Hidden Amplitude Zeros and Residues at Infinity

Shruti Paranjape
Brown University

QCD Meets Gravity 2024
LeCosPA, National Taiwan University

Based on arXiv:2403.10594 with C Bartsch, T Brown, K Kampf, U Oktem and J Trnka and arXiv:2501.xxxxx with C Jones

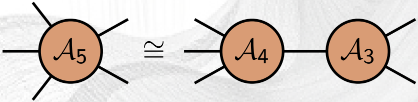
Tree Amplitudes

At tree-level, the amplitude is a rational function of kinematic variables,

$$\mathcal{A}_n = \frac{\text{numerator}}{\text{denominator}}$$

Additionally, a lot is known about the singularity structure from:

- Locality: Poles and branch cuts of the amplitudes
  $\lim_{p_I^2 \to 0} A_n \sim \frac{1}{p_I^2}$
- Unitarity: Factorization into lower-point amplitudes
  $\underset{p_I^2=0}{\text{Res}}\, A_n = A_L \times A_R$



What About Numerators?

Zeros and poles fully determine any rational function $\Rightarrow$ do amplitudes have zeros?

What About Numerators?

Zeros and poles fully determine any rational function $\Rightarrow$ do amplitudes have zeros?

Type	Example
Vanishing soft limits	Pion EFTs [Weinberg]
Radiation zeros	Gauge boson 4-point amplitudes in SM [Baur, Han, Ohnemus][Dixon, Kunszt, Signer]
Stringy zeros	Dual resonant amplitude [D'Adda, Sciuto,D'Auria, Gliozzi]
Helicity zeros	$\mathcal{N}=8$ SUGRA MHV amplitude [Koefler, Oktem, SP, Trnka, Zakovic]

What About Numerators?

Zeros and poles fully determine any rational function ⇒ do amplitudes have zeros?

Type	Example
Vanishing soft limits	Pion EFTs [Weinberg]
Radiation zeros	Gauge boson 4-point amplitudes in SM [Baur, Han, Ohnemus][Dixon, Kunszt, Signer]
Stringy zeros	Dual resonant amplitude [D'Adda, Sciuto,D'Auria, Gliozzi]
Helicity zeros	$\mathcal{N} = 8$ SUGRA MHV amplitude [Koefler, Oktem, SP, Trnka, Zakovic]
"Hidden Zeros"	?

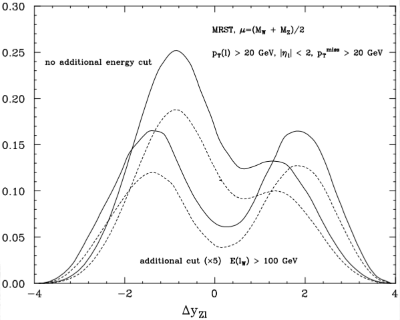
What About Numerators?

Zeros and poles fully determine any rational function $\Rightarrow$ do amplitudes have zeros?

Type	Example
Vanishing soft limits	Pion EFTs [Weinberg]
Radiation zeros	Gauge boson 4-point amplitudes in SM [Baur, Han, Ohnemus][Dixon, Kunszt, Signer]
Stringy zeros	Dual resonant amplitude [D'Adda, Sciuto,D'Auria, Gliozzi]
Helicity zeros	$\mathcal{N} = 8$ SUGRA MHV amplitude [Koefler, Oktem, SP, Trnka, Zakovic]
"Hidden Zeros"	?
Good large z behaviour	?

Example of a Zero

- Amplitudes of pions have an Adler zero i.e. they vanish in the limit of vanishing external pion momentum.
- Radiation zeros exist at tree-level in standard model processes like $q_1 \bar{q}_2 \to W^{\pm} Z$ at specific angles.



[Baur, Han, Ohnemus][Dixon, Kunszt, Signer]

Hidden Zeros

Hidden zeros are present in partial amplitudes of a certain class of theories: NLSM, Yang-Mills and $\mathrm{Tr}(\phi^3)$.

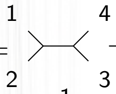
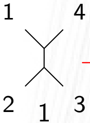
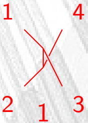
[Arkani-Hamed, Cao, Dong, Figueiredo, He]

In d-dimensional scalar theories, partial amplitudes vanish when a specific set of Mandelstam invariants $s_{ij} = (p_i + p_j)^2$ is zero,

$$\begin{array}{lcc} \text{4-point} & s_{13} = 0 & \\ \text{5-point} & s_{13} = s_{14} = 0 & \\ \text{6-point} & s_{13} = s_{14} = s_{15} = 0 & C_2 \\ & s_{14} = s_{15} = s_{24} = s_{25} = 0 & C_3 \end{array}$$

Note: The first type of zero is one Mandelstam away from being an Adler zero $p_1 \to 0$.

