

Scattering Amplitudes and Non-Invertible Symmetry

Shota Komatsu



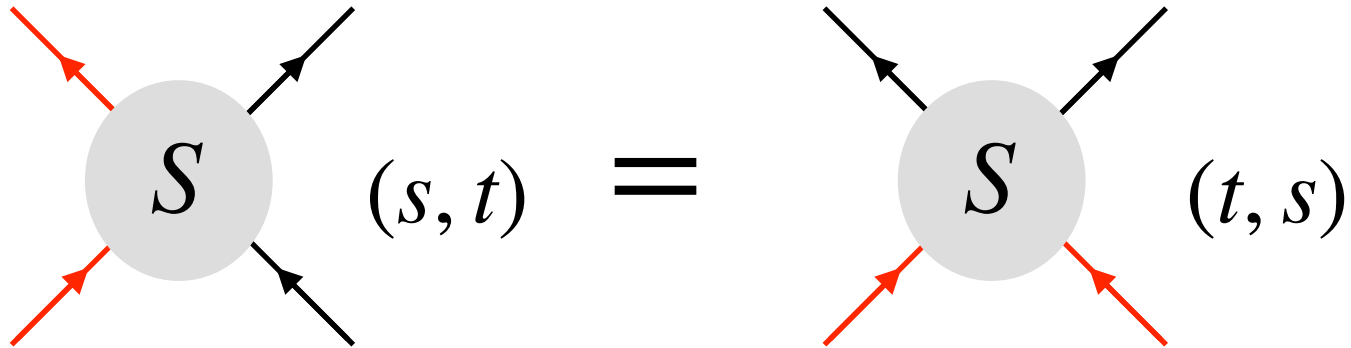
Based on works with **Christian Copetti** (Oxford) **Lucía Córdova** (CERN)

[2403.04835](#) : “Non-Invertible Symmetries, Anomalies and Scattering Amplitudes”

[2408.13132](#): “S-Matrix Bootstrap and Non-Invertible Symmetries”

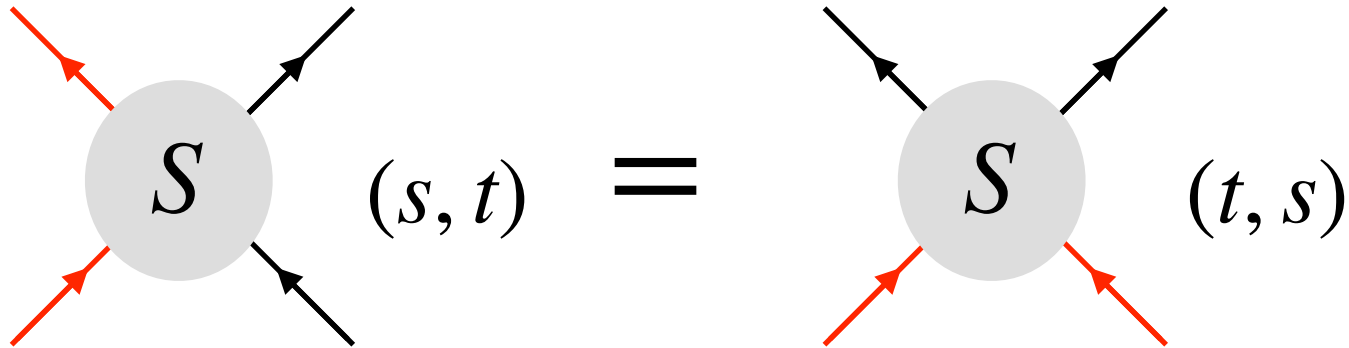
Crossing symmetry

- **Crossing symmetry:** One of important properties of S-matrix



Crossing symmetry

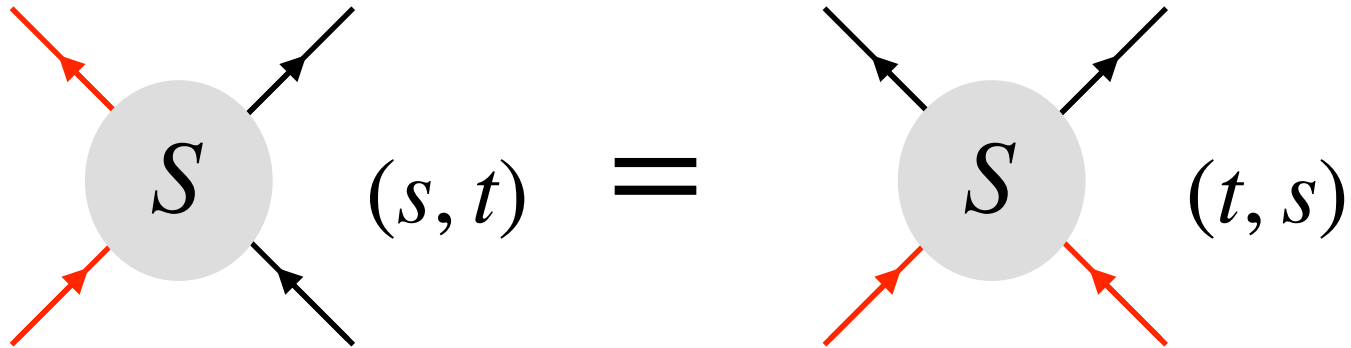
- **Crossing symmetry:** One of important properties of S-matrix



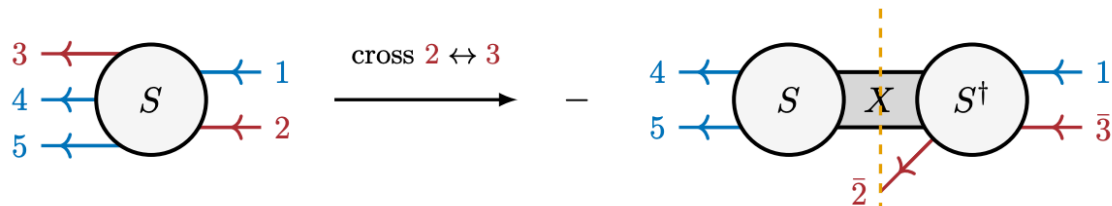
- First found through the computation (~50's).
- Useful in practice. No need to repeat the computation!
- Rigorous proof in some cases. [Bros, Epstein, Glaser],, [Mizera], ...

Crossing symmetry

- **Crossing symmetry:** One of important properties of S-matrix

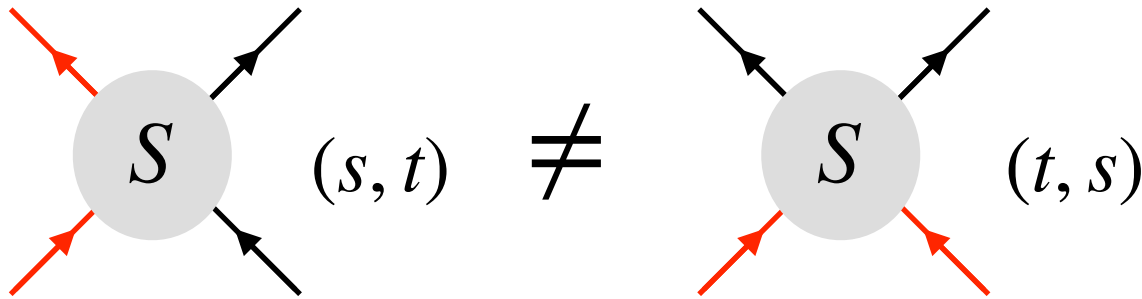


- First found through the computation (~50's).
- Useful in practice. No need to repeat the computation!
- Rigorous proof in some cases. [Bros, Epstein, Glaser],, [Mizera], ...
- Relates higher-pt S-matrix to in-in obs. Relevant for gravitational waveforms. [Caron-Huot, Giroux, Hannesdottir, Mizera]



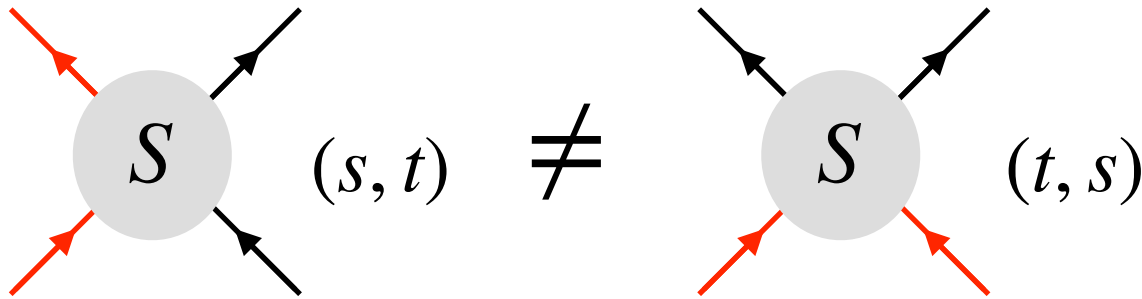
Takeaway of this talk

- In the presence of **long-range modes** (soft / topological), crossing symmetry of S-matrix is sometimes **modified**.



