

Nonrelativistic
Conformal Field Theory
and nuclear reactions

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December 11, 2024

Plan

- Nonrelativistic conformal symmetry
- Nonrelativistic CFTs
- Fermions at unitarity
- Neutrons and “UnNuclear Physics”

Refs.: Y. Nishida, DTS 0706.3746

H.-W. Hammer, DTS 2103.12610

S.D. Chowdhury, R. Mishra, DTS 2309.15177

Schrödinger group

- Symmetries of the free Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi$$

- Phase rotation $M = mN$: $\psi \rightarrow \psi' = e^{i\alpha}\psi$
- space \mathbf{P} translation: $\psi(t, \mathbf{x}) \rightarrow \psi'(t, \mathbf{x}) = \psi(t, \mathbf{x} + \mathbf{a})$
time translation, rotations J_{ij}
- Galilean boosts \mathbf{K} : $\psi(t, \mathbf{x}) \rightarrow e^{im\mathbf{v}\cdot\mathbf{x} - \frac{i}{2}mv^2t} \psi(t, \mathbf{x} - \mathbf{v}t)$
- Dilatation D : $\psi(t, \mathbf{x}) \rightarrow \lambda^{3/2}\psi(\lambda^2t, \lambda\mathbf{x})$

“Proper conformal transformation”

$$C : \psi(t, \mathbf{x}) \rightarrow \frac{1}{(1 + \alpha t)^{3/2}} \exp\left(\frac{i m \alpha x^2}{2(1 + \alpha t)}\right) \psi\left(\frac{t}{1 + \alpha t}, \frac{\mathbf{x}}{1 + \alpha t}\right)$$

Schrödinger algebra

$$[X, Y]$$

$X \backslash Y$	P_j	K_j	D	C	H
P_i	0	$-i\delta_{ij}M$	$-iP_i$	$-iK_i$	0
K_i	$i\delta_{ij}M$	0	iK_i	0	iP_i
D	iP_j	$-iK_j$	0	$-2iC$	$2iH$
C	iK_j	0	$2iC$	0	iD
H	0	$-iP_j$	$-2iH$	$-iD$	0

Schrödinger algebra

$$[X, Y]$$

$X \backslash Y$	P_j	K_j	D	C	H
P_i	0	$-i\delta_{ij}M$	$-iP_i$	$-iK_i$	0
K_i	$i\delta_{ij}M$	0	iK_i	0	iP_i
D	iP_j	$-iK_j$	0	$-2iC$	$2iH$
C	iK_j	0	$2iC$	0	iD
H	0	$-iP_j$	$-2iH$	$-iD$	0

$$[N, \text{anything}] = 0$$

Schrödinger algebra

$$[X, Y]$$

$X \backslash Y$	P_j	K_j	D	C	H
P_i	0	$-i\delta_{ij}M$	$-iP_i$	$-iK_i$	0
K_i	$i\delta_{ij}M$	0	iK_i	0	iP_i
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C	iK_j	0	$2iC$	0	iD
H	0	$-iP_j$	$-2iH$	$-iD$	0

SO(2,1)

$$[N, \text{anything}] = 0$$

Schrödinger generators and currents

$$n(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x}), \quad j_i(\mathbf{x}) = -\frac{i}{2}(\psi^\dagger(\mathbf{x})\partial_i\psi(\mathbf{x}) - \partial_i\psi^\dagger(\mathbf{x})\psi(\mathbf{x}))$$

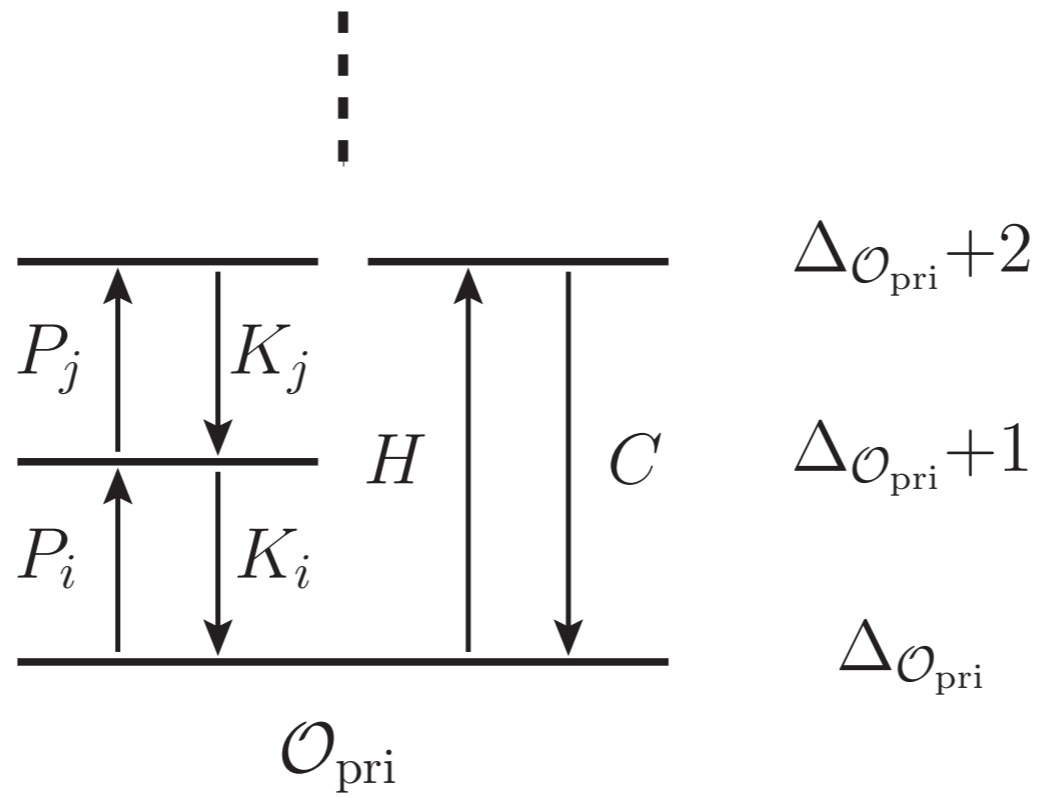
$$N = \int d\mathbf{x} n(\mathbf{x}), \quad P_i = \int d\mathbf{x} j_i(\mathbf{x}), \quad M_{ij} = \int d\mathbf{x} (x_i j_j(\mathbf{x}) - x_j j_i(\mathbf{x})),$$
$$K_i = \int d\mathbf{x} x_i n(\mathbf{x}), \quad C = \int d\mathbf{x} \frac{x^2}{2} n(\mathbf{x}), \quad D = \int d\mathbf{x} x_i j_i(\mathbf{x}),$$