Example in $\mathrm{Tr}(\phi^3)$

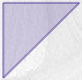
$A_4^{\mathrm{Tr}(\phi^3)}[1234] =$  $+$  $+$ 

$\frac{1}{s_{12}}$ $\frac{1}{s_{14}}$ $\frac{1}{s_{13}}$

We only include diagrams compatible with color ordering 1234:

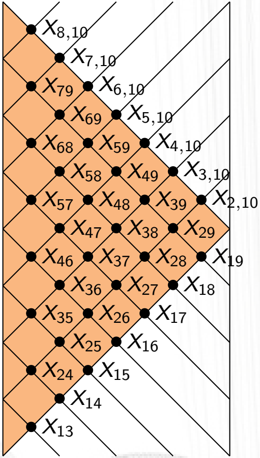
$$A_4^{\mathrm{Tr}(\phi^3)}[1234] = \frac{1}{s_{12}} + \frac{1}{s_{14}} = -\frac{s_{13}}{s_{12}s_{14}} \xrightarrow{s_{13}\to 0} 0$$

Hidden zeros do not occur diagram by diagram. They exist because $\mathrm{Tr}(\phi^3)$ amplitudes are canonical forms on associahedra that are Minkowski sums of components:

$\mathcal{A}_2 = \sum c_{24}$ $+ c_{14}$  $+ c_{13}$

Why do YM and NLSM have zeros?

Introducing the Kinematic Mesh

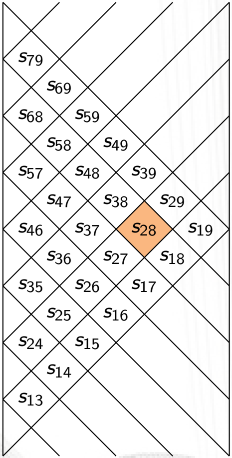


$$X_{ij} = (p_i + p_{i+1} + \cdots + p_{j-1})^2$$

The planar variables live on the vertices and the triangular region gives the variables that the amplitude depends on.

(see Song's talk)

Introducing the Kinematic Mesh



$$s = X_L + X_R - X_B - X_T$$

e.g. $s_{38} = X_{39} + X_{48} - X_{38} - X_{49}$

The non-planar variables (which can never be poles) live on the plaquettes.

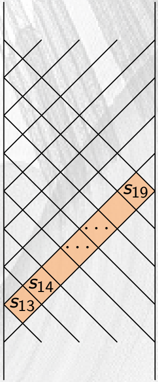
(see Song's talk)

Kinematic Mesh and n-Point Zeros

For scalar theories like NLSM and $\mathrm{Tr}(\phi^3)$, $A_n|_{C_m} = 0$ where the zero condition is given by

$$C_m = \begin{cases} s_{1m+1} = s_{1m+2} = \cdots = s_{1n-1} = 0 \\ s_{2m+1} = s_{2m+2} = \cdots = s_{2n-1} = 0 \\ \vdots \\ s_{m-1m+1} = \cdots = s_{m-1n-1} = 0 \end{cases}$$

where $m = 2, \cdots, \lfloor \frac{n}{2} \rfloor$.



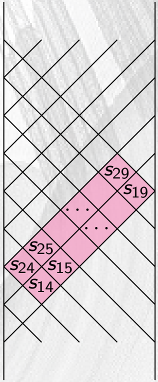
[Arkani-Hamed, Cao, Dong, Figueiredo, He]

Kinematic Mesh and n-Point Zeros

For scalar theories like NLSM and $\mathrm{Tr}(\phi^3)$, $A_n|_{C_m} = 0$ where the zero condition is given by

$$C_m = \begin{cases} s_{1m+1} = s_{1m+2} = \cdots = s_{1n-1} = 0 \\ s_{2m+1} = s_{2m+2} = \cdots = s_{2n-1} = 0 \\ \vdots \\ s_{m-1m+1} = \cdots = s_{m-1n-1} = 0 \end{cases}$$

where $m = 2, \cdots, \lfloor \frac{n}{2} \rfloor$.



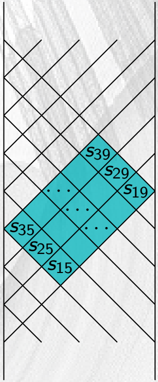
[Arkani-Hamed, Cao, Dong, Figueiredo, He]

Kinematic Mesh and n-Point Zeros

For scalar theories like NLSM and $\mathrm{Tr}(\phi^3)$, $A_n|_{C_m} = 0$ where the zero condition is given by

$$C_m = \begin{cases} s_{1m+1} = s_{1m+2} = \cdots = s_{1n-1} = 0 \\ s_{2m+1} = s_{2m+2} = \cdots = s_{2n-1} = 0 \\ \vdots \\ s_{m-1m+1} = \cdots = s_{m-1n-1} = 0 \end{cases}$$

where $m = 2, \cdots, \lfloor \frac{n}{2} \rfloor$.



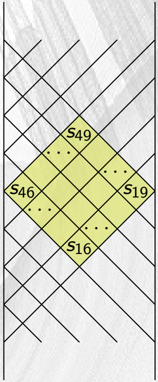
[Arkani-Hamed, Cao, Dong, Figueiredo, He]

Kinematic Mesh and n-Point Zeros

For scalar theories like NLSM and $\mathrm{Tr}(\phi^3)$, $A_n|_{C_m} = 0$ where the zero condition is given by

$$C_m = \begin{cases} s_{1m+1} = s_{1m+2} = \cdots = s_{1n-1} = 0 \\ s_{2m+1} = s_{2m+2} = \cdots = s_{2n-1} = 0 \\ \vdots \\ s_{m-1m+1} = \cdots = s_{m-1n-1} = 0 \end{cases}$$

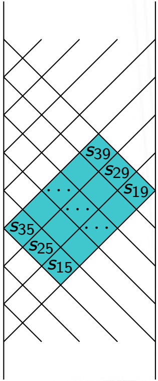
where $m = 2, \cdots, \lfloor \frac{n}{2} \rfloor$.