Takeaway of this talk

- In the presence of **long-range modes** (soft / topological), crossing symmetry of S-matrix is sometimes **modified**.

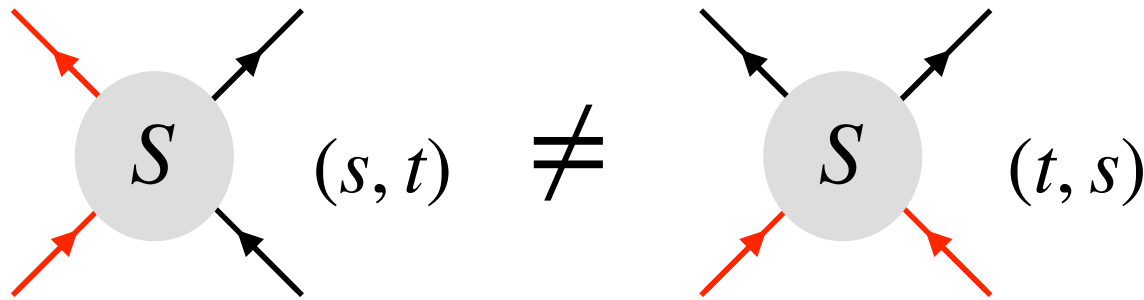


- Modification comes from (non-local) corrections to **norms of in- and out- states** due to **long-range modes**.

$$\langle \{p_i, Q_i\} | \{p_i, Q_i\} \rangle_{\text{full}} = f_{\text{long}}(\{p_i, Q_i\}) \times \langle \{p_i, Q_i\} | \{p_i, Q_i\} \rangle_{\text{short}}$$

Takeaway of this talk

- In the presence of **long-range modes** (soft / topological), crossing symmetry of S-matrix is sometimes **modified**.



- Modification comes from (non-local) corrections to **norms of in- and out- states** due to **long-range modes**.

$$\langle \{p_i, Q_i\} | \{p_i, Q_i\} \rangle_{\text{full}} = f_{\text{long}}(\{p_i, Q_i\}) \times \langle \{p_i, Q_i\} | \{p_i, Q_i\} \rangle_{\text{short}}$$

- Today I'll focus on simplest case:

Theories with **non-invertible sym** & **topological modes** in 1+1 d.

But we expect similar modifications in **higher d** & **soft modes**

(Relevant for **QCD & gravity** waves / charge-monopole....?)

Plan

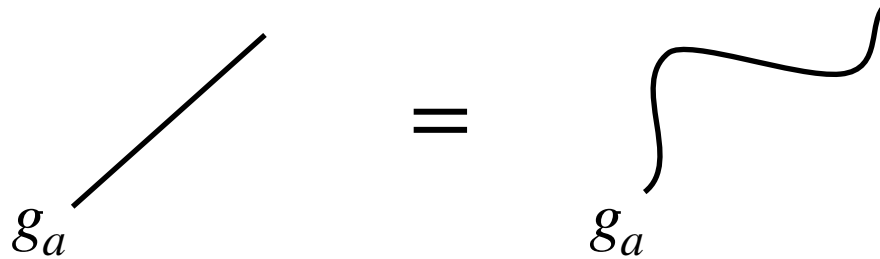
1. Non-invertible symmetries in 1+1 dim
2. Integrable flow from tricritical Ising and S-matrix
3. Derivation of modified crossing rules
4. Conclusion

Plan

1. Non-invertible symmetries in 1+1 dim
2. Integrable flow from tricritical Ising and S-matrix
3. Derivation of modified crossing rules
4. Conclusion

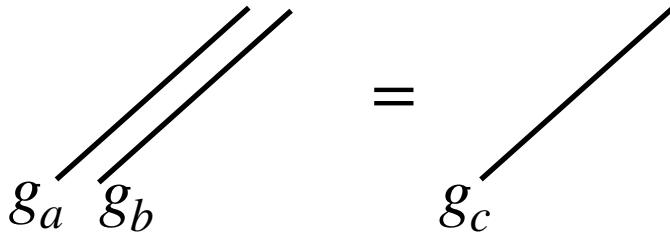
Non-Invertible Symmetries in 1+1 dim

- Symmetries in QFT \subset (d-1)-dim topological operators



A diagram illustrating a symmetry operator g_a . On the left, a straight diagonal line is labeled g_a at its bottom-left end. This is followed by an equals sign. On the right, a wavy diagonal line is labeled g_a at its bottom-left end.

- Usual symmetries (group): $g_a g_b = g_c$ $g_{a,b,c} \in G$



A diagram illustrating the composition of two symmetry operators g_a and g_b . On the left, two parallel diagonal lines are shown, with the left one labeled g_a and the right one labeled g_b . This is followed by an equals sign. On the right, a single diagonal line is labeled g_c .

Non-Invertible Symmetries in 1+1 dim

- Symmetries in QFT \subset (d-1)-dim topological operators

A diagram illustrating a symmetry operator g_a . On the left, a straight line is labeled g_a . This is followed by an equals sign, and on the right, a wavy line is also labeled g_a .

- Usual symmetries (group): $g_a g_b = g_c$ $g_{a,b,c} \in G$

A diagram illustrating the composition of two symmetries g_a and g_b . On the left, two parallel lines are labeled g_a and g_b . This is followed by an equals sign, and on the right, a single line is labeled g_c .

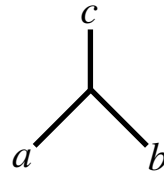
- Non-invertible symmetries (in 1+1 dim):

Fusion category $\mathcal{L}_a \mathcal{L}_b = \sum_{\mathcal{L}_c} N_{ab}^c \mathcal{L}_c$ $(N_{ab}^c \in \mathbb{Z}_{\geq 0})$

Data characterizing fusion categories

$$\mathcal{L}_a \mathcal{L}_b = \sum_{\mathcal{L}_c} N_{ab}^c \mathcal{L}_c$$

- Fusion coefficient N_{ab}^c :



If $N_{ab}^c \neq 0$

- Quantum dimension :

$$\langle \mathcal{L}_a \rangle = \text{tr}(\mathcal{L}_a) = d_a$$

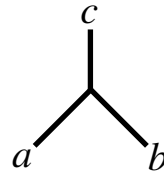
$\begin{matrix} \nearrow 1 & \text{group-like} \\ \searrow \sqrt{2} & \text{etc. for non-inv.} \end{matrix}$

Non-integer quantum dimension implies topological modes

Data characterizing fusion categories

$$\mathcal{L}_a \mathcal{L}_b = \sum_{\mathcal{L}_c} N_{ab}^c \mathcal{L}_c$$

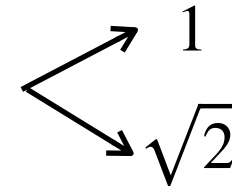
- Fusion coefficient N_{ab}^c :



If $N_{ab}^c \neq 0$

- Quantum dimension :

$$\langle \mathcal{L}_a \rangle = \text{tr}(\mathcal{L}_a) = d_a$$

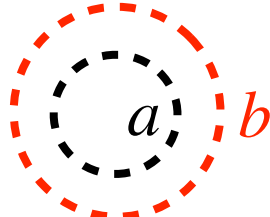


group-like

etc. for non-inv.

