BFKL/DGLAP Mixing and Light-ray Operators

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Motivation

- What is the space of detector operators in (perturbative) QCD?

- The Average Null Energy (ANEC) operator measures the flux of energy at infinity, and it is a special kind of detector.

$$\mathcal{E}(\widehat{n}) = \lim_{r\to\infty} r^{d-2} \int_0^\infty dt\ n^i T^0{}_i(t, r\widehat{n}) \qquad \widehat{n} \in S^{d-2}$$

- Its matrix elements (energy correlators) are IR safe, and can probe many different physics of QCD. [Ian's and Kai's talk]

- What else can we measure at infinity? More general detector operators appear in the light-ray OPE of ANEC operators.

- In a CFT, detectors can be identified with light-ray operators. In weakly-coupled theories, the detectors need to be renormalized, and they can mix in a non-trivial way [Caron-Huot, Kologlu, Kravchuk, Meltzer, Simmons-Duffin '22]. We will explore these phenomena in QCD.

Two important equations in QCD

- The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (**DGLAP**) equation describes the RG evolution of PDFs.

- Anomalous dimensions of twist-2 local operators (e.g., $F\partial^{J-2}F$):

$$\gamma_{gg}^{\text{twist-2}}(J) = \frac{\alpha_s}{2\pi}\left(-4C_A\left(\frac{1}{J(J-1)} + \frac{1}{(J+1)(J+2)} - \psi(J+1) - \gamma_E\right) - \frac{11}{3}C_A - \frac{2}{3}n_f\right)$$

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

- The Balitsky-Fadin-Kuraev-Lipatov (**BFKL**) equation controls scattering at high energies (Regge limit) $s \gg -t$.

- BFKL eigenvalue/anomalous spin:

$$\gamma^{\text{BFKL}}(\nu) = \frac{\alpha_s C_A}{\pi}\left(\psi\left(\frac{2+\nu}{2}\right) + \psi\left(-\frac{\nu}{2}\right) + 2\gamma_E\right)$$

- The two equations can mix with each other. DGLAP equation breaks down at small x, and BFKL resums $\log(1/x)$.

Two important equations in QCD

- The DGLAP anomalous dimension has a pole at $J = 1$.

$$\gamma_{gg}^{\text{twist-2}}(J) = -\alpha_s \frac{2C_A}{\pi} \frac{1}{J-1} + \dots$$

- Resummation [Ellis, Stirling, Webber '96]:

$$\gamma_{gg}^{\text{twist-2}}(J \to 1) = -\sqrt{\alpha_s \frac{2C_A}{\pi}} + \dots$$

- The BFKL eigenvalue has a pole at $\nu = -2$:

$$\gamma^{\text{BFKL}}(\nu) = -\alpha_s \frac{2C_A}{\pi} \frac{1}{\nu+2} + \dots$$

- These divergences are due to mixing between DGLAP and BFKL. The pole of DGLAP at $J = 1$ is related to the BFKL equation (and vice versa). Lots of exploration in QCD and $\mathcal{N} = 4$ SYM. [Jaroszewicz '82; Lipatov '96; Kotikov, Lipatov '00; Kotikov, Lipatov '02; Brower, Polchinski, Strassler, Tan '06; Kotikov, Lipatov, Rej, Staudacher, Velizhanin '07;…]

- This talk: understand these facts in the light-ray operator/detector language.

- By working in an appropriate basis, we can obtain a **finite** anomalous dimension matrix for the DGLAP and BFKL detectors (even when they mix).

Light-ray operators and detectors

- Light-ray operators [Kravchuk, Simmons-Duffin '18] are analytic continuation of null integral (light transform) of local operators.

$$\mathbf{L}[\mathcal{O}](x,z) = \int_{-\infty}^{\infty} d\alpha (-\alpha)^{-\Delta-J} \mathcal{O}(x - z/\alpha, z)$$

$$z^2 = 0$$

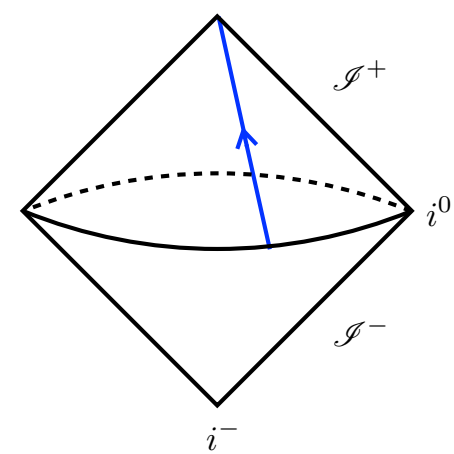
$$\mathcal{O}(x,z) = \mathcal{O}^{\mu_1 \cdots \mu_J}(x) z_{\mu_1} \ldots z_{\mu_J}$$

- $\mathbf{L}[\mathcal{O}]$ has quantum numbers $(\Delta_L, J_L) = (1-J, 1-\Delta)$

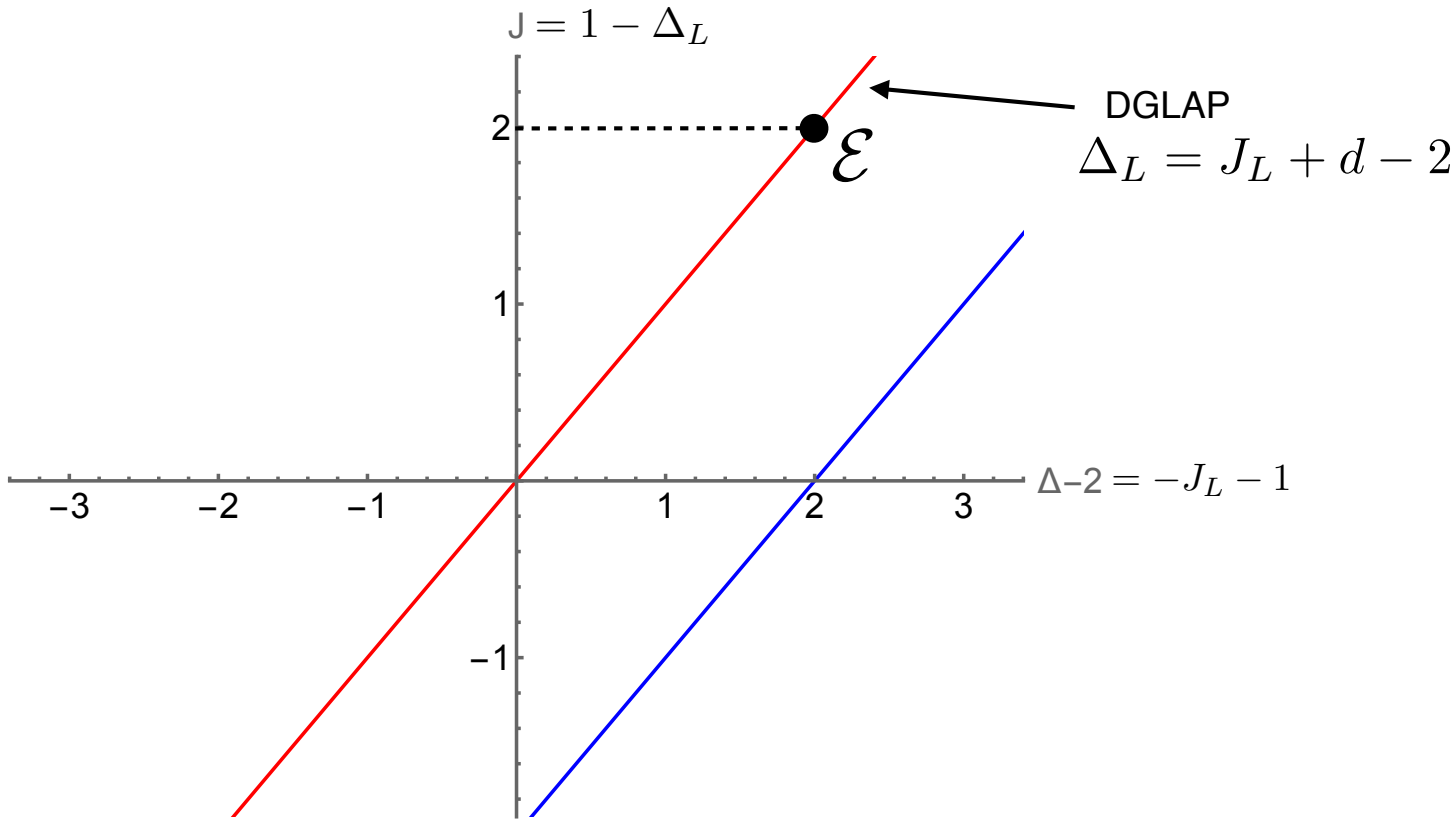
- Choosing $x = \infty$ gives an integral along future null infinity, and $z = (1,\vec{n})$ is the embedding space coordinate of the celestial sphere S^{d-2}. This is a "detector operator" $\mathcal{D}_{J_L}(z)$ (c.f. definition of ANEC).