Nonrelativistic CFTs

Y. Nishida, DTS, 2007

- NRCFTs are QFTs with Schrödinger symmetry
- local operators $O(\vec{x})$ characterized by charge (mass) and dimension $[D, O(0)] = i\Delta_o O(0)$, $[M, O(0)] = iN_o O(0)$
example: ψ $N_\psi = 1$, $\Delta_\psi = \frac{3}{2}$
- primary operators: $[K_i, O(\vec{0})] = [C, O(\vec{0})] = 0$
- Constraints from conformal invariance:

$$\langle TO(t, \vec{x}) O^\dagger(0,0) \rangle = \frac{c}{t^{\Delta_o}} \exp\left(\frac{im_o x^2}{2t}\right) \quad \begin{array}{l} [E] = 2 \\ [p] = 1 \end{array}$$



Example of NRCFTs

- Free particles

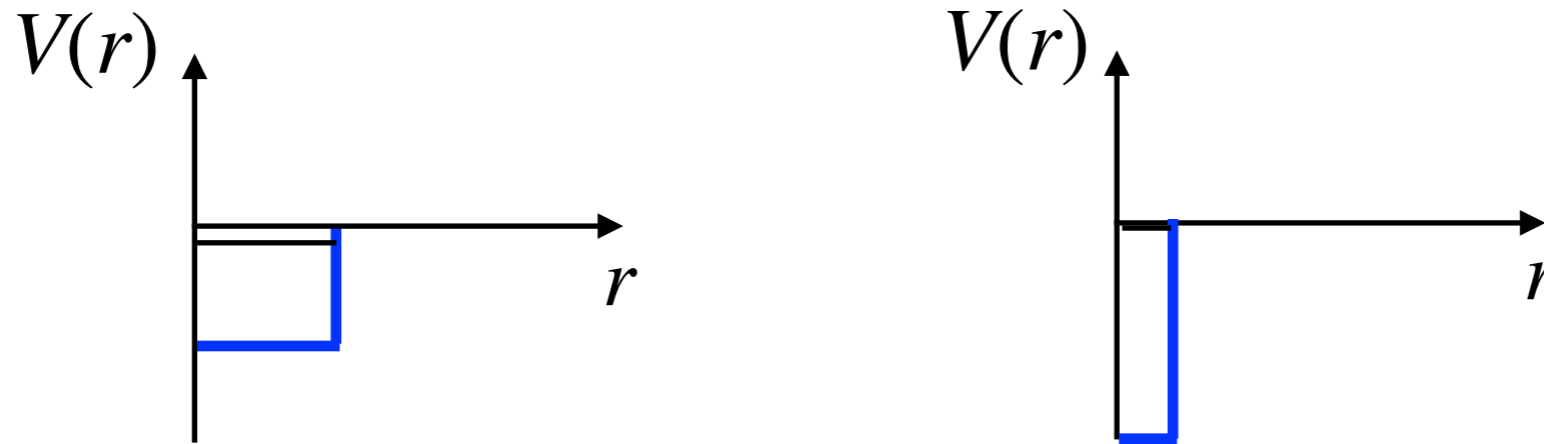
$$S = \int dt d^d x \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi \quad \Delta_\psi = \frac{d}{2} \quad N_\psi = -1$$

- nonrelativistic anyons (two spatial dimensions)
- Spin-1/2 fermions at unitarity
- realized in cold atom experiments, but also approximately by neutrons

Fermions at unitarity

- $S = \int dt d^d \mathbf{x} \left(i\psi^\dagger \partial_t \psi - \frac{1}{2} |\nabla \psi|^2 - \lambda \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \right)$
- NR power counting: $[E] = 2$, $[p] = 1$, $[\psi] = \frac{d}{2}$
- $d = 2 + \epsilon$: $\beta(\lambda) = \epsilon\lambda + \frac{1}{2\pi}\lambda^2$
- interacting CFT at $\lambda_* = -2\pi\epsilon$
- Does the fixed point survive at $d = 3$?
 - yes, this is the so-called “fermion at unitarity”

