Scattering of particles & strings from surfaces

Song He (Institute of Theoretical Physics, CAS)

based on works w. Nima Arkani-Hamed, Qu Cao, Jin Dong, Carolina Figueiredo

(2312.16282, 2401.05483) 2401.00041, 2408.11891 + in progress

w. Qu Cao, Jin Dong, Fan Zhu, 2412.xxxxx + in progress

QCD MEETS GRAVITY

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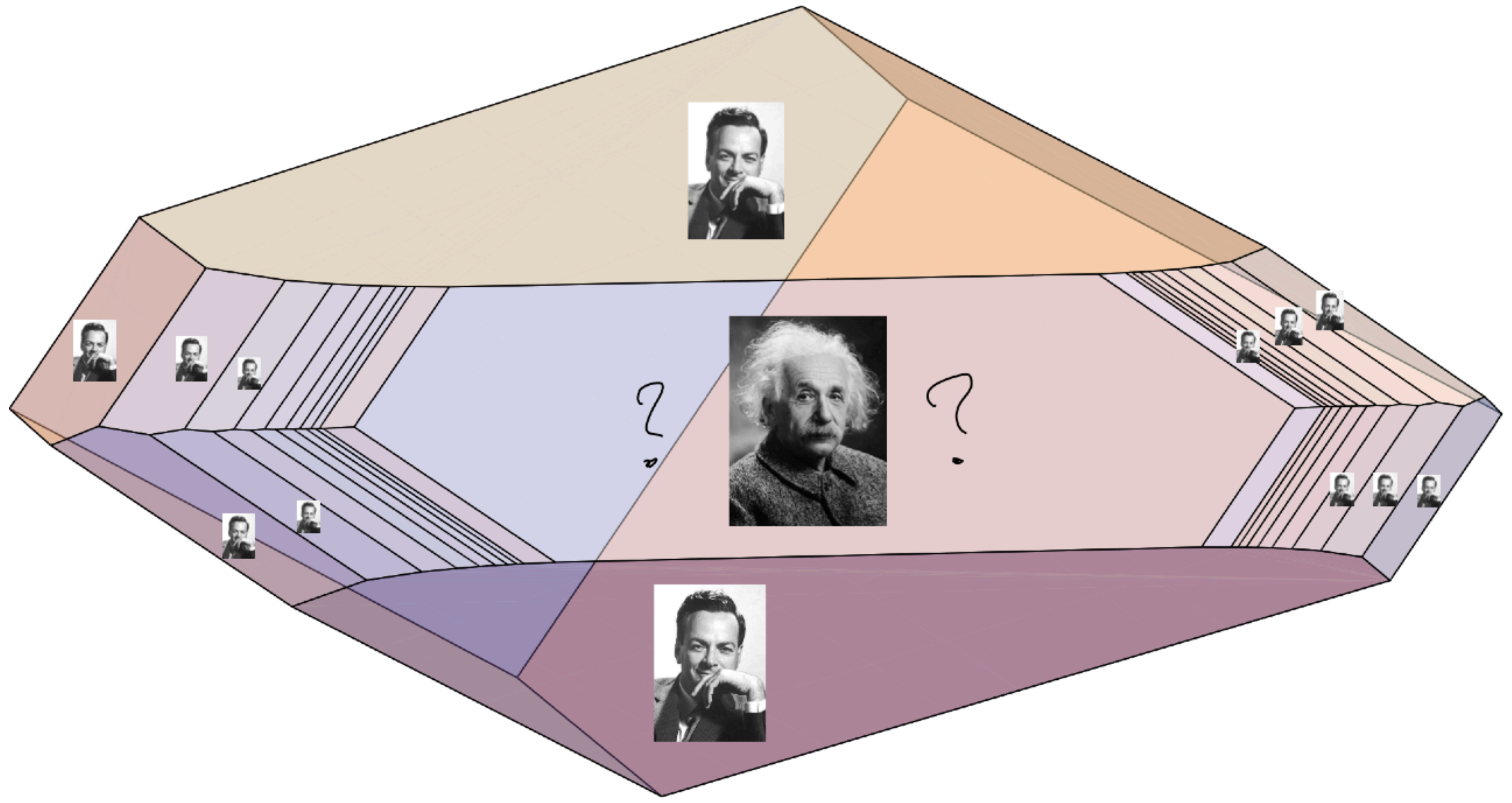
Positive Geometries underlying scattering amplitudes & beyond [see Jara's lecture]

- moduli space $\mathscr{M}_{g,n}$ for conventional & (ambi-)twistor strings [Witten, Berkovits 04; CHY 13; Mason, Skinner 14; …]
- positive Grassmannian $G_+(k,n)$, on-shell diagrams for planar N=4 SYM [Arkani-Hamed et al 12,…]
- Amplituhedron: map from $G_+(k,n)$ →all-loop integrands of N=4 SYM in momentum twistor space [Arkani-Hamed, Trnka 13 + Thomas 17;…]
  —> momentum amplituhedron [Ferro et al] [SH, Zhang]
- ABJM amplituhedron: reduction to D=3 from SYM amplituhedron —> all-loop integrand of ABJM! [SH, Kuo, Li, Zhang, 22; SH, Huang, Kuo, 23,…]
- Correlahedron for half-BPS correlator in SYM [Eden, Heslop, Mason; SH, Huang, Kuo, 24…] -> squared-amplituhedron (-> energy corrector) [SH et al 24]
- kinematic associahedra (bi-adjoint ϕ^3 tree) & worldsheet associahedra [Arkani-Hamed, Bai, SH, Yan, 17]
- surfacehedra + curve-integrals on surfaces… => "strings (+particles) without worldsheet"[Arkani-Hamed et al, 20, 23,…]
- cosmological polytopes +"kinematic flow" DE for wave function/correlator (cosmohedra?) [Arkani-Hamed et al 17 , 23,…]
- applications in wider context: tropical geometries for Feynman integrals, symbology etc. positive geometries in dS/AdS, …

Toy Models —> Real World



Curve integrals on surfaces (combinatorial): bosonic string —> all-loop Tr ϕ^3 (simplest colored scalars); amps determined by long-distance sing. or "denominators" !

[Arkani-Hamed, Salvatori, Frost, Plamondon, Thomas: 2309.15913, 2311.09284, …; Giulio's talk!]

More realistic theories: need "pole @ infinity" or numerators: Tr ϕ^3 vs. ϕ^p (general coupling), pions? gluons?
—> A new, unified formulation based on surfaces for colored theories: what is it good for?

- Manifest factorizations etc (finite α'): no need for blowup + all massive levels
- All-loop cuts ($\alpha' \to 0$) from surfaces-> all-loop recursion for Tr ϕ^3+ any colored-scalars
- The unity of ϕ^3, pions & gluons: e.g. manifest zeros & splits near zeros for Tr ϕ^3, NLSM, YM, …
- All-loop integrands of NLSM (+ mixed amps) in Tr ϕ^3
- "Combinatorial origin of YM": scalar-scaffolded gluons => all-loop YM in stringy Tr ϕ^3

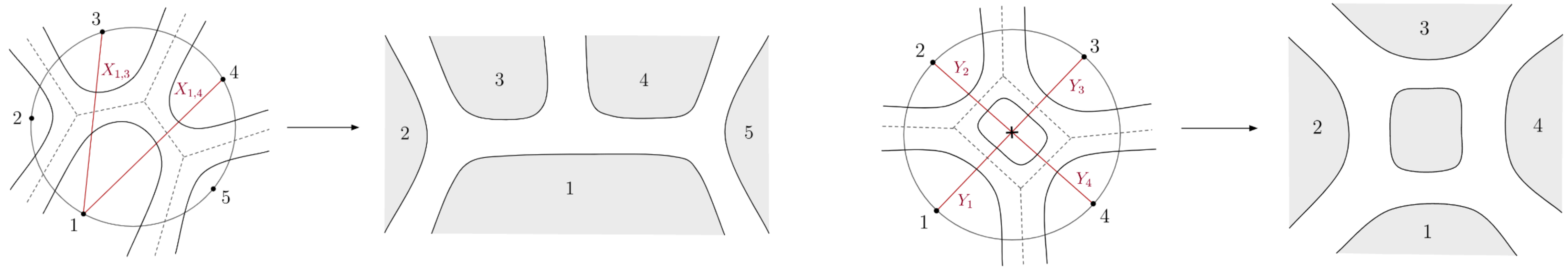
Tr ϕ^3 amplitudes [Arkani-Hamed, Bai, SH, Yan, '17; Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas, '23,...]

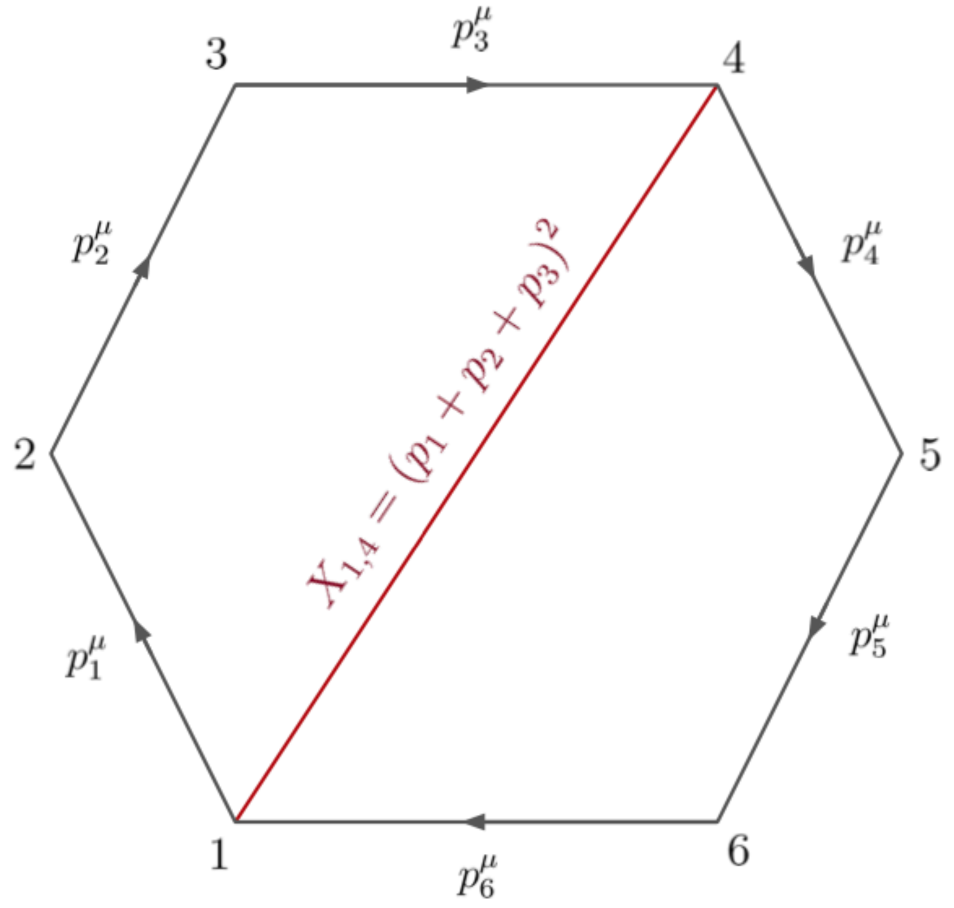
$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

ϕ : N by N matrix -> fat graphs, genus expansion (only planar graphs for $N \to \infty$)

kinematics: e.g. tree amps (planar)

