# Energy correlators in multi-collinear limit and beyond

Kai Yan

QCD Meets Gravity 2024.12

# Energy correlators: correlation function of flow operators

$$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\ldots\mathrm{E}(n_N)\rangle = \int d^4x e^{iq\cdot x}\frac{\langle O(x)\mathcal{E}(n_1)\cdots\mathcal{E}(n_N)O(0)\rangle}{\langle O(x)O(0)\rangle}$$

$$\mathcal{E}(n) = \int_{-\infty}^{+\infty} du \lim_{r\to\infty} r^2 T_{0i}(t=u+r, r\vec{n})n^i$$

Exhibits an OPE in the (multi-) collinear limit

key for understanding the conformal light-ray OPE [Chang et al, 2202.04090][Chen et al, 2202.04085]

Novel observables in collider physics measured at the LHC

new jet substructure calculations [Komiske et al, 2201.07800]

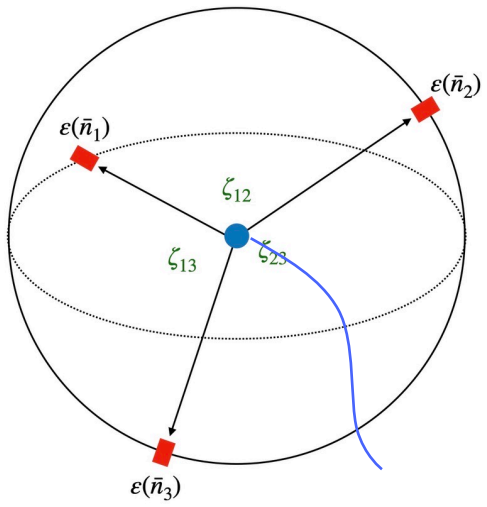
A potential playground for novel onshell methods in scattering amplitudes, as,e.g, in the context of cosmological correlator [Shounak De et al,2308.03753] [Arkani-Hamed,Yuan,1712.09991.] [Gong, Yuan, 2206.06507]

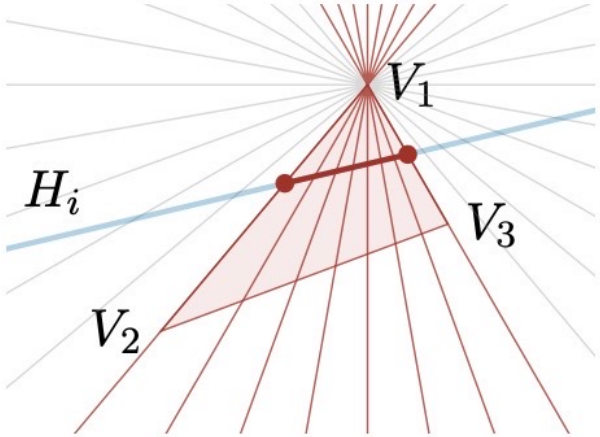
# N-point Energy correlators: at tree level

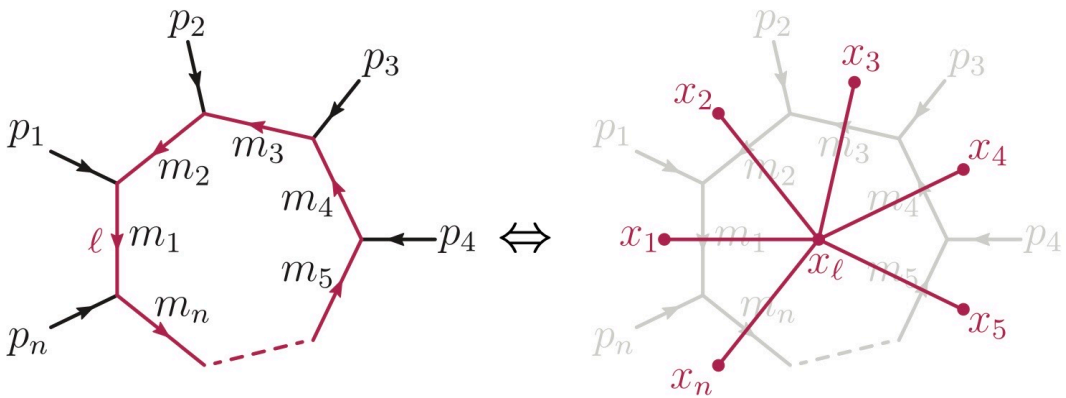
$\mathrm{E}^N C$: $N$-point energy correlator

$$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\ldots\mathrm{E}(n_N)\rangle = \sum_{k>N}\int d\,\Pi_k\,\delta^2(n_i-\Omega_i)\cdots\delta^2(n_N-\Omega_N)\frac{E_1\cdots E_N}{Q^N}\left|F_k\,(O)\right|^2$$

$k = N+1$: Leading order. Manifestly finite integration over tree-level matrix element

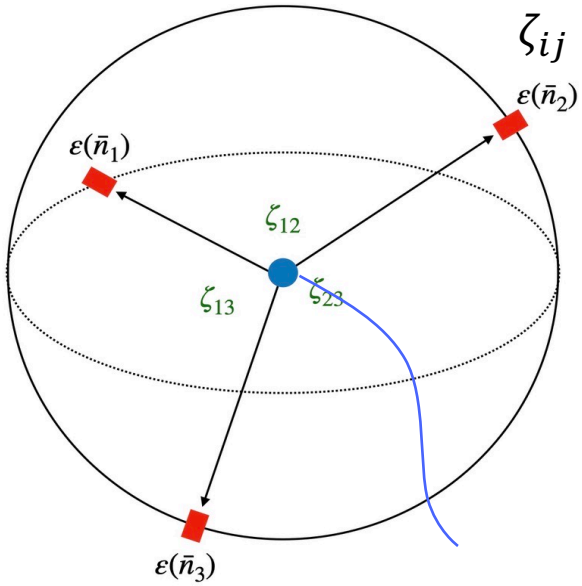


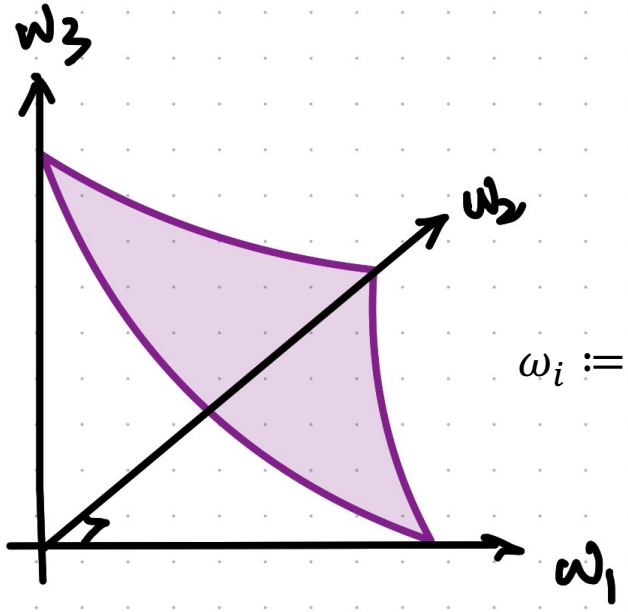



distances on the celestial sphere:

$$\zeta_{ij} = \frac{q^2(n_i \cdot n_j)}{2(n_i \cdot q)(n_j \cdot q)} \to \frac{1}{2}(1 - \cos\theta_{ij})$$




$$\int \prod d\, e_i\, \delta((q - \sum e_i n_i)^2)\theta(q^0 - \sum e_i)$$




$$\omega_i := \frac{2\, e_i}{q^0}$$

$$\int d\, PS_N = \int \Pi d\omega_i\, \delta\left(1 - \sum\omega_i - \sum\omega_i\, \omega_j \zeta_{ij}\right)\theta\left(1 - \frac{1}{2}\sum\omega_i\right)$$

$$= \int dvol_N\, (\hat{\omega}) \left(\frac{2}{2 + \sum\hat{\omega}_i}\right)^{N-1}$$

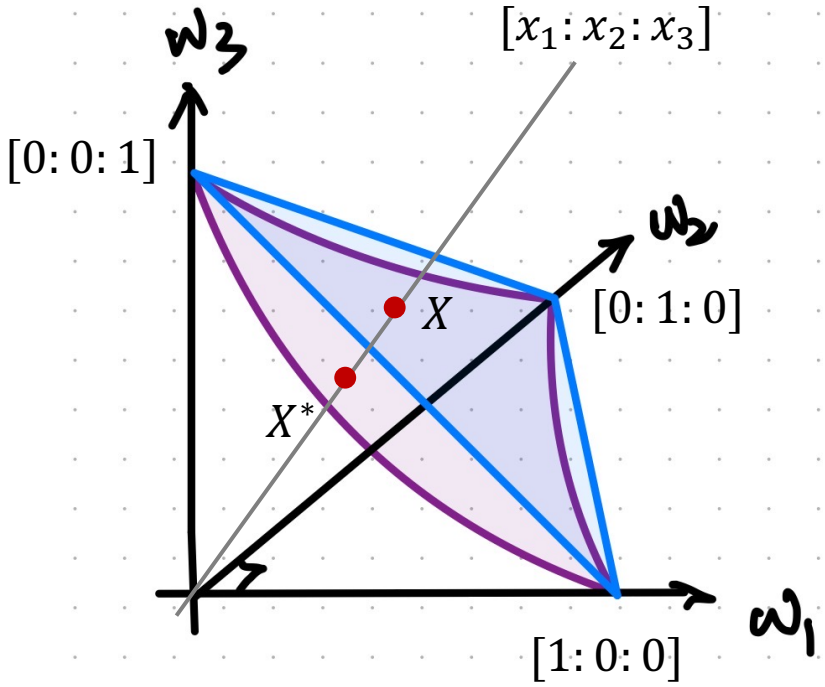
$$\boxed{\int dvol_N(x) := \int \prod d\, x_i\, \delta(1 - \langle xx \rangle)}$$

$$\langle \hat{\omega}\hat{\omega} \rangle := \frac{1}{4}\hat{\omega} A \hat{\omega} = \frac{1}{4}\left[\left(\sum\hat{\omega}_i\right)^2 - 4\sum\hat{\omega}_i\, \hat{\omega}_j \zeta_{ij}\right]$$

$$A_{ij} := \frac{\vec{n_i} \cdot \vec{n_j}}{|\vec{n_i}|\, |\vec{n_j}|} = 1 - 2\zeta_{ij}$$