- Each detector has a "spin shadow" $\mathbf{S}_J : J_L \to 2 - d - J_L$.

$$\mathbf{S}_J[\mathcal{D}_{J_L}](z) = \int D^{d-2} z' (-2z \cdot z')^{2-d-J_L} \mathcal{D}_{J_L}(z')$$



The Chew-Frautschi plot in free theory



space of detectors ⟷ Chew-Frautschi plot

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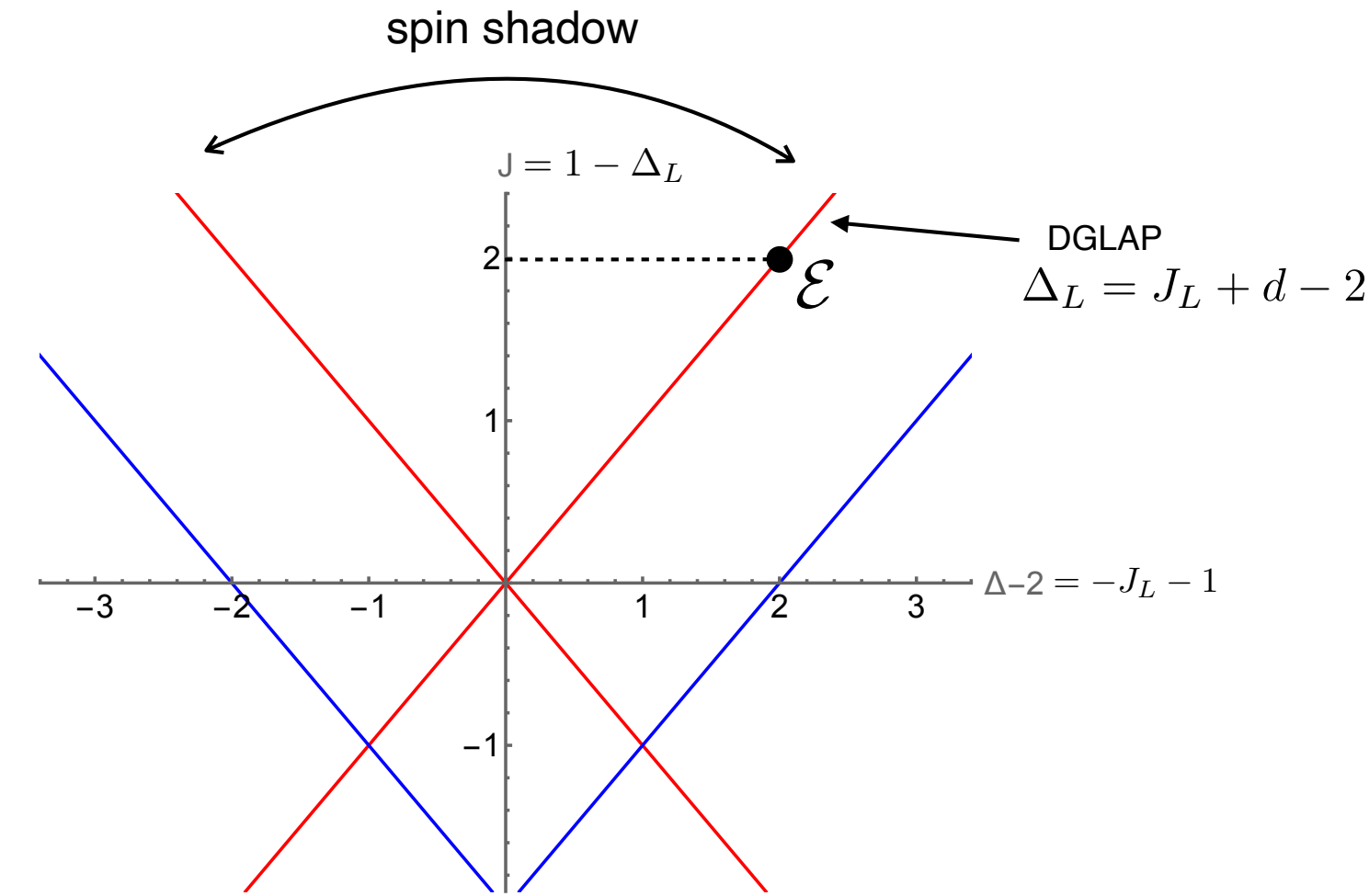