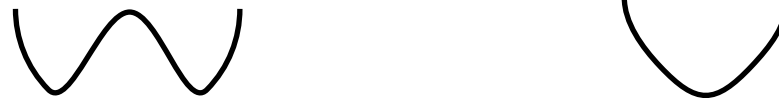
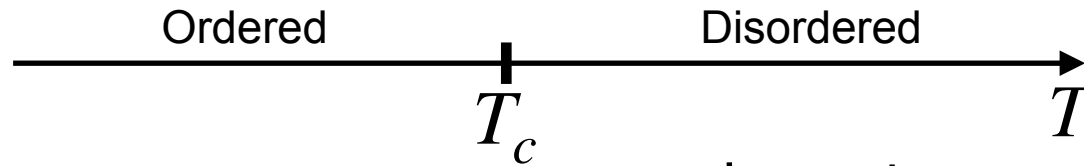
Non-integer quantum dimension implies topological modes

They satisfy consistency conditions.

e.g.  $\mapsto d_a d_b = \sum_c N_{ab}^c d_c$

Non-Invertible Kramers-Wannier

- 2d Ising model: $Z = \sum_{\{\sigma\}} \exp \left[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \right]$

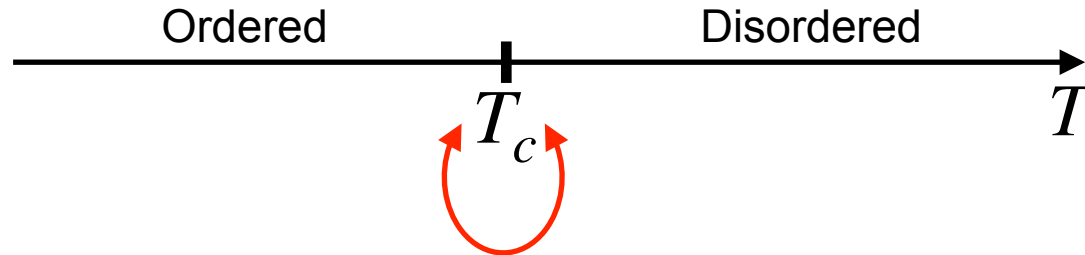


Kramers-Wannier duality

$$e^{-2\beta_{\text{low}}} \leftrightarrow \tanh \beta_{\text{high}}$$

Non-Invertible Kramers-Wannier

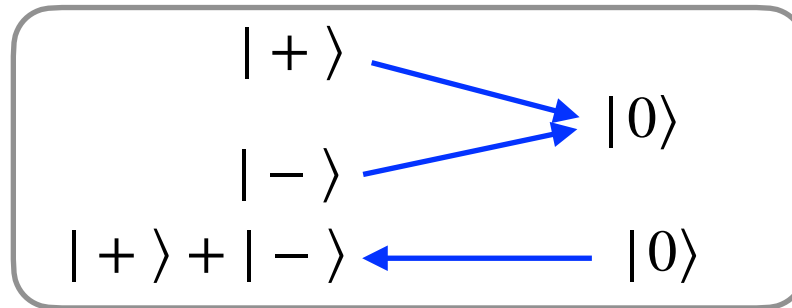
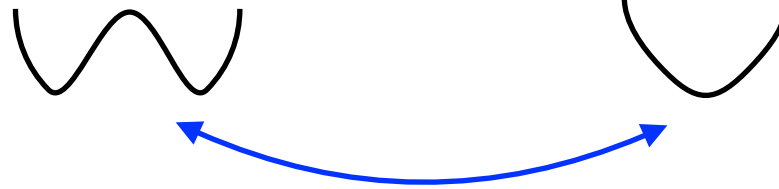
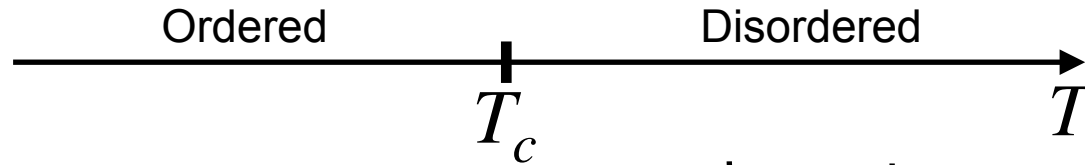
- 2d Ising model:
$$Z = \sum_{\{\sigma\}} \exp \left[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \right]$$



Kramers-Wannier **symmetry**

Non-Invertible Kramers-Wannier

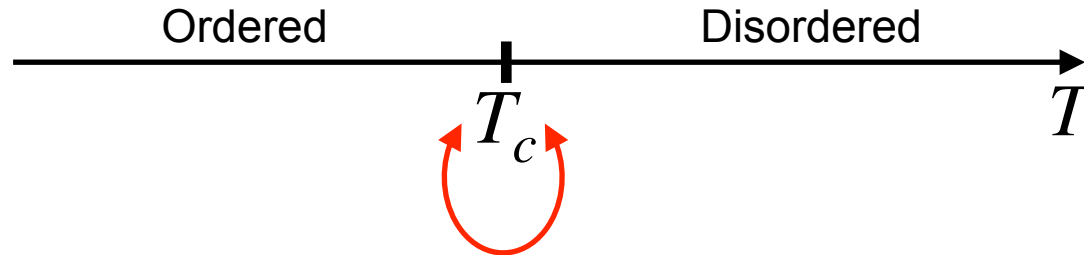
• 2d Ising model:
$$Z = \sum_{\{\sigma\}} \exp \left[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \right]$$



Non-invertible

Non-Invertible Kramers-Wannier

- 2d Ising model:
$$Z = \sum_{\{\sigma\}} \exp \left[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \right]$$

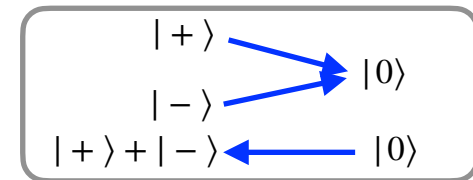


Kramers-Wannier **symmetry** \mathcal{N}

- Symmetry of 2d Ising CFT

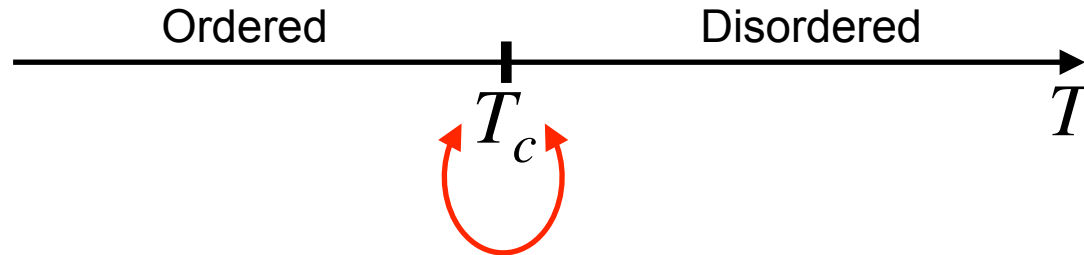
$$\eta^2 = 1, \quad \mathcal{N}^2 = 1 + \eta, \quad \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N}$$

$$(d_1 = d_\eta = 1, \quad d_{\mathcal{N}} = \sqrt{2}) \text{ topological modes!}$$



Non-Invertible Kramers-Wannier

- 2d Ising model:
$$Z = \sum_{\{\sigma\}} \exp \left[\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j \right]$$

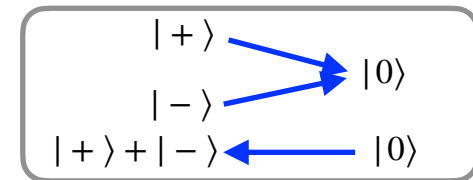


Kramers-Wannier **symmetry** \mathcal{N}

- Symmetry of 2d Ising CFT

$$\eta^2 = 1, \quad \mathcal{N}^2 = 1 + \eta, \quad \mathcal{N}\eta = \eta\mathcal{N} = \mathcal{N}$$

$$(d_1 = d_\eta = 1, \quad d_{\mathcal{N}} = \sqrt{2}) \text{ topological modes!}$$

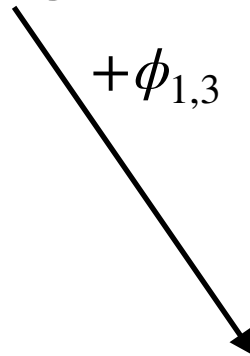


- In Ising CFT, KW symmetry **broken** by relevant perturbations
- RG flow preserving KW symmetry...?