Phase-space for the $E^N C$: $PS_N$ $\quad X = x_1 V_1 + .. x_N V_N$

codim-1 boundaries: $PS_{N-1}$

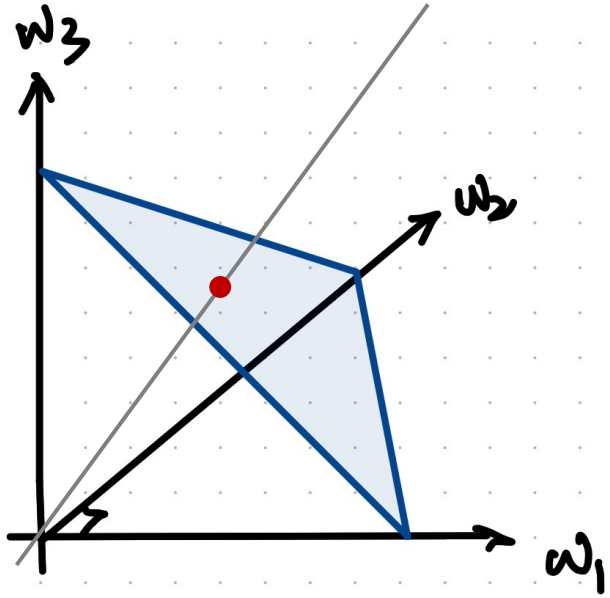



Multi-collinear limit

$\zeta_{ij} \sim 0$

$XAX \to (x_1 + .. + x_N)^2$

$$A = \begin{pmatrix} 1 & 1-2\zeta_{12} & .. \\ 1-2\zeta_{12} & 1 & .. \\ .. & .. & 1 \end{pmatrix}$$



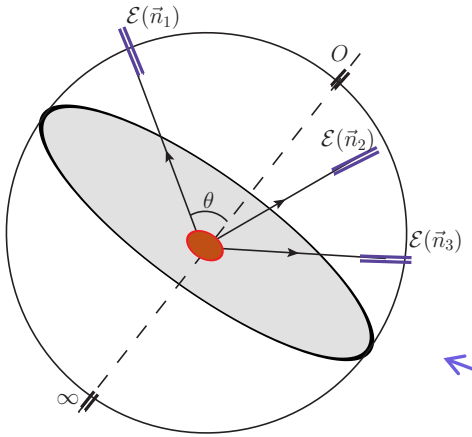

Canonical simplex $\Delta_{N-1}$

$$\int d\mathrm{PS}_N = \int \frac{2^{N-1}\langle X d^{N-1} X\rangle}{\sqrt{XAX}\left(x_1 + \cdots + x_N + \sqrt{XA\,X}\right)^{N-1}}$$

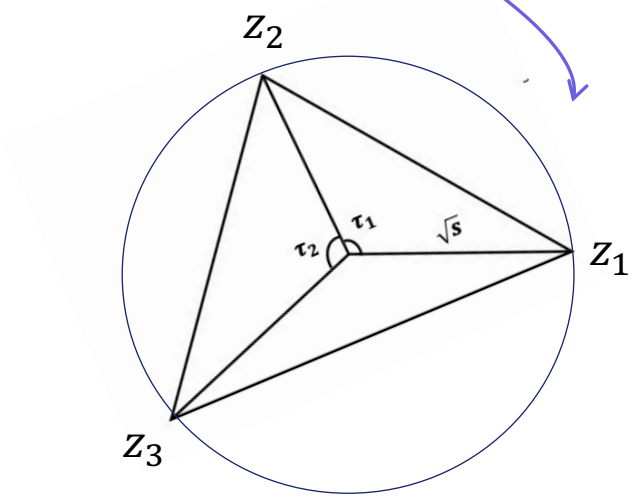
$$\int d\mathrm{PS}_N \to \int \frac{\langle X d^{N-1} X\rangle}{(x_1 + x_2 + x_3)^N}$$

## Integrand for the $ENC$

$(N+1)$-point form factor squared $|F_{N+1}(\phi^2)|$, symmetrized, summed over onshell super states




$$|F_4|^2_{\text{sym.}} = \frac{q^4}{s_{12}s_{23}s_{34}s_{41}}\left[\frac{1}{4} + 2\frac{s_{23}s_{41}}{s_{341}s_{412}} + \frac{s_{23}s_{34}}{s_{123}s_{341}}\right] + \text{perm}(1,2,3,4).$$



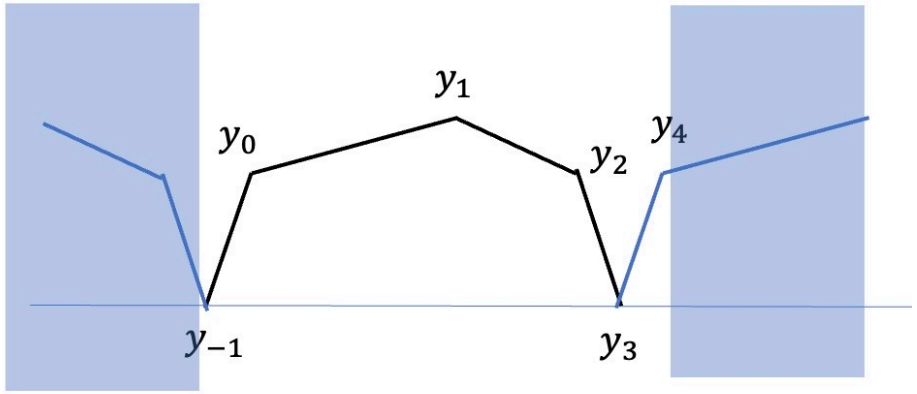

$$s_{I\ldots J} := \sum_{(i,j)\in[I,J]} \omega_i\omega_j\zeta_{ij}\,, I,J \in [1,N] \qquad \zeta_{IJ} := \frac{|z_{IJ}|^2}{(1+|z_I|^2)(1+|z_J|^2)},$$

$$s_{2\ldots N+1} = 1-\omega_1, \qquad s_{N+1,1} = \omega_1(1-\sum \omega_i\zeta_{1i})$$

Multi-collinear limit, $\zeta_{IJ} = |z_{IJ}|^2 \ll 1$, all denominators have homogenous degree 2 or 1.

Integrand for the $ENC$ in multi-collinear limit of : $1 \to N$ Splitting function

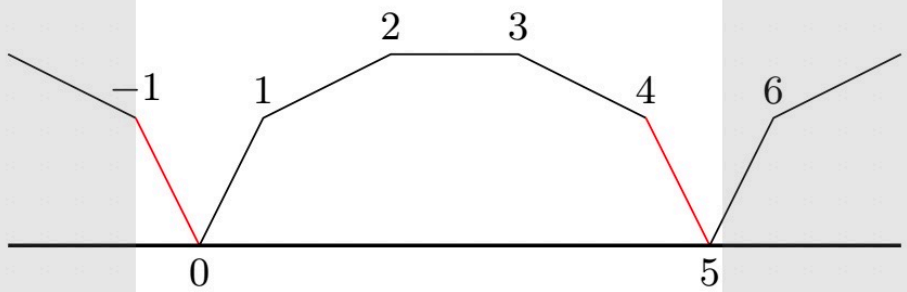
$$G_N = \lim_{(1..N)-coll} \frac{FF^{N+1}_{NMHV}}{FF^{N+1}_{MHV}}$$




$p_i = y_i - y_{i-1}$, $p_{i+4} := p_i$, $s_{i+1\ldots k} = y^2_{ik}$

$N = 3$: $(-1,1,2,4) + (-1,3,2,0) + (3,1,0,4)$

$$(a,b,c,d) \equiv \frac{y^2_{ab}y^2_{cd}}{y^2_{ac}y^2_{bd}}$$




$G_N$ is obtained up to $N = 11$ in 2408.04222

$y^2_{iN} \to x_{i+1} + \cdots + x_{N-1}$

$y^2_{-1i} \to x_1 + \cdots + x_i$

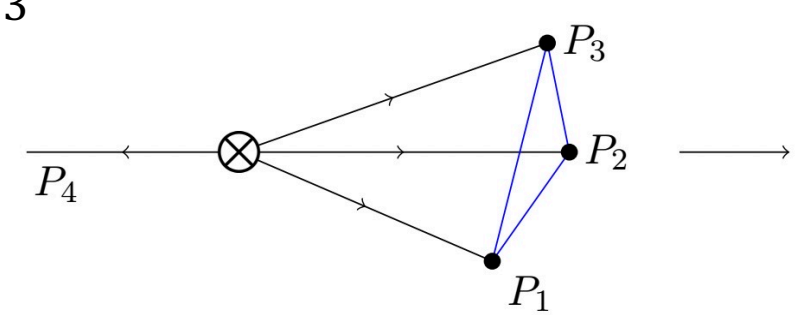
$$y^2_{I-1,k} = s_{I\ldots J} := \sum_{(i,j)\in[I,J]} x_i x_j |z_{ij}|^2$$

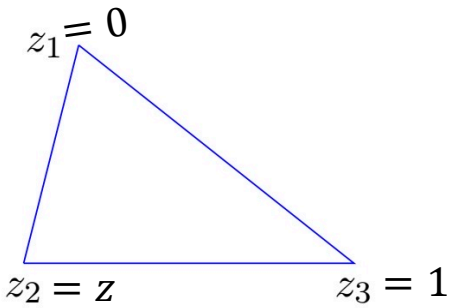
$N = 4$:

$$\begin{aligned}
&-1 + (-1,1,2,5) + (-1,2,3,5) + (-1,4,3,0) + (4,1,0,5) + (-1,3,2,0) + (4,2,1,5) \\
&+ (0,4,3,1) + (0,4,3,1)(-1,1,3,5) + (-1,4,3,1)(3,1,0,5) \\
&+ (-1,4,2,0)(0,2,3,5) + (-1,1,2,4)(4,2,0,5) + (-1,3,2,0)(4,2,0,5) + (-1,4,2,0)(4,2,1,5) \\
&+ (-1,4,3,0)(-1,1,2,4) + (4,1,0,5)(0,2,3,5) + (-1,4,3,1)(-1,4,2,0) + (3,1,0,5)(4,2,0,5) \\
&+ (-1,4,3,1)(-1,1,2,5) + (3,1,0,5)(-1,2,3,5) + (-1,1,2,4)(-1,1,3,5) + (0,2,3,5)(-1,1,3,5)
\end{aligned}$$