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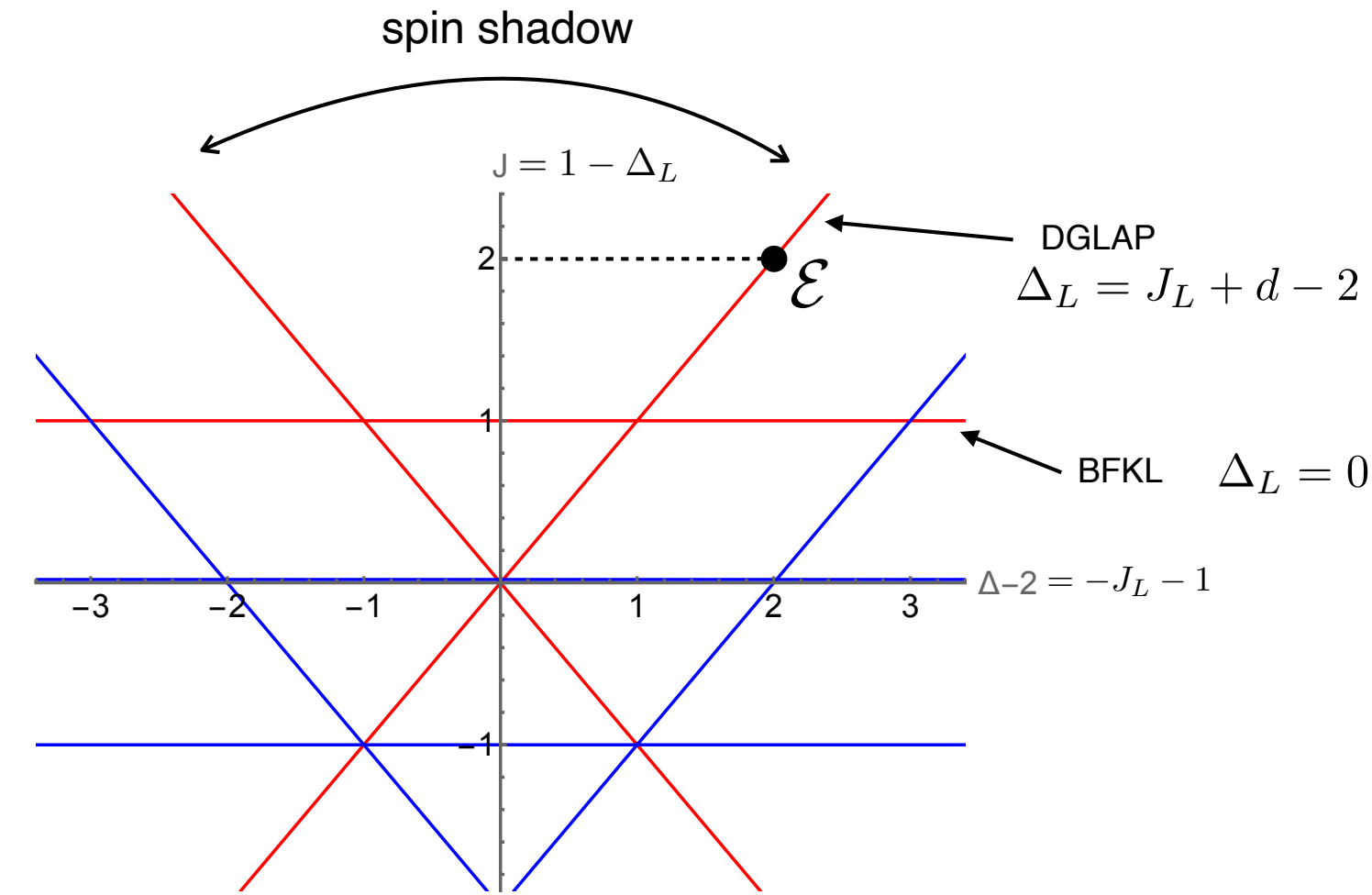
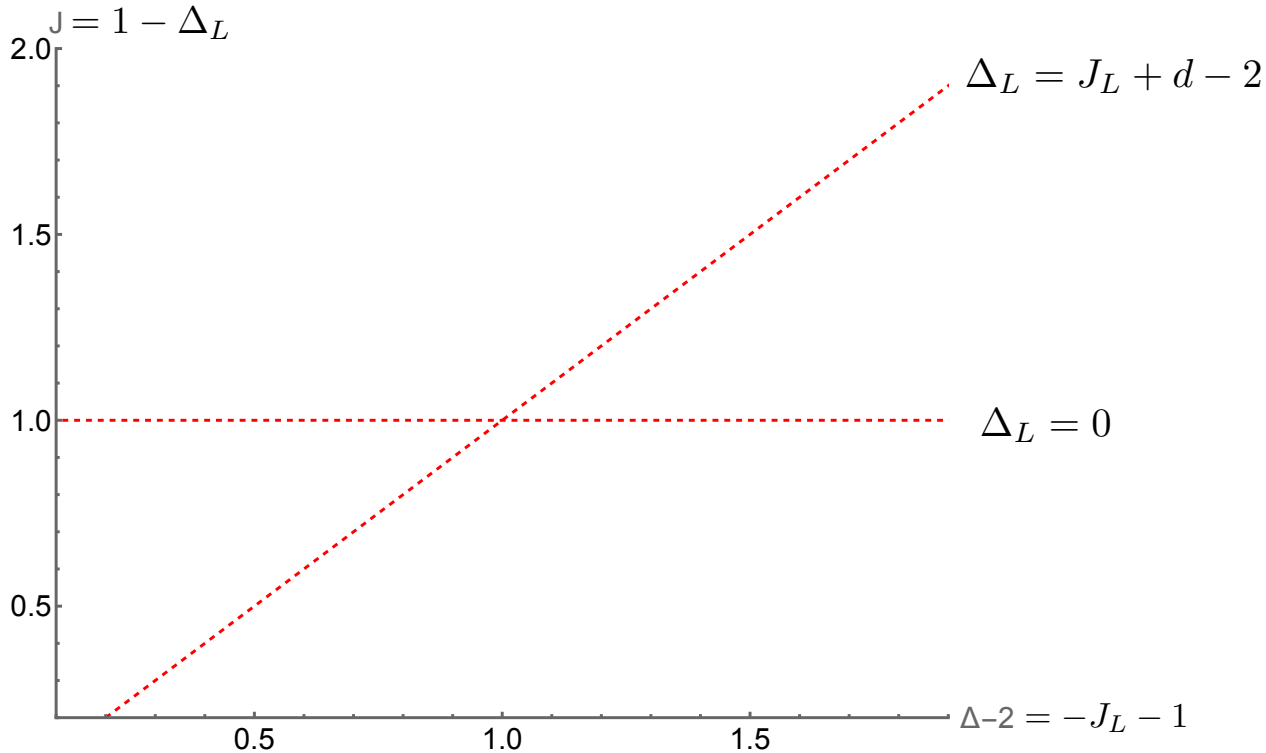


space of detectors ⟷ Chew-Frautschi plot

- The DGLAP trajectory and BFKL trajectory intersect at $J_L = 2 - d = -2 + 2\epsilon$.

The Chew-Frautschi plot in free theory



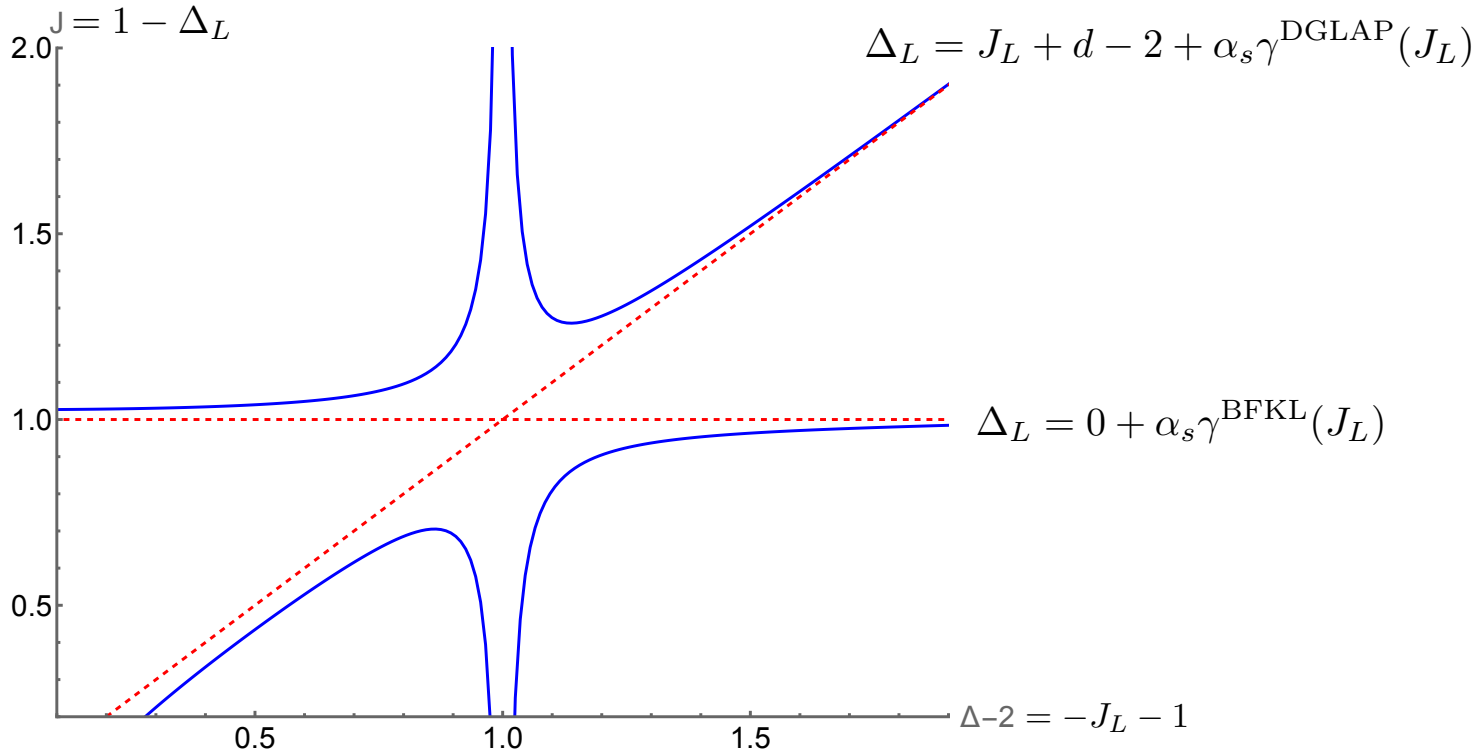
- Q: What does the C-F plot look like at one loop?