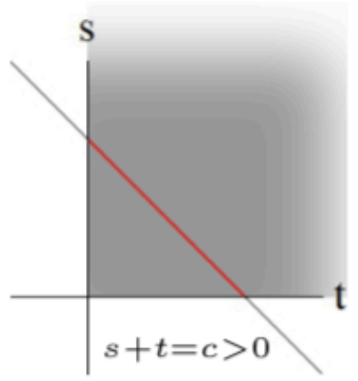
$$X_{i,j} = (p_i + \cdots + p_{j-1})^2.$$





tree amp = sum over n-gon triangulations = canonical form of associahedron

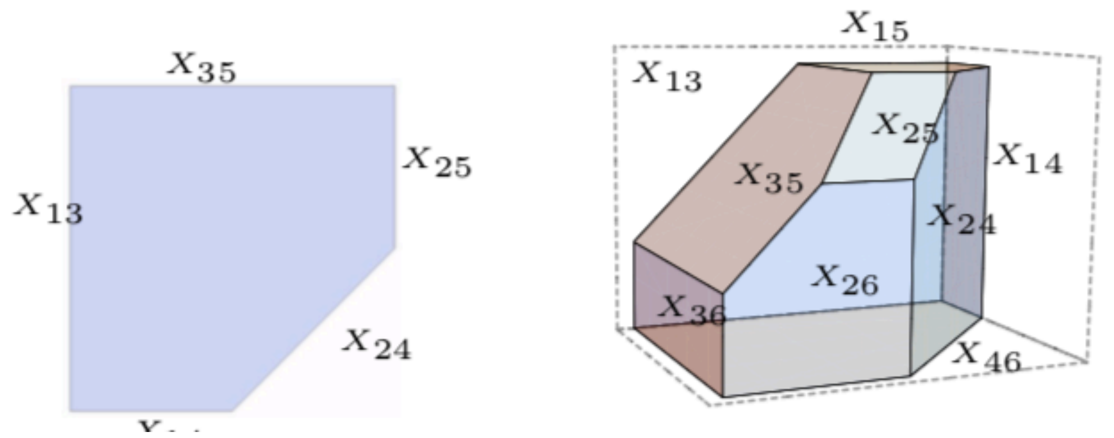
$$\mathcal{A}_4 = \frac{1}{X_{13}} + \frac{1}{X_{24}},$$



$$e.g.\ \mathcal{A}_1 = \{s > 0, t > 0\} \cap \{-u = \mathrm{const} > 0\}$$

$$\mathcal{A}_2 = \{X_{13}, \cdots, X_{25} > 0\} \cap \{-s_{13} = c_{13}, -s_{14} = c_{14}, -s_{24} = c_{24}\}$$

$$\mathcal{A}_5 = \frac{1}{X_{1,3}X_{1,4}} + \frac{1}{X_{2,4}X_{2,5}} + \frac{1}{X_{1,3}X_{3,5}} + \frac{1}{X_{1,4}X_{2,4}} + \frac{1}{X_{2,5}X_{3,5}}.$$



$$c_{i,j} := -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1},$$

Disk integral = stringy Tr ϕ^3 amp [c.f. Arkani-Hamed, SH, Lam, 19]

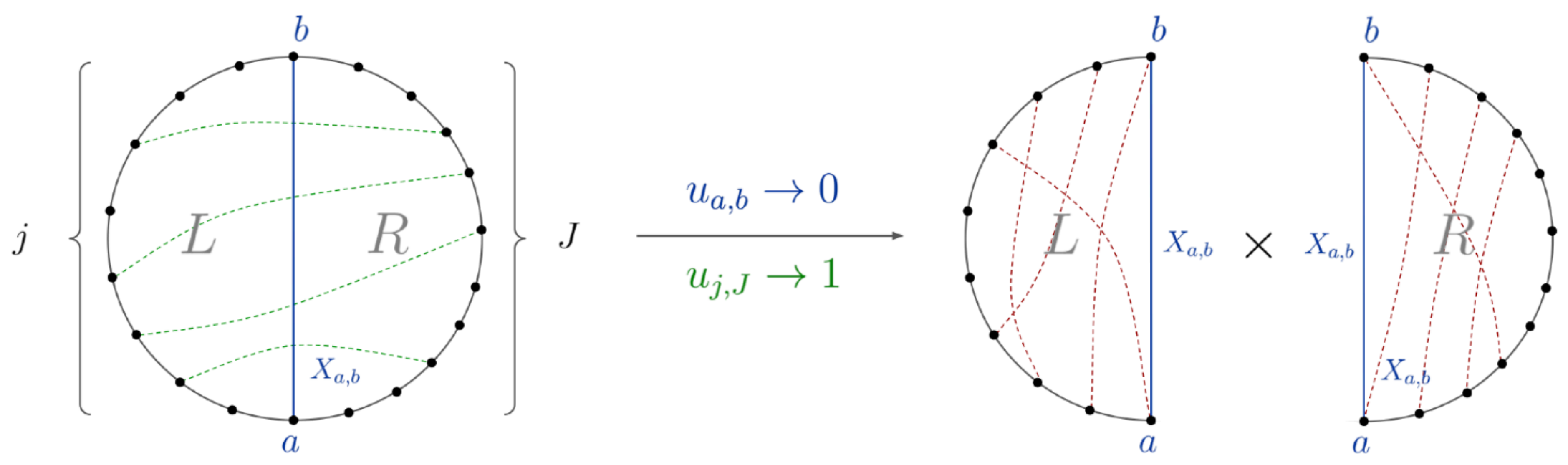
a la Veneziano-Koba-Nielsen; Z-theory [Carrasco, Mafra, Schlotterer 16',...]

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3}(1,2,...,n) = \int_{D(1\ldots n)} \frac{dz_1\ldots dz_n}{\text{vol SL}(2,\mathbb{R})} \underbrace{\frac{1}{z_{1,2}z_{2,3}\ldots z_{n,1}}}_{\mathrm{PT}(1,2,\ldots,n)} \times \underbrace{\prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}}_{\text{Koba-Nielsen factor}}$$

u variables: $u_{a,b} := \dfrac{z_{a,b-1}z_{a-1,b}}{z_{a,b}z_{a-1,b-1}}$

=> $\displaystyle\int \frac{d^{n-3}z}{z_{12}\cdots z_{n1}} \prod_{a<b} u_{a,b}^{\alpha' X_{a,b}}$ w. positive parametrization $y_{I=1,\cdots,n-3}$ => $\displaystyle\mathcal{I}_n^{\mathrm{Tr}\phi^3} = \int_0^\infty \prod_I \frac{dy_I}{y_I} \prod_C u_C(y)^{\alpha' X_C}$ curve-integral on the disk!

e.g. n=4: $\displaystyle\int_0^\infty \frac{dy}{y}\left(\frac{y}{1+y}\right)^{\alpha' X_{1,3}}\left(\frac{1}{1+y}\right)^{\alpha' X_{2,4}} = \frac{\Gamma(\alpha' X_{1,3})\Gamma(\alpha' X_{2,4})}{\Gamma(\alpha'(X_{1,3}+X_{2,4}))}$



u-var. manifest all factorizations @ $X_{a,b} = 0, -1, -2, \cdots,$ by $\underbrace{u_{a,b}}_{\to 0} + \prod_{j\in L, J\in R} \underbrace{u_{j,J}}_{\to 1} = 1 : \mathcal{I}(1,\cdots,n) \to \mathcal{I}_L(a,\cdots,b) \otimes \mathcal{I}_R(b,\cdots,a)$

beautiful surface-description of all cocycle (PT) + cycles (orderings) [Arkani-Hamed, SH, Lam, 19 ...]
=> IBP/BCJ+ monodromy for all Z integrals => surfaceology for closed strings [w. Cao, Dong, Li, Yang, in progress]

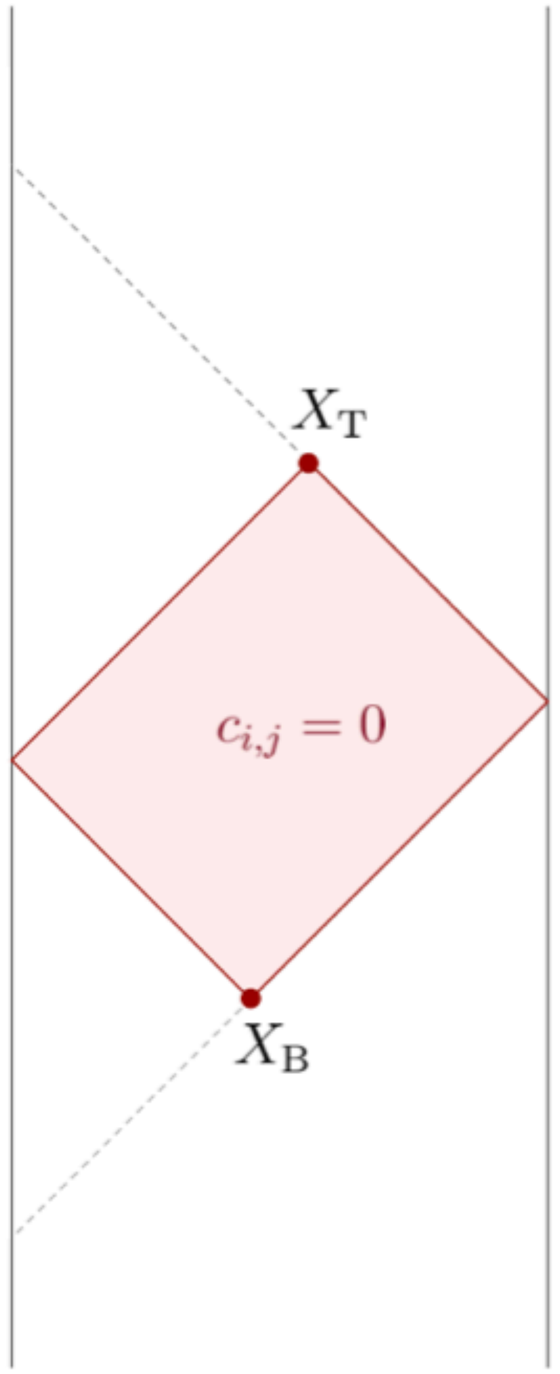
Zeros & splits of amps [ACDFH 2312; see D'Adda, Sciuto, D'Auria, Gliozzi, 71]

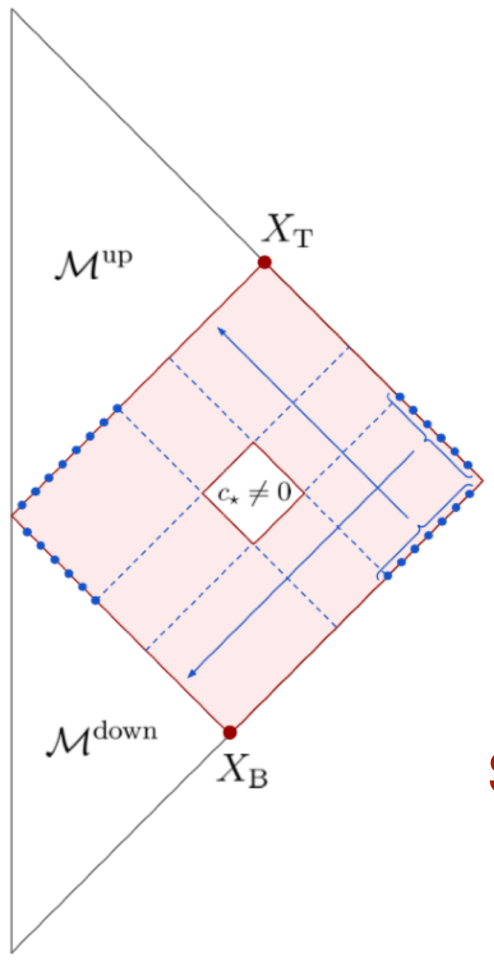
Zeros of Veneziano amp: by setting $\alpha' c_{1,3} = -n$, $\mathcal{I}_4^{\mathrm{Tr}(\phi^3)} \to \sum_{k=0}^{n} \underbrace{\int_{\mathbb{R}_{>0}} \frac{dy_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}+k}}_{=0} = 0.$

