

Total cross sections



Differential distributions

- **Still trivial analytic** integrations!
Nice feature to export at NNLO…

Backup slides

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})(1-\mathbf{SC}_{ijk})RR\,\mathcal{W}_{ijjk} \;=\; \text{finite}$$

Identities reduce the number of singular limits

$+$: 1

$-$: $\mathbf{S}_i$, $\mathbf{C}_{ij}$, $\mathbf{S}_{ij}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$

$+$: $\mathbf{S}_i\mathbf{C}_{ij}$, $\mathbf{S}_i\mathbf{S}_{ij}$, $\mathbf{S}_i\mathbf{C}_{ijk}$, $\mathbf{S}_i\mathbf{SC}_{ijk}$, $\mathbf{C}_{ij}\mathbf{S}_{ij}$, $\mathbf{C}_{ij}\mathbf{C}_{ijk}$, $\mathbf{C}_{ij}\mathbf{SC}_{ijk}$, $\mathbf{S}_{ij}\mathbf{C}_{ijk}$, $\mathbf{S}_{ij}\mathbf{SC}_{ijk}$, $\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$

$-$: $\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}$, $\mathbf{S}_i\mathbf{C}_{ij}\mathbf{C}_{ijk}$, $\mathbf{S}_i\mathbf{C}_{ij}\mathbf{SC}_{ijk}$, $\mathbf{S}_i\mathbf{S}_{ij}\mathbf{C}_{ijk}$, $\mathbf{S}_i\mathbf{S}_{ij}\mathbf{SC}_{ijk}$, $\mathbf{S}_i\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$, $\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}$, $\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{SC}_{ijk}$, $\mathbf{C}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$, $\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$

$+$: $\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}$, $\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{SC}_{ijk}$, $\mathbf{S}_i\mathbf{C}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$, $\mathbf{S}_i\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$, $\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$

$-$: $\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$

(Slide by S. Uccirati)

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})(1-\mathbf{SC}_{ijk})RR\,\mathcal{W}_{ijjk} \;=\; \text{finite}$$

Identities reduce the number of singular limits

$$\mathbf{C}_{ij}\,\mathbf{SC}_{ijk}\,RR\,\mathcal{W}_{ijjk} \;=\; \mathbf{C}_{ij}\,\mathbf{C}_{ijk}\,\mathbf{SC}_{ijk}\,RR\,\mathcal{W}_{ijjk} \;=\; \mathbf{S}_i\mathbf{C}_{ij}\,\mathbf{SC}_{ijk}\,RR\,\mathcal{W}_{ijjk} \;=\; \mathbf{S}_i\mathbf{C}_{ij}\,\mathbf{C}_{ijk}\,\mathbf{SC}_{ijk}\,RR\,\mathcal{W}_{ijjk}$$

$$\mathbf{C}_{ij}\,\mathbf{S}_{ij}\,\mathbf{SC}_{ijk}\,RR\,\mathcal{W}_{ijjk} \;=\; \mathbf{C}_{ij}\,\mathbf{S}_{ij}\,\mathbf{C}_{ijk}\,\mathbf{SC}_{ijk}\,RR\,\mathcal{W}_{ijjk} \;=\; \mathbf{S}_i\mathbf{C}_{ij}\,\mathbf{S}_{ij}\,\mathbf{SC}_{ijk}\,RR\,\mathcal{W}_{ijjk} \;=\; \mathbf{S}_i\mathbf{C}_{ij}\,\mathbf{S}_{ij}\,\mathbf{C}_{ijk}\,\mathbf{SC}_{ijk}\,RR\,\mathcal{W}_{ijjk}$$