Single-valued function of coordinates $(z_i, \bar{z}_i)$ on the celestial sphere

$N = 3$




$G(z)$ @ $LO$ [1912.11050 ]

$B_4 = \ln v,\ B_3 = \ln u,\quad B_2 = \Phi_2(z)$

$B_1 = Li_2(1-u) + \frac{1}{2}\ln u\ \ln v$

$B_5 = \zeta_2 \qquad B_6 = 1$

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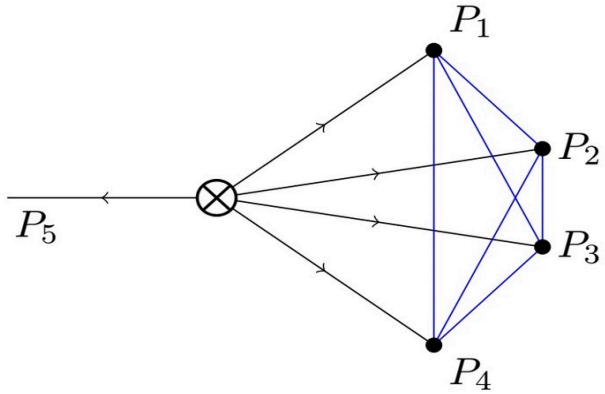
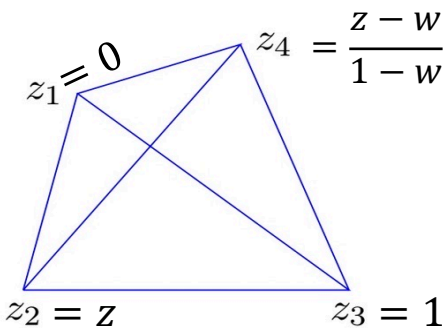
$N = 4$



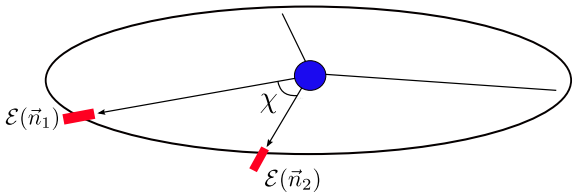




$\mathcal{G}(z,w)$ @ $LO$

41 $max.weight$ 3 $polylogarithms$

*Chicherin,Moult, Sokatchev,Yan,Zhu, Phys.Rev.D* 110 (2024) 9, L091901

$\langle E(n_1)E(n_2)\rangle$




$\mathcal{F}\left(\zeta = \sin^2\frac{\chi}{2}\right)$ @LO

$\ln(1-\zeta)$

@NLO

$Li_3@\{\zeta, \frac{\sqrt{\zeta}}{1\pm\sqrt{\zeta}}\}$

@NNLO

*Henn, Sokatchev,Yan, Zhiboedov, Phys.Rev.D 100 (2019) 3, 036010*

$MPL + Elliptic$

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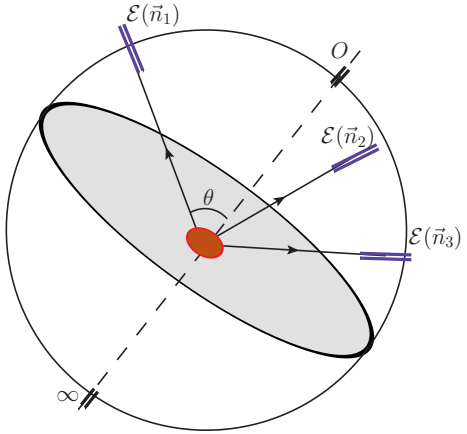
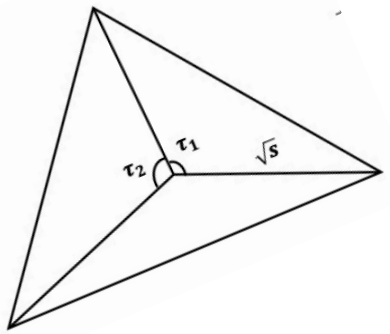
$\langle E(n_1)E(n_2)E(n_3)\rangle$




$\mathcal{F}(\zeta_{12}, \zeta_{13}, \zeta_{23})$ @LO

*Yan,Zhang, Phys.Rev.Lett.* 129 (2022) 2, 021602

$Li_2@\{s, \tau_1, \tau_2, \tau_1\tau_2, s\,\tau_1, \frac{\tau_1}{s}, s\,\tau_2, \frac{\tau_2}{s}, s\,\tau_1\tau_2, \frac{\tau_1\tau_2}{s}$
$\}$

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$\langle E(n_1)E(n_2)E(n_3)E(n_4)\rangle$

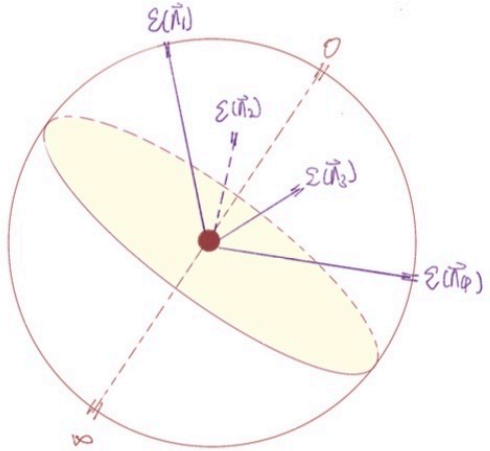
$\mathcal{F}(\{z_3, \bar{z}_3, z_4, \bar{z}_4, |z_2|\})$ @ LO

$$\zeta_{ij} = \frac{|z_i - z_j|^2}{(1+|z_i|^2)(1+|z_j|^2)} \qquad z_1 = 0,\ \bar{z}_2 = z_2$$

141 master integrals

$MPL$
$+ 3\ types\ of\ Elliptic\ curves$

*Lin,Ma,Gong,Yan,Zhang,To Appear*



# This talk

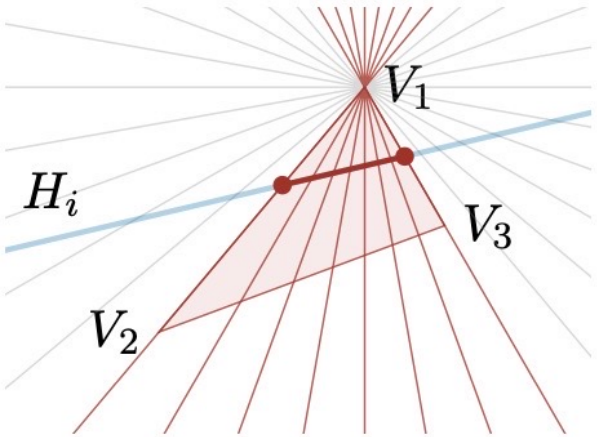
Simplex contour | quadric and linear singularity surfaces

Integration-by-parts algorithms + differential equations for finite integrals in feynman parameter space

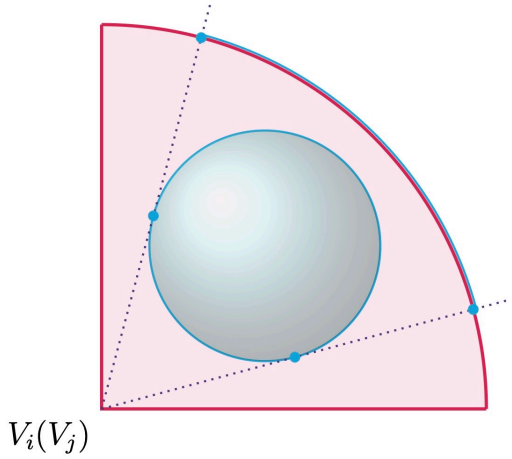
Apply to both calculations in multi-collinear limit and at generic scattering angle

$$E^N C\Big|_{coll,.} = c_i(z_i, \bar{z}_i) * B_i$$

A shortcut for determine the rational coefficients




Highest weight from sequential discontinuity (spherical contour algorithm)




Bootstrapping Lower Weight Functions From Cancellation of Spurious Poles

[Arkani-Hamed,Yuan,1712.09991.]
[Gong, Yuan, 2206.06507]

# Finite IBP+ DE

In collaboration with J.Lin, R.Ma, J.Gong, Y.Zhang

# Energy integration over splitting function

$$\mathrm{E^N C} \overset{\text{coll.}}{=} \frac{1}{|z_{12}\cdots z_{N-1N}|^2}\int \frac{d^N x}{\mathrm{GL}(1)}(x_1+\cdots+x_N)^{-N}\mathcal{G}_N + \text{perm}\,(z_1,\cdots,z_N)$$

$N=3$: $(3,1,0,4)=\frac{s_{23}x_{123}}{s_{123}x_{23}}$ $(-1,3,2,0)=\frac{s_{12}x_{123}}{s_{123}x_{12}}$ $(-1,1,2,4)=\frac{x_1x_{123}}{x_{12}x_{23}}$

$G_N$ contains only multi-particle poles, no two-particle pole

$$s_{I\ldots J} := \sum_{(i,j)\in[I,J]} x_i x_j |z_{ij}|^2 \qquad x_{I,\ldots,J} := x_I+\cdots+x_J$$

# N-point Integral Family

$N = 3$ :

$$B_{a_1,\cdots,a_6} \equiv \int \frac{d^3x}{\mathrm{GL}(1)} \frac{s_{12}^{-a_5}\, s_{23}^{-a_6}}{s_{123}{}^{a_1}\, x_{12}^{a_2}\, x_{23}^{a_3}\, x_{123}^{a_4}}$$

$N = 4$ :

$$A_{a_1,\cdots,a_{10}} \equiv \int \frac{d^4x}{\mathrm{GL}(1)} \frac{1}{\prod_i D_i^{a_i}}$$

$$D_1 = s_{1234}, \quad D_2 = s_{123}, \quad D_3 = s_{234},$$

$$D_4 = x_{1234}, \quad D_5 = x_{234}, \quad D_6 = x_{123}, \quad D_7 = x_{34},$$

$$D_8 = s_{12}, \quad D_9 = s_{23}, \quad D_{10} = s_{34}\,.$$

$D_{1-7}$ are multi-particle poles corresponding to physical singularities

# IBP identities for projective integrals

$$O_i = \frac{\partial}{\partial x_i} v \,,\; i = 1, \cdots, N$$

Differential operators acting on projective variables

$$\int \frac{d^N x}{\mathrm{GL}(1)} O_i \circ f = - \int \frac{d^{N-1} x}{\mathrm{GL}(1)} v \circ f \Big|_{x_i = 0}$$

$$v = \prod_k x_k^{-q_k} \,, \quad f = \frac{1}{\prod_j D_j^{a_j}} \,.$$

**1. Homogeneity:** integrand has the overall scaling dimension $-N$

**2. Finiteness**: free from IR divergences as any subset of energy variables go to zero.

$\boldsymbol{O_i \circ f}$ **must satisfy the power-counting** ***condition 1*** **and** ***condition 2***

**Boundary integrals on r.h.s automatically satisfy** ***condition 1*** **and** ***condition 2***

$$O_i \to \kappa^{\beta_S} O_i \,,\; \forall S \subset \{x_1, x_2, x_3, x_4\}$$

# IBP identities for projective integrals

$$O_{IBP} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (a_i \cdot \ )$$

$$\boxed{\sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0, \qquad j \in A}$$

$$a_i,\ b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3].$$

By solving syzygy equations, obtain IBP operators that do not increase the power of propagators that could vanish.