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \sum_{k_{a_1,b_1},\ldots,k_{a_N,b_N}=0}^{n_{a_1,b_1},\ldots,n_{a_N,b_N}} (\text{remaining integrals}) \times \underbrace{\int_{\mathbb{R}_{>0}} \frac{dy_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}+k_{a_1,b_1}+\cdots+k_{a_N,b_N}}}_{=0} = 0$$

$$c_{i,j} = -n_{ij}, \quad 1 \le i < a-1,\ a \le j < n$$

n(n-3)/2 infinite families of zeros





$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \mathcal{I}_i^{\mathrm{down},\mathrm{Tr}\,\phi^3} \times \mathcal{I}_{n-i+2}^{\mathrm{up},\mathrm{Tr}\,\phi^3} \times \mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,i}, \alpha'(c_{km} - X_{1,i})).$$

$$X_{l,i} \to X_{l,i} + X_{1,i} = X_{l,n}, \text{for} \quad l = 2,\ldots,k.$$
$$X_{i-1,j} \to X_{i-1,j} - X_{i-1,n} = X_{1,j}, \text{for} \quad j = m,\ldots,n-1.$$

$$\mathcal{M}_n(c_\star \neq 0) = \left(\frac{1}{X_{\mathrm{B}}} + \frac{1}{X_{\mathrm{T}}}\right) \times \mathcal{M}^{\mathrm{down}} \times \mathcal{M}^{\mathrm{up}}.$$

$$X = X_B + \tilde{X}. \quad \tilde{X} = X_{\mathrm{T}} + X,$$

shifted kinematics —> currents (with an off-shell leg)

"universal splits" (=> zeros) for NLSM + YM, colorless (DBI, sGal, GR), + bosonic/superstring [w. Cao, Dong, Shi, Zhu, 24] [Shruti's talk]

Deformed to the real world [ACDFH 2312]

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left(\frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha' \delta}, \qquad \mathcal{I}_{2n}^{\delta} = \mathcal{I}_{2n}^{\mathrm{Tr}\,\phi^3}[\alpha' X_{e,e} \to \alpha'(X_{e,e}+\delta), \alpha' X_{o,o} \to \alpha'(X_{o,o}-\delta)].$$

key: all $c_{i,j} = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$ are preserved => same zero + splits for deformed cases!

$\alpha'\delta = 0$

$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

$0 < \alpha'\delta < 1 \quad (\text{or } \mathbb{R}/\mathbb{Z})$ $\qquad \alpha' \to 0$

$$\mathcal{L}_{\mathrm{NLSM}} = \frac{1}{8\lambda^2} \mathrm{Tr}\left(\partial_\mu \mathrm{U}^\dagger \partial^\mu \mathrm{U}\right), \quad \text{with} \quad \mathrm{U} = (\mathbb{I} + \lambda\Phi)(\mathbb{I} - \lambda\Phi)^{-1}$$

$\alpha'\delta = \pm 1$

$$\mathcal{L}_{\mathrm{YMS}} = -\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\phi^I D_\mu\phi^I - \frac{g_{\mathrm{YM}}^2}{4}\sum_{I\neq J}[\phi^I, \phi^J]^2\right)$$

2n-pt Tr ϕ^3 string amps => 2n-pion in NLSM or 2n-scalar (n-gluon) in YMS: same function @ different pts!

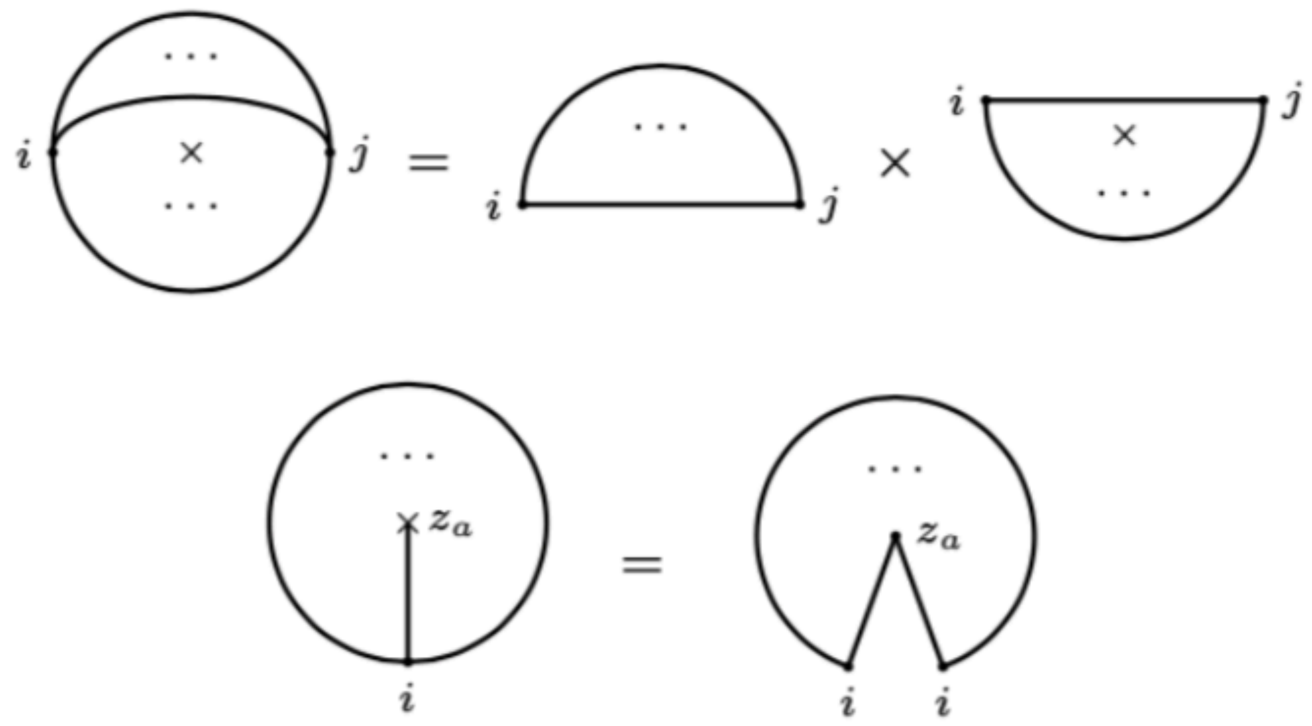
All-loop NLSM contained in Tr ϕ^3 [ACDFH 2401]

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(e,e)} u_{e,e}^{\alpha'(X_{e,e}+\delta)} \times \prod_{(o,o)} u_{o,o}^{\alpha'(X_{o,o}-\delta)} \times \prod_{(o,e)} u_{o,e}^{\alpha' X_{o,e}}$$
$$\to \mathcal{A}_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

Field-theory directly take $\delta \to \infty$: $$A_{2n}^{\mathrm{NLSM}} = \lim_{\delta\to\infty} \delta^{n-1} A_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

Same shift works for planar integrand of NLSM: $X_{e,e} \to X_{e,e} + \delta\,,\quad X_{o,o} \to X_{o,o} - \delta$ (inc. loop punctures)

$$\lim_{\delta\to\infty} \sum_{z_{a=1,\cdots,L}\ \mathrm{even/odd}}^{2^L} (\delta)^{n+2L-2} A_{n,L}^{\delta} = A_{n,L}^{\mathrm{NLSM}}\,.$$



"Adler zero": soft limit -> scaleless integrals! Very practical, e.g. 4-loop 4-pt NLSM integrand

Scalar-scaffolded gluons [ACDFH, 2312]

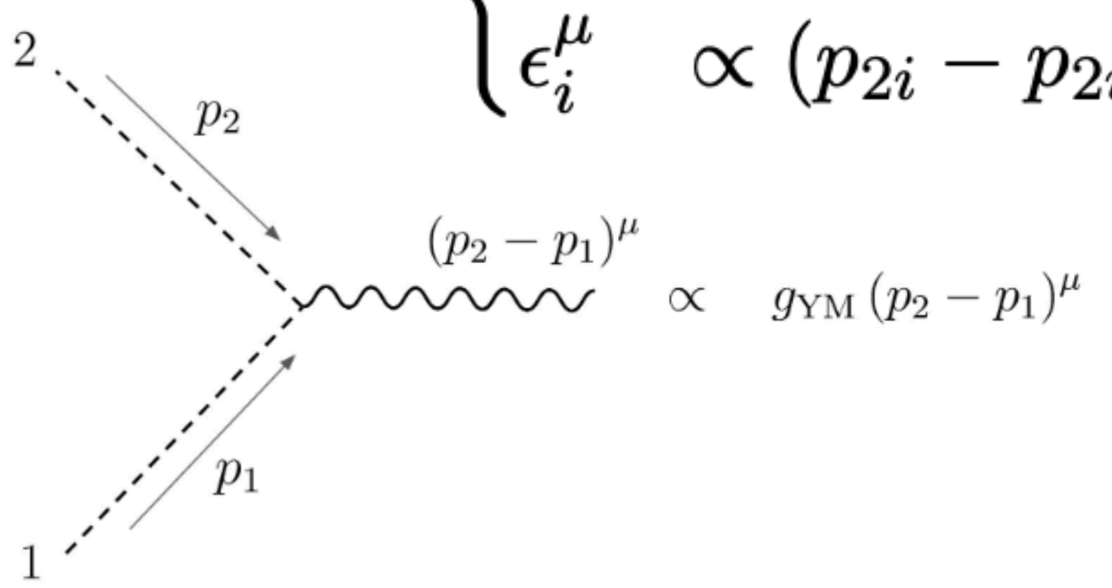
$\alpha'\delta = 1$ gives 2n-scalar stringy amplitude = 2n-scalar in bosonic string!

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,2n) = \int \frac{d^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \left(\prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}\right) \exp\left(\sum_{i\neq j} 2\frac{\epsilon_i\cdot\epsilon_j}{z_{i,j}^2} - \frac{\sqrt{\alpha'}\epsilon_i\cdot p_j}{z_{i,j}}\right)\Bigg|_{\text{multi-linear in }\epsilon_i},$$

$$p_i\cdot\epsilon_j = 0, \quad \forall\ (i,j)\in(1,\ldots,2n),$$

$$\epsilon_i\cdot\epsilon_j = \begin{cases} 1 & \text{if } (i,j)\in\{(1,2);(3,4);(5,6);\ldots;(2n-1,2n)\},\\ 0 & \text{otherwise.}\end{cases}$$

$$\begin{cases} q_i^\mu &= (p_{2i}+p_{2i-1})^\mu \\ \epsilon_i^\mu &\propto (p_{2i}-p_{2i-1})^\mu \end{cases},$$



$$\mathcal{A}_{2n}(1,2,\ldots,2n) \xrightarrow{\text{special kinematics}} \int \frac{d^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j} \frac{1}{z_{1,2}^2 z_{3,4}^2 z_{5,6}^2 \cdots z_{2n-1,2n}^2}$$