- When turning on interactions, J_L remains unchanged, and Δ_L receives corrections ("anomalous spin").

$$\Delta_L = J_L + d - 2 + \alpha_s \gamma^{\mathrm{DGLAP}}(J_L) \qquad \Delta_L = 0 + \alpha_s \gamma^{\mathrm{BFKL}}(J_L)$$

$$\gamma^{\mathrm{DGLAP}}(J_L) = \gamma^{\text{twist-2}}(-1 - J_L)$$

Naive Chew-Frautschi plot at one loop



- One expects Regge trajectories can't diverge in interacting theories. The singularity can be resolved by studying the non-trivial mixing of detectors at the intersection.

- Nonperturbative pictures known in planar $\mathcal{N} = 4$ SYM using integrability. [Gromov, Levkovich-Maslyuk, Sizov, Valatka '14; Alfimov, Gromov, Kazakov '15; Gromov, Kazakov, Leurent, Volin '15; Gromov, Levkovich-Maslyuk, Sizov '17]

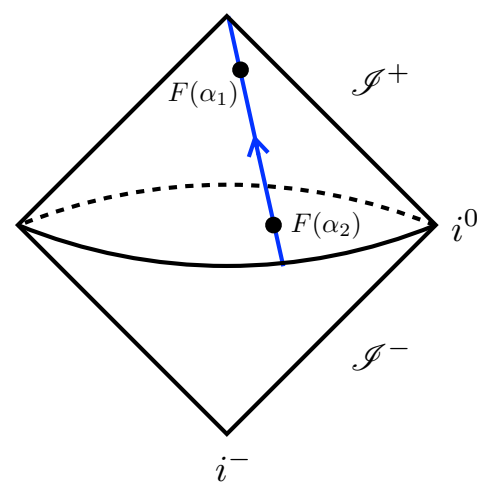
DGLAP detector in free theory

- For simplicity, let us consider pure Yang-Mills theory.
- DGLAP detector in free theory:

$$\mathcal{E}_{J_L}(z) \propto \delta^{ab} \sum_{\lambda} \int dE\ E^{-J_L-1} \left[a^{\dagger}_{\lambda,a}(p) a_{\lambda,b}(p) \right] \Big|_{p=Ez}$$

- A detector that measures flux of E^{2-d-J_L}. $J_L = 1-d$ gives the ANEC operator.
- Can be written as a bilocal integral along a null direction z at future null infinity.

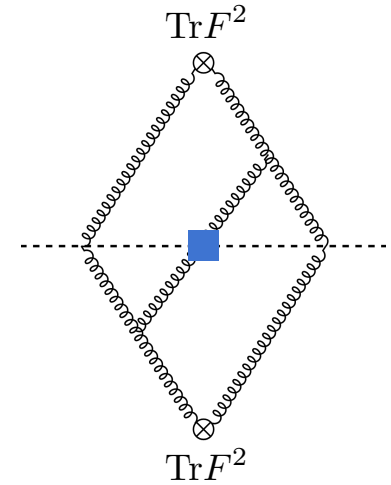
$$\mathcal{D}^{\mathrm{DGLAP}}_{J_L}(z) \propto \int d\alpha_1 d\alpha_2 |\alpha_1 - \alpha_2|^{d-2+J_L} F_a(\alpha_1, z) F^a(\alpha_2, z)$$



- Well-defined for all $J_L \in \mathbb{C}$.

DGLAP detector at one loop

- Vertex of the DGLAP detector:



$$V_{J_L}(p;z) = \int d\beta\ \beta^{-J_L-1}\delta^{(d)}(p-\beta z)$$

- Divergence of the one-loop matrix element (for $\epsilon \to 0$, generic J_L):

$$\langle \mathcal{D}^{\mathrm{DGLAP}}_{J_L}\rangle^{1-\mathrm{loop}} = \frac{g^2}{8\pi\epsilon}\gamma^{\mathrm{DGLAP}}(J_L)\langle \mathcal{D}^{\mathrm{DGLAP}}_{J_L}\rangle^{\mathrm{tree}} + (\mathrm{regular\ at}\ \epsilon = 0)$$

$$\gamma^{\mathrm{DGLAP}}(J_L) = \frac{2C_A}{\pi}\frac{1}{J_L+2} + \gamma^{\mathrm{DGLAP}}_{\mathrm{Reg}}(J_L)$$

- The pole at $J_L = -2$ is present even for finite ϵ, and it's proportional to the tree-level matrix element of a different detector — the BFKL detector.

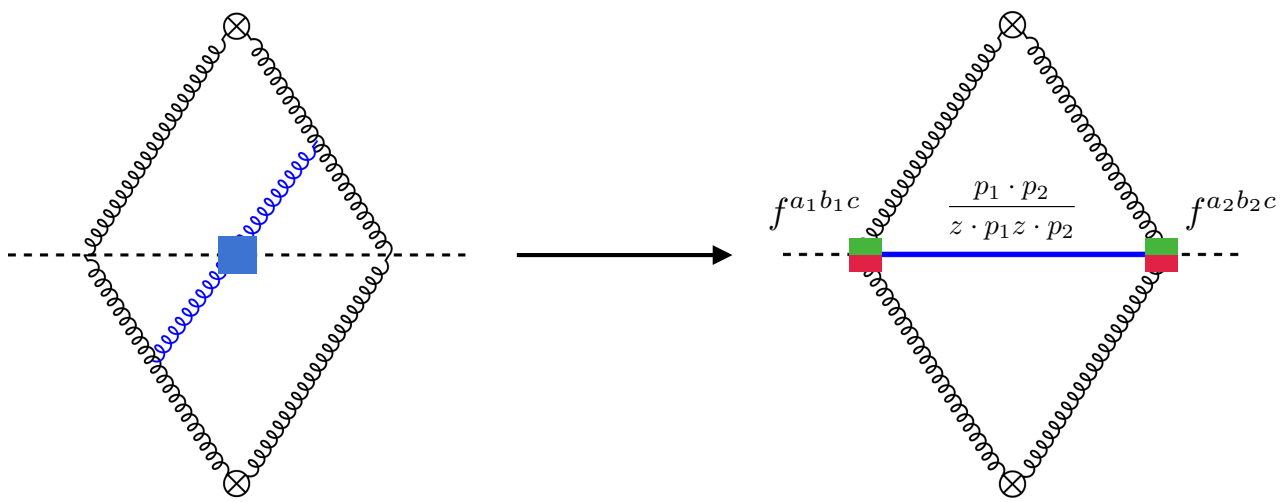
$$\langle \mathcal{D}^{\mathrm{DGLAP}}_{J_L}\rangle^{1-\mathrm{loop}} = \frac{g^2\mu^{2\epsilon}}{8\pi(J_L+2)}\left(\frac{2C_A}{\pi}\frac{1}{\epsilon} + \widetilde{\mathcal{R}}_1(\epsilon)\right)\langle \mathcal{D}^{\mathrm{BFKL}}_{J_L}\rangle^{\mathrm{tree}} + (\mathrm{regular\ at}\ J_L = -2)$$

DGLAP detector at one loop

- The pole at $J_L = -2$ comes from the leading soft theorem.

$$\langle \mathcal{D}_{J_L}^{\mathrm{DGLAP}}(z)\rangle^{1-\mathrm{loop}} = \int d\mathrm{PS}_{n-1} \int d\beta \beta^{-J_L-1} |\mathcal{F}_{\mathrm{tree}}(\beta z, p_2, \ldots, p_n)|^2$$

$$\sim \int d\mathrm{PS}_{n-1} \underbrace{\int d\beta \beta^{-J_L-3}}_{\frac{1}{J_L+2}} \underbrace{\sum_{i,j=2}^{n} \frac{p_i \cdot p_j}{z \cdot p_i z \cdot p_j} f^{a_i b_i c} f^{a_j b_j c} \left(\mathcal{F}_{\mathrm{tree}}^{a_2 \ldots b_j \ldots a_n}\right)^* \mathcal{F}_{\mathrm{tree}}^{a_2 \ldots b_i \ldots a_n}}_{\text{tree-level matrix element of BFKL detector}}$$



