Quantum Theory of the Lemaître Model for Gravitational Collapse

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LEMAITRE FOLLOWS TWO PATHS TO TRUTH

The Famous Physicist, Who Is Also a Priest, Tells Why He Finds No Conflict Between Science and Religion

By DUNCAN AIKMAN

PASADENA.

"THERE is no conflict between religion and science," the Abbé Lemaitre has been telling audiences over and over again in this country and then proving it by explaining the aims of both. His view is interesting and important not because he is a Catholic priest, not because he is one of the leading mathematical physicists of our time, but because he is both. Here is a man who believes firmly in the Bible as a revelation from on high, but who develops a theory of the universe without the slightest regard for the teachings of revealed religion on genesis. And there is no conflict!

Such an attitude would have been preposterous to a Victorian physicist. Either you accept the whole Book of Genesis and therefore shut yourself out of the world of science, or you accept science and repudiate the prophets as expositors of the manner in which the universe began. Today the physicist is meeker. Behind his formulas there is something that is still veiled. He is half mystic and ready to admit that the universe may reveal itself in other ways than in mathematical equations or the bands and lines of a spectograph. The abbé, therefore, follows the trend of modern thinking and derives from it more than ordinary satisfaction because he happens to be trained in theology as well as in mathematical physics.

Lemaitre, like Eddington, finds that science and religion supplement each other. Science can never study the universe as a whole. It selects a small portion, as much as it can handle, and then makes deductions. To a cosmologist the earth and Mars are only planets wheeling around the sun. Are they inhabited? Are they washed by air and water? Why were they created? Is there purpose in the universe? Science is indifferent to such questions, but not religion.

that there must be authentic religious dogma in the binomial theorem. Nevertheless a lot of otherwise intelligent and well-educated men do go on believing or at least acting on such a belief. When they find the Bible's scientific references wrong, as they often are, they repudiate it utterly. Should a priest reject relativity because it

that it took perhaps ten thousand million years to create what we think is the universe. Genesis is simply trying to teach us that one day in seven should be devoted to rest, worship and reverence—all necessary to salvation."

"And that story about Jonah and the big fish?

"I admit that a whale cannot

powers with which they are credited in the Bible.

"If scientific knowledge were necessary to salvation," he says, "it would have been revealed to the writers of the Scriptures and they would have set it down in their verses. For instance, the doctrine of the Trinity is much more abstruse than anything in relativity or

question of salvation. On other questions they were as wise or as ignorant as their generation. Hence it is utterly unimportant that errors of historic and scientific fact should be found in the Bible, especially if errors relate to events that were not directly observed by those who wrote about them. The idea that because they were right in their doctrine of immortality and salvation they must also be right on all other subjects is simply the fallacy of people who have an incomplete understanding of why the Bible was given to us at all."

Lemaitre tells of a classroom scene in which he figured. An old father was expounding at the desk. Before him sat the lad who was to discover the expanding universe and who, even then, was brimful of science. In his eagerness the lad read into a passage of Genesis an anticipation of modern science.

"I pointed it out," says Lemaitre, "but the old Father was skeptical. 'If there is a coincidence,' he decided, 'it is of no importance. Also if you should prove to me that it exists I would consider it unfortunate. It will merely encourage more thoughtless people to imagine that the Bible teaches infallible science, whereas the most we can say is that occasionally one of the prophets made a correct scientific guess.' "

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THERE is, the abbé admits, a varying sense of conflict between science and religion in the different branches of science. "The biologists seem to have peculiar difficulties," he reasons. "There is every reason for this. They have only recently discovered a few guiding laws and principles. Hence, in the past their studies have been confusing rather than enlightening. In a way their subject-matter has been gross.

"But give the biologist more laws like those of the Abbé Mendel and a new spirit is bound to awaken. The sense that this is a morally

Figure credit: Bueno Voz Católica

L'Univers en expansion

PAR

M. l'Abbé G. LEMAITRE

Introduction et résumé

Nous ne nous proposons pas dans ce travail de discuter les hypothèses sur lesquelles se fonde la théorie de l'expansion de l'Univers, ni la valeur des confirmations astronomiques qui l'étayent. Une telle discussion nous paraît actuellement prématurée et ne pourrait certes pas arriver à des conclusions définitives dans l'état actuel de la théorie et des observations.