+	−	+	−	+	−
1	$\mathbf{S}_i$	$\mathbf{S}_i\mathbf{C}_{ij}$	$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}$	$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}$	~~$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$~~
	$\mathbf{C}_{ij}$	$\mathbf{S}_i\mathbf{S}_{ij}$	$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{C}_{ijk}$	~~$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{SC}_{ijk}$~~	
	$\mathbf{S}_{ij}$	$\mathbf{S}_i\mathbf{C}_{ijk}$	~~$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{SC}_{ijk}$~~	~~$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$~~	
	$\mathbf{C}_{ijk}$	$\mathbf{S}_i\mathbf{SC}_{ijk}$	$\mathbf{S}_i\mathbf{S}_{ij}\mathbf{C}_{ijk}$	$\mathbf{S}_i\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$	
	$\mathbf{SC}_{ijk}$	$\mathbf{C}_{ij}\mathbf{S}_{ij}$	$\mathbf{S}_i\mathbf{S}_{ij}\mathbf{SC}_{ijk}$	~~$\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$~~	
		$\mathbf{C}_{ij}\mathbf{C}_{ijk}$	$\mathbf{S}_i\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$		
		~~$\mathbf{C}_{ij}\mathbf{SC}_{ijk}$~~	$\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}$		
		$\mathbf{S}_{ij}\mathbf{C}_{ijk}$	~~$\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{SC}_{ijk}$~~		
		$\mathbf{S}_{ij}\mathbf{SC}_{ijk}$	~~$\mathbf{C}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$~~		
		$\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$	$\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}$		

(Slide by S. Uccirati)

We are left with

$$(1-\mathbf{S}_i)(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})(1-\mathbf{C}_{ij}-\mathbf{SC}_{ijk})RR\,\mathcal{W}_{ijjk} = \text{finite}$$

$1\ \bullet$

$\mathbf{S}_i\ \bullet$
$\mathbf{C}_{ij}\ \bullet$
$\mathbf{S}_{ij}\ \bullet$
$\mathbf{C}_{ijk}\ \bullet$
$\mathbf{SC}_{ijk}\ \bullet$

$\mathbf{S}_i\mathbf{C}_{ij}\ \bullet$
$\mathbf{S}_i\mathbf{S}_{ij}\ \bullet$
$\mathbf{S}_i\mathbf{C}_{ijk}\ \bullet$
$\mathbf{S}_i\mathbf{SC}_{ijk}\ \bullet$
$\mathbf{C}_{ij}\mathbf{S}_{ij}\ \bullet$
$\mathbf{C}_{ij}\mathbf{C}_{ijk}\ \bullet$
$\mathbf{S}_{ij}\mathbf{C}_{ijk}\ \bullet$
$\mathbf{S}_{ij}\mathbf{SC}_{ijk}\ \bullet$
$\mathbf{C}_{ijk}\mathbf{SC}_{ijk}\ \bullet$

$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}\ \bullet$
$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{C}_{ijk}\ \bullet$
$\mathbf{S}_i\mathbf{S}_{ij}\mathbf{C}_{ijk}\ \bullet$
$\mathbf{S}_i\mathbf{S}_{ij}\mathbf{SC}_{ijk}\ \bullet$
$\mathbf{S}_i\mathbf{C}_{ijk}\mathbf{SC}_{ijk}\ \bullet$
$\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}\ \bullet$
$\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}\ \bullet$