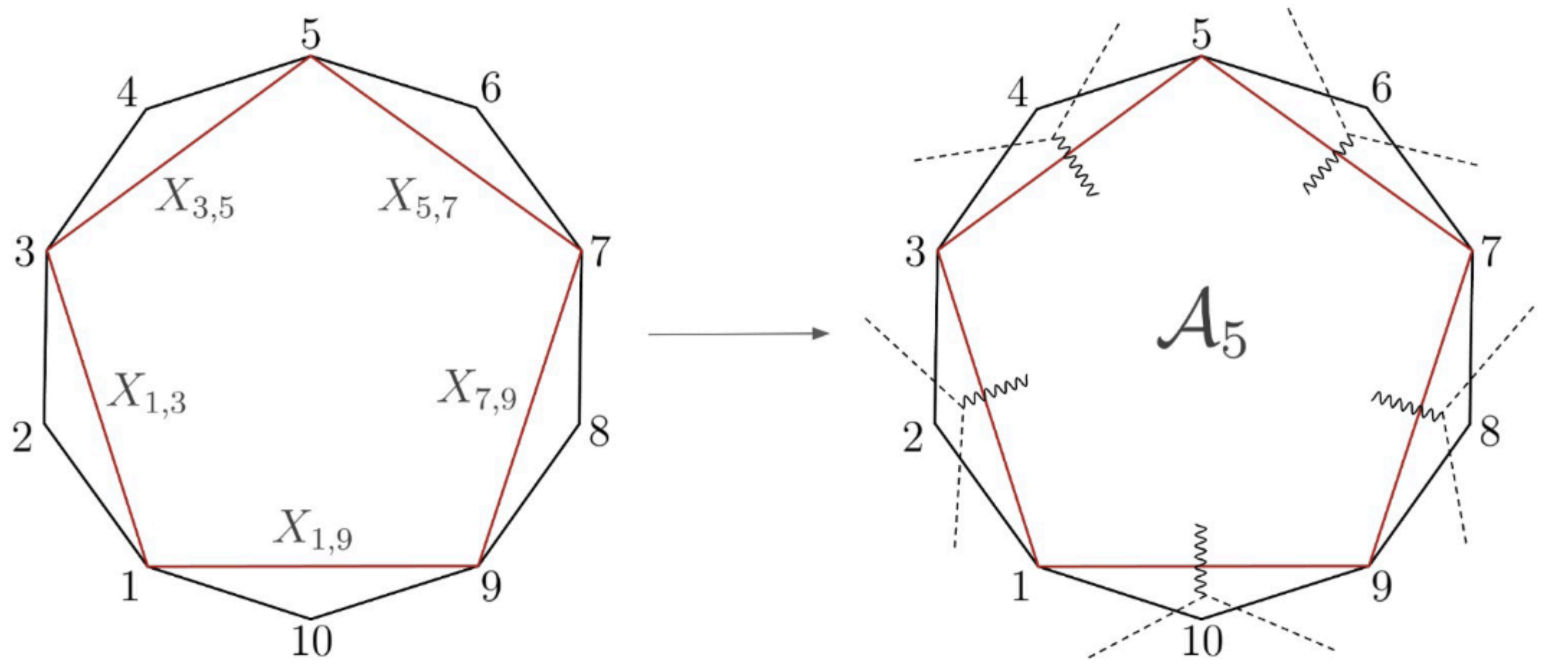
$$= \underbrace{\int \frac{d^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \frac{1}{z_{1,2}z_{2,3}z_{3,4}\cdots z_{2n,1}} \prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}}_{\text{Stringy Tr}\,\phi^3} \frac{z_{2,3}z_{4,5}z_{6,7}\cdots z_{2n,1}}{z_{1,2}z_{3,4}z_{5,6}\cdots z_{2n-1,2n}} = \left(\prod u_{e,e}/\prod u_{o,o}\right) \quad (\alpha'\delta = 1)$$

taking n ``scaffolding residues” $s_{1,2} = s_{3,4} = \cdots = 0$ => n-gluon bosonic string amps (in 2n-scalar language)

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \underbrace{\prod_{i=1}^{n} \frac{dy_{2i-1,2i+1}}{y_{2i-1,2i+1}^2} \prod_{I\in\mathcal{T}'} \frac{dy_I}{y_I^2} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}}}_{\Omega_{2n}},$$

$$X_{1,3} = X_{3,5} = \cdots = X_{1,2n-1} = 0$$

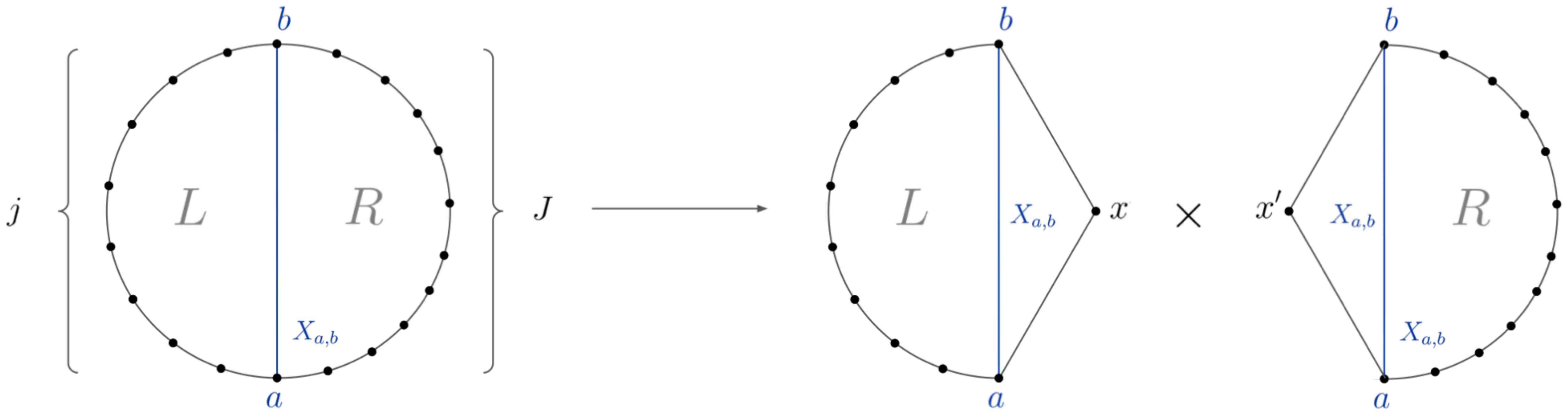
$$\mathcal{I}_{n}^{\text{gluon}} = \int_{\mathbb{R}_{>0}^{n-3}} \text{Res}_{y_{1,3}=0}\left(\text{Res}_{y_{3,5}=0}\left(\ldots\left(\text{Res}_{y_{1,2n-1}=0}\left(\Omega_{2n}\right)\right)\ldots\right)\right)\Big|_{X_{2i-1,2i+1}=0}$$



$$A_3^{\text{gluon}} = \alpha'^2\left(c_{1,3}c_{1,5} + c_{1,3}c_{2,5} + c_{1,3}c_{3,5} + c_{1,4}c_{3,5} + c_{1,5}c_{3,5} + c_{1,5}c_{3,6}\right) \quad + \\ - \alpha'^3\left(X_{1,4}X_{2,5}X_{3,6}\right)$$

surfaceology -> gauge invariance + gluon factorization (in X variables)

$$\mathcal{A}_n^{\text{gluon}} = \sum_{j\neq\{2i-1,2i,2i+1\}} \left(X_{2i,j} - X_{2i-1,j}\right) \times \mathcal{Q}_j \\ = \sum_{j\neq\{2i-1,2i,2i+1\}} \left(X_{2i,j} - X_{2i+1,j}\right) \times \tilde{\mathcal{Q}}_j \quad ,$$



$\mathcal{Q}_j = \tilde{\mathcal{Q}}_j := \partial_{X_{2i,j}} \mathscr{A}_n$ for gluon i (linear + gauge inv)

$$\text{Res}_{X_{a,b}=0}\,\mathcal{A}_n = -\sum_{j,J}\left(X_{j,J} - X_{j,b} - X_{a,J}\right) \underbrace{\prod_{X_L\in\mathcal{L}_j} \tilde{u}_{X_L}}_{Q_j^L} \cdot \underbrace{\prod_{X_R\in\mathcal{R}_J} \tilde{u}_{X_R}}_{Q_J^R}$$

All-loop YM in stringy Tr ϕ^3 [ACDFH]

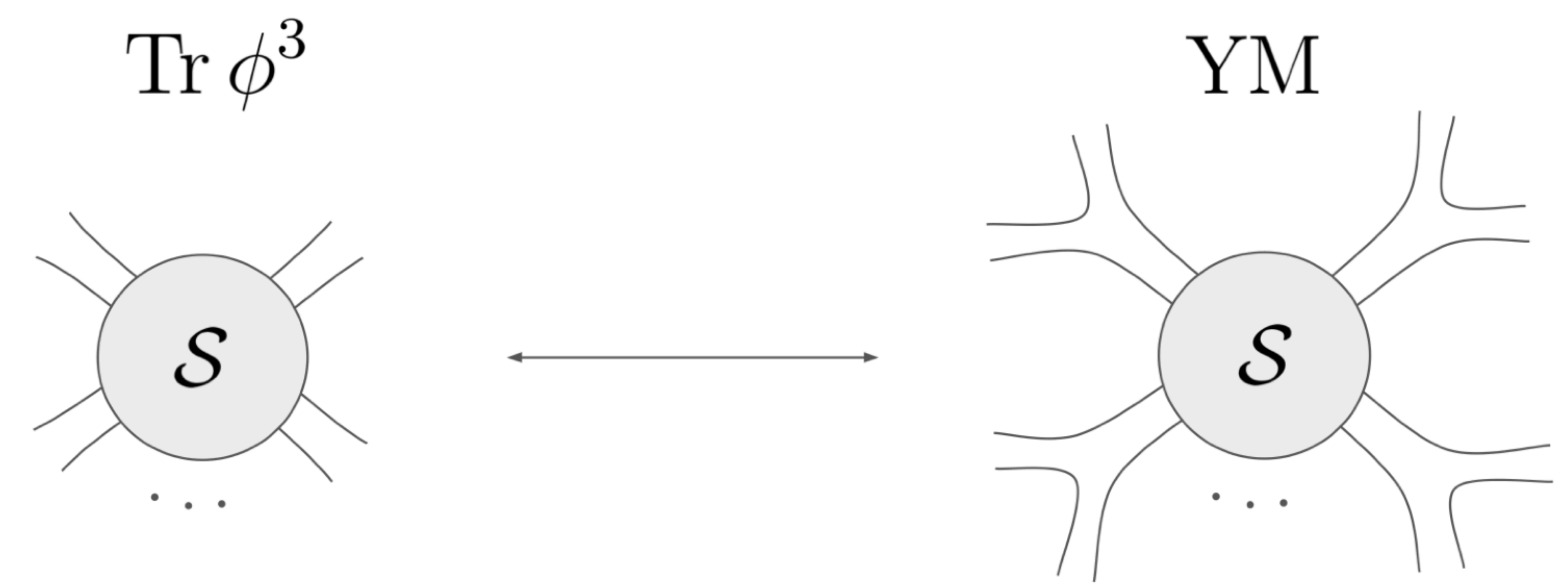
Surfaceology: generalize tree (disk) to loops (higher-genus surfaces):

$$A_n^{\text{gluon}} = \int_0^\infty \prod_i \frac{dy_i}{y_i^2} \text{Res}_{y_{s_1}=0} \left(\text{Res}_{y_{s_2}=0} \left(\dots \left(\text{Res}_{y_{s_n}=0}\Omega_{2n}\right) \dots\right)\right).$$

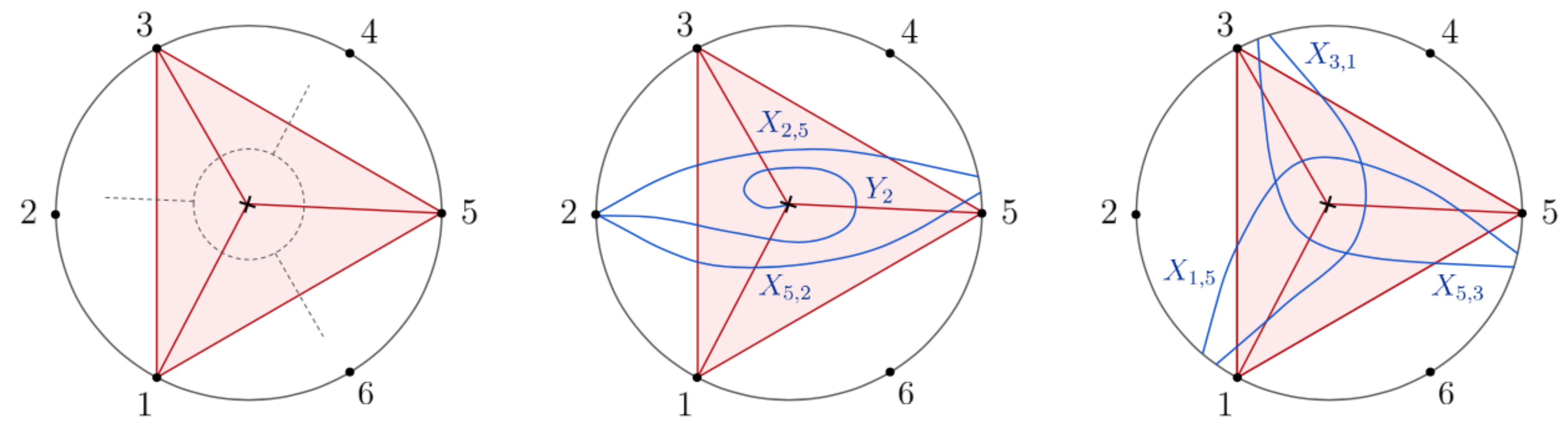
e.g. 1-loop w. self-inter. curves & closed curve Δ

$$\mathcal{I}_{2n}^{\text{1-loop}}(1,2,\dots,2n) = \int_0^\infty \prod_i \frac{dy_i}{y_i^2} \prod_C u_C^{\alpha' X_C} \times \prod_{C' \in \text{ s.i.}} u_{C'}^{\alpha' X_{C'}} \times u_\Delta^\Delta$$

simpler than bosonic string loops, but gives all-loop YM integrands!