La théorie peut être développée de deux façons : par l'étude de solutions exactes des équations de la gravitation, fournissant des modèles simplifiés, ou par le développement approché de la solution de problèmes plus complexes. Il nous paraît utile de ne pas mélanger ces deux méthodes, et dans ce travail nous ne nous occuperons que de solutions mathématiquement exactes. Lorsque nous voudrons les appliquer aux problèmes réels, nous aurons à faire appel à l'intuition physique pour réduire un problème trop complexe à un modèle simplifié, dont nous avons la solution. Plusieurs de nos résultats semblent pouvoir servir de points de départ à des méthodes de développement en série que nous espérons traiter dans un travail ultérieur.

Dans les deux premiers paragraphes, nous donnons en détail les calculs de tenseurs, dont nous aurons besoin, et que nous résumons au § 3, en introduisant des notations qui mettent en évidence l'analogie des résultats relativistes avec les formules classiques.

Nous introduisons ensuite la notion de champ quasi-statique qui permet de généraliser immédiatement des solutions statiques connues en y permettant des variations adiabatiques. Nous donnons une solution probablement nouvelle pour le cas d'une sphère à pression radiale constante, et nous en servons pour mettre en évidence le paradoxe de Schwarzschild et montrer que la limitation plus sévère du rayon d'une masse donnée introduite par la solution du problème intérieur s'évanouit lorsqu'on

Contents

Lemaître (1933)

Main result of Lemaître's 1933 paper *L'univers en expansion*: derivation of a spherically symmetric dust solution of Einstein's equations. In addition:

- Possible mechanism to describe the formation of clusters of galaxies (*nébuleuses*)
- Proof that the Schwarzschild horizon at $r = 2GM/c^2$ is only a coordinate singularity (Schwarzschild solution is the vacuum limit of the LTB solution)
- Introduction of Misner–Sharp mass thirty years before Misner and Sharp

The Lemaître–Tolman–Bondi (LTB) model

LTB model: spherically-symmetric solution of the Einstein equations with non-rotating dust of mass density ϵ as its source (for constant density we have the special case of the Oppenheimer–Snyder scenario).

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 + \frac{R'^2(\rho)}{1+2f(\rho)}\,\mathrm{d}\rho^2 + R^2(\rho)\,\mathrm{d}\Omega^2\,,$$

$$\text{with} \quad \frac{8\pi G}{c^2}\epsilon = \frac{F'}{R^2 R'} \quad \text{and} \quad \frac{\dot{R}^2}{c^2} = \frac{F}{R} + 2f,$$

where τ is the dust proper time and ρ the radial coordinate that labels the dust shells comprising the dust cloud; $F(\rho)$ is twice the active gravitational mass inside the shell with label ρ.

Topic here: Quantization – how to proceed?

réparties avec un univers d'Einstein parfaitement homogène, nous avons à considérer le réseau de cellules formé par les zones neutres séparant les condensations. L'univers homogène doit, pour ainsi dire, être tangent en ces points à l'univers présentant des condensations, et la pression normalement aux zones neutres, doit être la pression adoptée pour l'univers homogène. Alors l'équilibre, ou l'expansion de l'univers homogène tangent, nous fait connaître l'équilibre ou l'expansion du réseau de zones neutres.

Les deux univers peuvent avoir des masses différentes ou des volumes différents. On ne peut rien conclure de cela, le facteur déterminant est la pression à la zone neutre.

L'intérêt de ce résultat est qu'il est complètement indépendant du processus particulier suivant lequel se développent les condensations. Il donne le moyen pour tout processus particulier de prévoir l'effet de ce processus sur l'équilibre de l'univers.

En particulier, si la pression est nulle et reste nulle aux zones neutres, les condensations ne modifient pas l'équilibre. La pression radiale à la zone neutre est la densité d'énergie traversant cette zone, et mesure donc l'intensité des échanges entre les condensations. Nous avons appelé une diminution de ces échanges d'énergie, une « stagnation de l'univers ». Seul ce processus de stagnation peut déterminer la rupture de l'équilibre dans le sens de l'expansion.

8. Condensations dans l'univers en expansion.

Dans les applications à l'univers réel la pression est généralement négligeable vis-à-vis de la densité. Dans le cas de l'équilibre nous avons bien dû en tenir compte, puisque l'étude d'une rupture d'équilibre dépend naturellement de forces minimes, mais pour l'étude de l'expansion de l'univers et le développement de condensations au cours de l'expansion, nous pouvons la négliger.

Dans ce cas, l'équation (3.4) nous apprend que m n'est fonction que de χ, et l'équation (3.8), pour $p = \tau = 0$, que c n'est fonction que de t.