BFKL detector in free theory

- The BFKL detector is a product of two “color-interference” detectors smeared over the celestial sphere:

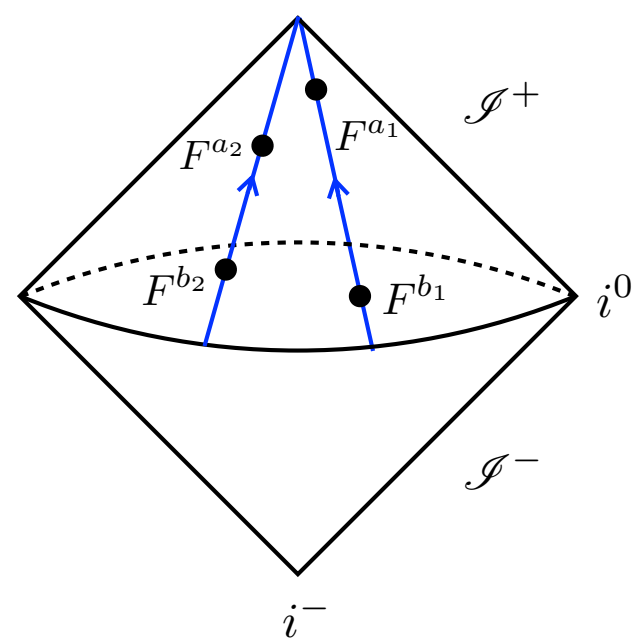
$$\mathcal{D}^{\mathrm{BFKL}}_{J_L}(z) = \int D^{d-2}z_1 D^{d-2}z_2 \; K_{J_L}(z_1, z_2, z)\mathcal{N}^c(z_1)\mathcal{N}^c(z_2)$$

$$\mathcal{N}^c(z) \propto i f^{abc} \sum_\lambda \int \frac{E^{d-2}dE}{2E} \left[a^\dagger_{\lambda,a}(p) a_{\lambda,b}(p)\right]\Big|_{p=Ez}$$

$$K_{J_L} \propto \left(\frac{-2z_1\cdot z_2}{(-2z_1\cdot z)(-2z_2\cdot z)}\right)^{-\frac{J_L}{2}}$$

- A Wilson loop at future null infinity decorated by four field strengths.

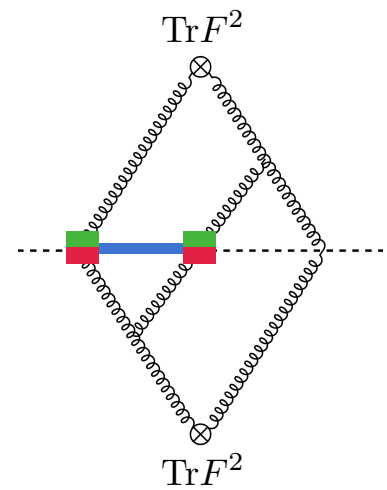
- $\Delta_L = 0$ for all values of $J_L \Rightarrow$ a horizontal trajectory at $J = 1$ on the C-F plot.



BFKL detector at one loop

- Divergences of the BFKL detector at one loop at $\epsilon = 0$ for generic J_L:

$$\langle \mathcal{D}_{J_L}^{\rm BFKL} \rangle^{1-\rm loop} = \frac{g^2}{8\pi\epsilon} \underbrace{\gamma^{\rm BFKL}(J_L)}_{\text{the BFKL eigenvalue}} \langle \mathcal{D}_{J_L}^{\rm BFKL} \rangle^{\rm tree} + (\text{regular at } \epsilon = 0)$$



$$\gamma^{\rm BFKL}(J_L) = -\frac{2C_A}{\pi}\frac{1}{J_L+2} + \gamma_{\rm Reg}^{\rm BFKL}(J_L)$$

- For finite ϵ, it has a pole at $J_L = -2 + 4\epsilon$ proportional to the tree-level matrix element of the DGLAP detector

$$\langle \mathcal{D}_{J_L}^{\rm BFKL} \rangle^{1-\rm loop} = \frac{g^2\mu^{2\epsilon}}{8\pi(J_L+2-4\epsilon)} \left(-\frac{2C_A}{\pi}\frac{1}{\epsilon} + \widetilde{\mathcal{R}}_2(\epsilon) \right) \langle \mathcal{D}_{J_L}^{\rm DGLAP} \rangle^{\rm tree} + (\text{regular at } J_L = -2+4\epsilon)$$

Tree-level basis for DGLAP/BFKL

- The DGLAP and BFKL trajectories intersect at $J_L = 2 - d = -2 + 2\epsilon$. At this intersection, the two detectors become exactly the same.

$$\mathcal{D}^{\rm BFKL}_{J_L}(z) = \int D^{d-2}z_1 D^{d-2}z_2 \; K_{J_L}(z_1, z_2, z)\mathcal{N}^c(z_1)\mathcal{N}^c(z_2)$$

$$K_{2-d}(z_1, z_2, z) = \delta(z - z_1) + \delta(z - z_2)$$

$$\lim_{\epsilon\to 0} \epsilon x^{-1+\epsilon}\theta(x) = \delta(x)$$

$$\Rightarrow \mathcal{D}^{\rm BFKL}_{2-d}(z) = \mathcal{N}^c(z) \int D^{d-2}z' \mathcal{N}^c(z') = \mathcal{D}^{\rm DGLAP}_{2-d}(z)$$

gauge invariance

- To make sure the basis is regular at all J_L (in particular at the intersection), one should consider the following tree-level basis:

$$\mathbb{D}_{J_L} = \begin{pmatrix} \mu^{J_L+2-2\epsilon}\mathcal{D}^{\rm DGLAP}_{J_L} \\ \frac{\mathcal{D}^{\rm BFKL}_{J_L} - \mu^{J_L+2-2\epsilon}\mathcal{D}^{\rm DGLAP}_{J_L}}{J_L+2-2\epsilon} \end{pmatrix}$$

Renormalizing the multiplet

- In the regular basis, the tree-level dilatation matrix has off-diagonal terms.

$$D\mathbb{D}_{J_L} = \begin{pmatrix} -2+2\epsilon-J_L & 0 \\ 1 & 0 \end{pmatrix} \mathbb{D}_{J_L}$$

a log-multiplet!

- Renormalized detector $[\mathbb{D}_{J_L}]_R \equiv \mathcal{Z}_{J_L}^{-1}\mathbb{D}_{J_L}$

- The renormalization factor should remove all divergences of the 1-loop matrix element, including $1/\epsilon,\ 1/(J_L+2),\ 1/(J_L+2-4\epsilon)$.

- One-loop dilatation operator $\mathscr{D} = \mathcal{Z}_{J_L}^{-1}\left(\mathscr{D}_0 + \beta(g)\frac{\partial}{\partial g}\right)\mathcal{Z}_{J_L}$

DGLAP meets BFKL

$$\mathscr{D} = \mathscr{D}_0 + \alpha_s \begin{pmatrix} -\gamma_{\rm Reg}^{\rm DGLAP}(J_L) & \frac{2C_A}{\pi} \\ \frac{1}{4}\left(\widetilde{\mathcal{R}}_1'(0) + \widetilde{\mathcal{R}}_2'(0) - 8\gamma_{21}(J_L)\right) & -\gamma_{\rm Reg}^{\rm BFKL}(J_L) \end{pmatrix} + O(\alpha_s^2)$$

$$\gamma_{21}(J_L) = \frac{\gamma_{\rm Reg}^{\rm BFKL}(J_L) - \gamma_{\rm Reg}^{\rm BFKL}(-2) - \gamma_{\rm Reg}^{\rm DGLAP}(J_L) + \gamma_{\rm Reg}^{\rm DGLAP}(-2)}{J_L + 2}$$

- The eigenvalues of this dilatation matrix ($-\Delta_L$) should recover the DGLAP and BFKL trajectories away from the intersection, and smoothly connect them when going through the intersection.