Flow from tricritical Ising

- Ising CFT is the simplest unitary minimal model ($\mathcal{M}_{3,4}$).
- Next simplest is **tricritical Ising** CFT ($\mathcal{M}_{4,5}$).
- Non-invertible symmetric RG flow from tricritical: [Chang, Lin, Shao, Wang, Yin]

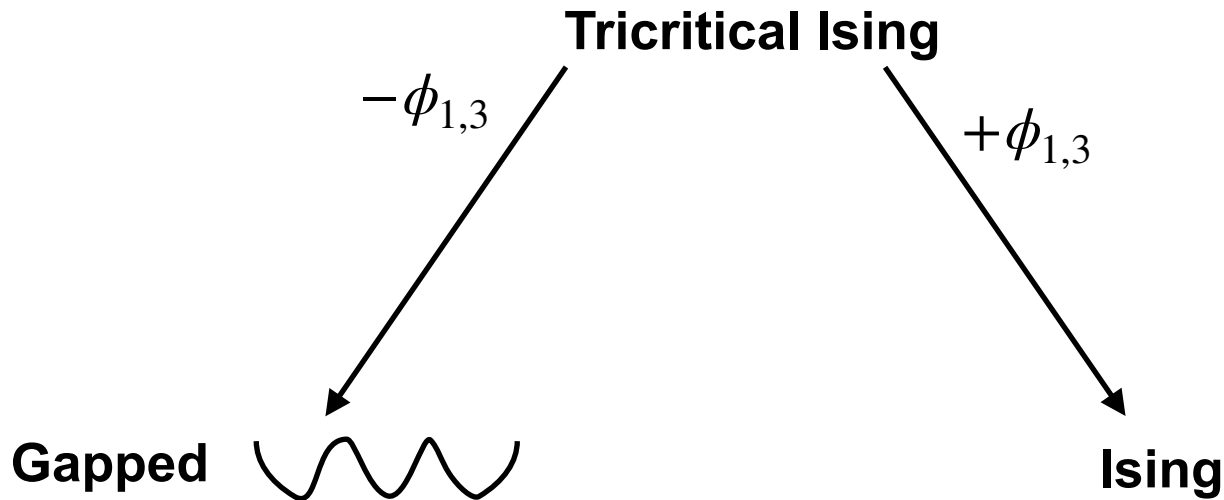
Tricritical Ising



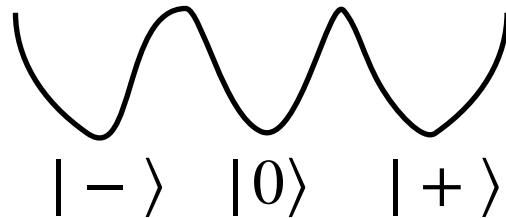
Ising

Flow from tricritical Ising

- Ising CFT is the simplest unitary minimal model ($\mathcal{M}_{3,4}$).
- Next simplest is **tricritical Ising** CFT ($\mathcal{M}_{4,5}$).
- Non-invertible symmetric RG flow from tricritical: [Chang, Lin, Shao, Wang, Yin]



Action on vacua



- \mathbb{Z}_2 -defect exchanges $|+\rangle$ and $|-\rangle$

$$\eta : |+\rangle \leftrightarrow |-\rangle$$

- \mathcal{N} -defect: $\mathcal{N}|0\rangle = |+\rangle + |-\rangle$

$$\mathcal{N}|+\rangle = \mathcal{N}|-\rangle = |0\rangle$$

- “Superposition” of disordered and ordered vacua in Ising

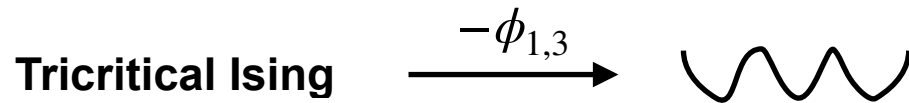
1. Non-Invertible symmetries in 1+1 dim

2. Integrable flow from tricritical Ising and S-matrix

3. Derivation of modified crossing rules

4. Conclusion


Integrability along the flow



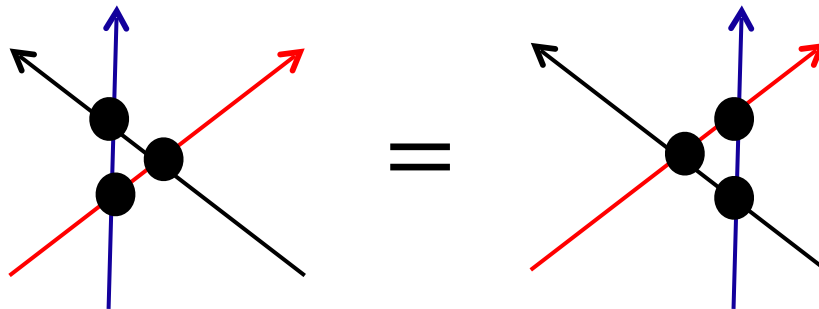
- At UV CFT fixed point, there exist ∞ many **higher spin** charges.
because of Virasoro
- Perturbation by $\phi_{1,3}$ preserve **higher spin** charges.

[Zamolodchikov 1989]

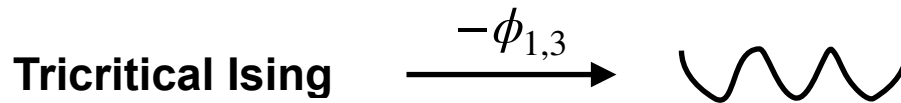
Integrability along the flow

Tricritical Ising $\xrightarrow{-\phi_{1,3}}$ 

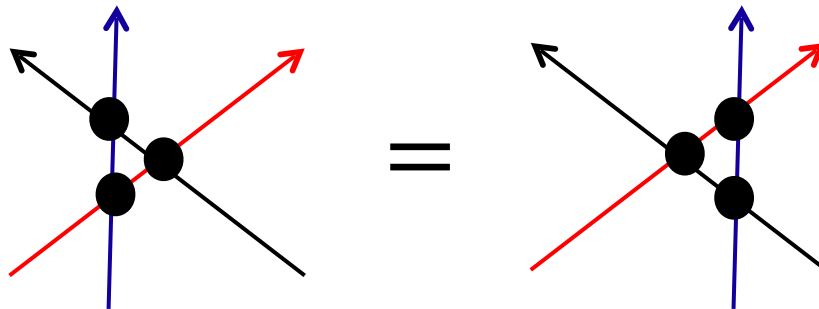
- At UV CFT fixed point, there exist ∞ many **higher spin** charges.
because of Virasoro
- Perturbation by $\phi_{1,3}$ preserve **higher spin** charges.
[Zamolodchikov 1989]
- Multi-particle S-matrix **factorizes** to $2 \rightarrow 2$ S-matrices.
[Shankar, Witten] [Parke]
- $2 \rightarrow 2$ S-matrices satisfy the **Yang-Baxter** equation.



Integrability along the flow



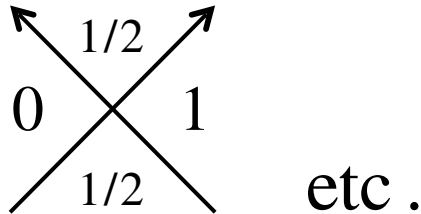
- At UV CFT fixed point, there exist ∞ many **higher spin** charges.
because of Virasoro
- Perturbation by $\phi_{1,3}$ preserve **higher spin** charges.
[Zamolodchikov 1989]
- Multi-particle S-matrix **factorizes** to $2 \rightarrow 2$ S-matrices.
[Shankar, Witten] [Parke]
- $2 \rightarrow 2$ S-matrices satisfy the **Yang-Baxter** equation.