“Unitarity regime”



- Take a potential of a certain shape, e.g.,
$$V(r) = -V_0 \text{ for } r < r_0, \quad 0 \text{ for } r > r_0$$
- fine-tune the depth so that there is one “bound state” at zero energy

$$V_0 = \frac{\pi^2 \hbar^2}{8m} \frac{1}{r_0^2}$$

- Then let $r_0 \rightarrow 0$: “unitarity regime” s-wave scattering saturates unitarity

Scattering length and unitarity regime

- This situation corresponds to low-energy resonant scattering in quantum mechanics
- s-wave scattering amplitude given by scattering length a and effective range r_0 :

$$f(k) = \frac{1}{-ik + \frac{1}{a} + \frac{1}{2}kr_0^2}$$

- Unitarity regime: $a \rightarrow \infty$
 $r_0 \rightarrow 0$
 - no dimensionful length scale

- The unitarity regime can be understood directly, without taking the limit $r_0 \rightarrow 0$

Quantum-mechanical approach

- Wave function of m spin-up and n spin-down fermions
 $\psi(\mathbf{x}_1, \dots, \mathbf{x}_m; \mathbf{y}_1, \dots, \mathbf{y}_n)$
- ψ antisymmetric under exchanging two \mathbf{x} 's or \mathbf{y} 's
- When one spin-up and one spin-down fermions approach each other:

$$\psi(\mathbf{x}, \mathbf{y}) = \frac{C}{|\mathbf{x} - \mathbf{y}|} + O(|\mathbf{x} - \mathbf{y}|) + \dots$$

- $H = -\frac{1}{2} \sum_a \frac{\partial^2}{\partial \mathbf{x}_a^2} - \frac{1}{2} \sum_a \frac{\partial^2}{\partial \mathbf{y}_a^2}$ (ignore Dirac's δ -function)

What are the local operators?

- First example: annihilation operator in second quantized formulation of QM
- $\langle 0 | \hat{\psi}(\mathbf{x}) | \Psi_{1\text{-body}} \rangle = \Psi(\mathbf{x})$
- This is a charge-1 operator, dimension=3/2

Charge-2 local operator

- Second-quantized formulation of QM:

$$\langle 0 | \psi_{\uparrow}(\mathbf{x})\psi_{\downarrow}(\mathbf{y}) | \Psi_{2\text{-body}} \rangle = \Psi(\mathbf{x}, \mathbf{y})$$

- Limit $\mathbf{y} \rightarrow \mathbf{x}$ does not exist:

$$\langle 0 | \psi_{\uparrow}(\mathbf{x})\psi_{\downarrow}(\mathbf{x}) | \Psi \rangle = \Psi(\mathbf{x}, \mathbf{x}) = \infty$$

- but one can define

$$O_2(\mathbf{x}) = \lim_{\mathbf{y} \rightarrow \mathbf{x}} |\mathbf{x} - \mathbf{y}| \psi_{\uparrow}(\mathbf{x})\psi_{\downarrow}(\mathbf{y})$$

- then

$$\langle 0 | O_2(\mathbf{x}) | \Psi \rangle = \lim_{\mathbf{y} \rightarrow \mathbf{x}} |\mathbf{x} - \mathbf{y}| \Psi(\mathbf{x}, \mathbf{y}) = \text{finite}$$

Dimension of O_2

- $O_2(\mathbf{x}) = \lim_{\mathbf{y} \rightarrow \mathbf{x}} |\mathbf{x} - \mathbf{y}| \psi_{\uparrow}(\mathbf{x}) \psi_{\downarrow}(\mathbf{y})$
- $\Delta[O_2] = 2\Delta[\psi] - 1 = 2$
- cf free theory: $\Delta[\psi\psi] = 3$

Charge-3 operator

- Need to know short distance behavior of $\Psi(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y})$
- 3-body problem solved by Efimov ~ 1970

$$\Psi(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}) \sim R^{-0.2273} f(\alpha, \hat{\rho}, \hat{r})$$

$$R^2 = |\mathbf{x}_1 - \mathbf{x}_2|^2 + |\mathbf{x}_1 - \mathbf{y}|^2 + |\mathbf{x}_2 - \mathbf{y}|^2$$

$\alpha, \hat{\rho}, \hat{r} = 5$ hyperangles

- Charge-3 operator

$$O_3(\mathbf{x}) \sim \lim_{\mathbf{x}_2 \rightarrow \mathbf{x}} \lim_{\mathbf{y} \rightarrow \mathbf{x}} R^{0.2273} \psi_{\uparrow}(\mathbf{x}) \psi_{\uparrow}(\mathbf{x}_2) \psi_{\downarrow}(\mathbf{y})$$

- $\Delta[O_3] = 4.2727$ cf free theory: $[\psi_{\downarrow} \psi_{\uparrow} \nabla \psi_{\uparrow}] = \frac{11}{2}$