**Pick $O_i \circ f$ that satisfy *condition 1* and *condition 2***

# IBP identities for projective integrals

$$\int \frac{d^N x}{\mathrm{GL}(1)} O_i \circ f = -\int \frac{d^{N-1}x}{\mathrm{GL}(1)} v \circ f \Big|_{x_i=0}$$

$$v = \prod_k x_k^{-q_k}, \quad f = \frac{1}{\prod_j D_j^{a_j}}.$$

Expanded over $A_{a,\ldots a_7;q_1..q_4}$
Each term is finite

boundary terms are generated on the surface of integration domain

$$B^{[j,k,l]}_{a_1,a_2,a_3,a_4;\ q_1,q_2,q_3} \equiv \int \frac{d^3x}{\mathrm{GL}(1)} \frac{x_j^{-q_1} x_k^{-q_2} x_l^{-q_3}}{s_{jkl}^{a_1}\, x_{jk}^{a_2}\, x_{kl}^{a_3}\, x_{jkl}^{a_4}}$$

Boundary integral family: lower-point integrals defined in $[x_1, \ldots \widehat{x_i}, \ldots, x_N]$

# Lift to differential equations

$$O_{IBP} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (a_i \cdot \ )$$

$$O_{\partial \zeta_{**}} = \frac{\partial}{\partial \zeta_{**}} + O_{IBP}$$

Differential operators acting on angle parameters

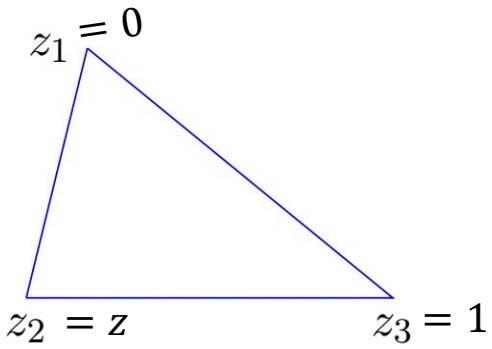
$$\frac{\partial}{\partial \zeta_{**}} D_j + \sum_{i=1}^{3} a_i \frac{\partial}{\partial x_i} D_j - b_j D_j = 0, \quad j \in \{1, \delta\}$$

$$a_i,\ b_j \in Q(\zeta_{12}, \zeta_{23}, \zeta_{13})[x_1, x_2, x_3]$$

$$\partial_{\zeta_{**}} f = \mathrm{B}_{**}.f + \text{ boundary integrals}$$

## $E^3C$ master integrals

$$B_{a_1,0,a_3,a_4,q_1,q_2,q_3} = \int \frac{d^3x}{GL(1)} \frac{x_1^{-q_1} x_2^{-q_2} x_3^{-q_3}}{s_{123}^{a_1} x_{23}^{a_3} x_{123}^{a_4}}$$




$$B_1: \frac{1}{s_{123} x_{123}} \qquad B_2: \frac{x_2}{s_{123} x_{23} x_{123}}$$

$$B_3: \frac{x_2}{s_{123} x_{123}^2} \qquad B_4: \frac{x_3}{s_{123} x_{123}^2}$$

$$d\,\vec{g} = dA\,\vec{g}$$

$$\begin{pmatrix} dlog\left(\frac{1-z}{1-\bar{z}}\right) & dlog\left(\frac{z}{\bar{z}}\right) & \\ dlog\left(\frac{|1-z|^2}{1-|z|^2}\right) & dlog|z|^2 & \\ & & dlog|z|^2 \\ & & dlog|1-z|^2 \end{pmatrix}$$

$C_1$: $\boxed{1}$

Boundary terms are kinematic independent two-point integrals integrating to rational numbers.

# Master Integrals of Three and Four-Point Any Angle

## Three Points

Into 3 families

**Propagators**

family 1 $\quad D_1 = -s_{134},\ D_2 = -s_{124},\ D_3 = -s_{123},$

family 2 $\quad D_1 = -s_{134},\ D_2 = -s_{123},\ D_3 = \frac{s_{24}}{x_2},$

family 3 $\quad D_1 = -s_{124},\ D_2 = \frac{s_{34}}{x_3},\ D_3 = \frac{s_{24}}{x_2}.$

$$D_\delta = 1 - x_{123} - s_{123}$$

subfamily 1 $\quad D_1 = \frac{s_{24}}{x_2}|_{x_3\to 0},\ D_2 = \frac{s_{34}}{x_3}|_{x_3\to 0},\ D_\delta = (1 - x_{123} - s_{123})|_{x_3\to 0}\,,$

subfamily 2 $\quad D_1 = \frac{s_{14}}{x_1}|_{x_2\to 0},\ D_2 = \frac{s_{34}}{x_3}|_{x_2\to 0},\ D_\delta = (1 - x_{123} - s_{123})|_{x_2\to 0}\,,$

subfamily 3 $\quad D_1 = \frac{s_{24}}{x_2}|_{x_1\to 0},\ D_2 = \frac{s_{34}}{x_3}|_{x_1\to 0},\ D_\delta = (1 - x_{123} - s_{123})|_{x_1\to 0}\,.$

**Master Integrals**

Family 1

$$\left\{\frac{x_2}{D_2\,D_3\,D_4},\ \frac{D_1}{D_3\,D_4},\ \frac{D_2}{D_3\,D_4},\ \frac{1}{D_3\,D_4},\ \frac{1}{D_2\,D_4},\ \frac{D_2}{D_1\,D_4},\ \frac{1}{D_1\,D_4},\ \frac{1}{D_1\,D_2\,D_4},\ \frac{1}{D_{\text{sub1},3}},\ \frac{1}{D_{\text{sub2},3}},\ \frac{1}{D_{\text{sub3},3}},\ 1\right\}$$

Family 2

$$\left\{\frac{D_2}{D_1\,D_3\,D_4},\ \frac{D_1}{D_2\,D_3\,D_4},\ \frac{1}{D_1\,D_3\,D_4},\ \frac{1}{D_2\,D_3\,D_4},\ \frac{D_3}{D_2\,D_4},\ \frac{D_1}{D_2\,D_4},\ \frac{1}{D_1\,D_4},\ \frac{1}{D_2\,D_4},\ \frac{1}{D_{\text{sub1},3}},\ \frac{1}{D_{\text{sub2},3}},\ \frac{1}{D_{\text{sub2},2}\,D_{\text{sub2},3}},\ \frac{1}{D_{\text{sub3},3}},\ 1\right\}$$

Family 3

$$\left\{\frac{D_3}{D_1\,D_2\,D_4},\ \frac{D_1}{D_2\,D_3\,D_4},\ \frac{1}{D_1\,D_2\,D_4},\ \frac{1}{D_2\,D_3\,D_4},\ \frac{1}{D_1\,D_4},\ \frac{1}{D_{\text{sub1},3}},\ \frac{1}{D_{\text{sub1},2}\,D_{\text{sub1},3}},\ \frac{1}{D_{\text{sub2},3}},\ \frac{1}{D_{\text{sub3},3}},\ 1\right\}$$

Also depends on subfamily integrals!
The subscripts describes which subfamily the propagtor belongs to.

## Four Points

13 propagators, 141 master integrals

$$D_1 = \frac{s_{45}}{x_4},\quad D_2 = s_{123},\quad D_3 = \frac{s_{15}}{x_1},\quad D_4 = -s_{125},\quad D_5 = -s_{1234},$$

$$D_6 = -s_{1235},\quad D_7 = -s_{145},\quad D_8 = -s_{1245},\quad D_9 = -s_{345},\quad D_{10} = -s_{1345},$$

$$D_{11} = -s_{2345},\quad D_{12} = s_{234},\quad D_{13} = 1 - x_{1234} - s_{1234}$$

$$\left\{\frac{1}{D_9\,D_{13}},\ \frac{1}{D_7\,D_{12}\,D_{13}},\ \frac{x_1}{D_7\,D_{12}\,D_{13}},\ \frac{x_2}{D_7\,D_{12}\,D_{13}},\ \frac{x_2^2}{D_7\,D_{12}\,D_{13}},\ \frac{x_3^2}{D_7\,D_{12}\,D_{13}},\ \frac{x_4}{D_7\,D_{12}\,D_{13}},\ \frac{x_2\,x_4}{D_7\,D_{12}\,D_{13}},\ \frac{1}{D_6\,D_{10}\,D_{13}},\ \frac{1}{D_6\,D_7\,D_{13}},\ \frac{x_2}{D_6\,D_7\,D_{13}},\ \frac{x_3}{D_6\,D_7\,D_{13}},\ \frac{1}{D_5\,D_{11}\,D_{13}},\ \frac{1}{D_5\,D_7\,D_{13}},\right.$$

$$\frac{x_1}{D_5\,D_7\,D_{13}},\ \frac{x_1^2}{D_5\,D_7\,D_{13}},\ \frac{x_2}{D_5\,D_7\,D_{13}},\ \frac{x_3}{D_5\,D_7\,D_{13}},\ \frac{1}{D_5\,D_6\,D_{13}},\ \frac{x_2}{D_5\,D_6\,D_{12}\,D_{13}},\ \frac{x_1\,x_2}{D_5\,D_6\,D_{12}\,D_{13}},\ \frac{x_3}{D_5\,D_6\,D_{12}\,D_{13}},\ \frac{1}{D_4\,D_{10}\,D_{13}},\ \frac{x_3}{D_4\,D_{10}\,D_{13}},\ \frac{x_4}{D_4\,D_{10}\,D_{13}},\ \frac{1}{D_4\,D_9\,D_{13}},$$