- Imposing unitarity, crossing & YB, the S-matrix can be almost uniquely **“bootstrapped”**.
[Zamolodchikov 1991]

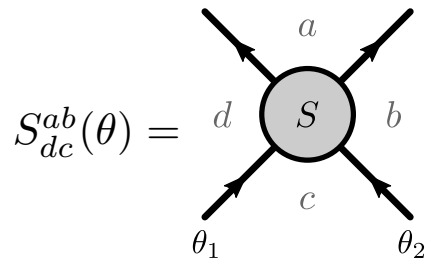
S-matrix by Zamolodchikov

- Particles in IR = **kinks** interpolating between adjacent vacua.



*Warning: Change of notation!

- It also depends on total energy: $s = (p_1 + p_2)^2 = 4m^2 \cosh^2(\theta/2)$

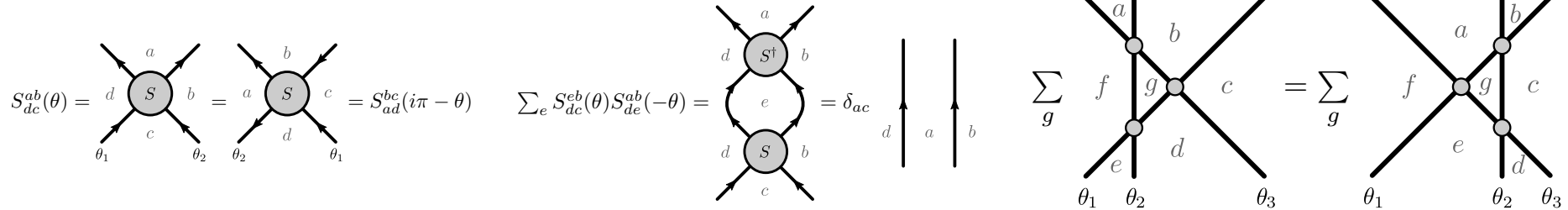


$$\theta = \theta_1 - \theta_2$$

$$s + t = 4m^2 \quad t = -4m^2 \sinh^2(\theta/2)$$

S-matrix by Zamolodchikov

- Imposing crossing & unitarity & YB,



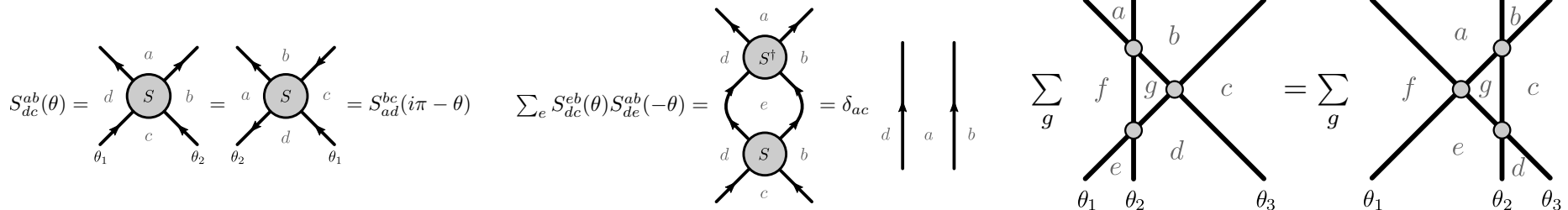
we can fix S-matrix uniquely as

$$\widehat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

$$(n = 3, \quad d_0 = d_1 = 1, \quad d_{1/2} = \sqrt{2})$$

S-matrix by Zamolodchikov

- Imposing crossing & unitarity & YB,

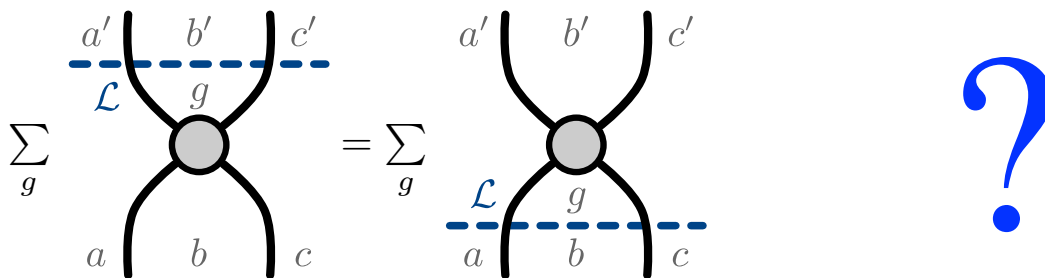


we can fix S-matrix uniquely as

$$\widehat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

$(n = 3, \quad d_0 = d_1 = 1, \quad d_{1/2} = \sqrt{2})$

- We expect that it also preserves noninvertible symmetries:



Paradox

$$\sum_g \text{Diagram 1} = \sum_g \text{Diagram 2} \quad ?$$

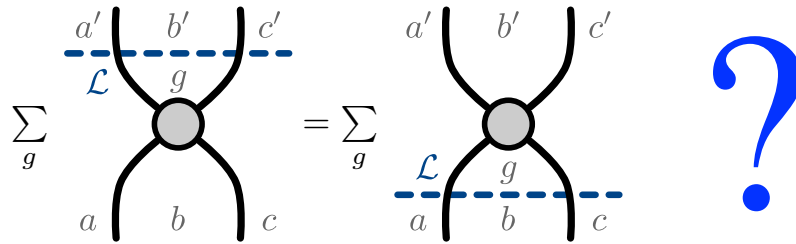
$$\widehat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

- We found that \widehat{S} commutes with $\eta (\mathbb{Z}_2)$, but **not with \mathcal{N}** .

$$\text{Diagram 1} \stackrel{?}{=} \text{Diagram 2} + \text{Diagram 3}$$

$$\mathcal{N} : S_{0\frac{1}{2}}^{\frac{1}{2}0}(\theta) \stackrel{?}{=} S_{\frac{1}{2}0}^{0\frac{1}{2}}(\theta) + S_{\frac{1}{2}1}^{0\frac{1}{2}}(\theta)$$

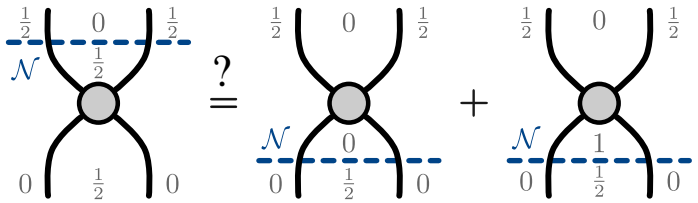
Paradox



$$\sum_g \text{[Crossing with } \mathcal{L} \text{ above]} = \sum_g \text{[Crossing with } \mathcal{L} \text{ below]}$$

$$\widehat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

- We found that \widehat{S} commutes with $\eta (\mathbb{Z}_2)$, but **not with \mathcal{N}** .

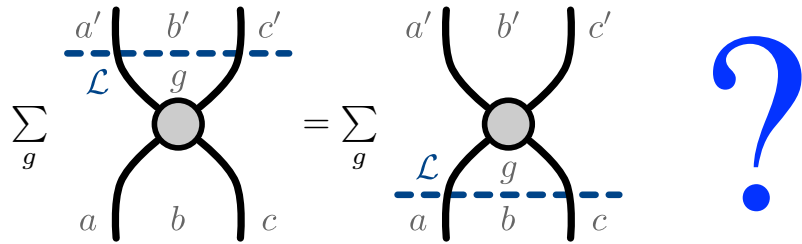


$$\text{[Crossing with } \mathcal{N} \text{ above]} \stackrel{?}{=} \text{[Crossing with } \mathcal{N} \text{ below, left]} + \text{[Crossing with } \mathcal{N} \text{ below, right]}$$

$$\mathcal{N} : \quad S_{0\frac{1}{2}}^{\frac{1}{2}0}(\theta) \stackrel{?}{=} S_{\frac{1}{2}0}^{0\frac{1}{2}}(\theta) + S_{\frac{1}{2}1}^{0\frac{1}{2}}(\theta)$$

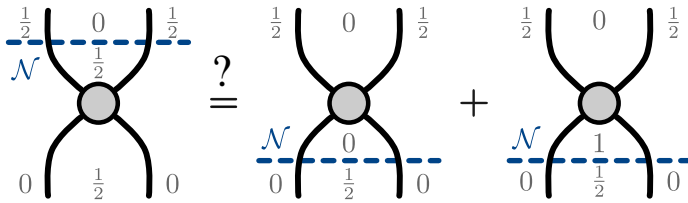
- The following 4 properties are mutually incompatible.
 - Unitarity
 - Crossing
 - Integrability (YB)
 - Noninvertible symmetry

Paradox




$$\hat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

- We found that \hat{S} commutes with $\eta (\mathbb{Z}_2)$, but **not with \mathcal{N}** .