Dimensions of charge-3 operators

$$\Delta = \frac{5}{2} + s \quad \text{where } s \text{ solves an equation}$$

$$l = 0: \quad s \cos\left(\frac{\pi}{2}s\right) + \frac{4}{\sqrt{3}} \sin\left(\frac{\pi}{6}s\right) = 0$$

$$\Delta = 4.666, 7.627, 9.614, \dots, 2n + \frac{7}{2}$$

$$l = 1: \quad (s^2 - 1)\sin\left(\frac{\pi}{2}s\right) + \frac{4}{\sqrt{3}}s \cos\left(\frac{\pi}{6}s\right) - 4 \sin\left(\frac{\pi}{6}s\right) = 0$$

$$\Delta = 4.273, 6.878, 8.216, \dots, 2n + \frac{5}{2}$$

Charge-4 operator

- Dimension of operator with particle number $N > 3$ can only be obtained numerically
- Nishida and DTS 2007: each primary operator O corresponds to an eigenstate of N_O unitary fermions in a harmonic trapping potential

$$\underbrace{(H + \omega^2 C)}_{H_{\text{osc}}} e^{-H/\omega} O^\dagger(\vec{0}) |0\rangle = \Delta_O \omega e^{-H} O^\dagger(\vec{0}) |0\rangle$$

- Numerics: $\Delta[O_4] = 5.0 \pm 0.1$ (cf. free theory: 8)

Two-point functions

- One can compute two-point functions by inserting a complete set of states

$$\langle 0 | O(t, \mathbf{x}) O^\dagger(0) | 0 \rangle = \sum_n \langle 0 | O(0) | n \rangle e^{-iE_n t + i\mathbf{P}_n \cdot \mathbf{x}} \langle n | O^\dagger(0) | 0 \rangle$$

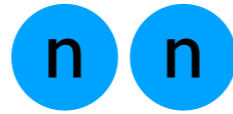
- Also constrained by Schrödinger symmetry
- 2-point function

$$\langle O(t, \mathbf{x}) O^\dagger(0, \mathbf{0}) \rangle = \frac{C}{t^{\Delta_o}} \exp\left(\frac{iM_o x^2}{2t}\right)$$

- In momentum space

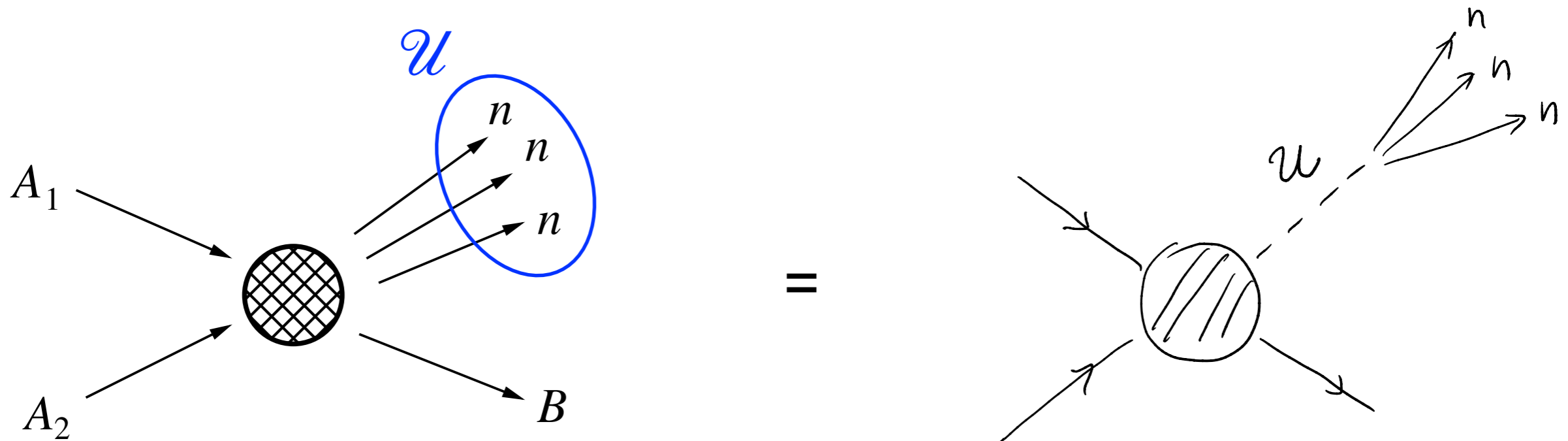
$$\langle OO^\dagger \rangle(\omega, \mathbf{p}) \sim \left(\frac{\mathbf{p}^2}{2M_o} - \omega\right)^{\Delta_o - 5/2}$$

NRCFT in real world: neutrons



- $a \approx -19$ fm, $r_0 \approx 2.8$ fm
- NRCFT in energy range between $\hbar^2/ma^2 \sim 0.1$ MeV and $\hbar^2/mr_0^2 \sim 5$ MeV
- Consequence: power-law behavior in processes with final state neutrons
- “Unnuclear Physics” [Hammer, DTS 2021](#) nonrelativistic version of Georgi’s “unparticle physics”

“UnNuclear physics”