- DGLAP: $\Delta_L = J_L + 2 + \alpha_s \left(\frac{2C_A}{\pi(J_L+2)} + \gamma_{\rm Reg}^{\rm DGLAP}(J_L) \right) + O(\alpha_s^2)$

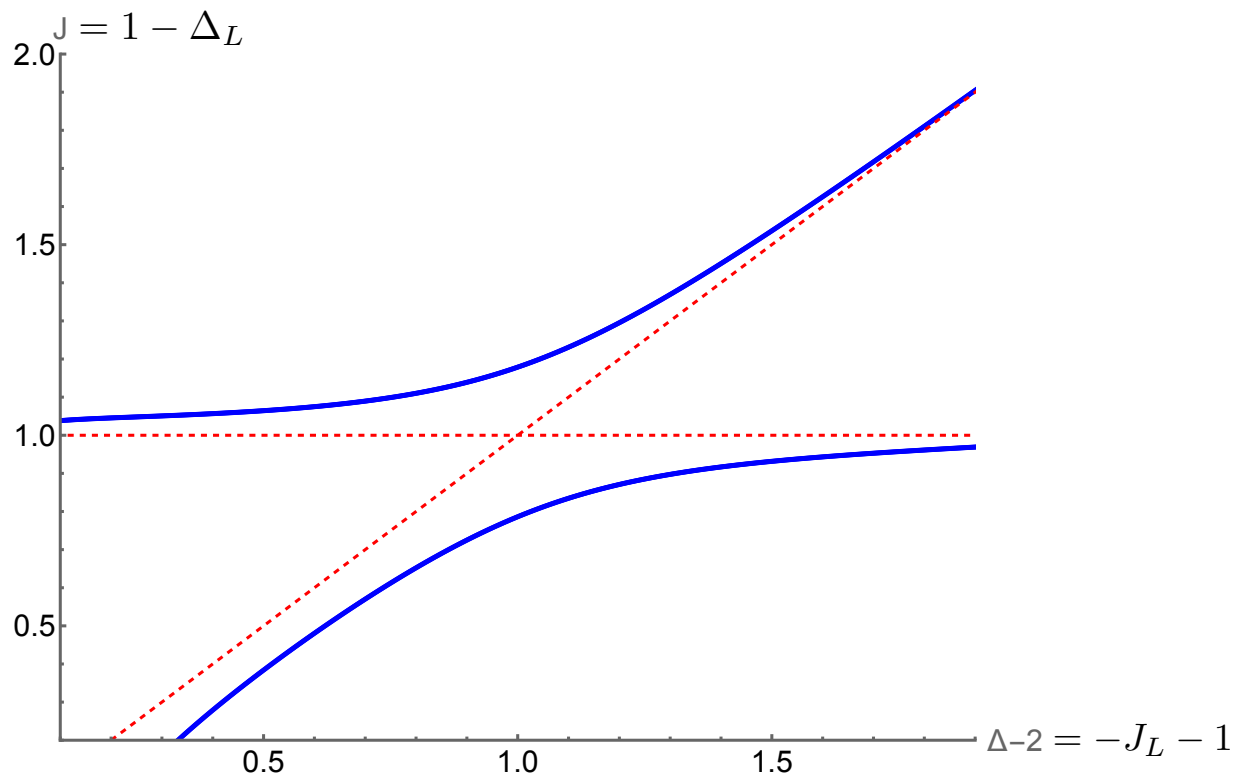
- BFKL: $\Delta_L = \alpha_s \left(-\frac{2C_A}{\pi(J_L+2)} + \gamma_{\rm Reg}^{\rm BFKL}(J_L) \right) + O(\alpha_s^2)$

- Intersection: $\mathscr{D}_{J_L=-2} = \begin{pmatrix} -\alpha_s \gamma_{\rm Reg}^{\rm DGLAP}(-2) & \alpha_s \frac{2C_A}{\pi} \\ 1 + O(\alpha_s) & -\alpha_s \gamma_{\rm Reg}^{\rm BFKL}(-2) \end{pmatrix} + O(\alpha_s^2)$

$$\Rightarrow \ \Delta_L = \pm\sqrt{\alpha_s \frac{2C_A}{\pi}} + \alpha_s \frac{11C_A}{12\pi} + O(\alpha_s^{\frac{3}{2}})$$

DGLAP meets BFKL

$$\mathscr{D} = \mathscr{D}_0 + \alpha_s \begin{pmatrix} -\gamma_{\text{Reg}}^{\text{DGLAP}}(J_L) & \frac{2C_A}{\pi} \\ \frac{1}{4}\left(\widetilde{\mathcal{R}}_1'(0) + \widetilde{\mathcal{R}}_2'(0) - 8\gamma_{21}(J_L)\right) & -\gamma_{\text{Reg}}^{\text{BFKL}}(J_L) \end{pmatrix} + O(\alpha_s^2)$$

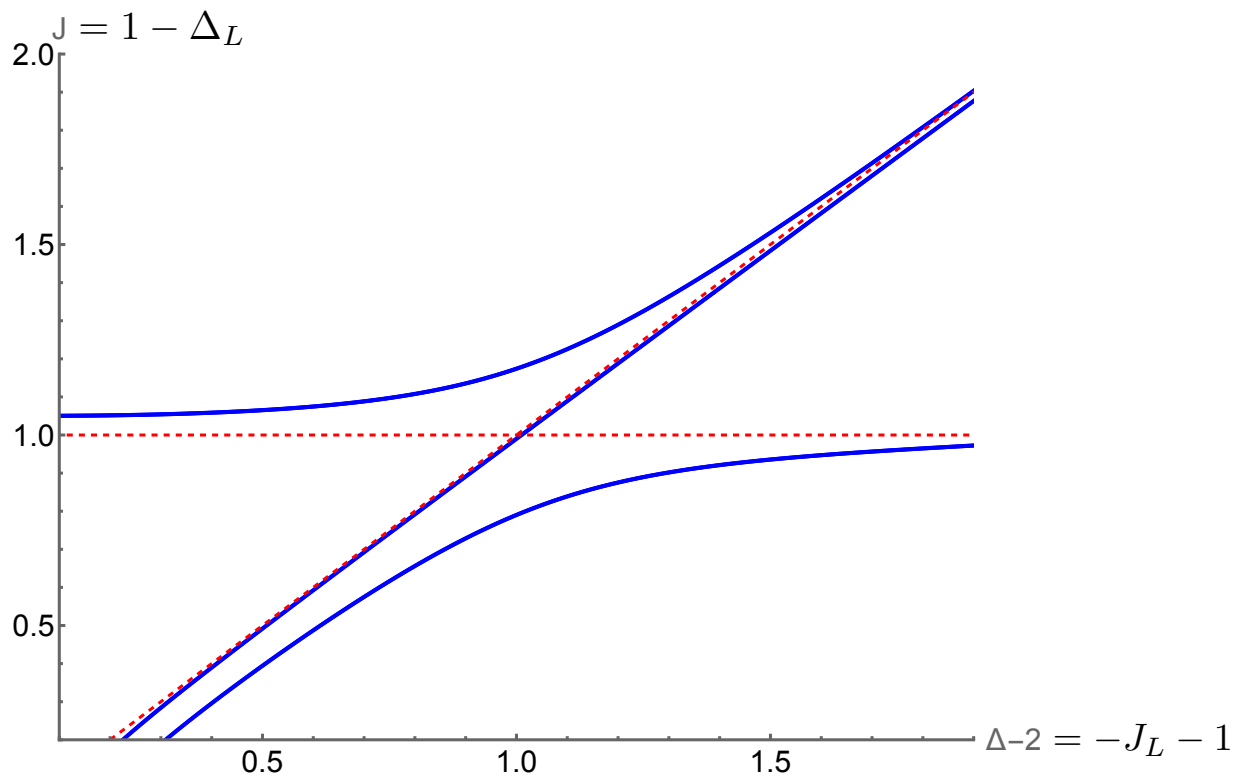


DGLAP meets BFKL in QCD

- In QCD, there are two DGLAP detectors, but only one of them will mix with BFKL.