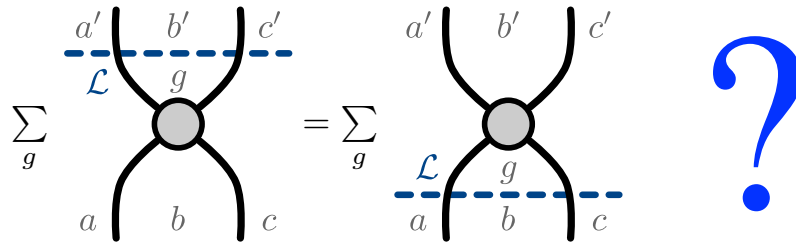


$$\mathcal{N} : S_{0\frac{1}{2}}^{\frac{1}{2}0}(\theta) \stackrel{?}{=} S_{\frac{1}{2}0}^{0\frac{1}{2}}(\theta) + S_{\frac{1}{2}1}^{0\frac{1}{2}}(\theta)$$

- The following 4 properties are mutually incompatible.

- Unitarity  basic principle
- Crossing
- Integrability (YB)
- Noninvertible symmetry

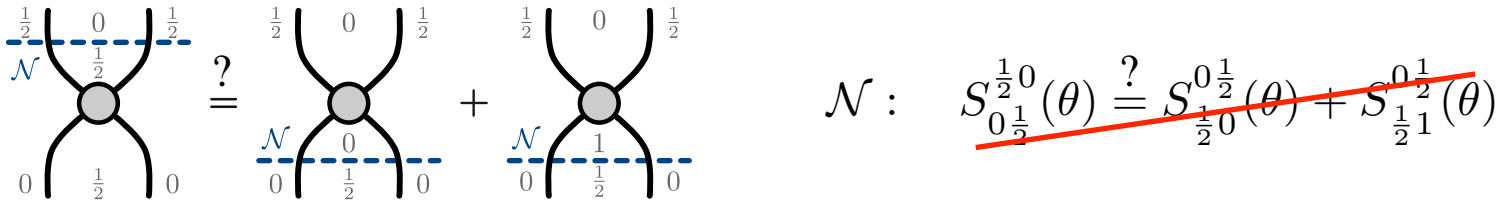
Paradox



$$\sum_g \text{Diagram 1} = \sum_g \text{Diagram 2} \quad ?$$

$$\widehat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$




- We found that \widehat{S} commutes with $\eta (\mathbb{Z}_2)$, but **not with \mathcal{N}** .



$$\sum_g \text{Diagram 1} = \sum_g \text{Diagram 2} + \sum_g \text{Diagram 3} \quad ?$$

$$\mathcal{N} : \quad S_{0\frac{1}{2}}^{\frac{1}{2}0}(\theta) \stackrel{?}{=} S_{\frac{1}{2}0}^{0\frac{1}{2}}(\theta) + S_{\frac{1}{2}1}^{0\frac{1}{2}}(\theta)$$

- The following 4 properties are mutually incompatible.

- Unitarity  basic principle
- Crossing
- Integrability (YB)  deformation of CFT
- Noninvertible symmetry  deformation of CFT




Paradox

$$\sum_g \text{Diagram} = \sum_g \text{Diagram} \quad ? \quad \widehat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d} \right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

- We found that \widehat{S} commutes with $\eta (\mathbb{Z}_2)$, but **not with \mathcal{N}** .

$$\text{Diagram} \stackrel{?}{=} \text{Diagram} + \text{Diagram} \quad \mathcal{N} : \quad S_{0\frac{1}{2}}^{\frac{1}{2}0}(\theta) \stackrel{?}{=} S_{\frac{1}{2}0}^{0\frac{1}{2}}(\theta) + S_{\frac{1}{2}1}^{0\frac{1}{2}}(\theta)$$

- The following 4 properties are mutually incompatible.

- Unitarity  basic principle
- **Crossing**
- Integrability (YB)  deformation of CFT
- Noninvertible symmetry  deformation of CFT
- The only viable option is to give up **crossing**.

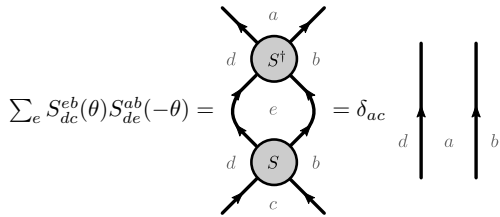
New proposal

$$S_{dc}^{ab}(\theta) = Z(\theta) \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

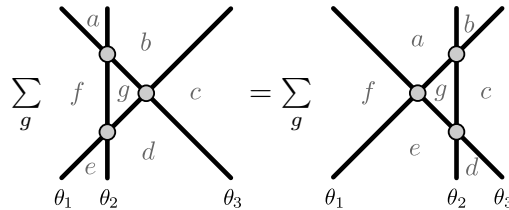
$$\widehat{S}_{dc}^{ab}(\theta) = Z(\theta) \left(\frac{d_a d_c}{d_b d_d}\right)^{\frac{i\theta}{2\pi}} \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$

New proposal

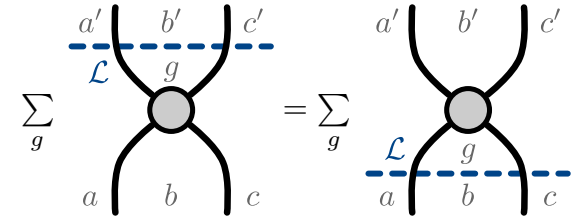
$$S_{dc}^{ab}(\theta) = Z(\theta) \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$



Unitarity



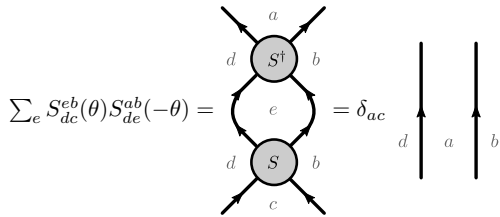
Yang-Baxter



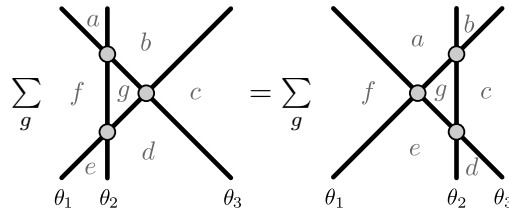
Non-invertible

New proposal

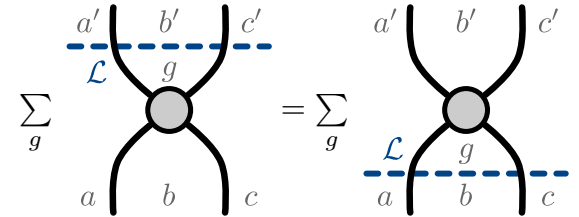
$$S_{dc}^{ab}(\theta) = Z(\theta) \left[\sqrt{\frac{d_a d_c}{d_b d_d}} \sinh\left(\frac{\theta}{n}\right) \delta_{bd} + \sinh\left(\frac{i\pi - \theta}{n}\right) \delta_{ac} \right]$$



Unitarity



Yang-Baxter



Non-invertible

- Crossing symmetry is modified:

$$S_{dc}^{ab}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{ad}^{bc}(i\pi - \theta)$$

- Physical origin? (The topic of the rest of the talk)

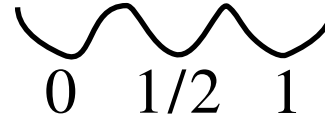
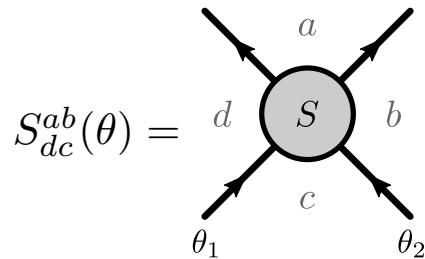
1. Non-invertible symmetries in 1+1 dim

2. Integrable flow from tricritical Ising and S-matrix

3. Derivation of modified crossing rules

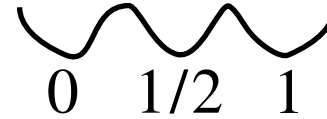
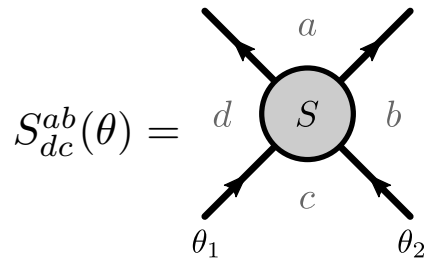
4. Conclusion

Key physical input



- The IR dynamics is described by a **nontrivial TQFT**.