[Arkani-Hamed, Cao, Dong, Figueiredo, He]

Factorization Near Zeros

Like near poles, the amplitude factorizes near zeros into lower-point amplitudes, when all-but-one Mandelstam in C_m vanishes.



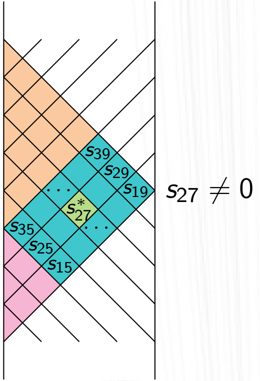
Unlike near poles, there is no physical principle that tells us why this should be the case.

[Arkani-Hamed, Cao, Dong, Figueiredo, He]

related: [Cachazo, Early, Umbert]

Factorization Near Zeros

Like near poles, the amplitude factorizes near zeros into lower-point amplitudes, when all-but-one Mandelstam in C_m vanishes.



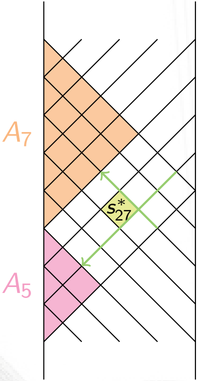
Unlike near poles, there is no physical principle that tells us why this should be the case.

[Arkani-Hamed, Cao, Dong, Figueiredo, He]

related: [Cachazo, Early, Umbert]

Factorization Near Zeros

Like near poles, the amplitude factorizes near zeros into lower-point amplitudes, when all-but-one Mandelstam in C_m vanishes.



$$A_{10} \xrightarrow{C_4, s^*_{27}} \left(\frac{1}{s_{1234}} + \frac{1}{s_{3456789}} \right) \times A_5 \times A_7$$

Unlike near poles, there is no physical principle that tells us why this should be the case.

[Arkani-Hamed, Cao, Dong, Figueiredo, He]

related: [Cachazo, Early, Umbert]

Spinning Zeros

For gluons, we need to extend the zero conditions slightly to now include polarization vectors as well i.e.

$$\text{if } s_{ij} = 0 \text{ on } C_m\,,$$
$$(p_i \cdot p_j) = (\varepsilon_i \cdot p_j) = (p_i \cdot \varepsilon_j) = (\varepsilon_i \cdot \varepsilon_j) = 0 \text{ on } C_m^{\text{spinning}}$$

4-point YM has contact+pole terms:

$$(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) \xrightarrow{(\varepsilon_1 \cdot \varepsilon_3) \to 0} 0$$

Spinning Zeros

For gluons, we need to extend the zero conditions slightly to now include polarization vectors as well i.e.

$$\text{if } s_{ij} = 0 \text{ on } C_m\,,$$
$$(p_i \cdot p_j) = (\varepsilon_i \cdot p_j) = (p_i \cdot \varepsilon_j) = (\varepsilon_i \cdot \varepsilon_j) = 0 \text{ on } C_m^{\text{spinning}}$$

4-point YM has contact+pole terms:

$$(\varepsilon_1 \cdot \varepsilon_3)(\varepsilon_2 \cdot \varepsilon_4) \xrightarrow{(\varepsilon_1 \cdot \varepsilon_3) \to 0} 0$$

$$\frac{1}{s_{12}}\left((\varepsilon_1 \cdot \varepsilon_2) p_1^\mu + (\varepsilon_1 \cdot p_2)\varepsilon_2^\mu + (\varepsilon_2 \cdot p_1)\varepsilon_1^\mu\right)$$
$$\times \left((\varepsilon_3 \cdot \varepsilon_4) p_3^\nu + (\varepsilon_3 \cdot p_4)\varepsilon_4^\nu + (\varepsilon_4 \cdot p_3)\varepsilon_3^\nu\right) \eta_{\mu\nu} + \text{cyc} \xrightarrow{C_2} 0$$

Can this be seen from any of the many constructions we have for YM amplitudes?

In This Talk

1. See the zeros as a consequence of color-kinematic duality / BCJ relations
2. Understand the relation between vanishing residues at infinity and splitting near zeros
3. Extend both these observations to theories without color via KLT

In This Talk

1. See the zeros as a consequence of color-kinematic duality / BCJ relations
2. Understand the relation between vanishing residues at infinity and splitting near zeros
3. Extend both these observations to theories without color via KLT

Theories with Hidden Zeros

So far, we've seen that the following theories have hidden zeros:

- Tr(ϕ^3) theory of adjoint scalars
- SU(N) non-linear sigma model
- Yang-Mills theory
- Yang-Mills + scalar

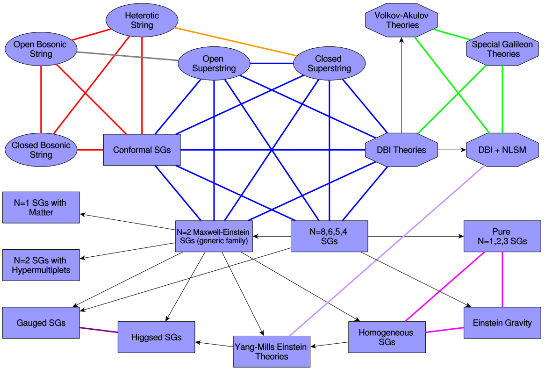
Theories with Hidden Zeros

So far, we've seen that the following theories have hidden zeros:

- Tr(ϕ^3) theory of adjoint scalars
- SU(N) non-linear sigma model
- Yang-Mills theory
- Yang-Mills + scalar

They all "play a role" in the double copy.