$$\frac{x_1}{D_4\,D_9\,D_{13}},\ \frac{x_1^2}{D_4\,D_9\,D_{13}},\ \frac{x_2}{D_4\,D_9\,D_{13}},\ \frac{x_3}{D_4\,D_9\,D_{13}},\ \frac{x_1\,x_3}{D_4\,D_9\,D_{13}},\ \frac{x_2\,x_3}{D_4\,D_9\,D_{13}},\ \frac{x_4}{D_4\,D_9\,D_{13}},\ \frac{x_1\,x_4}{D_4\,D_9\,D_{13}},\ \frac{1}{D_4\,D_7\,D_{13}},\ \frac{x_1}{D_4\,D_7\,D_{13}},\ \frac{x_2}{D_4\,D_7\,D_{13}},\ \frac{x_3^2}{D_4\,D_7\,D_{13}},\ \frac{x_4}{D_4\,D_7\,D_{13}},$$

$$\frac{1}{D_4\,D_5\,D_{13}},\ \frac{x_1}{D_4\,D_5\,D_{13}},\ \frac{x_1^2}{D_4\,D_5\,D_{13}},\ \frac{x_2}{D_4\,D_5\,D_{13}},\ \frac{x_3}{D_4\,D_5\,D_{13}},\ \frac{1}{D_3\,D_{13}},\ \frac{1}{D_3\,D_7\,D_{13}},\ \frac{x_4}{D_3\,D_7\,D_{13}},\ \frac{x_4^2}{D_3\,D_7\,D_{13}},\ \frac{1}{D_3\,D_5\,D_{13}},\ \frac{1}{D_3\,D_5\,D_7\,D_{13}},\ \frac{x_3}{D_3\,D_5\,D_7\,D_{13}},\ \frac{x_4}{D_3\,D_5\,D_7\,D_{13}},$$

$$\frac{x_4^2}{D_3\,D_5\,D_7\,D_{13}},\ \frac{1}{D_3\,D_5\,D_6\,D_{13}},\ \frac{x_1}{D_3\,D_5\,D_6\,D_{13}},\ \frac{x_1\,x_2}{D_2\,D_6\,D_{12}\,D_{13}},\ \frac{x_2^2}{D_2\,D_6\,D_{12}\,D_{13}},\ \frac{x_1}{D_2\,D_6\,D_{10}\,D_{13}},\ \frac{x_2}{D_2\,D_6\,D_{10}\,D_{13}},\ \frac{x_1}{D_2\,D_6\,D_7\,D_{13}},\ \frac{x_1^2}{D_2\,D_6\,D_7\,D_{13}},\ \frac{x_2}{D_2\,D_6\,D_7\,D_{13}},\ \frac{x_1\,x_2}{D_2\,D_6\,D_7\,D_{13}},$$

$$\frac{x_1}{D_2\,D_5\,D_7\,D_{13}},\ \frac{x_1^2}{D_2\,D_5\,D_7\,D_{13}},\ \frac{x_1\,x_2}{D_2\,D_5\,D_7\,D_{13}},\ \frac{x_2\,x_4}{D_2\,D_5\,D_7\,D_{13}},\ \frac{x_1^2}{D_2\,D_4\,D_5\,D_{13}},\ \frac{x_1\,x_2}{D_2\,D_4\,D_5\,D_{13}},\ \frac{x_2^2}{D_2\,D_4\,D_5\,D_{13}},\ \frac{x_3}{D_2\,D_3\,D_{13}},\ \frac{x_1\,x_3}{D_2\,D_3\,D_{13}},\ \frac{x_4}{D_2\,D_3\,D_{13}},\ \frac{x_3\,x_4}{D_2\,D_3\,D_{13}},\ \frac{x_4^2}{D_2\,D_3\,D_{13}},$$

$$\frac{1}{D_2\,D_3\,D_{10}\,D_{13}},\ \frac{x_1}{D_2\,D_3\,D_{10}\,D_{13}},\ \frac{1}{D_2\,D_3\,D_7\,D_{13}},\ \frac{x_1}{D_2\,D_3\,D_7\,D_{13}},\ \frac{x_3}{D_2\,D_3\,D_7\,D_{13}},\ \frac{x_4}{D_2\,D_3\,D_7\,D_{13}},\ \frac{x_1}{D_2\,D_3\,D_6\,D_{13}},\ \frac{x_2}{D_2\,D_3\,D_6\,D_{13}},\ \frac{x_1}{D_2\,D_3\,D_5\,D_{13}},\ \frac{x_1^2}{D_2\,D_3\,D_5\,D_{13}},\ \frac{x_2}{D_2\,D_3\,D_5\,D_{13}},$$

$$\frac{x_3}{D_2\,D_3\,D_5\,D_{13}},\ \frac{x_1}{D_2\,D_3\,D_4\,D_{13}},\ \frac{x_2}{D_2\,D_3\,D_4\,D_{13}},\ \frac{x_3}{D_2\,D_3\,D_4\,D_{13}},\ \frac{x_3^2}{D_2\,D_3\,D_4\,D_{13}},\ \frac{1}{D_1\,D_7\,D_{12}\,D_{13}},\ \frac{x_1}{D_1\,D_7\,D_{12}\,D_{13}},\ \frac{x_2}{D_1\,D_7\,D_{12}\,D_{13}},\ \frac{x_4}{D_1\,D_7\,D_{12}\,D_{13}},\ \frac{1}{D_1\,D_6\,D_{13}},\ \frac{1}{D_1\,D_6\,D_{10}\,D_{13}},$$

$$\frac{x_1}{D_1\,D_6\,D_{10}\,D_{13}},\ \frac{1}{D_1\,D_6\,D_7\,D_{13}},\ \frac{x_1}{D_1\,D_6\,D_7\,D_{13}},\ \frac{x_1^2}{D_1\,D_6\,D_7\,D_{13}},\ \frac{x_2}{D_1\,D_6\,D_7\,D_{13}},\ \frac{1}{D_1\,D_5\,D_{13}},\ \frac{x_2}{D_1\,D_5\,D_{12}\,D_{13}},\ \frac{x_1\,x_2}{D_1\,D_5\,D_{12}\,D_{13}},\ \frac{x_3}{D_1\,D_5\,D_{12}\,D_{13}},\ \frac{x_4}{D_1\,D_5\,D_{12}\,D_{13}},\ \frac{1}{D_1\,D_5\,D_7\,D_{13}},$$

$$\frac{x_1}{D_1\,D_5\,D_7\,D_{13}},\ \frac{x_1^2}{D_1\,D_5\,D_7\,D_{13}},\ \frac{x_2}{D_1\,D_5\,D_7\,D_{13}},\ \frac{1}{D_1\,D_4\,D_{13}},\ \frac{x_1}{D_1\,D_4\,D_{13}},\ \frac{x_2}{D_1\,D_4\,D_{13}},\ \frac{1}{D_1\,D_4\,D_{10}\,D_{13}},\ \frac{x_1}{D_1\,D_4\,D_{10}\,D_{13}},\ \frac{1}{D_1\,D_4\,D_9\,D_{13}},\ \frac{x_1}{D_1\,D_4\,D_9\,D_{13}},\ \frac{x_2}{D_1\,D_4\,D_9\,D_{13}},$$

$$\frac{x_4}{D_1\,D_4\,D_9\,D_{13}},\ \frac{1}{D_1\,D_4\,D_7\,D_{13}},\ \frac{x_1}{D_1\,D_4\,D_7\,D_{13}},\ \frac{x_2}{D_1\,D_4\,D_7\,D_{13}},\ \frac{x_4}{D_1\,D_4\,D_7\,D_{13}},\ \frac{1}{D_1\,D_3\,D_{13}},\ \frac{1}{D_1\,D_3\,D_{10}\,D_{13}},\ \frac{x_1}{D_1\,D_3\,D_{10}\,D_{13}},\ \frac{1}{D_1\,D_3\,D_6\,D_{13}},\ \frac{x_1}{D_1\,D_3\,D_6\,D_{13}},\ \frac{1}{D_1\,D_3\,D_4\,D_{13}},$$

$$\left.\frac{x_1}{D_1\,D_3\,D_4\,D_{13}},\ \frac{x_1^2}{D_1\,D_3\,D_4\,D_{13}},\ \frac{x_2}{D_1\,D_3\,D_4\,D_{13}},\ \frac{x_2}{D_1\,D_2\,D_{12}\,D_{13}},\ \frac{x_3}{D_1\,D_2\,D_{12}\,D_{13}},\ \frac{x_1}{D_1\,D_2\,D_6\,D_{13}},\ \frac{x_2}{D_1\,D_2\,D_6\,D_{13}},\ \frac{x_1}{D_1\,D_2\,D_4\,D_{13}},\ \frac{x_2}{D_1\,D_2\,D_4\,D_{13}},\ \frac{1}{D_1\,D_2\,D_3\,D_{13}},\ \frac{x_1}{D_1\,D_2\,D_3\,D_{13}}\right\}$$

# Analytic Results of Master Integrals

## Three Points

All Master Integrals are MPLs

### Symbol Letters

Family 3 for instance

$$|\Delta_1| \equiv \left|\left|\vec{n}_1 \wedge \vec{n}_2 \wedge \vec{n}_3\right|\right| = \left|\sqrt{(1-u_1-u_2-u_3)^2 - 4u_1u_2u_3}\right|$$

$$|\Delta_2| \equiv \left|\left|\vec{n}_1 \wedge \vec{n}_2 + \vec{n}_2 \wedge \vec{n}_3 + \vec{n}_3 \wedge \vec{n}_1\right|\right| = \left|\sqrt{\lambda(\zeta_{12}, \zeta_{23}, \zeta_{31})}\right|$$

## Four Points

Most of Integrals are Elliptic Integrals

Some cannot be integrated directly

Symbol Integration

### MPLs Propagators

$$\frac{1}{D_3\, D_7\, D_{13}}, \quad \frac{1}{D_1\, D_3\, D_4\, D_{13}}, \quad \frac{1}{D_1\, D_3\, D_6\, D_{13}}, \quad \frac{1}{D_1\, D_3\, D_{10}\, D_{13}}$$

$$\frac{1}{D_4\, D_7\, D_{13}}, \quad \frac{1}{D_4 D_{10} D_{13}}, \quad \frac{1}{D_6 D_{10} D_{13}}$$

### Elliptics Origination

Subscipts indicate the highest power of the polynomial

$$\int \mathrm{d}x \frac{\log\left(P_1(x) + \sqrt{P_2(x)}\right) \log\left(Q_1(x) + \sqrt{Q_2(x)}\right)}{\sqrt{Q_2(x)}}$$

From a quadric delta function

$$\int \mathrm{d}x\mathrm{d}y \frac{\log\left(P_1(x,y) + \sqrt{Q_2(x,y)}\right)}{\sqrt{Q_2(x,y)}}$$

$Q_2$ degree 2 polynomials both in x and y

In two dim integrals, square roots emerge. Quadric of two variables inside.