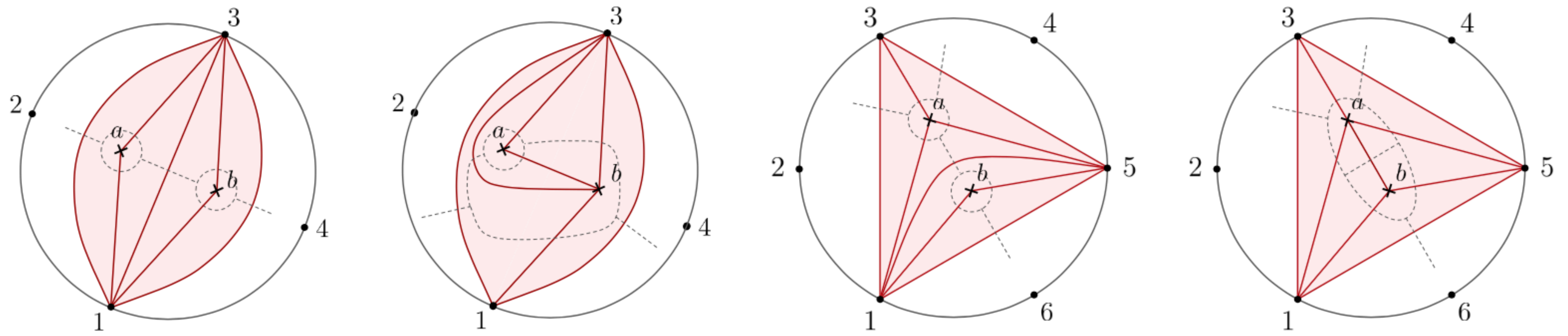


$$\int_0^\infty \prod \frac{dy}{y} \prod u_X^{\alpha' X} \qquad \int_0^\infty \prod \frac{dy}{y^2} \prod u_X^{\alpha' X}$$



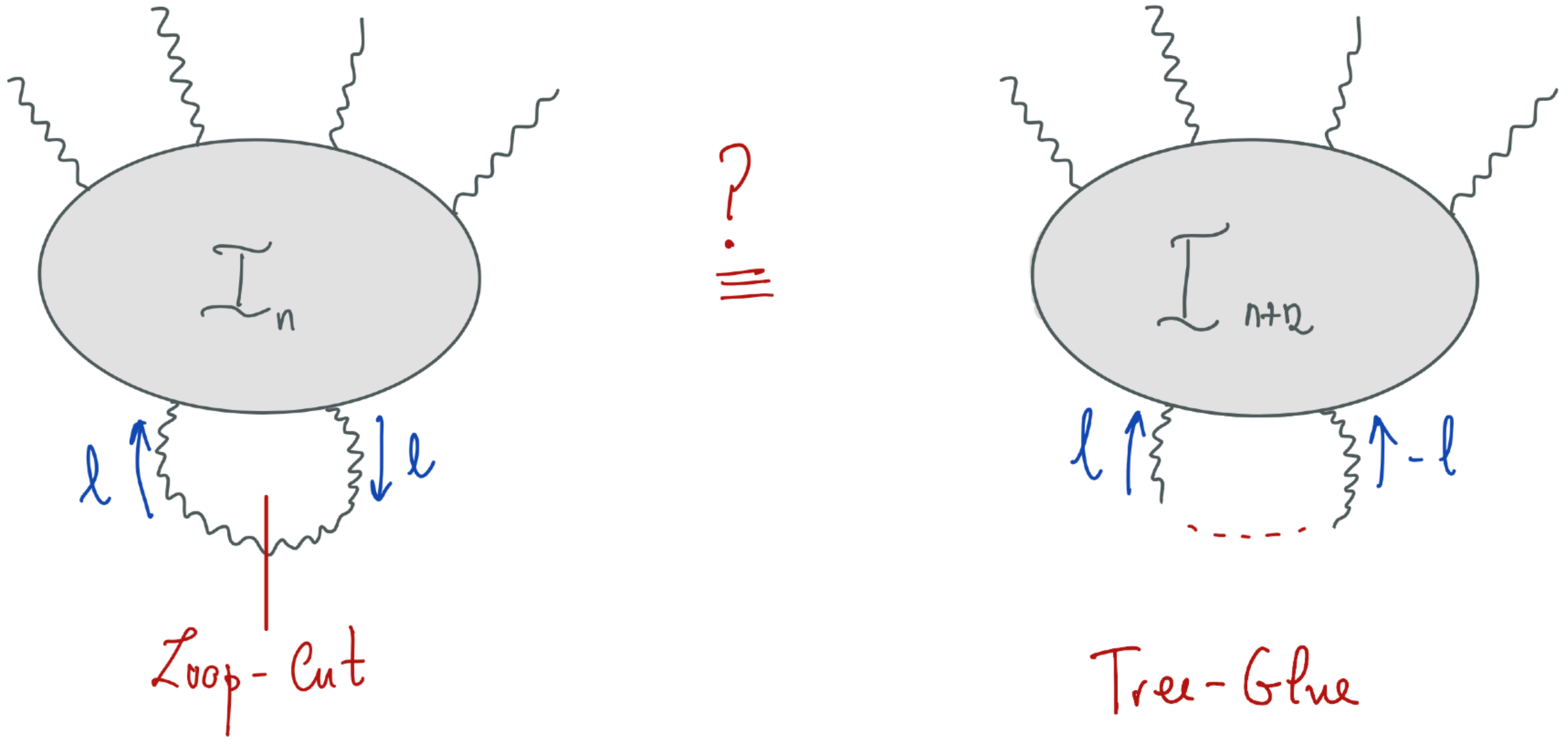
- extend the surface notion of gauge invariance + factorization/cuts
- e.g. 1-loop YM integrands computed from surface
- proof for all-loop leading singularities (max. residue) with $\Delta = 1 - D$:

residue of $\int \prod \frac{dy}{y^2} \prod u^X$ = gluing of 3pt YM+ F^3 (in X space)



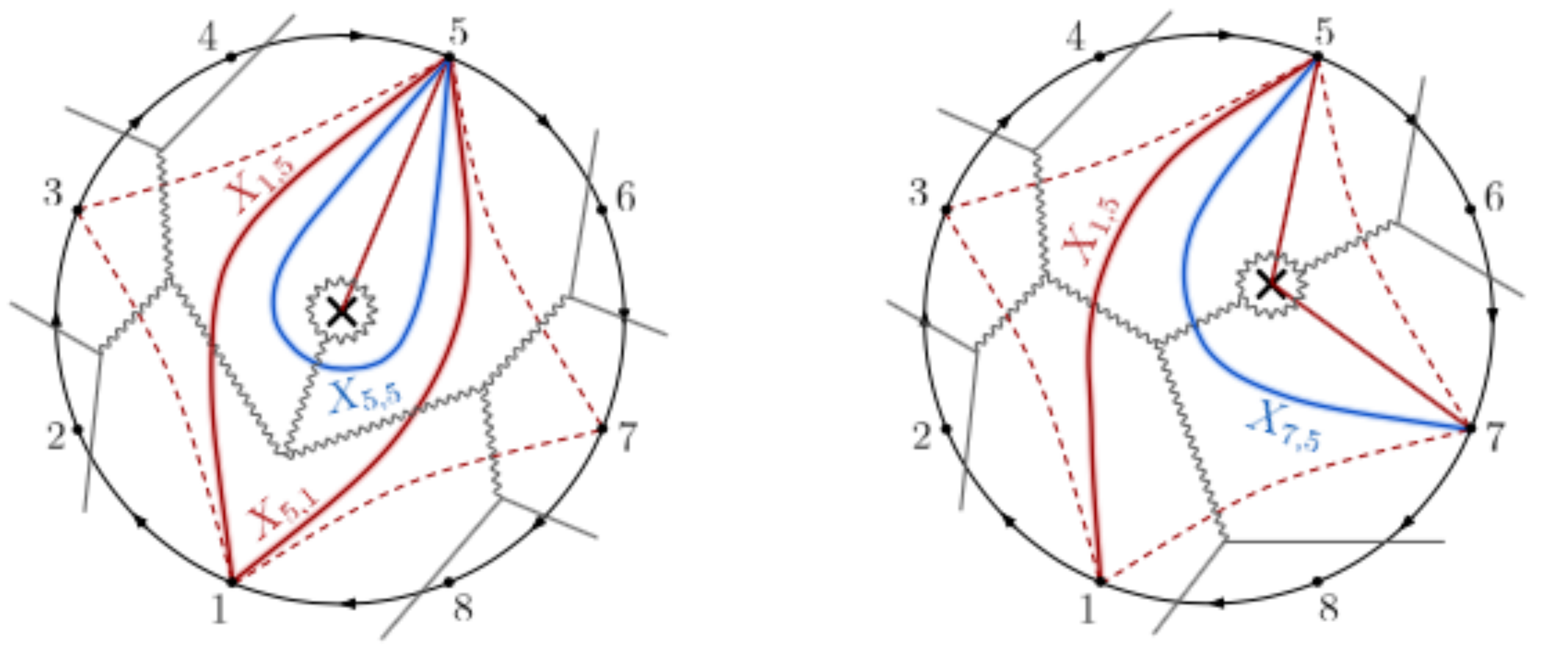
How to determine “the integrand” of YM?