Web of Theories



For example: [Bern, Carrasco, Chiodaroli, Johansson, Roiban]

$$\mathrm{BI} = \mathrm{YM} \otimes \mathrm{NLSM} \quad \text{e.g.} \quad \mathcal{M}_4^{\mathrm{BI}}(1234) = \frac{us}{t} A_4^{\mathrm{YM}}[1234] A_4^{\mathrm{NLSM}}[1234]$$

All theories with hidden zeros are related to this map, including

$$A_n^{\phi^3_{a\tilde{a}}}[\alpha|\alpha] = A_n^{\mathrm{Tr}(\phi^3)}[\alpha]$$

BCJ Relations at 6-Point

$$A_6[123456] = \frac{1}{s_{12}s_{123}s_{56}}\Big[s_{13}s_{25}(s_{56}-s_{24})\ A_6[162543]$$
$$+ s_{15}(s_{12}+s_{23})(s_{14}-s_{56})\ A_6[162345]$$
$$- s_{14}(s_{12}+s_{23})(s_{25}+s_{35})\ A_6[162354]$$
$$+ s_{13}s_{15}s_{24}\ A_6[162435]$$
$$+ s_{13}s_{24}(s_{15}+s_{35})\ A_6[162453]$$
$$- s_{14}s_{25}(s_{12}+s_{23})\ A_6[162534]\Big]$$

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

BCJ Relations at 6-Point

$$A_6[123456] = \frac{1}{s_{12}s_{123}s_{56}}\Big[s_{13}s_{25}(s_{56}-s_{24})\,A_6[162543]$$
$$+\,s_{15}(s_{12}+s_{23})(s_{14}-s_{56})\,A_6[162345]$$
$$-\,s_{14}(s_{12}+s_{23})(s_{25}+s_{35})\,A_6[162354]$$
$$+\,s_{13}s_{15}s_{24}\,A_6[162435]$$
$$+\,s_{13}s_{24}(s_{15}+s_{35})\,A_6[162453]$$
$$-\,s_{14}s_{25}(s_{12}+s_{23})\,A_6[162534]\Big]$$

$$C_2 = \{s_{13} = s_{14} = s_{15} = 0\}$$

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

BCJ Relations at 6-Point

$$A_6[123456] = \frac{1}{s_{12}s_{123}s_{56}}\Big[s_{13}s_{25}(s_{56} - s_{24})\ A_6[162543]$$
$$+ s_{15}(s_{12} + s_{23})(s_{14} - s_{56})\ A_6[162345]$$
$$- s_{14}(s_{12} + s_{23})(s_{25} + s_{35})\ A_6[162354]$$
$$+ s_{13}s_{15}s_{24}\ A_6[162435]$$
$$+ s_{13}s_{24}(s_{15} + s_{35})\ A_6[162453]$$
$$- s_{14}s_{25}(s_{12} + s_{23})\ A_6[162534]\Big]$$

$$C_3 = \{s_{14} = s_{15} = s_{24} = s_{25} = 0\}$$

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

BCJ Relations for $\mathrm{Tr}(\phi^3)$

$$
\begin{aligned}
&A_6[123456|123456] \\
&\quad = \frac{1}{s_{12}s_{123}s_{56}}\Big[s_{13}s_{25}(s_{56}-s_{24})\, A_6[162543|123456] \\
&\quad + s_{15}(s_{12}+s_{23})(s_{14}-s_{56})\, A_6[162345|123456] \\
&\quad - s_{14}(s_{12}+s_{23})(s_{25}+s_{35})\, A_6[162354|123456] \\
&\quad + s_{13}s_{15}s_{24}\, A_6[162435|123456] \\
&\quad + s_{13}s_{24}(s_{15}+s_{35})\, A_6[162453|123456] \\
&\quad - s_{14}s_{25}(s_{12}+s_{23})\, A_6[162534|123456]\Big]
\end{aligned}
$$

The amplitudes on the RHS are doubly color-ordered bi-adjoint scalar amplitudes, while the one on the LHS is that of $\mathrm{Tr}(\phi^3)$.