Key physical input

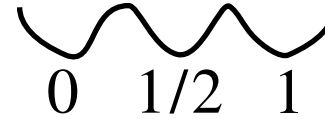
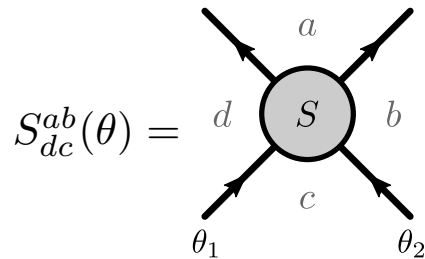


- The IR dynamics is described by a **nontrivial TQFT**.

1. In the IR, the worldline of a **kink = symmetry line** $\mathcal{N} (=:\nu)$

$$\mathcal{N}|0\rangle = |1/2\rangle, \quad \mathcal{N}|1/2\rangle = |0\rangle + |1\rangle, \quad \mathcal{N}|1\rangle = |1/2\rangle.$$

Key physical input



- The IR dynamics is described by a **nontrivial TQFT**.

1. In the IR, the worldline of a **kink** = **symmetry line** \mathcal{N} ($=: \nu$)

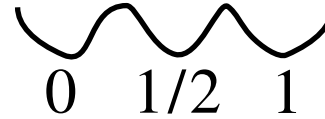
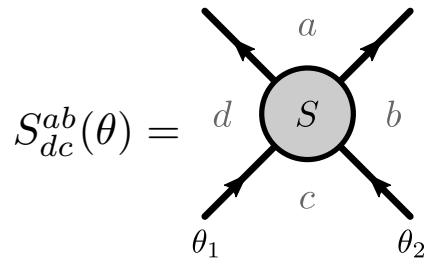
2. The **vacua** are in 1-to-1 correspondence with **symmetry lines**.

$$|0\rangle \leftrightarrow \mathbf{1} \quad |1/2\rangle \leftrightarrow \mathcal{N} \quad |1\rangle \leftrightarrow \eta$$

(“Adjoint” representation of fusion category)

$$\mathcal{N} |1/2\rangle = |0\rangle + |1\rangle \quad \Leftrightarrow \quad \mathcal{N}^2 = \mathbf{1} + \eta$$

Key physical input



- The IR dynamics is described by a **nontrivial TQFT**.

1. In the IR, the worldline of a **kink** = **symmetry line** \mathcal{N} ($=: \nu$)

2. The **vacua** are in 1-to-1 correspondence with **symmetry lines**.

$$|0\rangle \leftrightarrow \mathbf{1} \quad |1/2\rangle \leftrightarrow \mathcal{N} \quad |1\rangle \leftrightarrow \eta$$

(“Adjoint” representation of fusion category)

3. All the vacua can be obtained from $|0\rangle$ by acting symmetry lines.

$$\mathcal{N}|0\rangle = |1/2\rangle$$

$$\eta|0\rangle = |1\rangle$$

Key physical input

1. In the IR, the worldline of a kink = symmetry line $\mathcal{N} (=: v)$

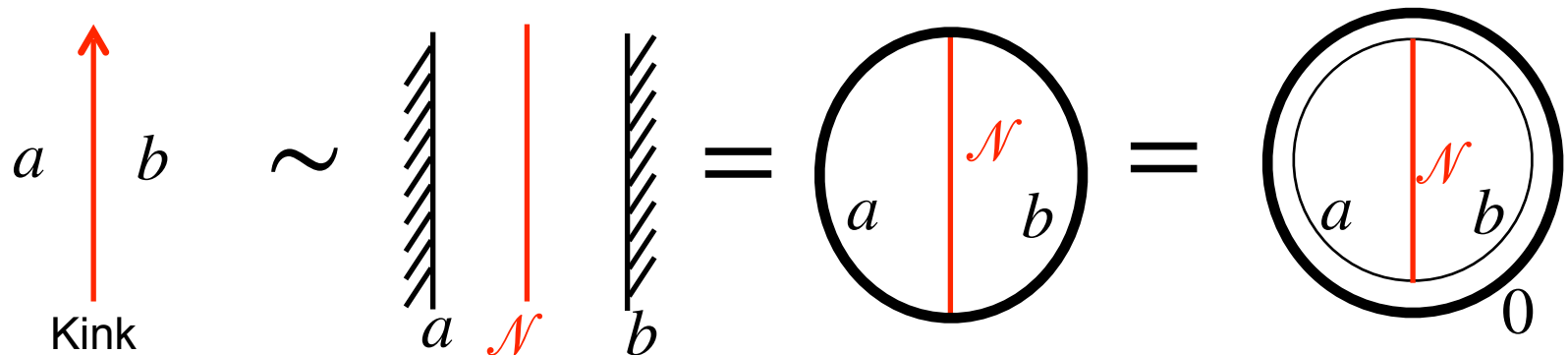
2. The vacua are in 1-to-1 correspondence with symmetry lines.

$$|0\rangle \leftrightarrow \mathbf{1} \quad |1/2\rangle \leftrightarrow \mathcal{N} \quad |1\rangle \leftrightarrow \eta$$

3. All the vacua can be obtained from $|0\rangle$ by acting symmetry lines.

$$\mathcal{N}|0\rangle = |1/2\rangle$$

$$\eta|0\rangle = |1\rangle$$



Key physical input

1. In the IR, the worldline of a kink = symmetry line $\mathcal{N} (=: v)$

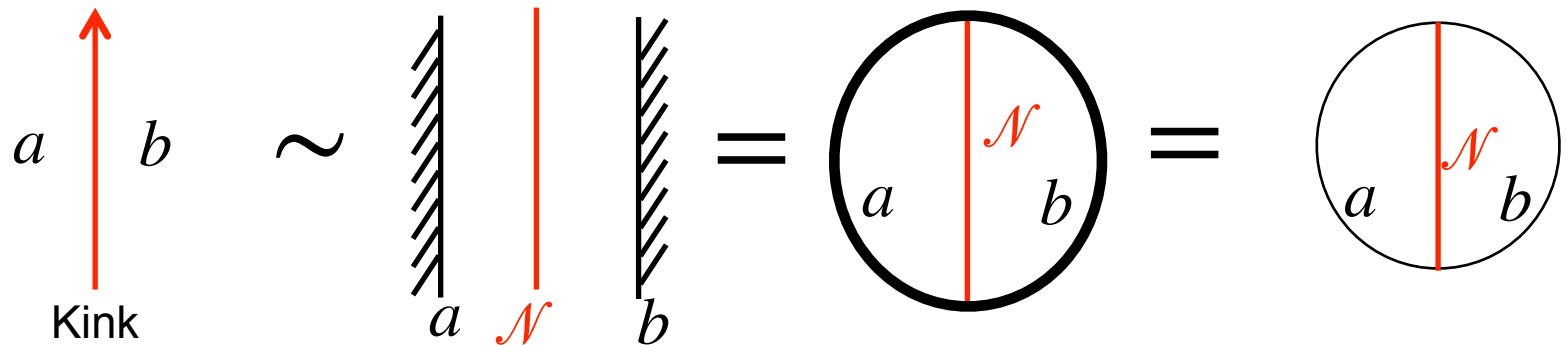
2. The vacua are in 1-to-1 correspondence with symmetry lines.