1. Propagator $D_2$ and $D_{12}$ separately
2. Combinations of $D_4 D_9 = s_{125} s_{345}$
3. Propagator $D_5$

From Maximal Cut

$$\int \frac{N(x)\mathrm{d}x}{\sqrt{P_4(x)}}$$

$P_4$ degree 4 polynomials

# A bootstrap approach

In collaboration with J.Gong, A.Pokraka, H.Hannesdottir, X.Zhang

# $E^3C$ mapped onto one-loop Feynman integrals

$$\mathrm{E^NC} \overset{\text{coll.}}{=} \frac{1}{|z_{12}\cdots z_{N-1N}|^2}\int \frac{d^N x}{\mathrm{GL}(1)}(x_1+\cdots+x_N)^{-N}\mathcal{G}_N + \text{perm}\,(z_1,\cdots,z_N)$$

$$s_{I\ldots J} := \sum_{(i,j)\in[I,J]} x_i x_j |z_{ij}|^2$$

$$x_{I,\ldots,J} := x_I + \cdots + x_J$$

$N = 3$:

$$(3,1,0,4) = \frac{s_{23}x_{123}}{s_{123}x_{23}} \qquad (-1,3,2,0) = \frac{s_{12}x_{123}}{s_{123}x_{12}} \qquad (-1,1,2,4) = \frac{x_1 x_{123}}{x_{12}x_{23}}$$

Introduce two more variables, combine into a single quadric $Q_5 = s_{123} + x_5 x_{23} + x_4 x_{123}$

$$u = \frac{|z_{12}|^2}{|z_{23}|^2} \qquad v = \frac{|z_{13}|^2}{|z_{23}|^2}$$

$$Q_5 = \begin{vmatrix} 0 & u & v & 1 & 0 \\ u & 0 & 1 & 1 & 1 \\ v & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{vmatrix}$$

$$E_{5,3} = \frac{\langle X d^4 X\rangle T[X^3]}{(XQ_5X)^{\frac{5+3}{2}}} \longrightarrow \partial_{u,v}E_{5,1} \longrightarrow \partial_{u,v}\circ d_X \circ E_{5,0}$$

$$\partial_{u,v}\circ d_X \circ E_{5,0} \longrightarrow \partial_{u,v}\circ E_{4,0}$$

Pentagon integral with
Rank-3 tensor numerator

4d scalar box integrals

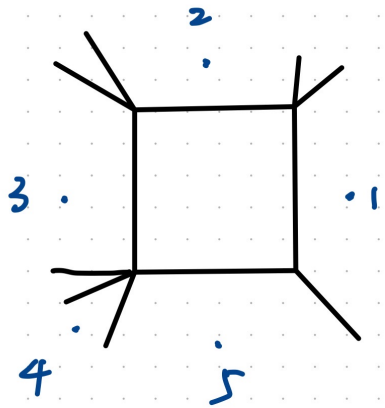
A. Nima, E.Y. Yuan 17'

$$E^3C = \frac{1}{|z_{12}|^2|z_{23}|^2} \times$$

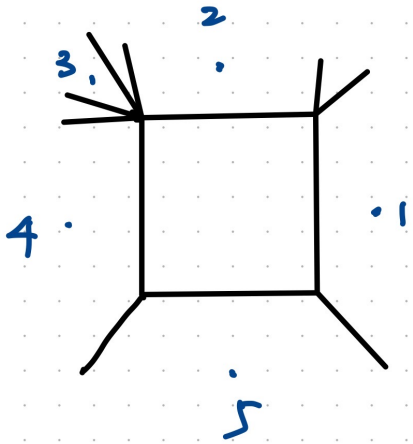
$$I = \int_\Delta \frac{\langle X \mathrm{d}X^{n-1}\rangle \sqrt{-\det Q}}{(XQX)^{\frac{n}{2}}}$$

$$n = 4, Q = Q_5^{(i)}, i = 2,3,4,5$$

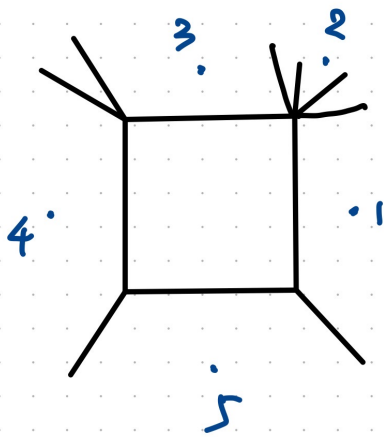
$$-(u\,\partial_u + v\,\partial_v\,) \circ \Bigg[ \frac{u-v}{2}$$