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

BCJ Relations for Tr(ϕ^3)

$$
\begin{aligned}
&A_6[123456|123456] \\
&\quad = \frac{1}{s_{12}s_{123}s_{56}}\Big[s_{13}s_{25}(s_{56}-s_{24})\, A_6[162543|123456] \\
&\quad + s_{15}(s_{12}+s_{23})(s_{14}-s_{56})\, A_6[162345|123456] \\
&\quad - s_{14}(s_{12}+s_{23})(s_{25}+s_{35})\, A_6[162354|123456] \\
&\quad + s_{13}s_{15}s_{24}\, A_6[162435|123456] \\
&\quad + s_{13}s_{24}(s_{15}+s_{35})\, A_6[162453|123456] \\
&\quad - s_{14}s_{25}(s_{12}+s_{23})\, A_6[162534|123456]\Big]
\end{aligned}
$$

The amplitudes on the RHS are doubly color-ordered bi-adjoint scalar amplitudes, while the one on the LHS is that of Tr(ϕ^3).

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

2-Particle Poles?

- NLSM+h.d.: Only 4-particle interactions i.e. 3-particle poles
- Tr(ϕ^3)+h.d.: Choosing second ordering to be [123456] will give no poles at the location of the zeros

2-Particle Poles?

- NLSM+h.d.: Only 4-particle interactions i.e. 3-particle poles
- Tr(ϕ^3)+h.d.: Choosing second ordering to be [123456] will give no poles at the location of the zeros

- YM: The amplitude factorizes on 2-particle poles into

$$A_n \xrightarrow{p_1\cdot p_2=0} A_3(p_1, p_2, -(p_1+p_2)) \times A_{n-1}$$
$$= \Big[(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot p_1)+(\varepsilon_2\cdot\varepsilon_3)(\varepsilon_1\cdot p_2)+(\varepsilon_3\cdot\varepsilon_1)(\varepsilon_2\cdot p_3)\Big]\times A_{n-1}$$

2-Particle Poles?

- NLSM+h.d.: Only 4-particle interactions i.e. 3-particle poles
- Tr(ϕ^3)+h.d.: Choosing second ordering to be [123456] will give no poles at the location of the zeros
- YM: The amplitude factorizes on 2-particle poles into

$$A_n \xrightarrow{p_1 \cdot p_2 = 0} A_3(p_1, p_2, -(p_1+p_2)) \times A_{n-1}$$
$$= \Big[(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot p_1) + (\varepsilon_2 \cdot \varepsilon_3)(\varepsilon_1 \cdot p_2) + (\varepsilon_3 \cdot \varepsilon_1)(\varepsilon_2 \cdot p_3)\Big] \times A_{n-1}$$

- $(DF)^2 + h.d.$: Non-local extension of YM also has an A_3 that cancels 2-particle poles.
- $(F)^3 + h.d.$: Higher-derivative extension of YM also has an A_3 that cancels 2-particle poles.

BCJ Relation at n-Point

$$A_n[123\cdots n] = (-1)^n \sum_{\sigma(3\ldots n-1)} A_n[1n2\sigma] \times \prod_{k=3}^{n-1} \frac{\mathcal{F}_k[2\sigma 1]}{s_{kk+1\ldots n}}$$

[Bern, Carrasco, Johansson]

The factors $\mathcal{F}_k$ are given by

$$\mathcal{F}_k[\rho] = \left\{ \begin{array}{ll} \sum_{l=t_k}^{n-1} \mathcal{S}_{k,\rho_l} & \text{if } t_k > t_{k+1} \\ -\sum_{l=1}^{t_k} \mathcal{S}_{k,\rho_l} & \text{if } t_k < t_{k+1} \end{array} \right\} + \begin{cases} s_{kk+1\ldots n} & \text{if } t_{k-1} > t_k > t_{k+1} \\ -s_{kk+1\ldots n} & \text{if } t_{k-1} < t_k < t_{k+1} \\ 0 & \text{else,} \end{cases}$$

where t_k is the position of leg k in the ordered list $\rho = \{2\sigma 1\}$ and ρ_l denotes its l-th element and

$$t_2 = 0, \qquad t_n = t_{n-2}$$

$$\mathcal{S}_{i,j} = \begin{cases} s_{ij} & \text{if } i > j \text{ or } j = 1, 2 \\ 0 & \text{else} \end{cases}$$

BCJ + absence of 2-particle poles $\Rightarrow$ Hidden zeros

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

BCJ Relation at n-Point

$$A_n[123\cdots n] = (-1)^n \sum_{\sigma(3\ldots n-1)} A_n[1n2\sigma] \times \prod_{k=3}^{n-1} \frac{\mathcal{F}_k[2\sigma 1]}{s_{kk+1\ldots n}}$$

[Bern, Carrasco, Johansson]

The factors $\mathcal{F}_k$ are given by

$$\mathcal{F}_k[\rho] = \begin{Bmatrix} \sum_{l=t_k}^{n-1} \mathcal{S}_{k,\rho_l} & \text{if } t_k > t_{k+1} \\ -\sum_{l=1}^{t_k} \mathcal{S}_{k,\rho_l} & \text{if } t_k < t_{k+1} \end{Bmatrix} + \begin{cases} s_{kk+1\ldots n} & \text{if } t_{k-1} > t_k > t_{k+1} \\ -s_{kk+1\ldots n} & \text{if } t_{k-1} < t_k < t_{k+1} \\ 0 & \text{else,} \end{cases}$$

where t_k is the position of leg k in the ordered list $\rho = \{2\sigma 1\}$ and ρ_l denotes its l-th element and

$$t_2 = 0, \qquad t_n = t_{n-2}$$

$$\mathcal{S}_{i,j} = \begin{cases} s_{ij} & \text{if } i > j \text{ or } j = 1,2 \\ 0 & \text{else} \end{cases}$$

BCJ + absence of 2-particle poles ⇒ Hidden zeros

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

Works at n-point

1. See the zeros as a consequence of color-kinematic duality / BCJ relations
2. Understand the relation between vanishing residues at infinity and splitting near zeros
3. Extend both these observations to theories without color via KLT

Zero and Infinity

Recursion relations have showed us that many properties of amplitudes at finite kinematic points can be understood via their behaviour at infinity.