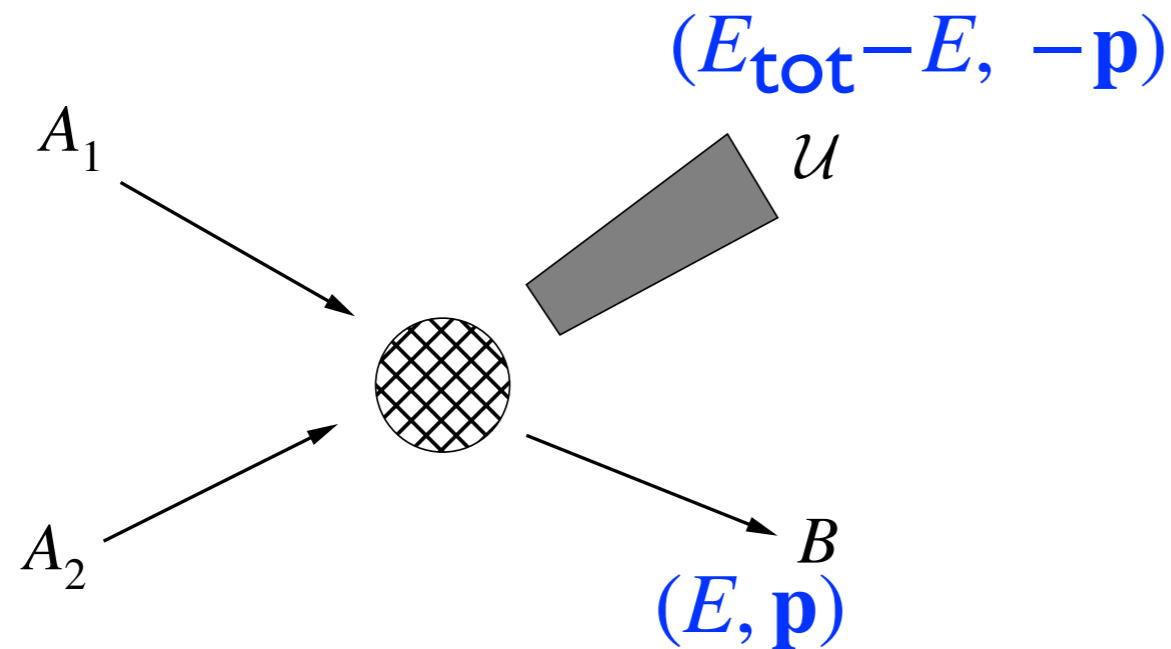


$$P(A_1 + A_2 \rightarrow B + 3n) = P(A_1 + A_2 \rightarrow B + \mathcal{U})P(\mathcal{U} \rightarrow 3n)$$

when energy scale of primary reaction is larger than $\mathcal{U} \rightarrow 3n$

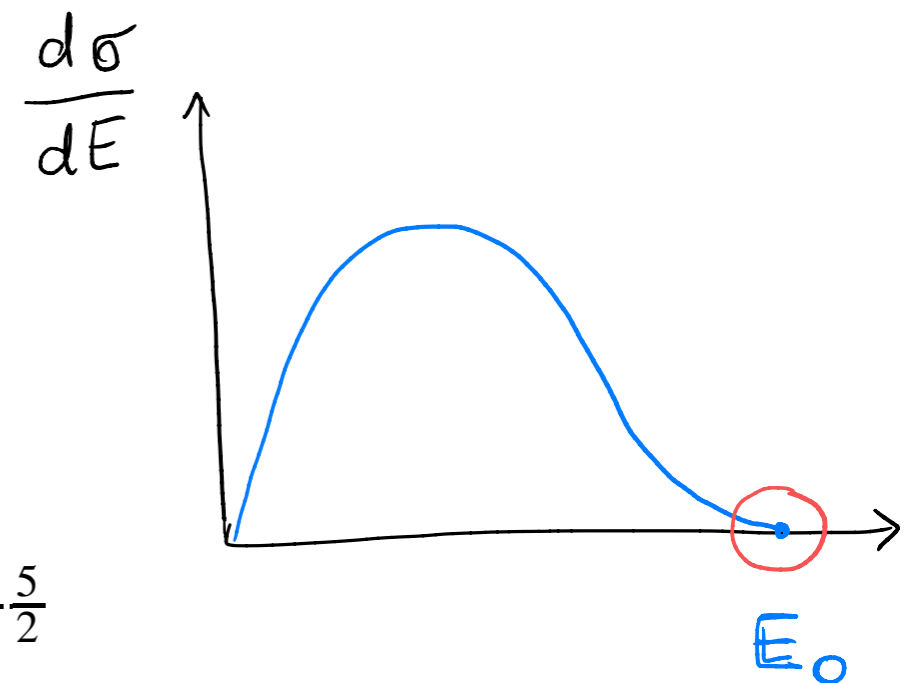
\mathcal{U} = “unnucleus” = field in NRCFT

Rates of UnNuclear Processes



$$E_{\text{tot}} = E + E_{\mathcal{U}}$$

- $$\frac{d\sigma}{dE} \sim |\mathcal{M}|^2 \sqrt{E} \times \underbrace{\text{Im } G_{\mathcal{U}}(E_{\text{tot}} - E, \mathbf{p})}_{(E_0 - E)^{\Delta - \frac{5}{2}}}$$



- Near end point: $\frac{d\sigma}{dE} \sim (E_0 - E)^{\Delta - \frac{5}{2}}$, $\Delta = \text{dimension of } \mathcal{U}$