Similar to tree factorizations, all we need are cuts: e.g. 1-loop single-cut = forward limit (gluing tree)



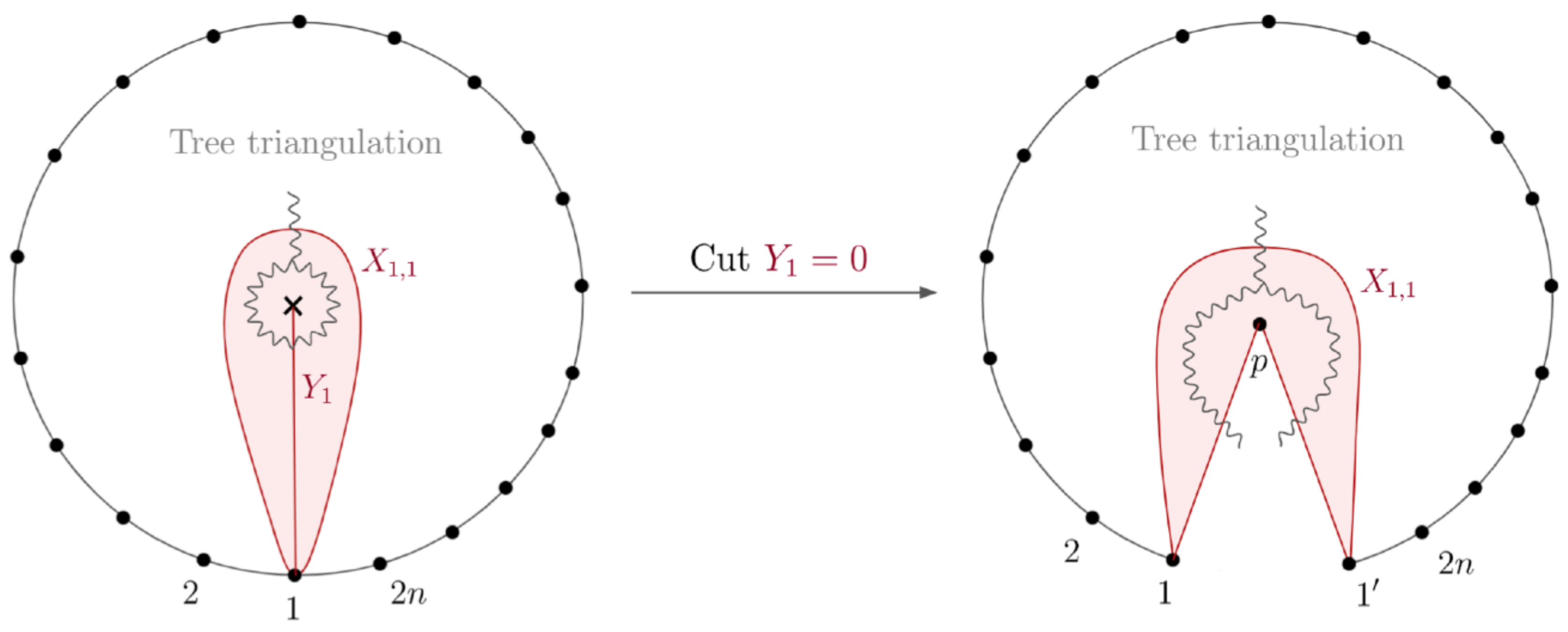
naively divergent => “the integrand”
(e.g. Adler zero, gauge inv.) ill defined!

no issues for scalars, but for gluons 1/0 !
(cancels in super-Yang-Mills)



surface provides a natural way out: curves without standard momentum (e.g. tadpoles) => “the integrand”

“doubling” variables: similar to Lorentzian -> complex in 4d tree kinematics



Loop recursions in YM [ACDFH, 2408.11891, work in progress]

Surface makes it clear “the integrand” to all loops exist (beyond planar limit)
=> the notion of surface gauge invariance+ cuts

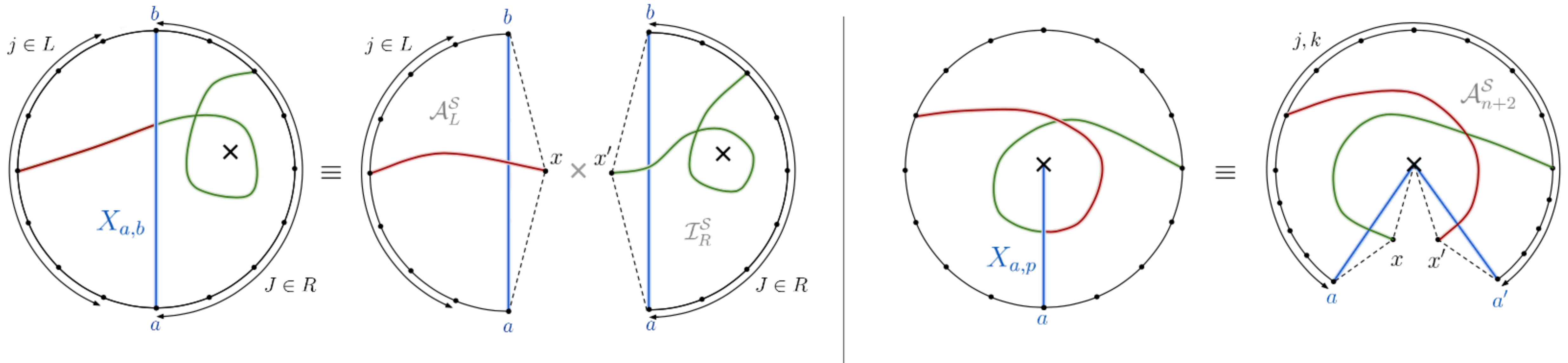
Loop integrands reconstructed from “residues” e.g. single-cut
=> forward-limit recursions for “the integrand” @ 1-loop & higher

Gauge-Invariance and Linearity (in gluon 1):

$$\mathcal{I}_n^{\mathcal{S}} = \sum_{j\neq 2}\left[(X_{2,j}-X_{1,j})\frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{2,j}} + (X_{j,2}-X_{j,1})\frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{j,2}}\right] + X_{2n,1}\times\left[\frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{2n,2}} + \frac{\partial \mathcal{I}_n^{\mathcal{S}}}{\partial X_{2n,1}}\Big|_{2\to 1}\right]$$

$$\underset{X_{a,p}=0}{\mathrm{Res}}\left(\mathcal{I}_n^{\mathcal{S}}\right) = \sum_{j,k}\left[(X_{j,p}+X_{k,p}-X_{k,j})\frac{\partial^2 \mathcal{A}_{n+2}^{\mathcal{S}}}{\partial X_{j,x}\partial X_{k,x'}}\right]\Bigg|_{a'\to a} - d\frac{\partial \mathcal{A}_{n+2}^{\mathcal{S}}}{\partial X_{x',x}}; \quad j,k\in(a,a+1,\cdots,a-1,a'),$$

$$\mathcal{I}_n^{\mathcal{S}} = \sum_i \frac{\mathrm{Res}_{X_{i,p}=0}\left(\mathcal{I}_n^{\mathcal{S}}(X_{j,p}\to X_{j,p}-X_{i,p})\right)}{X_{i,p}}\,.$$



I. Tree-Loop cut   II. Loop cut

Discard scaleless integrals => explicit results up to 2-loop 6-pt (-> 3-loop 4-pt in progress) [w. Cao, Dong, Zhu]

—> correct amplitudes after loop integration (in D dim), e.g. 1-loop helicity amps checked up to 5-pt; all-plus to all n?

enormous simplifications when reducing to 4d spinor-helicity: new results for higher loops?

Fermions, general gauge theories & SYM [w. Cao, Dong, Zhu, to appear]

How to include matters e.g. fermions in the loop? nice structure @ 1-loop from tree amps obtained via worldsheet [w. Edison et al '20, '22]

"universal expansion" of gluon tree + 2 gluons/fermions/scalars
FL (surface) => 1-loop gluon amps in ANY gauge theory

$$\mathcal{A}^{\rm YM}_{n+2}(-^g,+^g)=\sum_{m,\alpha\in S_m}\epsilon_-\cdot f_{\alpha_1}\cdot f_{\alpha_2}\cdots f_{\alpha_m}\cdot\epsilon_+\times\mathcal{A}^{\rm mixed}(+,\alpha,-)$$

$$\mathcal{A}^{\rm gauge}_{n+2}(-^f,+^f)=\sum_{m,\alpha\in S_m}\bar{\chi}_-\not{f}_{\alpha_1}\not{f}_{\alpha_2}\cdots\not{f}_{\alpha_m}\xi_+\times\mathcal{A}^{\rm mixed}(+,\alpha,-)$$

$$\epsilon_-\cdot\epsilon_+\xrightarrow{\rm F.L.}D-2,$$

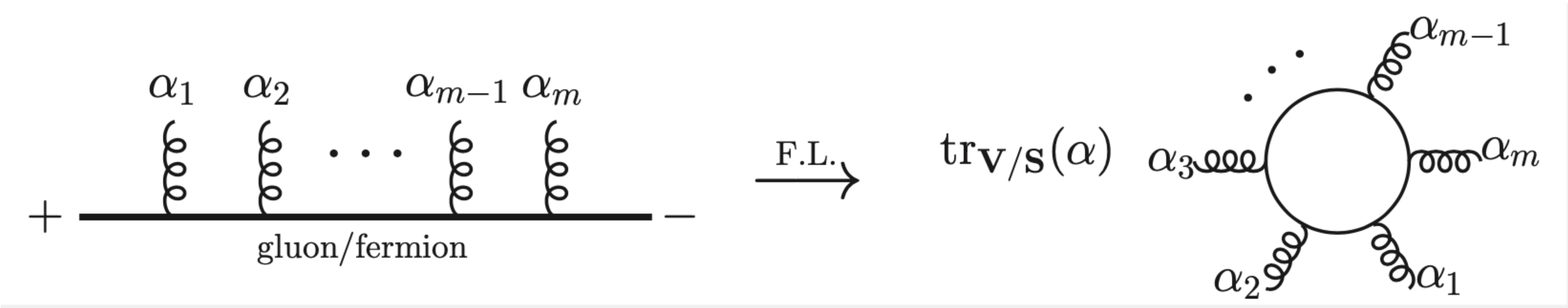
$$\epsilon_-\cdot f_{\alpha_1}\cdot f_{\alpha_2}\cdots f_{\alpha_m}\cdot\epsilon_+\xrightarrow{\rm F.L.}{\rm tr}_{\rm V}(f_{\alpha_1}f_{\alpha_2}\cdots f_{\alpha_m}).$$

$${\rm Res}_{Y_i=0}\mathcal{I}^{\rm YM}_{\text{gluon-loop}}(1,2,\ldots,n)=\sum_{m,\alpha}{\rm tr}_{\rm V}(f_{\alpha_1}\cdots f_{\alpha_m}){\rm Res}_{Y_i=0}\mathcal{I}^{\rm YMS}_{\text{scalar-loop}}(\alpha|1,\ldots,n)$$

$$\bar{\chi}_-\xi_+\xrightarrow{\rm F.L.}2^{D/2-1},$$

$$\bar{\chi}_-\not{f}_{\alpha_1}\not{f}_{\alpha_2}\cdots\not{f}_{\alpha_m}\xi_+\xrightarrow{\rm F.L.}{\rm tr}_{\rm S}(\not{f}_{\alpha_1}\not{f}_{\alpha_2}\cdots\not{f}_{\alpha_m}).$$

$${\rm Res}_{Y_i=0}\mathcal{I}^{\rm gauge}_{\text{fermion-loop}}(1,2,\ldots,n)=\sum_{m,\alpha}{\rm tr}_{\rm S}(\not{f}_{\alpha_1}\cdots\not{f}_{\alpha_m}){\rm Res}_{Y_i=0}\mathcal{I}^{\rm YMS}_{\text{scalar-loop}}(\alpha|1,\ldots,n)$$



New formula/relations for n-gluon amps: pure YM 1-loop = sum of mixed scalar-loop amps with Lorentz traces

$$\mathcal{A}_n^{\rm YM} = \sum_{m,\alpha} {\rm tr}_{\rm V}(f_{\alpha_1}\cdots f_{\alpha_m})\mathcal{A}_\alpha^{\rm scalar-loop}$$

e.g.