Let us revisit the splitting relation:

$$A_6[1,2,3,4,5,6] \xrightarrow{s_{14}=s_{24}=s_{25}=0} \left(\frac{1}{X_{14}}+\frac{1}{X_{36}}\right) A_4[1,2,3,4]\, A_4[3,4,5,6]$$
$$= \left(\frac{s_{15}}{X_{14}X_{36}}\right) A_4[1,2,3,4]\, A_4[3,4,5,6]$$

This appears to be a sum of two residues: at $X_{14} \to 0$ and $X_{36} \to 0$.
Also, the splitting relation $\Rightarrow$ existence of zero.

Can we design a shift of our variables such that Cauchy's theorem gives us the splitting relation?

Zero and Infinity

Recursion relations have showed us that many properties of amplitudes at finite kinematic points can be understood via their behaviour at infinity.

Let us revisit the splitting relation:

$$A_6\left[1,2,3,4,5,6\right] \xrightarrow{s_{14}=s_{24}=s_{25}=0} \left(\frac{1}{X_{14}}+\frac{1}{X_{36}}\right) A_4\left[1,2,3,4\right] A_4\left[3,4,5,6\right]$$
$$= \left(\frac{s_{15}}{X_{14}X_{36}}\right) A_4\left[1,2,3,4\right] A_4\left[3,4,5,6\right]$$

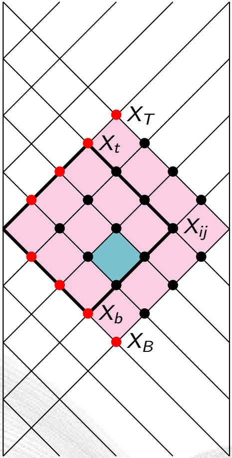
This appears to be a sum of two residues: at $X_{14} \to 0$ and $X_{36} \to 0$.
Also, the splitting relation $\Rightarrow$ existence of zero.

Can we design a shift of our variables such that Cauchy's theorem gives us the splitting relation?

$$\hat{X}_{14} = X_{14} - z, \qquad \hat{X}_{36} = X_{36} + z$$

[Jones, SP]

Shifting X



$X_{ij} = X_b + X_t - c$

The only X_{ij}'s that we need to shift are a subset of the black vertices on the mesh.

$$\hat{X}_{15} = X_{14} + X_{35} - z,$$
$$\hat{X}_{26} = X_{24} + X_{36} + z$$

We have poles at z_{ij} where $\hat{X}_{ij}(z_{ij}) = 0$, in addition to $\hat{X}_{14}$ and $\hat{X}_{36}$.

[Jones, SP]

Residues on Poles

Cauchy's theorem gives

$$\oint \frac{dz}{2\pi i}\frac{A_n(z)}{z} = 0 \overset{?}{\Rightarrow} A_6 \xrightarrow{s_{14}=s_{24}=s_{25}=0} \left(\frac{1}{X_{14}}+\frac{1}{X_{36}}\right) A_4 \times A_4$$

where the residues are at

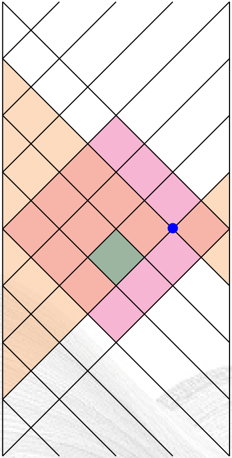
1. $z{=}0$: $A_n(0)$ is the amplitude on split kinematics
2. $z = -X_{13}$ and $z = X_{46}$: These residues give the splitting relation
3. $z = \infty$: Vanishing residues at infinity
4. $z = z_{ij}$: The residues on the poles z_{ij} vanish due to lower-point zeros.

$$A_6\,[1,2,3,4,5,6] \xrightarrow{X_{26}\to 0} \frac{A_3 \times A_5\,[p_{61},2,3,4,5]}{X_{26}}$$

[Jones, SP]

Residue Theorem from Kinematic Mesh

This always happens:



For every pole, one of the subamplitudes is evaluated on a zero and vanishes.

$$\underset{z=0}{\mathrm{Res}} A_n(z) = \underset{z=X_{13}}{\mathrm{Res}} A_n(z) + \underset{z=X_{46}}{\mathrm{Res}} A_n(z) + \underset{z=\infty}{\mathrm{Res}} A_n(z)$$

[Jones, SP]

Residues at Infinity

$$\operatorname*{Res}_{z=\infty} \text{ vanishing} + \text{lower-point zeros} \Leftrightarrow \text{splitting and zeros at all } n$$

Residue vanishes in all checks - particularly surprising in NLSM due to derivative counting! Related to the existence of a c-expansion.