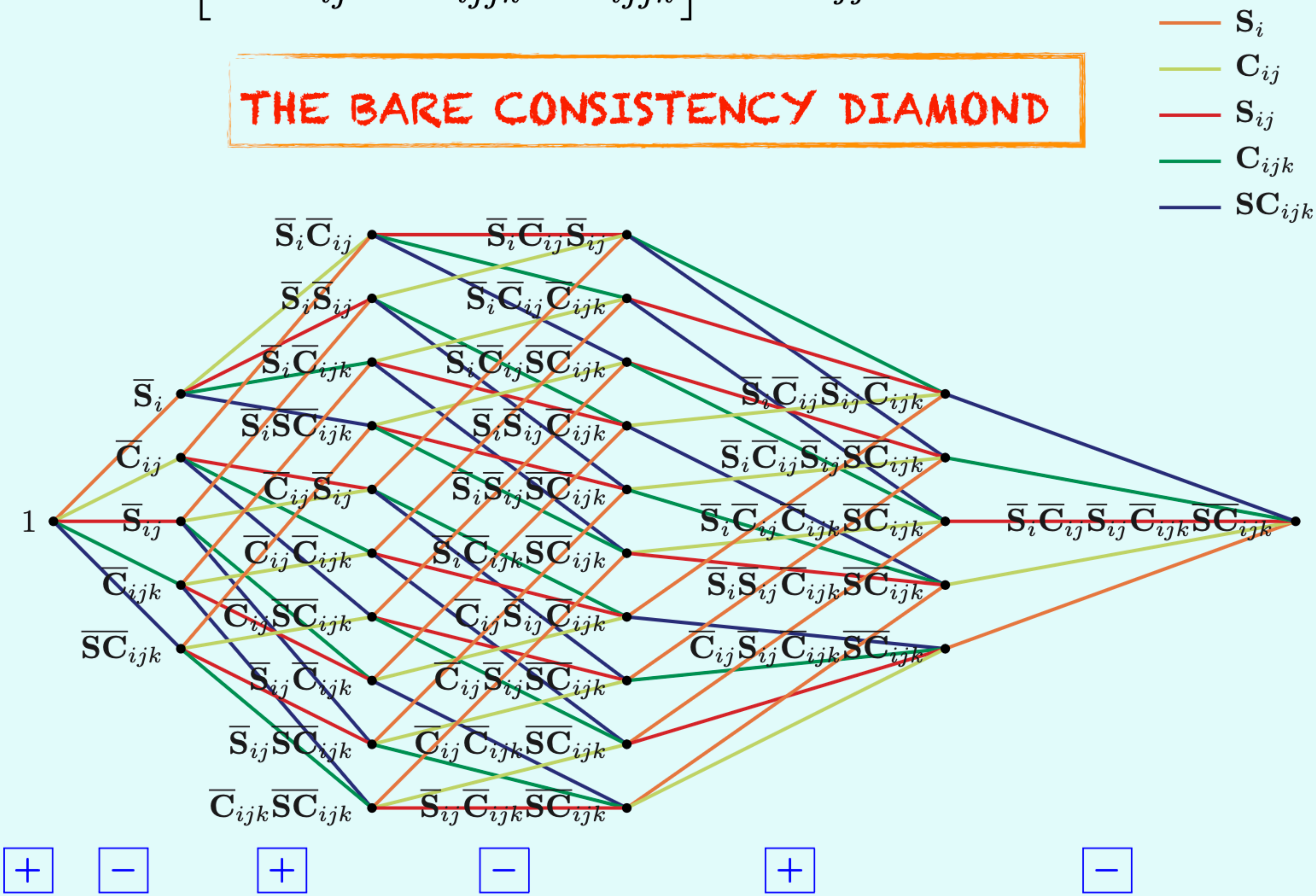
$\mathbf{S}_i\mathbf{C}_{ij}\mathbf{S}_{ij}\mathbf{C}_{ijk}\ \bullet$
$\mathbf{S}_i\mathbf{S}_{ij}\mathbf{C}_{ijk}\mathbf{SC}_{ijk}\ \bullet$

+	−	+	−	+	−

(Slide by S. Uccirati)

$$\left[1-\overline{\mathbf{L}}^{(\mathbf{1})}_{ij}-\overline{\mathbf{L}}^{(\mathbf{2})}_{ijjk}+\overline{\mathbf{L}}^{(\mathbf{12})}_{ijjk}\right] RR\,\mathcal{W}_{ijjk} \quad = \quad \text{finite}$$

THE BARE CONSISTENCY DIAMOND



NNLO subtraction formula

Massless QCD final-state radiation

$$\frac{d\sigma_{NNLO}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathrm{RV})}\right)\delta_{X_n}$$

$$+ \int d\Phi_{n+1}\left[\left(RV + I^{(1)}\right)\delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)}\right)\delta_{X_n}\right]$$