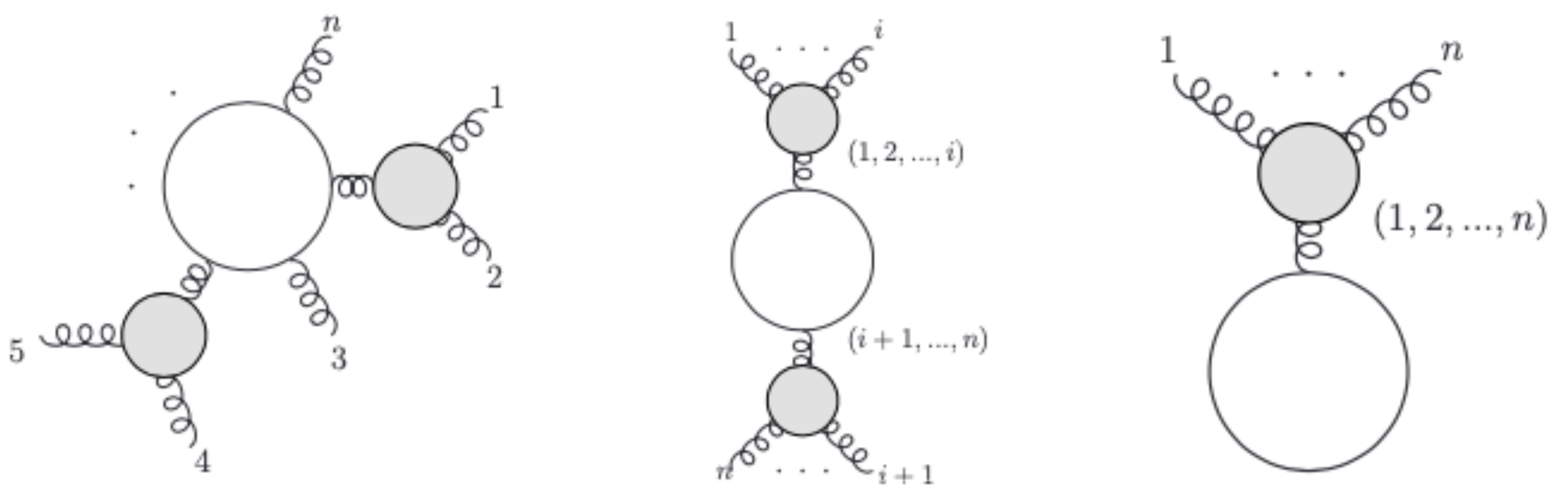
$$n=2: \quad (D-2)\mathcal{A}_\emptyset + {\rm tr}_{\rm V}(f_1 f_2)\mathcal{A}_{1,2},$$

$$n=3: \quad (D-2)\mathcal{A}_\emptyset + [{\rm tr}_{\rm V}(f_1 f_2)\mathcal{A}_{1,2} + 2 \ \text{perms}] + {\rm tr}_{\rm V}(f_1 f_2 f_3)\mathcal{A}_{1,2,3}.$$

universal 1-loop objects: scalar loop with tree blobs of m scalars (in α ordering) + (n-m) gluons attached

most important (m=0): $\mathscr{A}_\varnothing$ as the coefficient of "D-2" (large D limit), simply attaching n gluons to scalar loop

—>all $\mathscr{A}^{\rm mixed}_{\alpha,m>0}$ obtained by differential operators w.r.t polarizations $\epsilon_{\alpha_1}, \cdots, \epsilon_{\alpha_m}$



(a) [(1,2),3,(4,5),...,n] (b) bubble (c) tadpole

$\mathscr{I}^{\rm mixed}_{\alpha,m>0} = D^{(m)}_\alpha \mathscr{I}_\varnothing$: degree-m polynomial of $\partial_{\epsilon_a \cdot k}$ $(a = \alpha_1, \cdots, \alpha_m)$
(derived from tree-level transmuted operators)

$$D^{(3)}_{132} = (\partial_1^+ \partial^+_{(3,2)} + \partial_2^- \partial^-_{(3,1)} + \text{cyclic}) + \partial^+_{(1,3,2)} + \partial^-_{(2,3,1)} + \partial_2^- \partial_3^- \partial_1^-$$

With $\partial_a^\pm := \mp\, \partial_{\epsilon_a \cdot \ell_a}, \quad \partial^\pm_{(b,a)} := \mp\, \partial_{\epsilon_b \cdot k_a}$

Almost identical for any gauge theory! any multiplet in the loop = sum of mixed scalar-loop amps with “vector/spinor trace”

$$\mathcal{A}_n^{\text{gauge}} = \sum_{m,\alpha} \mathcal{T}_{\alpha_1,\cdots,\alpha_m} \mathcal{A}_\alpha^{\text{scalar-loop}}$$

$\mathscr{T}^{\mathbf{n}_{\mathbf{v,f,s}}}_{\alpha_1,\cdots\alpha_m} := \mathbf{n}_v \text{tr}_V(\alpha_1\cdots\alpha_m) - \frac{\mathbf{n}_f}{2}\text{tr}_S(\alpha_1\cdots\alpha_m)$ for $\mathbf{n}_v, \mathbf{n}_f, \mathbf{n}_\mathbf{s}$ vectors, Weyl fermions, scalars

(m=0: $\mathscr{T}_\varnothing := (D-2)\mathbf{n}_v + \mathbf{n}_s - 2^{(D-2)/2}\frac{\mathbf{n}_f}{2}$, counts # of on-shell d.o.f)

effective SUSY Ward identities! In particular, huge simplifications for SYM, e.g. w, maximal SUSY

$\mathscr{T}_{m<4} = 0$ since it is proportional to $\mathbf{n}_v - 2^{D/2-5}\mathbf{n}_f = 0$ (D=10, $\mathbf{n}_f = \mathbf{n}_v, \mathbf{n}_s = 0$, or D=4, $\mathbf{n}_f = 8\mathbf{n}_v, \mathbf{n}_s = 6\mathbf{n}_v$)

“no triangle/bubble/tadpole” + correct power-counting ℓ^{m-4} for the m-gon!

e.g. m=4: t_8 tensor for box numerator, $\text{tr}_V - \frac{1}{2}\text{tr}_S|_{D=10} = \frac{1}{2}[\text{tr}_V(1,2,3,4) - \frac{1}{4}\text{tr}_V(1,2)\text{tr}_V(3,4) + \text{cyc.}]$

Q: higher loops? forward-limit recursion for pure YM, but more works needed for fermion loop/SUSY;
SYM leading singularities from surfaces?

Q: BCJ & double copy to (super-)gravity? e.g. [Edison et al ’22] “bootstrap”@ 1-loop => up to n=6 for pure YM & n=8 for SYM;
Can we directly forward-limit reconstruct gravity integrands?

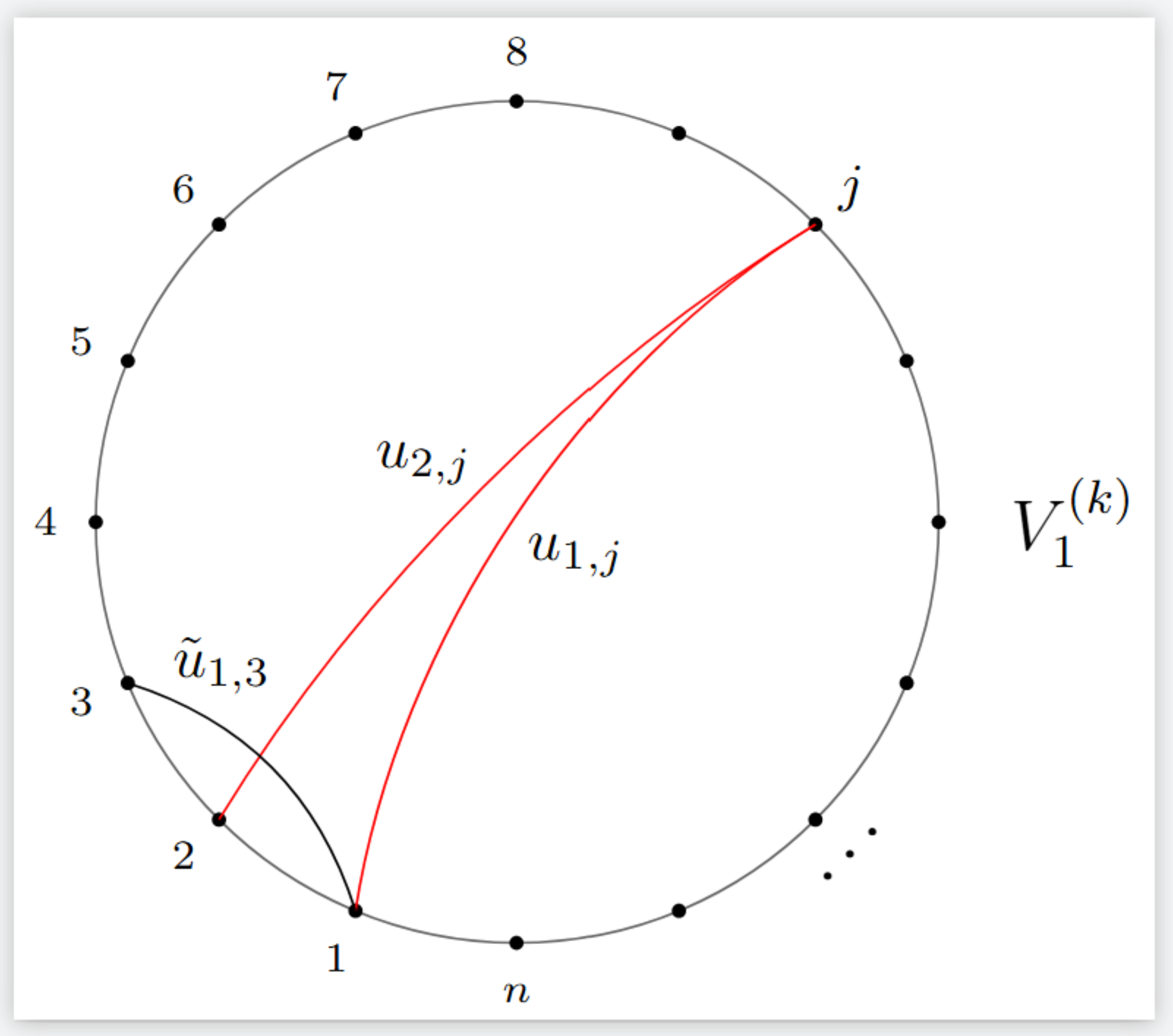
Massive string states

(combinations of) massive string states from residues! $X_{i,i+2} = -k \rightarrow$ k-th derivatives @ $u_{i,i+2} = y_{i,i+2}\tilde{u}_{i,i+2} = 0$

e.g. level-1, residue @ $X_{1,3} = -1 \rightarrow \frac{\partial}{\partial y_{1,3}} : \int \frac{dy_{1,3}}{y_{1,3}^2}(\tilde{u}_{1,3})^{X_{1,3}} \prod_{i,j} u_{i,j}^{X_{i,j}} \implies X_{1,3}\frac{\partial \log \tilde{u}_{1,3}}{y_{1,3}} + \sum_{j>3}\left(X_{2,j}\frac{\partial log u_{2,j}}{\partial y_{1,3}} + X_{1,j}\frac{\partial \log u_{1,j}}{\partial y_{1,3}}\right)$

u equations (surfaceology): $\text{Res}_{X_{1,3}=-1}\mathcal{I}_n = V_1^{(1)}\mathcal{I}_{n-1}$ with $V_1^{(1)} := \sum_j \underbrace{\left(X_{2,j} - X_{1,j} + X_{1,3}\right)}_{\equiv \epsilon_1 \cdot x_{3,j}} \frac{\partial \log u_{2,j}}{\partial y_{1,3}},$

exactly the same formula as gluon except for $X_{1,3} = -1$ (no gauge invariance, massive spin-1 + spin-0)