$$|0\rangle \leftrightarrow \mathbf{1} \quad |1/2\rangle \leftrightarrow \mathcal{N} \quad |1\rangle \leftrightarrow \eta$$

3. All the vacua can be obtained from $|0\rangle$ by acting symmetry lines.

$$\mathcal{N}|0\rangle = |1/2\rangle$$

$$\eta|0\rangle = |1\rangle$$



kink states \leftrightarrow network of lines

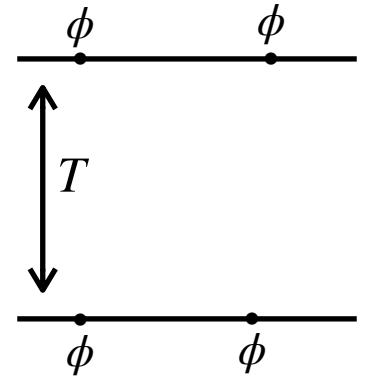
Modified crossing for S-matrix

- Kink creation operators are **non-local** (boundary cond. changing op)

Unclear how to get **S-matrix** from **LSZ** reduction.

- Use alternative (discussed in Itzykson-Zuber Ch.5)

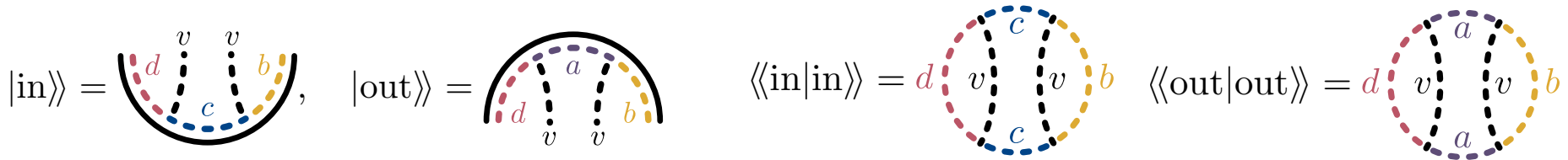
$$S(\{p_i\}) = \lim_{T \rightarrow \infty} \int \prod_j dv_j e^{ip_j x(v_j)} \prod_k (n_k \cdot \overleftrightarrow{\partial}_{x_k}) G(\{x(v_k)\})$$



- In our case,

$$S_{dc}^{ab}(\theta) \propto \left(\text{Diagram of a circle with a central grey disk and four radial lines labeled a, b, c, d} \right) \Big|_{\text{analyt. cont.}}$$

- But we need to take into account normalization.



Modified crossing for S-matrix

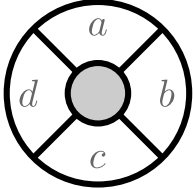
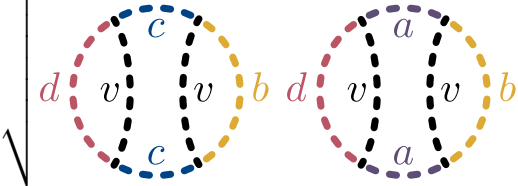
- Thus the correct expression is

$$S_{dc}^{ab}(\theta) = \frac{\text{Diagram 1}}{\sqrt{\text{Diagram 2} \times \text{Diagram 3}}} \Big|_{\text{analyt. cont.}}$$

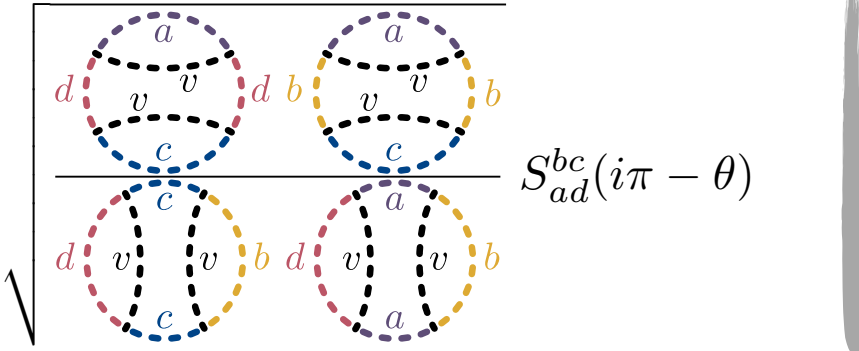
- Numerator : disk correlation function, **crossing symmetric**.
- Denominator : depend on the **channel** we consider.

Modified crossing for S-matrix

- Thus the correct expression is

$$S_{dc}^{ab}(\theta) = \frac{\text{disk correlation function}}{\sqrt{\text{channel diagrams}}} \Big|_{\text{analyt. cont.}}$$



- Numerator : disk correlation function, **crossing symmetric**.
- Denominator : depend on the **channel** we consider.
- Modified crossing:

$$S_{dc}^{ab}(\theta) = \frac{\text{channel diagrams}}{\sqrt{\text{channel diagrams}}} S_{ad}^{bc}(i\pi - \theta) \quad \Big| \quad S_{dc}^{ab}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{ad}^{bc}(i\pi - \theta)$$


1. Non-invertible symmetries in 1+1 dim

2. Integrable flow from tricritical Ising and S-matrix

3. Derivation of modified crossing rules

4. Conclusion

Conclusion & Future

$$S_{dc}^{ab}(\theta) = \sqrt{\begin{array}{c} \text{Diagram 1} \quad \text{Diagram 2} \\ \hline \text{Diagram 3} \quad \text{Diagram 4} \end{array}} S_{ad}^{bc}(i\pi - \theta) \quad S_{dc}^{ab}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{ad}^{bc}(i\pi - \theta)$$

- Topological modes in 1+1 d leads to **modified crossing symmetry**.
- Physically, it comes from **corrections to norms** of in- and out-states due to TQFT dynamics.
- Similar modified crossing observed in Chern-Simons matter in 2+1 d.

[Mehta, Patel, Prakash, Minwalla, Sharma],...

In our examples, importance of norms is clearer & TQFT d.o.f is more hidden.

- Soft modes in gauge theories and gravity? Charge-monopole scattering? Gravity waves?

Backups

Future

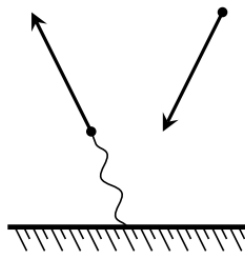
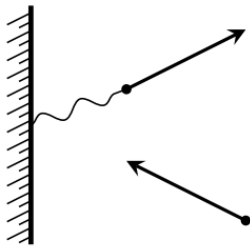
$$S_{dc}^{ab}(\theta) = \sqrt{\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array}} S_{ad}^{bc}(i\pi - \theta)$$

The diagram shows two configurations of four particles in a square arrangement. Each particle is represented by a dashed circle with a label 'a', 'b', 'c', or 'v'. The top-left configuration has 'a' at the top, 'c' at the bottom, 'd' on the left, and 'b' on the right. The top-right configuration has 'a' at the top, 'c' at the bottom, 'b' on the left, and 'd' on the right. The bottom-left configuration has 'c' at the top, 'a' at the bottom, 'd' on the left, and 'b' on the right. The bottom-right configuration has 'c' at the top, 'a' at the bottom, 'b' on the left, and 'd' on the right. The labels 'd' and 'b' are colored red and yellow respectively, while 'a' and 'c' are blue.

$$S_{dc}^{ab}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}} S_{ad}^{bc}(i\pi - \theta)$$

- S-matrix bootstrap with categorical symmetry (Haagerup fusion category) [WIP]

- Modified crossing in monopole scattering?



[Csaki, Hong, Shirman, Telem, Terning, Waterbury]

[van Beest, Boyle Smith, Delmastro, Komargodski, Tong]

[van Beest, Boyle Smith, Delmastro, Mouland, Tong]

- Toy model for IR effects (Faddeev-Kulish) in gravity and gauge theory?