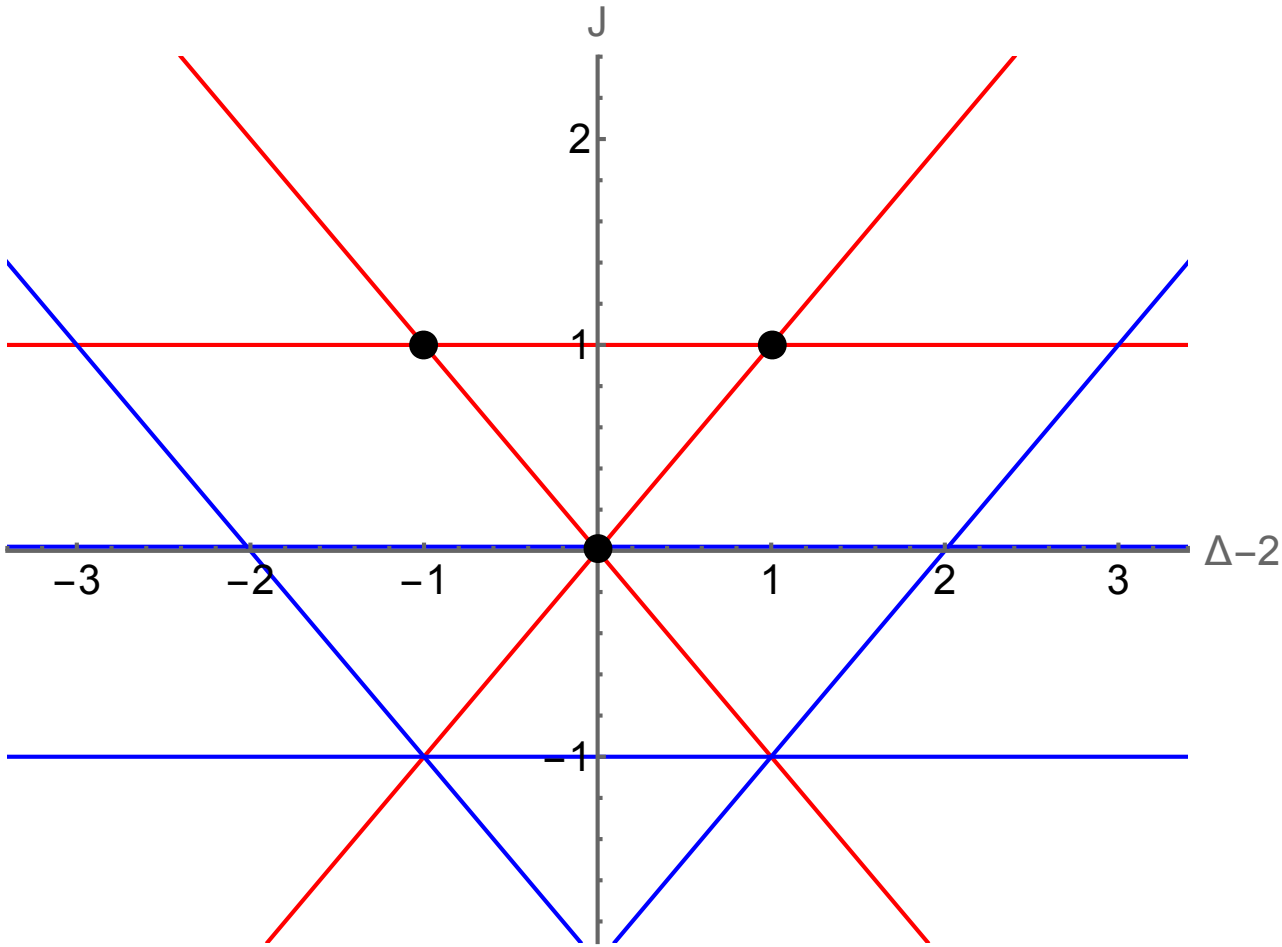
$$\gamma^{\mathrm{DGLAP}} = \begin{pmatrix} \gamma_{gg} & \gamma_{gq} \\ \gamma_{qg} & \gamma_{qq} \end{pmatrix} \sim \frac{1}{J_L + 2} \begin{pmatrix} \frac{2C_A}{\pi} & \frac{2C_F}{\pi} \\ 0 & 0 \end{pmatrix} + \dots$$

$$\mathcal{D}_{2-d}^{\mathrm{BFKL}} = C_A \mathcal{D}_{2-d}^{\mathrm{DGLAP_g}} + C_F \mathcal{D}_{2-d}^{\mathrm{DGLAP_q}}$$



Subleading trajectories

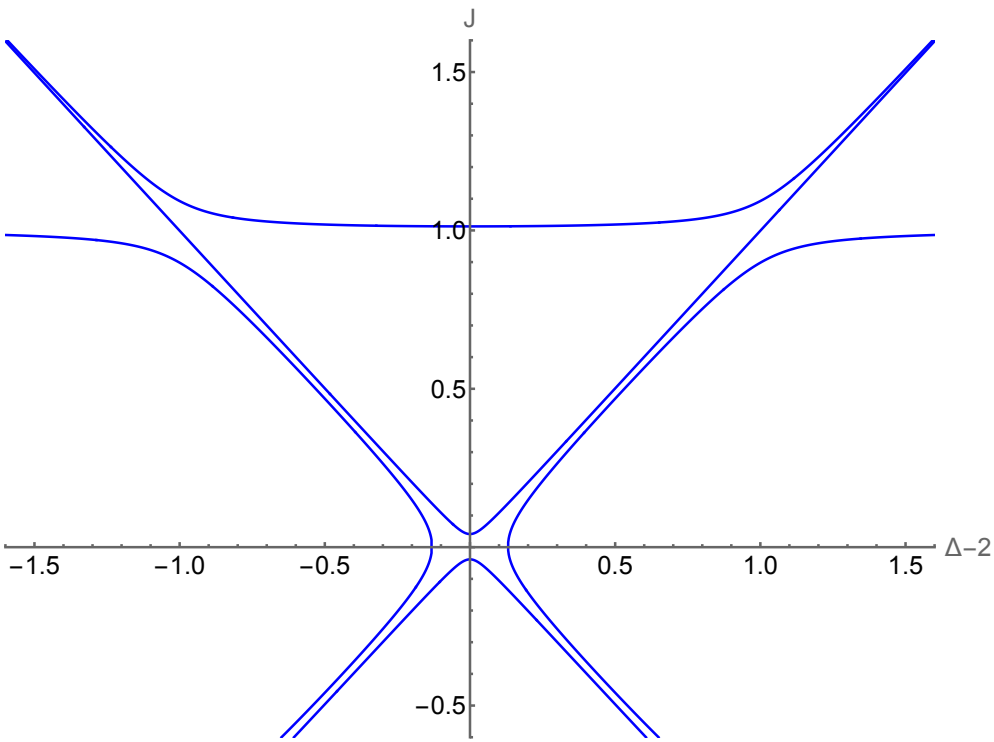
- The DGLAP anomalous dimension also has poles at $J = 0, -1, -2, \ldots$.
- The pole at $J = 0$ is due to the mixing between DGLAP and its spin shadow (subleading soft theorem [Casali '14]). The two detectors also become exactly the same at the intersection.