This means that

$$\text{BCJ} \;\Rightarrow\; \text{Hidden zeros} \xrightarrow[]{\operatorname*{Res}_{z=\infty}=0} \text{Splitting}$$

Residues at Infinity

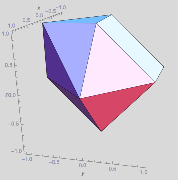
$$\underset{z=\infty}{\text{Res}} \text{ vanishing} + \text{lower-point zeros} \Leftrightarrow \text{splitting and zeros at all } n$$

Residue vanishes in all checks - particularly surprising in NLSM due to derivative counting! Related to the existence of a c-expansion.

This means that

$$\text{BCJ} \Rightarrow \text{Hidden zeros} \xrightarrow[z=\infty]{\text{Res}=0} \text{Splitting}$$

In Tr(ϕ^3), behaviour at infinity is given by a subset of simplices of the ABHY associahedron.



[SP, Skowronek, Spradlin, Volovich]

1. See the zeros as a consequence of color-kinematic duality / BCJ relations
2. Understand the relation between vanishing residues at infinity and splitting near zeros
3. Extend both these observations to theories without color via KLT

Doubling Zeros

Remember that the KLT relation at 5-point:

$$\mathcal{M}_5 = \sum_{\alpha\beta} A_5[152\alpha(34)]\ S[\alpha|\beta]\ A_5[1\beta(34)25]$$

where $S[\alpha|\beta]$ is the KLT kernel. Consider the matrix

$$\begin{bmatrix} s_{13}s_{14} & (s_{13}+s_{34})s_{14} \\ s_{13}(s_{14}+s_{34}) & s_{13}s_{14} \end{bmatrix}$$

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

Doubling Zeros

Remember that the KLT relation at 5-point:

$$\mathcal{M}_5 = \sum_{\alpha\beta} A_5[152\alpha(34)]\ S[\alpha|\beta]\ A_5[1\beta(34)25]$$

where $S[\alpha|\beta]$ is the KLT kernel. Consider the matrix

$$\begin{bmatrix} s_{13}s_{14} & (s_{13}+s_{34})s_{14} \\ s_{13}(s_{14}+s_{34}) & s_{13}s_{14} \end{bmatrix}$$

$$C_2 = \{s_{13} = s_{14} = 0\}$$

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

Doubling Zeros

Remember that the KLT relation at 5-point:

$$\mathcal{M}_5 = \sum_{\alpha\beta} A_5[152\alpha(34)]\ S[\alpha|\beta]\ A_5[1\beta(34)25]$$

where $S[\alpha|\beta]$ is the KLT kernel. Consider the matrix

$$\begin{bmatrix} s_{13}s_{14} & (s_{13}+s_{34})s_{14} \\ s_{13}(s_{14}+s_{34}) & s_{13}s_{14} \end{bmatrix}$$

$$C_2 = \{s_{13} = s_{23} = 0\}$$

Thus it is important to choose the correct KLT basis in order to manifest a particular zero.

Since the BCJ relations imply basis independence, we expect the zeros to survive through the double copy.

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

KLT Relation at n-Point

At n-point,

$$\mathcal{M}_n = \sum_{\alpha\beta} A_n[1mn\alpha]\ S[\alpha|\beta]\ A_n[1\beta mn]$$

where the kernel is

$$S[i_1, \cdots i_m | j_1 \cdots j_m]_p = \left(\frac{1}{2}\right)^{-m} \prod_{t=1}^{m} \left(p \cdot k_{i_t} + \sum_{q>t}^{m} \theta(i_t, i_q) k_{i_t} \cdot k_{i_q} \right)$$

where $\theta(i_t, i_q) = 1$ if the ordering of i_t and i_q is the opposite in $\{i_1, \cdots i_m\}$ and $\{j_1, \cdots j_m\}$ and 0 if the same.

This manifests the m^{th} type of zero i.e.
KLT + absence of 2-particle poles $\Rightarrow$ hidden zeros

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

KLT Relation at n-Point

At n-point,

$$\mathcal{M}_n = \sum_{\alpha\beta} A_n[1mn\alpha]\ S[\alpha|\beta]\ A_n[1\beta mn]$$

where the kernel is

$$S[i_1, \cdots i_m | j_1 \cdots j_m]_p = \left(\frac{1}{2}\right)^{-m} \prod_{t=1}^{m} \left(p \cdot k_{i_t} + \sum_{q>t}^{m} \theta(i_t, i_q) k_{i_t} \cdot k_{i_q} \right)$$

Works at n-point

where $\theta(i_t, i_q) = 1$ if the ordering of i_t and i_q is the opposite in $\{i_1, \cdots i_m\}$ and $\{j_1, \cdots j_m\}$ and 0 if the same.

This manifests the m^{th} type of zero i.e.
KLT + absence of 2-particle poles $\Rightarrow$ hidden zeros

[Bartsch, Brown, Kampf, Oktem, SP, Trnka]

2-Particle Poles?