Rates of nuclear reactions

- $\frac{d\sigma}{dE} \sim (E_0 - E)^\alpha \quad \alpha = \Delta - \frac{5}{2}$

	Δ	α	
• ${}^3\text{H} + {}^3\text{H} \rightarrow {}^4\text{He} + 2\text{n}$	2	-0.5	Watson-Migdal 1950's
• $\pi^- + {}^3\text{H} \rightarrow \gamma + 3\text{n}$	4.27	1.77	
• ${}^4\text{He} + {}^8\text{He} \rightarrow {}^8\text{Be} + 4\text{n}$	5.0	2.5	

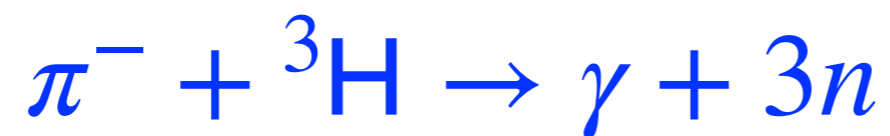
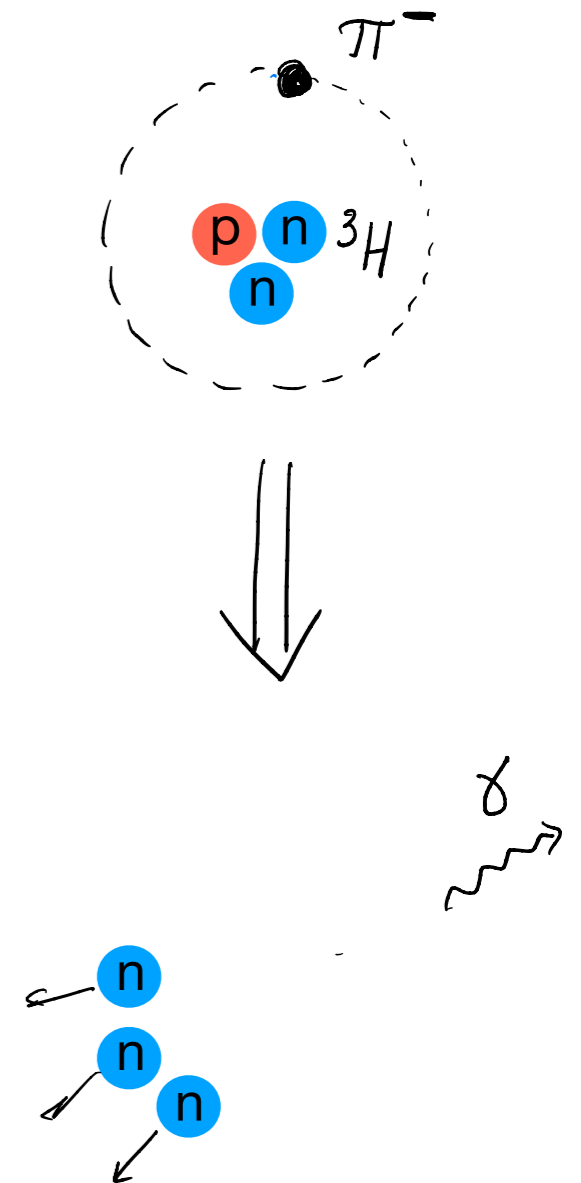
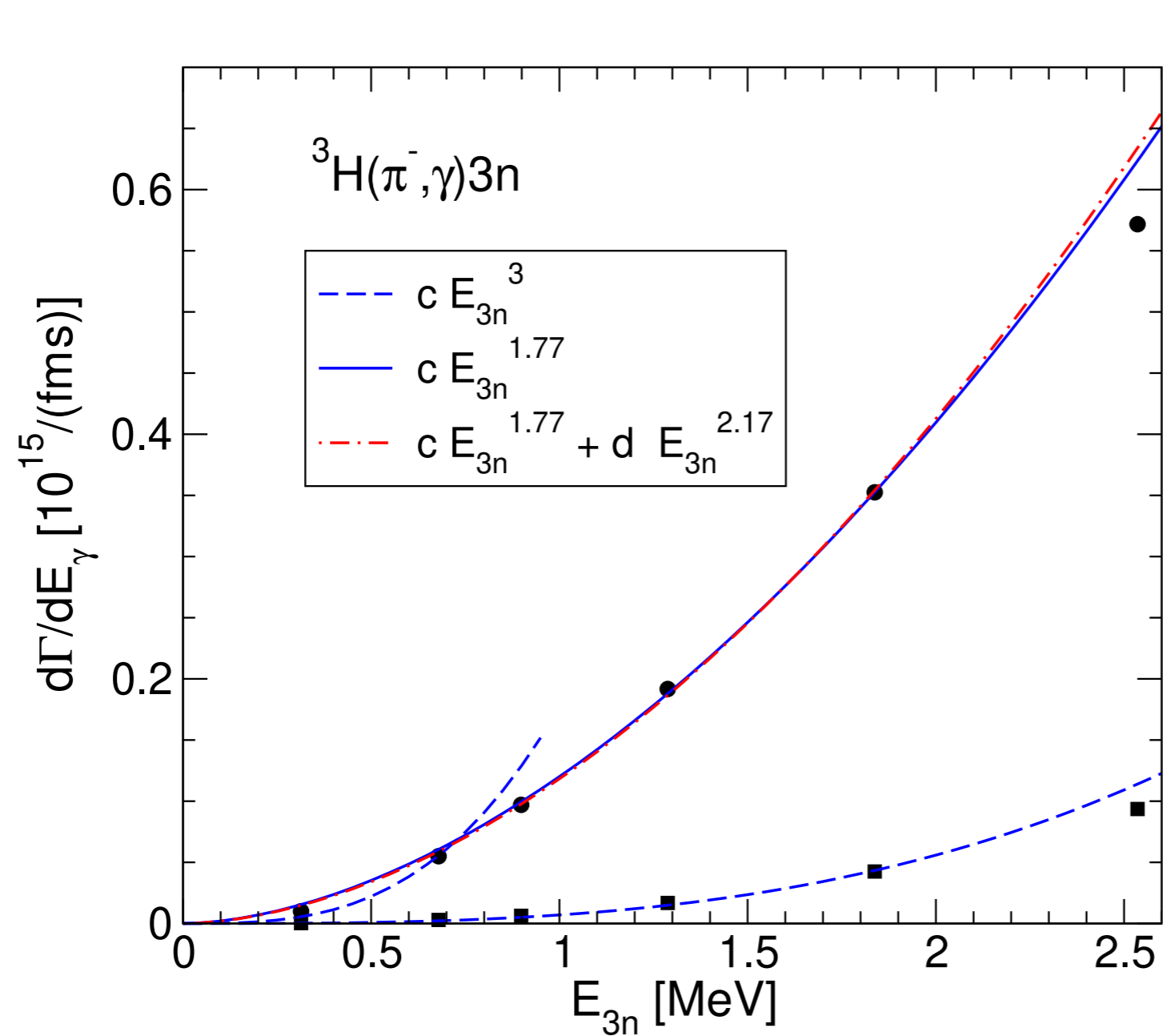
Watson-Migdal