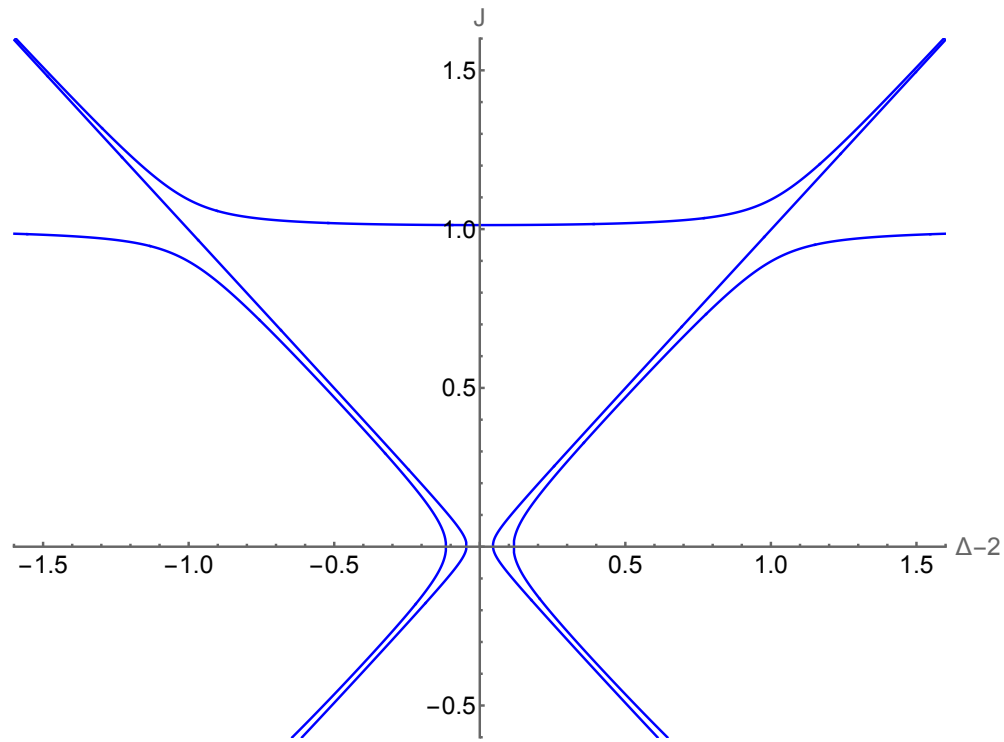
C-F plot including subleading trajectories

- The same method gives a 5x5 dilatation matrix that is finite near all the intersections.

$n_f = 2$



$n_f = 4$



C-F plot including subleading trajectories

- Depending on n_f, one gets different behaviors near the $J = 0$ intersection due to different signs of the residue of DGLAP.

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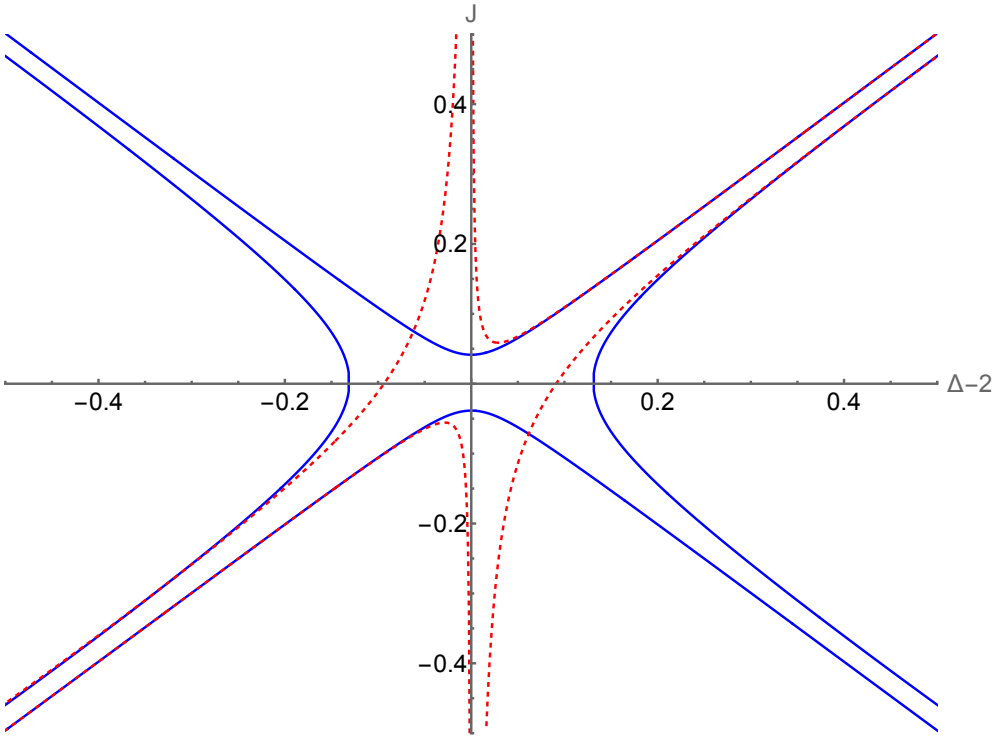

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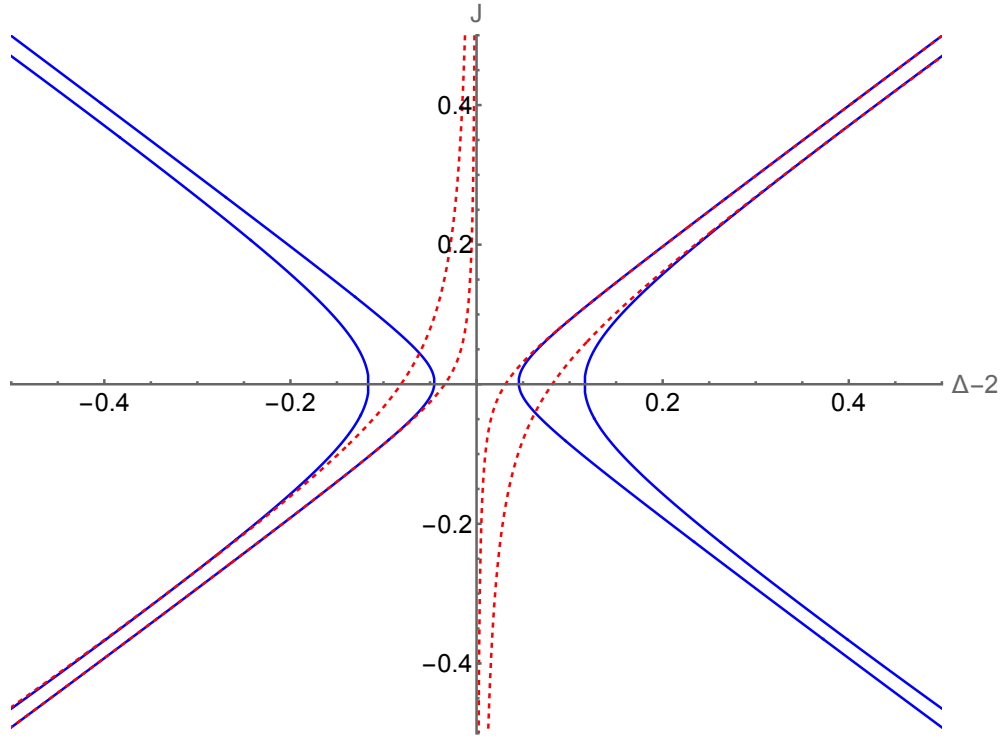

Outlook

- Higher-order; include more intersections
- Odd-spin or nonzero transverse spin sector.
- Connection between divergences of DGLAP anomalous dimension and soft theorems (for squared form factors)?
- What does the shape of the subleading trajectories near $J = 0$ tell us about the Regge limit at subleading power?
- Relation to other approaches for BFKL physics: null Wilson lines [Caron-Huot '13], color dipole [Balitsky, Kazakov, Sobko '13], Glauber SCET [Rothstein, Stewart '16].
- What is the OPE between general detector operators?
- Can one study detector operators in gravity? [Herrmann, Kologlu, Moult '24]
- Can one formulate matching relations between hadron detectors and gluon/quark detectors?

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