Moyennant un changement de variable, nous pouvons donc supposer c constant et poser

$$c = c_0$$

Nous avons alors, par (3.6)

$$\frac{1}{a}\frac{\partial r}{\partial \chi} = f(\chi),$$

et (3.1) devient

$$ds^2 = -\left(\frac{\partial r}{\partial \chi}\right)^2 \frac{d\chi^2}{f^2(\chi)} - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + c_0^2 dt^2 \tag{8.1}$$

où r est une fonction de χ et de t satisfaisant à (3.5)

(8.2) $$\left(\frac{\partial r}{\partial t}\right)^2 = -c^2[1 - f^2(\chi)] + \frac{2Km}{r} + \frac{\lambda c^2}{3} r^2$$

où par (3.3)

(8.3) $$4\pi\rho\, r^2 \frac{\partial r}{\partial \chi} = \frac{dm}{d\chi}.$$

Enfin, l'équation (3.7) devient

(8.4) $$\frac{\partial^2 r}{\partial t^2} = -\frac{Km}{r^2} + \frac{\lambda c^2}{3} r.$$

L'élément de longueur à un instant t est d'après (8.1)

$$d\sigma^2 = \frac{dr^2}{f^2(\chi)} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

Lorsque $f(\chi) = 1$, la géométrie est donc euclidienne. Les équations ne diffèrent alors des équations de la mécanique classique que par l'introduction de la répulsion cosmique et, en outre, par le fait que la constante d'énergie dans (8.2) qui, au point de vue classique, pourrait avoir une valeur arbitraire, est maintenant nulle.

Dans le cas général, on peut encore considérer r comme la distance à l'origine, et la constante d'énergie en chaque point matériel, c'est-à-dire pour chaque valeur de χ, peut être choisie arbitrairement. Mais alors la géométrie n'est plus euclidienne. On peut en faire une carte dans un espace euclidien où les longueurs normales au rayon vecteur sont représentées en vraie grandeur. Les longueurs suivant le rayon vecteur sont alors représentées à une échelle

$$\frac{dr}{d\sigma} = f(\chi).$$

L'échelle des longueurs radiales ne dépend que de χ, c'est-à-dire reste la même pour chaque point matériel pendant tout son mouvement, et elle est liée à la constante d'énergie dans l'équation du mouvement de ce point d'après l'équation (8.2).

La coordonnée χ peut naturellement être choisie arbitrairement. Lorsque $f(\chi)$ est inférieur ou égal à un, on pourra choisir la coordonnée χ de telle sorte que

$$f(\chi) = \cos\chi,$$

alors (8.2) s'écrira plus simplement

(8.21) $$\left(\frac{\partial r}{\partial t}\right)^2 = -c^2\sin^2\chi + \frac{2Km}{r} + \frac{\lambda c^2}{3} r^2.$$

Ce choix des coordonnées convient lorsque l'espace est fermé. Pour un espace du type simplement elliptique, tout l'espace est décrit lorsque χ varie de 0 à $\frac{\pi}{2}$.

Main Approaches to Quantum Gravity

No question about quantum gravity is more difficult than the question, "What is the question?"
(John Wheeler 1984)

- Quantum general relativity
  - Covariant approaches (perturbation theory, path integrals, spin foam, ...)
  - Canonical approaches (geometrodynamics, connection dynamics, loop dynamics, ...)
- String theory
- Other approaches
  (Causal sets, group field theory, ...)

Approach used here: Canonical quantum geometrodynamics

(For more details on all approaches, see e.g. C.K., *Quantum Gravity*, 3rd ed., Oxford 2012)

Collapse of a thin dust shell

- Spherically-symmetric thin shell consisting of particles with zero rest mass (“null dust shell”);
- Classical theory: collapse to a black hole, or expansion from a white hole (usually excluded for thermodynamical reasons)
- Our quantization will lead to a singularity-free quantum state (“superposition of black and white hole”)

(Hájíček and C.K. 2001)

Dynamics of a null dust shell

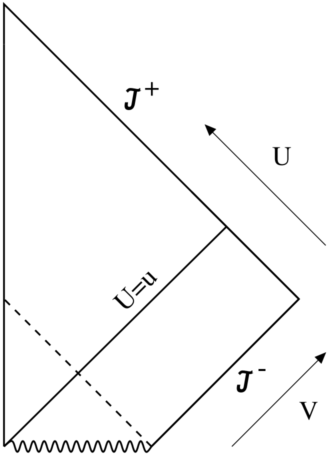

Figure: Penrose diagram for the outgoing shell in the classical theory. The shell is at $U = u$.

Our approach: reduced quantization

- Separation of variables into pure gauge degrees of freedom (‘embedding variables’) and physical degrees of freedom (plus the respective canonical momenta)
- General existence of this ‘Kuchař decomposition’ can be shown by making a transformation to the standard ADM phase space of general relativity (Hájíček and Kijowski 2000)
- In this construction, a formal ‘background manifold’ plays a crucial role.