$$\mathrm{L}_{ab} = \log\frac{s_{ab}}{\mu^2}$$

$$\begin{aligned}
VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ & \left[I^{(0)} + \sum_j I_j^{(1)}\,\mathbf{L}_{jr} + \sum_j I_j^{(2)}\,\mathbf{L}_{jr}^2 + \frac{1}{2}\sum_{j,l\neq j}\gamma_j^{\mathrm{hc}}\,\gamma_l^{\mathrm{hc}}\,\mathbf{L}_{jr'}\mathbf{L}_{lr'}\right]\mathbf{B} \\
& + \sum_j \left[I_{jr}^{(0)} + I_{jr}^{(1)}\,\mathbf{L}_{jr}\right]\mathbf{B}_{jr} - 2\,(1-\zeta_2)\sum_{j,c\neq j,r}\gamma_j^{\mathrm{hc}}\,(2-\mathbf{L}_{cr})\,\mathbf{B}_{cr} \\
& + \sum_{c,d\neq c}\mathbf{L}_{cd}\left[I_{cd}^{(0)} + I_{cd}^{(1)}\,\mathbf{L}_{cd} + \frac{\beta_0}{12}\mathbf{L}_{cd}^2 + (4-\mathbf{L}_{cd})\sum_j\gamma_j^{\mathrm{hc}}\,\mathbf{L}_{jr}\right]\mathbf{B}_{cd} \\
& + \sum_{c,d\neq c}\left[-2+\zeta_2+2\,\zeta_3-\frac{5}{4}\,\zeta_4+2(1-\zeta_3)\,\mathbf{L}_{cd}\right]\mathbf{B}_{cdcd} \\
& + (1-\zeta_2)\sum_{\substack{c,d\neq c\\ e\neq d}}\mathbf{L}_{cd}\,\mathbf{L}_{ed}\,\mathbf{B}_{cded} + \sum_{\substack{c,d\neq c\\ e,f\neq e}}\mathbf{L}_{cd}\,\mathbf{L}_{ef}\left[1-\frac{1}{2}\,\mathbf{L}_{cd}\left(1-\frac{1}{8}\mathbf{L}_{ef}\right)\right]\mathbf{B}_{cdef} \\
& + \pi\sum_{\substack{c,d\neq c\\ e\neq c,d}}\left[\ln\frac{s_{ce}}{s_{de}}\,\mathbf{L}_{cd}^2 + \frac{1}{3}\ln^3\frac{s_{ce}}{s_{de}} + 2\,\mathrm{Li}_3\left(-\frac{s_{ce}}{s_{de}}\right)\right]\mathbf{B}_{cde}\Bigg\} \\
& + \left(\frac{\alpha_s}{2\pi}\right)\left\{\left[\Sigma_\phi - \sum_j\gamma_j^{\mathrm{hc}}\,\mathbf{L}_{jr}\right]\mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c}\mathbf{L}_{cd}\left(2-\frac{1}{2}\mathbf{L}_{cd}\right)\mathbf{V}_{cd}^{\mathrm{fin}}\right\} + \mathbf{VV}^{\mathrm{fin}}
\end{aligned}$$

Analytic and compact!

NNLO subtraction formula

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$$+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\mathrm{RV})} + I^{(12)} \right) \delta_{X_n} \right]$$

$$\mathrm{L}_{ab} = \log \frac{s_{ab}}{\mu^2}$$

Analytic and compact!

$$VV + I^{(2)} + I^{(\mathrm{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg\{ \Bigg[I^{(0)} + \sum_j I_j^{(1)} \mathbf{L}_{jr} + \sum_j I_j^{(2)} \mathbf{L}_{jr}^2 + \ldots$$

$$+ \sum_j \left[I_{jr}^{(0)} + I_{jr}^{(1)} \mathbf{L}_{jr} \right] \mathbf{B}_{jr} - 2\,(1-\zeta_2 \ldots$$

$$+ \sum_{c,d\neq c} \mathbf{L}_{cd} \left[I_{cd}^{(0)} + I_{cd}^{(1)} \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}_{cd}^2 + \ldots \right.$$

$$+ \sum_{c,d\neq c} \left[-2 + \zeta_2 + 2\,\zeta_3 - \frac{5}{4}\,\zeta_4 + 2(1 \ldots \right.$$

$$+ (1-\zeta_2) \sum_{\substack{c,d\neq c\\ e\neq d}} \mathbf{L}_{cd}\,\mathbf{L}_{ed}\,\mathbf{B}_{cded} + \sum_{\substack{c,d\neq c\\ e,f\neq e}} \mathbf{L} \ldots$$

$$+ \pi \sum_{\substack{c,d\neq c\\ e\neq c,d}} \left[\ln\frac{s_{ce}}{s_{de}}\, \mathbf{L}_{cd}^2 + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2\,\mathbf{L} \ldots \right.$$

$$+ \left(\frac{\alpha_s}{2\pi}\right) \left\{ \left[\Sigma_\phi - \sum_j \gamma_j^{\mathrm{hc}}\, \mathbf{L}_{jr} \right] \mathbf{V}^{\mathrm{fin}} + \sum_{c,d\neq c} \mathbf{L}_{cd} \ldots \right.$$