- E_{nn} is the kinetic energy of 2 final-state neutrons in their center-of-mass frame

$$\frac{d\sigma}{dE_{nn}} \sim \frac{\sqrt{E_{nn}}}{E_{nn} + \epsilon_n}$$

$$\epsilon_n = \frac{\hbar^2}{m_n a_{nn}^2} \approx 0.12 \text{ MeV}$$

Comparison with “experiment”



Accurate nucleon-nucleon potential with charge-independence breaking

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(Received 15 August 1994)

We present a new high-quality nucleon-nucleon potential with explicit charge dependence and charge asymmetry, which we designate Argonne v_{18} . The model has a charge-independent part with 14 operator components that is an updated version of the Argonne v_{14} potential. Three additional charge-dependent and one charge-asymmetric operators are added, along with a complete electromagnetic interaction. The potential has been fit directly to the Nijmegen pp and np scattering database, low-energy nn scattering parameters, and deuteron binding energy. With 40 adjustable parameters it gives a χ^2 per datum of 1.09 for 4301 pp and np data in the range 0–350 MeV.

Nuclear potential used

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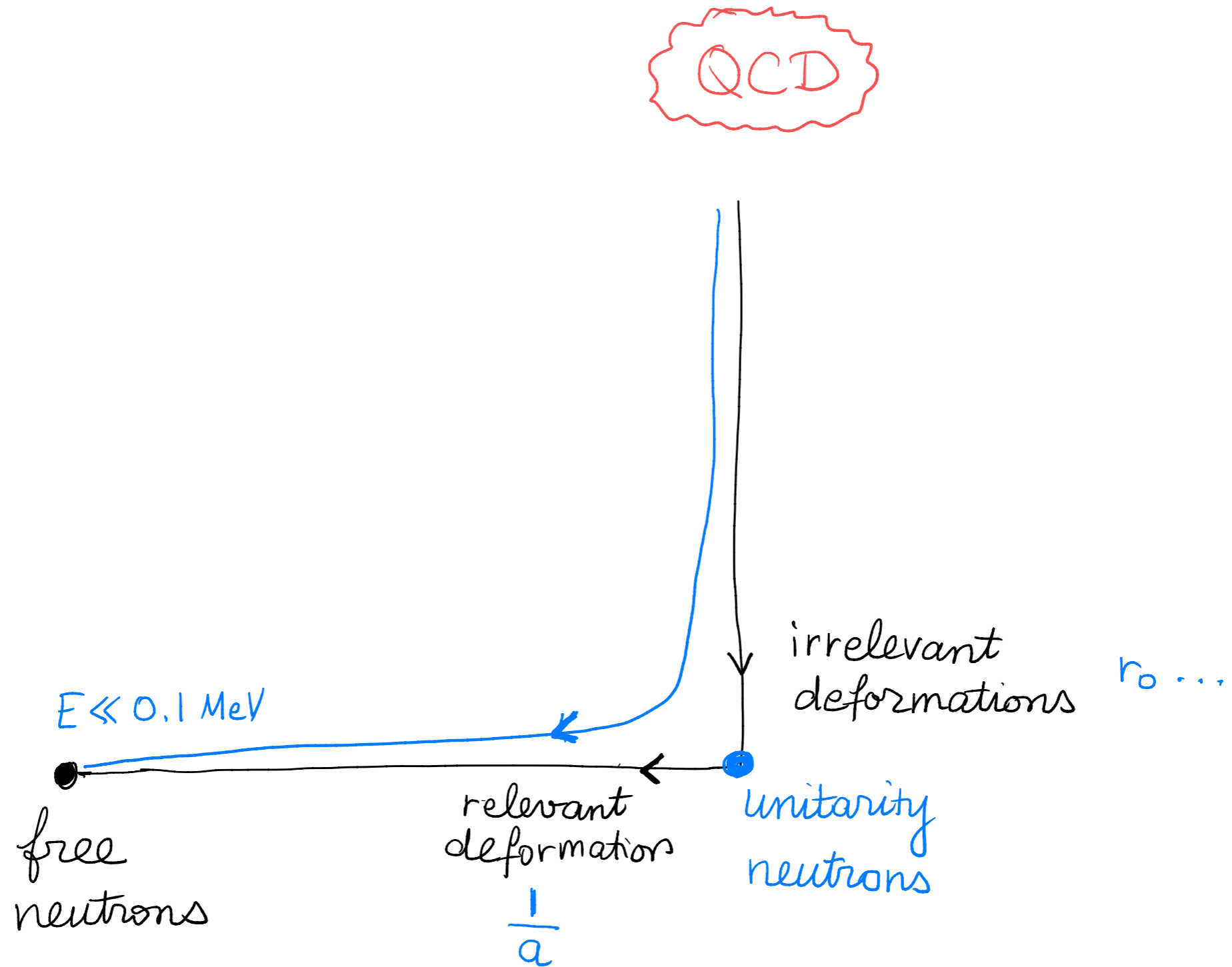
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Nuclear potential used