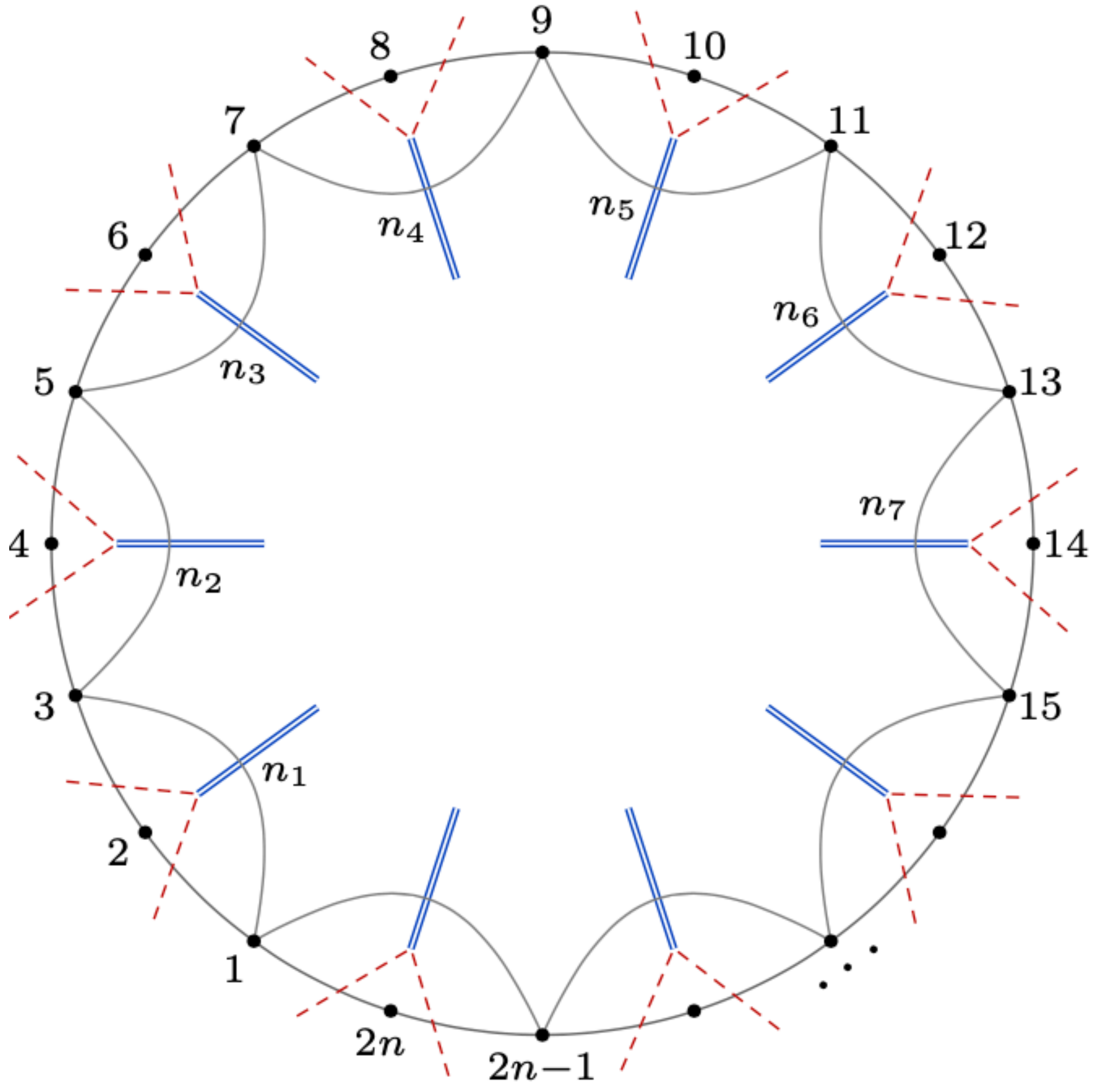
same for higher levels, e.g. residue @ $X_{1,3} = -2 \rightarrow \frac{\partial^2}{\partial y_{1,3}^2}$:

$V_1^{(2)} := \left(V_1^{(1)}\right)^2 + \sum_j \epsilon_1 \cdot x_{3,j}\frac{\partial^2 \log u_{2,j}}{\partial y_{1,3}^2}$ (massive spin-2, 1, 0)

“vertex operators” for any k: $V_1^{(k)} \supset \left(V_1^{(1)}\right)^k, \cdots, \sum_j \epsilon_1 \cdot x_{3,j}\frac{\partial^k \log u_{2,j}}{\partial y_{1,3}^k}$

more massive string states e.g. $X_{1,3} = X_{5,7} = -1 \to V_1^{(1)} V_3^{(1)} + \partial_{y_{5,7}} V_1^{(1)} \implies \underbrace{(X_{1,5} + X_{2,6} - X_{1,6} - X_{2,6})}_{c_{1,5}} \frac{\partial^2 log u_{2,6}}{\partial y_{1,3} \partial y_{5,7}} \equiv W_{1,3}^{(1,1)}$

Just like scaffolded gluons: 2n-gon massless => massive states with $X_{1,3} = -n_1,\ X_{3,5} = -n_2,\ X_{5,7} = -n_3, \ldots$



A sum of products of $V_a^{(m_a)}, W_{a,b}^{(m_a,m_b)}$ with total degree $n_1, n_2, \cdots,$

e.g. $\mathscr{A}_3^{(1,1,1)} = V_1^{(1)} V_2^{(1)} V_3^{(1)} + \left(W_{1,2}^{(1,1)} V_3^{(1)} + \text{cyc.} \right)$

explicit 4-pt (any level): $\mathscr{A}_4^{(n_1,n_2,n_3,n_4)} = \underbrace{\left(\sum \prod \tilde{V}\ \tilde{W} \right)}_{\text{polynomial in X}} \times \frac{\Gamma(X_{1,5})\Gamma(X_{3,7})}{\Gamma(X_{1,5} + X_{3,7})}$

compare residues with worldsheet calculation: leading trajectories

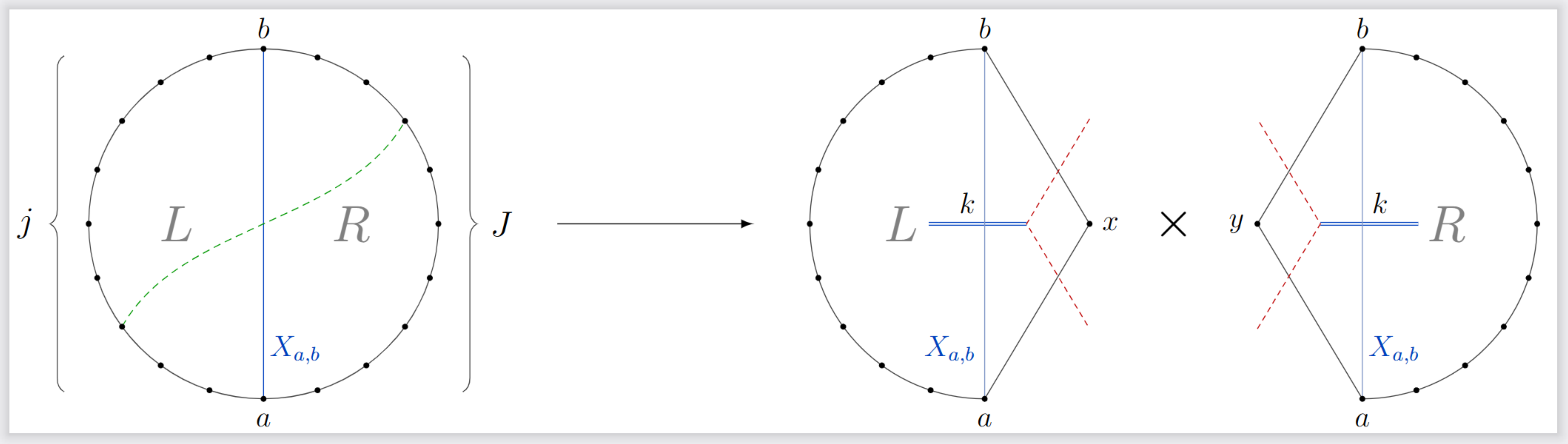
Q: can we produce & label all string states in this way?

Massive factorizations: open & closed

explicit formula for massive factorizations: gluing of massive states @ $X_{a,b} = -k \to$ k-th derivatives @ $u_{a,b} = y_{a,b}\tilde{u}_{a,b} = 0$

residue @ $X_{i,j} = -1 \to \frac{\partial}{\partial y_{i,j}}$: $\mathrm{Res}_{X_{a,b}=-1}\mathcal{A}_n = -\sum_{j,J}(X_{j,J} + X_{a,b} - X_{j,b} - X_{a,J})Q_j^L Q_J^R$ (same as gluon fan) with

$Q_j^L := \frac{\partial \mathcal{A}^L(a,\cdots,b,x)}{\partial X_{x,j}}$, $Q_J^R := \frac{\partial \mathcal{A}^R(b,\cdots,a,y)}{\partial X_{y,J}}$; obvious generalization to k-th level!



Even more interesting for closed strings: already $\int_{\mathbb{C}} \prod |\frac{dy}{y}|^2$ => detect massive poles for all possible orderings!

e.g. non-planar poles starts @ $s_{1,3} = -1$ for $n \geq 4$, $s_{1,3,5} = -2$ for $n \geq 6$, $s_{1,3,5,7} = -3$ for $n \geq 8$, same factorization formula

Towards superstring?

Scaffolded n-gluon in superstring: $\mathcal{I}_{2n} = \int d^{2n-3}z\, d^{2n-2}\theta \prod_{i<j} |z_i - z_j + \theta_i\theta_j|^{2\alpha' p_i\cdot p_j} \times \frac{1}{(12)(34)\cdots(2n-12n)}$

=> $\mathcal{I}_{2n} = \int d^{2n-3}z \prod_{i<j} |z_{i,j}|^{2\alpha' p_i\cdot p_j} \times \frac{\mathrm{Pf}' A_{2n\times 2n}}{z_{1,2}z_{3,4}\cdots z_{2n-1,2n}}$ $\quad$ ($A_{i<j} = \frac{s_{i,j} - \delta_{i,j}}{z_{i,j}}$ w. $\delta_{1,2} = \delta_{3,4} = \cdots = 1$ otherwise 0)

u variables => sum over pairings as "corrections" to bosonic string: $\prod \frac{dy}{y^2}(1 + ss\prod u + \ldots + s\ldots s\prod u)$

e.g. n=3: $\mathcal{I}_6 = \int \prod \frac{dy}{y^2} u^X \left(-1 + \frac{\alpha'^2 s_{1,4}s_{2,3}u_{2,4}}{u_{1,3}u_{3,5}u_{3,6}} + \frac{\alpha'^2 s_{1,3}s_{2,4}}{u_{1,3}u_{3,5}u_{3,6}} \right)$

$$\propto \underbrace{\frac{-1}{z_{1,2}^2 z_{3,4}^2 z_{5,6}^2}}_{\mathrm{YM}+F^3} + \underbrace{\frac{\alpha'^2 s_{1,4}s_{2,3}}{z_{1,2}z_{2,3}z_{3,4}z_{4,1}z_{5,6}^2} + \frac{\alpha'^2 s_{1,3}s_{2,4}}{z_{1,2}z_{2,4}z_{4,3}z_{3,1}z_{5,6}^2}}_{-F^3}$$

(with $s_{1,2} = s_{3,4} = s_{5,6} = 0$)

Q: surface origin of worldsheet SUSY? e.g. systematic cancellation of F^3 even for tree-level LS?

Q: connections to spacetime fermions? How to see SUSY cancellation for loop-level LS?

Outlook

surfacehedra: Tr ϕ^3; curve integrals: string(y) => all-loop amps of pions, gluons, etc. (real world)

- Hidden zeros + splits: physical implications for trees & loops (scalars, YM, even gravity)?
- All-loop recursion for YM: automated code -> practically usable form -> spinor-helicity? integration? inclusion of fermions: e.g. surface origin of SUSY cancellation for LS?
- (scaffolded) Gravity: trees & all-loop LS from squaring; extend (loop-level) curve-integrals to colorless case => surface diff. invariance + manifest ALL factorizations/cuts?
- Revisiting tree-level string amps: monodromy + KLT from surfaceology; worldsheet SUSY?
- Massive string states/amplitudes: how to produce all string states via scalar-scaffolding?

Thank you!