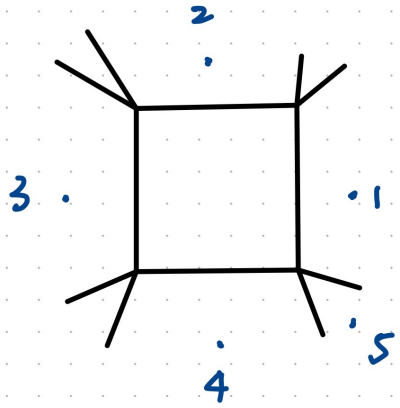

$$+\frac{v-u}{2}$$




$$+\frac{u-v}{2}$$

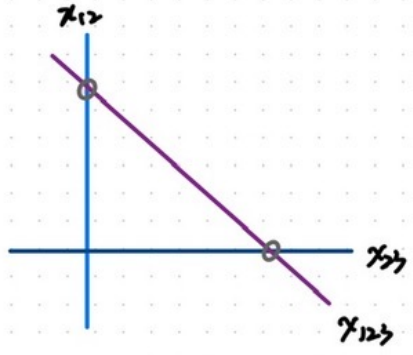



$$+\frac{-u-v+u^2+v^2-2uv}{2\sqrt{\Delta(u,v)}}$$

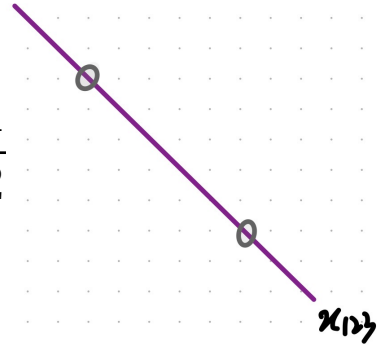



$$\Bigg]$$

$$+\ \frac{1}{2}$$




$$+\ \frac{1}{2}$$

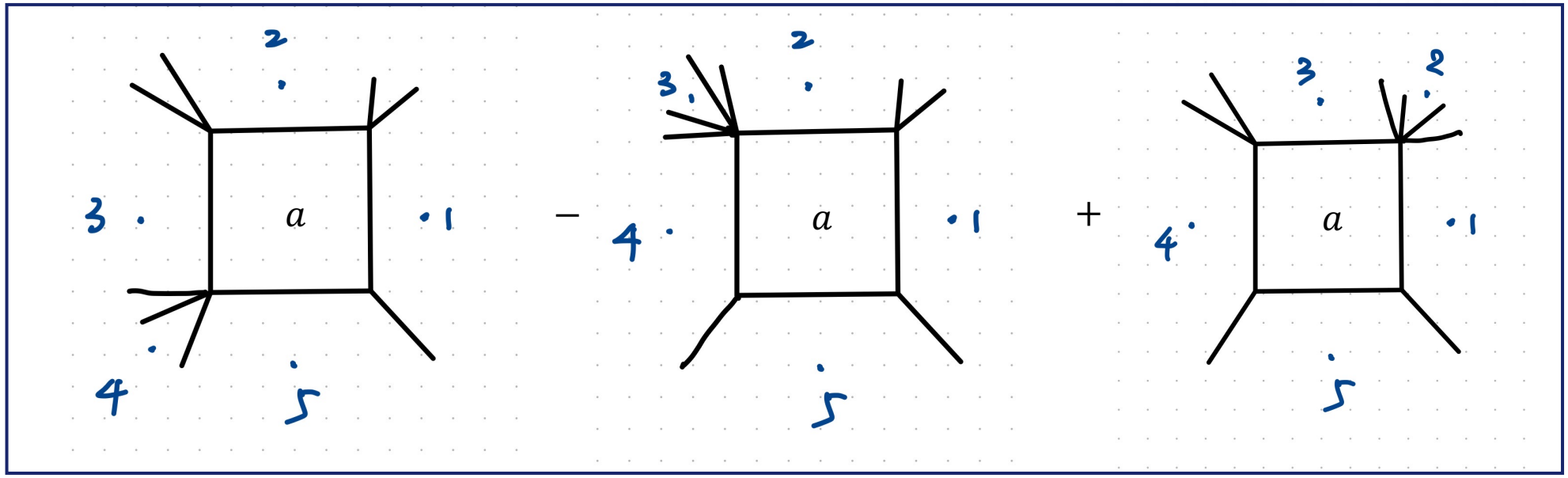



$+\ permutations$

$$E^3 C = \sum c_i(u,v)\, B_i$$

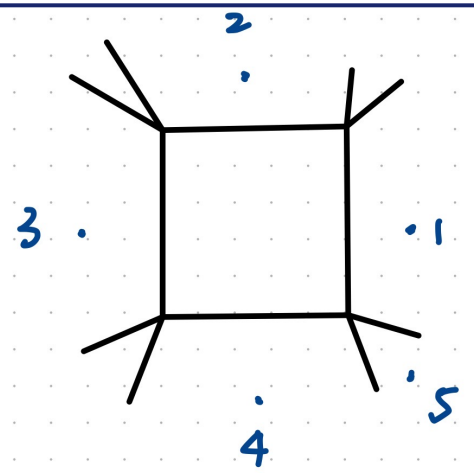
$B_i$: UT master integrals

$B_1$




$$Li_2\left(1-\frac{v}{u}\right)+\frac{1}{2}\ln\frac{v}{u}\ln\frac{1}{u}$$

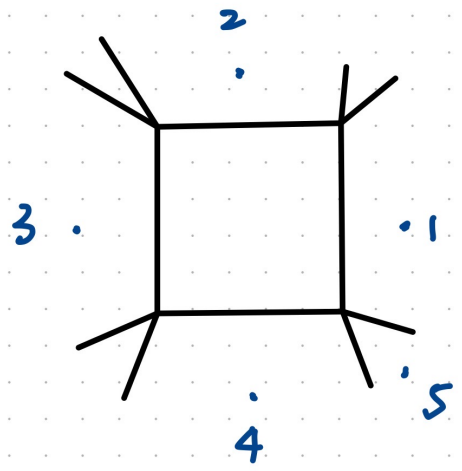
$B_2$



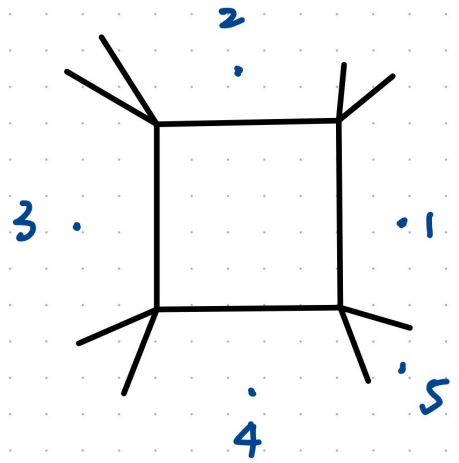

$\Phi_2(z,\bar{z})$

$B_{3,4}$

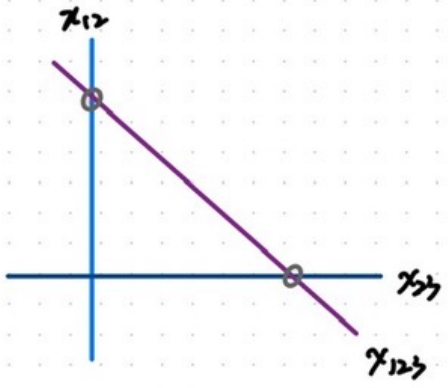
$\partial_u \circ$



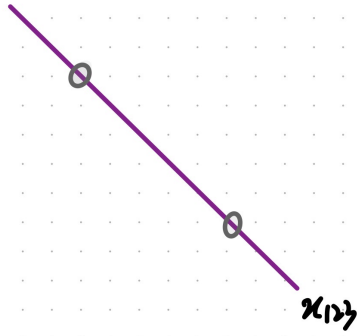

$\partial_v \circ$




$B_{5,6}$

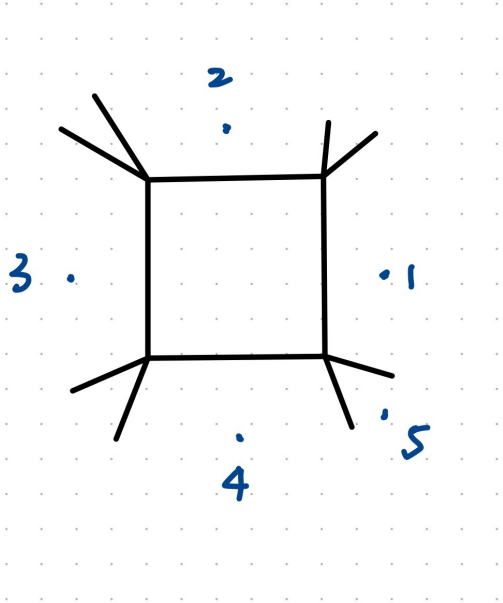



$$\int \frac{1}{x_{12}x_{23}x_{123}} = \zeta_2$$




$$\int \frac{2}{x_{123}^3} = 1$$

$c_2(\mathrm{u},\mathrm{v})$




$$\frac{\Phi_2(z)}{(z-\bar{z})^3} + \ \ldots$$

Hidden relation between highest-weight and lower weight coefficients

e.g. Spurious pole at $z = zb$ must cancel

$$E^3 C = \sum c_i(u,v)\, B_i$$

Compute the symbol of highest weight functions via taking sequential discontinuities --> fix all highest-weight coefficients

Impose physical constraints --> fix all lower-weight coefficients

# Bootstrapping: Algorithms Overview

Obtain the Integrands

↓

Using Feynman Parametrization to Combine All the Propagators

↓

Spherical Contours for Highest Weight

↓

Bootstrapping Lower Weight Functions From Cancellation of Spurious Poles

# Bootstrapping: Integrands

- Three Point EEC, We need Splitting Functions

$$\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon} \frac{4g_s^4}{s_{123}^2} P_{1\to 3}(z_1, z_2, z_3)$$

- N =4 SYM,

$$P_{1\to 3}(z_1, z_2, z_3) = N_c^2 \left[\frac{s_{123}^2}{2s_{13}s_{23}}\left(\frac{1}{z_1 z_2} + \frac{1}{(1-z_1)(1-z_2)}\right) + \frac{s_{123}}{s_{12}z_3}\left(\frac{1}{z_1} + \frac{1}{1-z_1}\right) + \text{perms}\right]$$

- QCD, quark jet for instance

J. M. Campbell and E. W. N. Glover 97'
S. Catani and M. Grazzini 98'

$$P_{\bar{q}q'q} = C_F T_F \frac{s_{123}}{2s_{12}}\left[-\frac{\left[z_1\left(s_{12}+2s_{23}\right) - z_2\left(s_{12}+2s_{13}\right)\right]^2}{\left(z_1+z_2\right)^2 s_{12}s_{123}} + \frac{4z_3 + \left(z_1 - z_2\right)^2}{z_1+z_2} + (1-2\epsilon)\left(z_1 + z_2 - \frac{s_{12}}{s_{123}}\right)\right]$$

# Rewrite into Projective Space

- Purpose: One-denominator quadric singularity to perform Spherical Contour
- Match the homogenity, into projective measure

$$\int_\Delta \frac{\langle X \mathrm{d}X^4\rangle s_{12}x_3}{(s_{123}(X)+x_4(x_5+x_2+x_3)+x_4(x_1+x_2))^4} = \int_\Delta \frac{\langle X \mathrm{d}X^4\rangle N\left[X^3\right]}{(XQX)^4}$$

Where Q is a 4x4 matrix

with the volume element $\langle X \mathrm{d}X^4\rangle = \frac{1}{5}\epsilon_{I_1\cdots I_5}X^{I_1}\mathrm{d}X^{I_2}\wedge\cdots\wedge\cdots\mathrm{d}X^{I_5}$

- Requirement: Quadric Singularities!

Feyman Parameters of quadric term should be set to 1!

# Feynman Parametrization

- Generally (used in four points bootstrapping or higher)

**projective integrals**

$$\frac{1}{D_1^{c_1} D_2^{c_2} \cdots D_k^{c_k}} = \frac{\Gamma(c_1 + c_2 + \cdots + c_k)}{\Gamma(c_1)\Gamma(c_2)\cdots\Gamma(c_k)} \int_0^1 \mathrm{d}a_1 \int_0^1 \mathrm{d}a_2 \cdots \int_0^1 \mathrm{d}a_k \delta(1 - \sum_{i=1}^{k} a_k) \frac{a_1^{c_1-1} a_2^{c_2-1} \cdots a_k^{c_k-1}}{(a_1 D_1 + a_2 D_2 + \cdots + a_k D_k)^{\sum_i c_i}}$$

- For N=4, at most seven integration parameters.

$$(t) s_{1234} + (1-t)\ s_{123} = s_{123} + t(s_{14} + s_{24} + s_{34})$$

→ max. weight-3, 6d hexagon

$$\downarrow\ tx_4 \to x_{4'}, x_4 \to x_5 + x_{4'}$$

$$s_{1234'}$$

Cannot combine $s_{123}, s_{234}$ without encountering cubic curve, keep t as a parameter

$$\int \frac{d^4 x}{GL(1)} \frac{x_2}{s_{123} s_{234} x_{1234}} \longrightarrow \int dt\, S(t,z,w) Li_2(z,w,t)$$

# Highest Weight: Spherical Contours

- Algorithms of Spherical Contour

$$I = \int_\Delta \frac{\langle X \mathrm{d}X^{n-1}\rangle \sqrt{-\det Q}}{(XQX)^{\frac{n}{2}}}$$

$$Q = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$$\mathcal{S}e_n = C_n \sum_\rho \overline{\rho_1\rho_2} \otimes \overline{\rho_3\rho_4} \otimes \cdots \otimes \overline{\rho_{n-1}\rho_n}$$

- Consider n is even
- Q is symmetric matrix
- Q is not degenerate
- Δ is a canonical simplex defined by $x_i = 0$

- Using

$$\mathcal{S}[I] = a_1 \otimes a_2 \otimes \cdots \otimes a_k$$
$$\mathcal{S}[\mathrm{Disc}_{a_1}[I]] = a_2 \otimes \cdots \otimes a_k$$

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- First Entries

$$\overline{ij}_{\text{first entry}} = \begin{cases} r(Q^{-1}_{\{i,j\},\{i,j\}}), & Q_{ii} \neq 0, Q_{jj} \neq 0, \\ \left(\frac{Q_{ij}^2}{Q_{jj}}\right)^{-\mathrm{sign}(Q_{ij})}, & Q_{ii} = 0, Q_{jj} \neq 0, \\ \left(\frac{Q_{ij}^2}{Q_{ii}}\right)^{-\mathrm{sign}(Q_{ij})}, & Q_{ii} \neq 0, Q_{jj} = 0, \\ Q_{ij}^{-2\,\mathrm{sign}(Q_{ij})}, & Q_{ii} = Q_{jj} = 0. \end{cases}$$

- Discontinuities

$$\frac{\sqrt{-1}}{2(n-2)} \int_{\Delta^{(ij)}} \frac{\sqrt{-\det Q^{(ij)}}\,\langle X_{(ij)} \mathrm{d}^{n-3} X_{(ij)}\rangle}{(X_{(ij)} Q^{(ij)} X_{(ij)})^{\frac{n-2}{2}}},$$

where $Q^{(ij)} = Q_{\widehat{\{i,j\}},\widehat{\{i,j\}}} - Q_{\widehat{\{i,j\}},\{i,j\}} (Q_{\{i,j\},\{i,j\}})^{-1} Q_{\{i,j\},\widehat{\{i,j\}}}.$