NRCFT as a transient fixed point



Deformations of NRCFT

- Dimensional counting

$$S = \int \underbrace{dt}_{-2} \underbrace{d^3x}_{-3} (\mathcal{L}_{\text{CFT}} + \text{deformations})$$

- operators with $\text{dim} < 5$: relevant; $\text{dim} > 5$: irrelevant
- one relevant deformation: $[O_2^\dagger O_2] = 4$
- Leading Galilean-invariant irrelevant deformation:

$$O_2^\dagger \left(\partial_t + \frac{\nabla^2}{4} \right) O_2 \quad \text{dim} = 6$$

Away from conformality

- Finite scattering length and effective range can be treated as perturbation away from NRCFT

$$L = L_{\text{CFT}} + \frac{1}{a} O_2^\dagger O_2 - r_0 O_2^\dagger \left(i\partial_t + \frac{1}{4} \nabla^2 \right) O_2$$

- Contribution to $\langle O_3 O_3^\dagger \rangle$ can be computed using [conformal perturbation theory](#)
S.D. Chowdhury, R. Mishra, DTS 2309.15177

- $$\frac{d\sigma}{dE} \sim \omega^{\Delta-5/2} \left(1 + \frac{c_1}{a_0 \sqrt{m\omega}} + c_2 r_0 \sqrt{m\omega} \right)$$

Conformal perturbation theory

- $$\langle O_3(x)O_3^\dagger(0) \rangle = \frac{1}{Z} \int \mathcal{D}\psi O_3(x)O_3^\dagger(0) e^{iS_{\text{CFT}} + \frac{i}{a} \int_y O_2^\dagger(y)O_2(y)}$$

$$= \langle O_3(x)O_3^\dagger(0) \rangle_0 + \frac{1}{a} \int dy \langle O_3(x)O_2^\dagger(y)O_2(y)O_3^\dagger(0) \rangle_0$$

\nearrow
 $|1\rangle\langle 1|$

\uparrow
 $|3\rangle\langle 3|$

$$\frac{d\sigma}{dE} \sim \omega^{\Delta-5/2} \left(1 + \frac{c_1}{a\sqrt{m\omega}} \right) \quad \begin{array}{l} \Delta = 4.273 \\ c_1 = 2.642 \end{array}$$

Effective-range correction

- Effective range correction proportional to

$$\frac{\Gamma\left(\frac{d}{2} + s - 2\right)}{\Gamma\left(3 - \frac{d}{2}\right)\Gamma(d - 3)} = 0$$

- Vanishes at physical dimension $d = 3$, but not in fractional spatial dimension
- we do not understand this
- Also observed for large-charge operators [Beane Orlando Reffert \(2024\)](#)

Open questions

- Can one resum all $1/a$ corrections and determine the correlator along the whole RG flow?
- Can be done for O_2 (Watson-Migdal)

$$\langle O_2 O_2^\dagger \rangle = \left(\sqrt{\frac{p^2}{4} - p_0} - \frac{1}{a} \right)^{-1}$$

but needs to be done for charge-3 operators

- Charge-4 operators and higher?
- Large-charge limit? [Beane Orlando Reffert 2403.18898](#)

Conclusion

- NR conformal field theories; example: fermions at unitarity
- Approximately realized by neutrons in nuclear physics
- Leads to a power-law behavior of differential cross sections of certain processes near threshold
- Nonrelativistic conformal perturbation theory: full power of NRCFT still to be explored
- factorization theorems?
- final-state correlators?