Wave packets

Exact time evolution for a wave packet describing the shell:

$$\Psi_{\kappa\lambda}(t,r) = \frac{1}{\sqrt{2\pi}} \frac{\kappa!(2\lambda)^{\kappa+1/2}}{\sqrt{(2\kappa)!}} \left[\frac{\mathrm{i}}{(\lambda + \mathrm{i}t + \mathrm{i}r)^{\kappa+1}} - \frac{\mathrm{i}}{(\lambda + \mathrm{i}t - \mathrm{i}r)^{\kappa+1}} \right]$$

Important consequence:

$$\lim_{r\to 0} \Psi_{\kappa\lambda}(t,r) = 0$$

This means that the probability of finding the shell at vanishing radius is zero! In this sense, the singularity is avoided in the quantum theory. The quantum shell bounces and re-expands, and no event horizon forms.

Expectation value and variance of the shell radius:

$$\langle R_0 \rangle_{\kappa\lambda} := 2G\langle E \rangle_{\kappa\lambda} = (2\kappa + 1)\frac{l_{\mathrm{P}}^2}{\lambda},$$

$$\Delta(R_0)_{\kappa\lambda} = 2G\Delta E_{\kappa\lambda} = \sqrt{2\kappa + 1}\frac{l_{\mathrm{P}}^2}{\lambda}$$

It turns out that the wave packet can be squeezed below its Schwarzschild radius if its energy is greater than the Planck energy—a genuine quantum effect!

“Superposition of black and white hole”

Astrophysical relevance?

Central question: what is the timescale t_b for shell collapse and re-expansion?

- Ambrus and Hájíček (2005): t_b is of order M, which would be too short for an observational significance of the model;
- later investigations (e.g. in loop quantum gravity) led to other timescales,[1] e.g. $t_\mathrm{b} \propto M^2$;
- question also relevant for the LTB model, see below.

[1] See e.g. D. Malafarina, *Universe* **3** (2017) 2,48 for a review.

Quantization of the LTB model

- Wheeler–DeWitt quantization: semiclassical solutions can be found from which Hawking radiation and corrections can be calculated
- Similar attempts in loop quantum gravity
- here: reduced quantization in analogy to treatment of thin shells

Assumption: the different shells in the cloud decouple, so we can focus on a single shell. The Hamiltonian for the outermost shell (with radius R_o) turns out to read

$$H = -\frac{P_o^2}{2R_o},$$

which is the negative of the ADM energy. (P_o is the momentum conjugate to R_o.) Restriction is made to the marginally bound case.

C.K. and T. Schmitz, *Phys. Rev. D* **99**, 126010 (2019)

As in the case of the collapsing shell, we seek for a unitary evolution (here with respect to the dust proper time τ).
Schrödinger quantization:

$$P_o \to \hat{P}_o = -\mathrm{i}\hbar \frac{d}{dR_o}.$$

The operator $\hat{R}_o$ acts by multiplication. (In the following we will suppress the subscript o.)
Hamilton operator:

$$\hat{H} = \frac{\hbar^2}{2}\, R^{-1+a+b} \frac{d}{dR} R^{-a} \frac{d}{dR} R^{-b},$$

where a and b encode factor ordering ambiguities. Schrödinger equation:

$$\mathrm{i}\hbar \frac{\partial \Psi(R,\tau)}{\partial \tau} = \hat{H}\Psi(R,\tau)$$

We impose square-integrability on wave functions and let them evolve unitarily according to a self-adjoint Hamiltonian. This corresponds to enforcing probability conservation in dust proper time.

Singularity avoidance for wave packets

For a wide class of wave packets, the probability for the outermost dust shell to be in the classically singular configuration $R = 0$ is zero.

One explicit example:

$$\Psi(R,\tau) = \sqrt{3}\left(\frac{\sqrt{2}}{3}\right)^{\frac{1}{3}|1+a|+1} \frac{\Gamma\left(\frac{1}{6}|1+a| + \frac{\kappa}{2} + 1\right)}{\sqrt{\Gamma(\kappa+1)}\Gamma\left(\frac{1}{3}|1+a|+1\right)} R^{\frac{1}{2}(1+a+|1+a|+2b)}$$
$$\times \frac{\lambda^{\frac{1}{2}(\kappa+1)}}{\left(\frac{\lambda}{2} - i\tau\right)^{\frac{1}{6}|1+a|+\frac{\kappa}{2}+1}} \, {}_1F_1\left(\frac{1}{6}|1+a| + \frac{\kappa}{2} + 1; \frac{1}{3}|1+a|+1; -\frac{2R^3}{9(\frac{\lambda}{2} - i\tau)}\right)$$