- Recursively read "First Entries" and discontinuities

# Highest Weight: Spherical Contours Explanation

- Why Spherical Contour Works?

$$I = \int_\Delta \frac{\langle X \mathrm{d}X^{n-1}\rangle \sqrt{-\det Q}}{(XQX)^{\frac{n}{2}}} \xrightarrow{\text{choose i,j}} \text{ij First Entries} \xrightarrow[\text{Discontinuities}]{\text{Calculating}} I' = \int_\Delta \frac{\langle X \mathrm{d}X^{n-3}\rangle \sqrt{-\det Q'}}{(X'Q'X')^{\frac{n-2}{2}}}$$

How to? $\downarrow$

- Perform Change of Variables $\begin{pmatrix} x_i \\ x_j \end{pmatrix} = R \begin{pmatrix} w_i \\ w_j \end{pmatrix} - Q^{-1}_{\{i,j\},\{i,j\}} Q_{\{i,j\},\widehat{\{i,j\}}} X_{(ij)}$ $\qquad R^{\mathrm{T}} Q_{\{i,j\},\{i,j\}} R = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$

- After change of Variables $I = \sqrt{-\det Q} \int_{\Delta_{ij}} \langle X \mathrm{d}X^{n-3}\rangle \int \mathrm{d}w_i \mathrm{d}w_j \frac{1}{(w_i w_j + X_{(ij)} Q_{(ij)} X_{(ij)})^{\frac{n}{2}}}$

- Discontinuity Contours: Spherical Contours

$$\mathrm{Disc}[I] = \sqrt{-\det Q} \int_{\Delta_{ij}} \langle X \mathrm{d}X^{n-3}\rangle \int_{S^2} \mathrm{d}w_i \mathrm{d}w_j \frac{1}{(w_i w_j + X_{(ij)} Q_{(ij)} X_{(ij)})^{\frac{n}{2}}}$$

$\downarrow$

$$w_i = re^{i\theta},\ w_j = re^{-i\theta}, \quad r \in [0, +\infty],\ \theta \in [0, 2\pi].$$

This is related to that the discontinuities are similar to residue contours.

# Highest Weight: Spherical Contours with Numerators

- More information: Rational coefficient of symbols. Mixed weights.
- Symbol Entries will not change.
- Discontinuities can be calculated by the same spherical contour!

$$I = \int_\Delta \frac{\langle X \mathrm{d}X^{n-1}\rangle N\left[X^k\right]}{(XQX)^{\frac{n+k}{2}}} \xrightarrow{\text{choose i,j}} \text{ij First Entries} \xrightarrow[\text{Discontinuities}]{\text{Calculating}} \cdots\cdots \longrightarrow \boxed{\text{Last integral Rational}}$$

Coeffecient

- Consider ij variables $I = \int_\Delta \langle X \mathrm{d}X^{n-3}\rangle N \int \mathrm{d}x_i \mathrm{d}x_j \frac{x_i^{p_i} x_j^{p_j}}{(XQX)^{\frac{n+k}{2}}}$
- After performing change of variables and taking spherical contours

$$\mathrm{Disc}[I] = \sqrt{-\det Q} \int_{\Delta_{ij}} \langle X \mathrm{d}X^{n-3}\rangle N \boxed{\int_{S^2} \mathrm{d}w_i \mathrm{d}w_j \frac{x_i(w_i,w_j)^{p_i} x_j(w_i,w_j)^{p_j}}{(w_i w_j + X_{(ij)} Q_{(ij)} X_{(ij)})^{\frac{n}{2}}}}$$

This part can be integrated in a general form! Still a recursive relation.

# Lower weight Functions: Subtraction

- Solving Mixed Weight Problems:
  - Subtract the highest weight
  - Bootstrap the lower weight
- From Spherical Contour, we know the rational coefficient of highest weight

$$I = \int_\Delta \frac{\langle X \mathrm{d}X^{n-1}\rangle N\left[X^k\right]}{(XQX)^{\frac{n+k}{2}}} \xrightarrow{\text{Spherical Contour}} \mathcal{S}[I]_{\max} = C(z_1, \cdots, z_n)\, a_1 \otimes a_2 \otimes \cdots \otimes a_k$$

(z is kinematics)

- The integrals related to "no-coefficient symbol" $I = \int_\Delta \frac{\langle X \mathrm{d}X^{n-1}\rangle \sqrt{-\det Q}}{(XQX)^{\frac{n}{2}}}$
- Subtractions will give only lower weight integral!

$$I_{\text{lower}} = \int_\Delta \frac{\langle X \mathrm{d}X^{n-1}\rangle N\left[X^k\right]}{(XQX)^{\frac{n+k}{2}}} - \frac{C(z_1, \cdots, z_n)}{C_n} \int_\Delta \frac{\langle X \mathrm{d}X^{n-1}\rangle \sqrt{-\det Q}}{(XQX)^{\frac{n}{2}}} \xrightarrow{\text{Spherical Contour}} \text{Lower Weight}$$

Usually this means more remaining integration variables will give rational terms.

# Lower weight Functions: Symbol Bootstrapping

- Energy Correlations: Spurious poles must cancel
- Example: From E3C

$$F_1 = \int_\Delta \frac{\langle \omega \mathrm{d}\omega^3 \rangle \omega_2 \omega_3}{[\omega_1\omega_2 + |z|^2\omega_2\omega_3 + |1-z|^2\omega_1\omega_3 + \omega_1 + \omega_2 + \omega_3]^4}$$

- Highest Weight

$$\frac{|1-z|^2\left(-1-|z|^2+2|z|^4+2|1-z|^2-|z|^2|1-z|^2-|1-z|^4\right)}{(z-\bar{z})^5}\left(z\bar{z}\otimes\frac{1-z}{1-\bar{z}}+(1-z)\otimes\bar{z}-z\otimes(1-\bar{z})\right)$$

- Assumptions: $\frac{1}{(z-\bar{z})^4}\sum_{i,j=0}^{n}\left[a_{ij}z^i\bar{z}^j + b_{ij}z^i\bar{z}^j\left[\otimes(z\bar{z})\right] + c_{ij}z^i\bar{z}^j\left[\otimes(1-z)(1-\bar{z})\right]\right]$
  - The weight two function when $z-\bar{z}\to\epsilon$ is order $\epsilon$ , so we assume the pole is $(z-\bar{z})^{-4}$.
  - The heighest power of $z$ and $\bar{z}$ in the weight two numerators are power 3.
  - By this assumption we can recover all the solutions.

QCD results, H. Chen, M.-X. Luo, I. Moult, T.-Z. Yang, X. Zhang, and H. X. Zhu 19'

# New Phenomena in Four-Points

- Feynman Parametrization will give more than quadrics: Cubics

For instance $$I = \int \frac{\langle X \mathrm{d}X^3 \rangle x_2 \zeta_{23}}{s_{123} s_{234} x_{1234}} = \int \mathrm{d}t \int \frac{\langle X \mathrm{d}X^4 \rangle x_2 \zeta_{23}}{(s_{123} + \boxed{t s_{234}} + x_5 x_{1234})^3}$$

Performing Spherical Contour with parameter t $$\mathcal{S}[I] = \int \mathrm{d}t \frac{\mathcal{S}[t; z, \bar{z}, w, \bar{w}] R(t)}{\boxed{a_3 t^3 + a_2 t^2 + a_1 t + a_0}},$$

$a_3 = w\bar{w}(1-z)^2(1-\bar{z})^2,\ a_2 = -(1-z)(1-\bar{z})(w\bar{w}(1+z)(1+\bar{z}) - (w+\bar{w})(w\bar{w}+z\bar{z})),$
$a_1 = -(1-w)(1-\bar{w})(z\bar{z}(1+w)(1+\bar{w}) - (z+\bar{z})(w\bar{w}+z\bar{z})),\ a_0 = (1-w)^2(1-\bar{w})^2 z\bar{z},$

$\mathcal{S}[t; z, \bar{z}, w, \bar{w}]$ can still be calculated by Spherical Contour

- Symbol Integrations: Do the quadrics with spherical contour, integration over the remaining variables.
- Generalized Projections: Symbol Calculations from projective geometries

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- More kinematic variables:
  - We assume that all the spurious poles conditions are enough for bootstrapping all the lower weight functions.

# Outlook

- E^NC @ Generic Angle: hyperbolic simplex
  - Multiple quadrics,  complication from boundaries
  - Generalized Projections: Symbol Calculations from projective geometries
- E^NC in QCD
  - Complication from  integrands
- E^NC @ Higher loop
  - IR Regulators/cutoffs

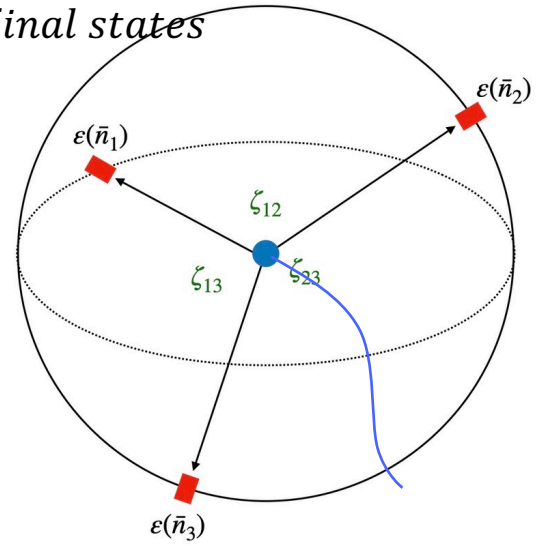
# Onshell definition

$$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\ldots\mathrm{E}(n_N)\rangle = \sum_{k>N}\int d\,\Pi_k\,\delta^2(n_i-\Omega_i)\cdots\delta^2(n_N-\Omega_N)\frac{E_1\cdots E_N}{Q^N}FF_k(O)$$

$FF_k(O):\ |\langle 0|O|1,2,\ldots,k\rangle|^2$ *summed over helicity, color and permutations on final states*

Analytic function of distances on the celestial sphere:

$$\zeta_{ij}=\frac{q^2(n_i\cdot n_j)}{2(n_i\cdot q)(n_j\cdot q)}\to\frac{1}{2}(1-\cos\theta_{ij})$$




$k=N+1$: Leading order. Manifestly finite integration over tree-level matrix element $FF^{(0)}_{N+1}$