- SGal+h.d. = NLSM + h.d. $\otimes$ NLSM + h.d. : NLSM only has 4-particle interactions so no 2-particle poles
- BI+h.d. = NLSM + h.d. $\otimes$ YM + h.d. : A_3^{YM} vanishes on the polarization conditions, leading to no 2-particle poles

2-Particle Poles?

- SGal+h.d. = NLSM + h.d. $\otimes$ NLSM + h.d. : NLSM only has 4-particle interactions so no 2-particle poles
- BI+h.d. = NLSM + h.d. $\otimes$ YM + h.d. : A_3^{YM} vanishes on the polarization conditions, leading to no 2-particle poles

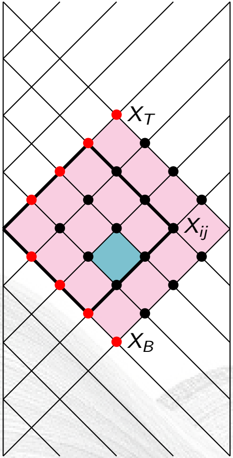
- Dilaton gravity = YM $\otimes$ YM: Only graviton polarizations of the type $\varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu$ satisfy the polarization conditions.

2-Particle Poles?

- SGal+h.d. $=$ NLSM $+$ h.d. $\otimes$ NLSM $+$ h.d. : NLSM only has 4-particle interactions so no 2-particle poles
- BI+h.d. $=$ NLSM $+$ h.d. $\otimes$ YM $+$ h.d. : A_3^{YM} vanishes on the polarization conditions, leading to no 2-particle poles
- Dilaton gravity $=$ YM $\otimes$ YM: Only graviton polarizations of the type $\varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu$ satisfy the polarization conditions.
- Conformal gravity $= (DF)^2 \otimes (DF)^2$: Same polarization selection rule as above
- Higher-derivative gravity $=$ YM+h.d. $\otimes$ YM $+$ h.d.: Same polarization selection rule as above

c's and X's

Let us revisit our shift $\hat{X}_B$ and $\hat{X}_T$, along with the other X's.



Under this shift, none of the $\hat{c} = c$ and so there are no extra poles in these non-planar variables in the case of uncolored theories $\Rightarrow$ the residue theorem also works for uncolored theories.

$$A_6\left(1,2,3,4,5,6\right) \xrightarrow{s_{14}=s_{24}=s_{25}=0} \left(\frac{1}{X_{14}}+\frac{1}{X_{36}}\right) A_4\left(1,2,3,4\right) A_4\left(3,4,5,6\right)$$

(see Giulio's talk?)

[Jones, SP]

Residues at Infinity and KLT

Like large-z BCFW scaling, large-z behaviour under these splitting shifts is also implied by the double copy.

Choosing the same basis that guarantees the presence of zeros, most of the elements of the KLT matrix vanish.
For the ones that don't: The scaling of the corresponding ordered amplitudes in NLSM at 6-point e.g. behave as $\mathcal{O}\left(\frac{1}{z}\right)$.

Thus the double copy behaves as

$$M_6^{\mathrm{sGal}} \sim \mathcal{O}\left(z^0\right) \times \mathcal{O}\left(\frac{1}{z}\right) \times \mathcal{O}\left(\frac{1}{z}\right) \sim \mathcal{O}\left(\frac{1}{z^2}\right)$$

i.e. Double copy $\Rightarrow$ good z-scaling $\Rightarrow$ splitting and zeros.

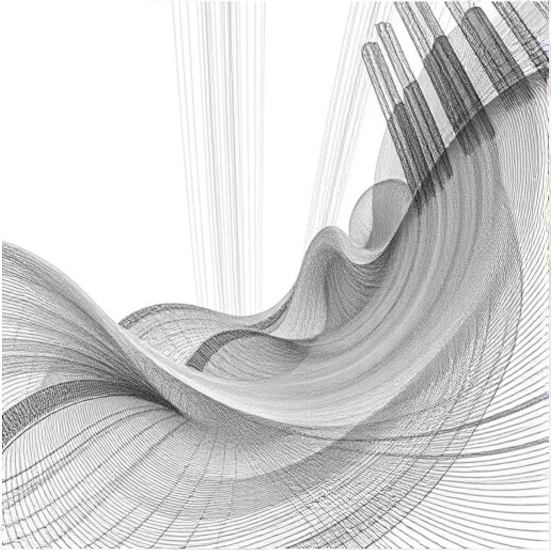
[Jones, SP]

Next Questions

- The BCJ relations guarantee the presence of hidden zeros in single-copy amplitudes.
- Theories have zeros and split near them if and only if their residue at infinity vanishes.
- The KLT relations guarantees the presence of hidden zeros in double-copy amplitudes.
- The z scaling at infinity double copies via the KLT relations, extending these results to uncolored theories.

Next Questions

- The BCJ relations guarantee the presence of hidden zeros in single-copy amplitudes.
- Theories have zeros and split near them if and only if their residue at infinity vanishes.
- The KLT relations guarantees the presence of hidden zeros in double-copy amplitudes.
- The z scaling at infinity double copies via the KLT relations, extending these results to uncolored theories.
- Is there a polytopal description of the function at infinity?
- Which theories have vanishing residues at infinity?
- To what extent do gravitational theories exhibit zeros and factorization?
- Do hidden zeros in double copy theories also have geometric origin?



Thank You