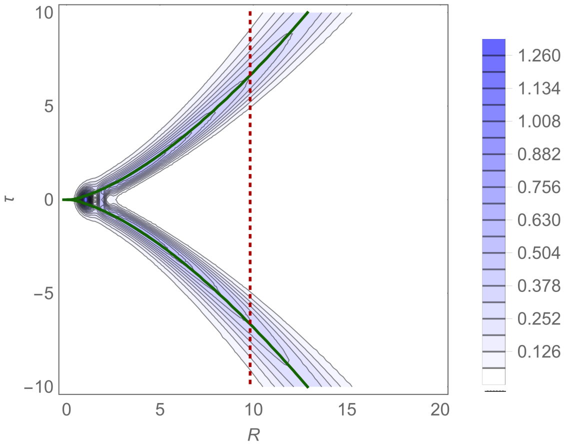

Figure: Probability amplitude for R as given by $R^{1-a-2b}\,|\Psi(R,\tau)|^2$, compared to the classical trajectories (full green line) and the exterior apparent horizon (dotted red line), with $a = 2$ and $b = 1$

- Discussion of Oppenheimer–Snyder models with flat[2] and non-flat[3] Friedmann sections describing the interior of the dust cloud: Again, for certain parameter values, there is a bounce of wave packets as seen by a stationary observer.
- Lifetime of bouncing solution (for an exterior observer) turns out to be proportional to M^3 (same order as black-hole evaporation time); but more recent investigations suggest $\tau_{\rm b} \propto M$ (Schmitz 2020) as in some models of loop quantum gravity

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[2]T. Schmitz, *Phys. Rev. D* **101**, 026016 (2020)
[3]C. Kiefer and H. Mohaddes, *Phys. Rev. D* **107**, 126006 (2023)

Quantum Oppenheimer–Snyder scenario

Closed slices: Bounce and oscillations inside the horizons: eternal lifetime for the comoving observer

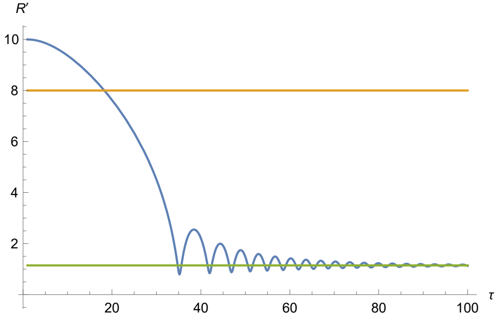

Figure: Graph in the $R - \tau$ space. The orange line represents the Schwarzschild radius $R = 2GM/c^2$, the green line the location where equilibrium is reached after oscillations, where $\dot{R} = \ddot{R} = 0$.

Reflections

- Lemaître's model from 1933 is well suited not only for applications in classical cosmology, but also for addressing fundamental issues in quantum gravity
- One can construct quantum models for gravitational collapse which are singularity-free. There is a unitary evolution from a collapsing to an expanding wave packet (bouncing solution). If generally true, this would solve the cosmic-censorship problem.
- Lifetime of black-and-white hole? Compatible with observations? Relevance for primordial black holes?
- Fate of naked singularities?
- Implementation of Hawking radiation? Fate of horizon and relevance for information-loss problem? Most likely, horizon disappears.
- Role of decoherence?

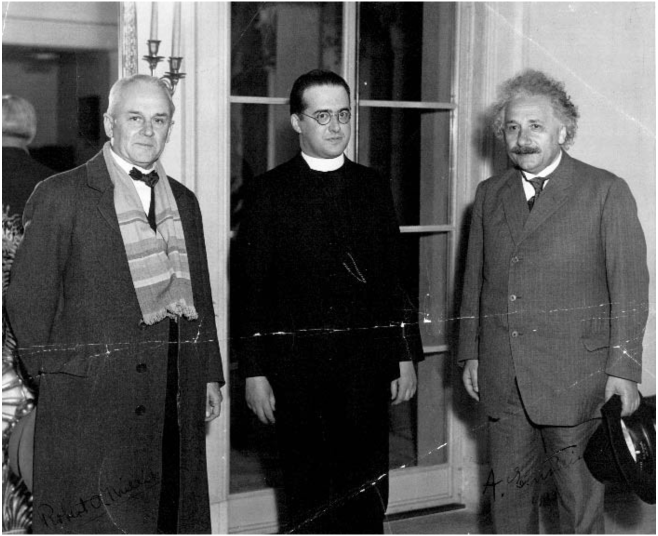

Figure credit: Wikimedia Commons