$$\begin{aligned}
I^{(0)} = {} & N_q^2 C_F^2 \left[\frac{101}{8} - \frac{141}{8}\zeta_2 + \frac{245}{16}\zeta_4 \right] + N_g N_q C_F \left[C_A \left(\frac{13}{3} - \frac{125}{6}\zeta_2 + \frac{245}{8}\zeta_4 \right) + \beta_0 \left(\frac{77}{12} - \frac{53}{12}\zeta_2 \right) \right] \\
& + N_g^2 \left[C_A^2 \left(\frac{20}{9} - \frac{13}{3}\zeta_2 + \frac{245}{16}\zeta_4 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{1}{8}\zeta_2 \right) + C_A \beta_0 \left(-\frac{1}{9} - \frac{11}{3}\zeta_2 \right) \right] \\
& + N_q C_F \left[C_F \left(\frac{53}{32} - \frac{57}{8}\zeta_2 + \frac{1}{2}\zeta_3 + \frac{21}{4}\zeta_4 \right) + C_A \left(\frac{677}{432} + \frac{5}{3}\zeta_2 - \frac{25}{2}\zeta_3 + \frac{47}{8}\zeta_4 \right) \right. \\
& \qquad \left. + \beta_0 \left(\frac{5669}{864} - \frac{85}{24}\zeta_2 - \frac{11}{12}\zeta_3 \right) \right] \\
& + N_g \left[C_F C_A \left(-\frac{737}{48} + 11\zeta_3 \right) + C_F \beta_0 \left(\frac{67}{16} - 3\zeta_3 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{3}{8}\zeta_2 \right) \right. \\
& \qquad \left. + C_A^2 \left(-\frac{4289}{216} + \frac{15}{2}\zeta_2 - 14\zeta_3 + \frac{89}{8}\zeta_4 \right) + C_A \beta_0 \left(\frac{647}{54} - \frac{53}{8}\zeta_2 - \frac{11}{12}\zeta_3 \right) \right] \\
I_j^{(1)} = {} & \delta_{f_a\{q,\bar{q}\}} C_F \left[N_q C_F \left(\frac{5}{2} - \frac{7}{4}\zeta_2 \right) + N_g C_A \left(\frac{1}{3} - \frac{7}{4}\zeta_2 \right) + \frac{2}{3} N_g \beta_0 \right. \\
& \qquad \left. + C_F \left(-\frac{3}{8} - 4\zeta_2 + 2\zeta_3 \right) + C_A \left(\frac{25}{12} - 3\zeta_2 + 3\zeta_3 \right) + \beta_0 \left(\frac{1}{24} + \zeta_2 \right) \right] \\
& + \delta_{f_a g} \left[N_q C_F C_A (10 - 7\zeta_2) - N_q C_F \beta_0 \left(\frac{5}{2} - \frac{7}{4}\zeta_2 \right) + N_g C_A^2 \left(\frac{4}{3} - 7\zeta_2 \right) + N_g C_A \beta_0 \left(\frac{7}{3} + \frac{7}{4}\zeta_2 \right) \right. \\
& \qquad \left. - \frac{2}{3}(N_g + 1)\beta_0^2 + \frac{11}{4} C_F C_A - \frac{3}{4} C_F \beta_0 + C_A^2 \left(\frac{28}{3} - \frac{23}{2}\zeta_2 + 5\zeta_3 \right) - C_A \beta_0 \left(\frac{2}{3} - \frac{5}{2}\zeta_2 \right) \right] \\
I_j^{(2)} = {} & \frac{1}{8}(15\,C_A - 7\,\beta_0 - 15) C_{f_j} - \frac{1}{4}(5\,C_A - 2\,\beta_0)\gamma_j + 2\,\zeta_2\, C_{f_j}^2 \\
I_{jr}^{(0)} = {} & (-1 + 3\zeta_2 - 2\zeta_3) C_A - \frac{1}{2}(13 + 10\zeta_2 + 2\zeta_3) C_{f_j} + (5 + 2\zeta_3)\gamma_j \\
I_{jr}^{(1)} = {} & (1 - \zeta_2) C_A + \frac{1}{2}(4 + 7\zeta_2) C_{f_j} - (2 + \zeta_2)\gamma_j \\
I_{cd}^{(0)} = {} & \left(\frac{20}{9} - 2\zeta_2 - \frac{7}{2}\zeta_3 \right) C_A + \frac{31}{9}\,\beta_0 + 2\,\Sigma_\phi + 8\,(1 - \zeta_2)\, C_{f_d} \\
I_{cd}^{(1)} = {} & -\left(\frac{1}{3} - \frac{1}{2}\zeta_2 \right) C_A - \frac{11}{12}\,\beta_0 - \frac{1}{2}\,\Sigma_\phi
\end{aligned}$$