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Again about singularity crossing in gravitation and cosmology.

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Introduction

- General relativity connects the geometrical properties of the spacetime to its matter content.
  Matter tells spacetime how to curve itself, the spacetime geometry tells matter how to move.
- Cosmological singularities constitute one of the main problems of modern cosmology.
- The discovery of cosmic acceleration stimulated the development of “exotic” cosmological models of dark energy; some of these models possess the so called soft or sudden singularities characterized by a finite value of the radius of the universe and its Hubble parameter.

- The geodesics in these cases are regular and one can describe the passing through the singularities in a simple way.
- In some models with soft singularities an interplay between the geometry and matter forces matter to change some of its basic properties, such as the equation of state for fluids and even the form of the Lagrangian.
- "Traditional" or "hard" singularities are associated with a zero volume of the universe (or of its scale factor), and with infinite values of the Hubble parameter, of the energy density and of the pressure –Big Bang and Big Crunch
- One can describe the transition through the singularity in Friedmann-Lemaître cosmological models including scalar fields
- The procedure includes a change of a field parametrization (for example, the transition between Einstein and Jordan frames)

- Is it possible to describe the singularity crossing in anisotropic universes?
- Can quantum cosmology eliminate the singularities arising in classical General Relativity?
- What happens with quantum particles at the singularity crossing?
- When is it possible and when it is not possible to describe the transition through the singularity?
  An attempt of a general approach.

Big Bang – Big Crunch crossing

- The idea that the Big Bang - Big Crunch singularity can be crossed appears very counterintuitive.
- Some approaches to the description of this crossing were elaborated during the last couple of decades (I. Bars, S.H. Chen, P.J. Steinhardt and N. Turok, C. Wetterich, P. Dominis Prester).

- There is an analogy with the horizon which arises due to a certain choice of the spacetime coordinates: the singularity arises because of some choice of the field parametrization.
- On choosing some convenient field parametrization one can provide a matching between the characteristics of the universe before and after the singularity crossing.
- Analogy to the Kruskal coordinates for the Schwarzschild metric.

- On choosing appropriate combinations of the field variables we can describe the passage through the Big Bang - Big Crunch singularity, but this does not mean that the presence of such a singularity is not essential. Indeed, extended objects cannot survive this passage.

Singularity crossing in a Bianchi - I universe

$$d\tilde{s}^2 = \tilde{N}(\tau)^2 d\tau^2 - \tilde{a}^2(\tau)(e^{2\beta_1(\tau)}dx_1^2 + e^{2\beta_2(\tau)}dx_2^2 + e^{2\beta_3(\tau)}dx_3^2),$$

$$ds^2 = N(\tau)^2 d\tau^2 - a^2(\tau)(e^{2\beta_1(\tau)}dx_1^2 + e^{2\beta_2(\tau)}dx_2^2 + e^{2\beta_3(\tau)}dx_3^2),$$

$$\beta_1 + \beta_2 + \beta_3 = 0.$$

$$\dot{\beta}_i = \frac{\beta_{i0}}{\tilde{a}^3},\ \ \theta_0 = \beta_{10}^2 + \beta_{20}^2 + \beta_{30}^2.$$

$$\dot{\phi} = \frac{\phi_0}{\tilde{a}^3},\ \ \phi = \frac{\phi_0}{\left(\frac{3\theta_0}{2} + \frac{3\phi_0^2}{4U_1}\right)^{\frac{1}{2}}} \ln \tilde{t}.$$

In the vicinity of the singularity in the Einstein frame

$$\tilde{a} \sim \tilde{t}^{\frac{1}{3}}.$$

In the Jordan frame

$$a \sim \tilde{t}^{\frac{1}{3}}(\tilde{t}^{\gamma} + \tilde{t}^{-\gamma}) \to 0,$$

because

$$\gamma = \frac{\phi_0}{3\sqrt{\phi_0^2 + 2\theta_0 U_1}} < \frac{1}{3}.$$

Thus, one also encounters the Big Bang singularity in the Jordan frame.

Mixing between geometrical and matter degrees of freedom and the singularity crossing

The Friedmann-Lemaîthe model with a massless scalar field can be described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{y}^2,$$

where

$$x = \frac{4\sqrt{U_1}}{\sqrt{3}}\tilde{a}^{\frac{3}{2}} \cosh \frac{\sqrt{3}}{4\sqrt{U_1}}\phi, \;\; y = \frac{4\sqrt{U_1}}{\sqrt{3}}\tilde{a}^{\frac{3}{2}} \sinh \frac{\sqrt{3}}{4\sqrt{U_1}}\phi,$$

and the Friedmann equation is

$$\dot{x}^2 - \dot{y}^2 = 0.$$

Inversely,

$$\tilde{a}^3 = \frac{3(x^2 - y^2)}{16U_1},$$

$$\phi = \frac{4\sqrt{U_1}}{\sqrt{3}} \mathrm{arctanh} \frac{x}{y}.$$

Initially

$$x > |y|.$$

The solution

$$x = x_1 \tilde{t} + x_0, \;\; y = y_1 \tilde{t} + y_0, \;\; x_1^2 = y_1^2.$$

Choosing the constants as

$$x_0 = y_0 = A > 0, \;\; x_1 = -y_1 = B > 0,$$

we have

$$\tilde{a}^3 = \frac{3AB\tilde{t}}{4U_1}.$$

We can make a continuation in the plane (x, y), to $x < |y|$ or, in other words, to $\tilde{t} < 0$. Such a continuation implies an antigravity regime and the transition to the phantom scalar field, just as in the more complicated schemes, discussed before.

How can we generalize these considerations to the case when the anistropy term is present ?

$$L = \frac{1}{2}\dot{r}^2 - \frac{1}{2}r^2(\dot{\varphi}^2 + \dot{\varphi}_1^2 + \dot{\varphi_2}^2),$$

$$\varphi_1 = \sqrt{\frac{3}{8}}\alpha_1, \ \varphi_2\sqrt{\frac{3}{8}}\alpha_2,$$

$$\beta_1 = \frac{1}{\sqrt{6}}\alpha_1 + \frac{1}{\sqrt{2}}\alpha_2, \ \beta_2 = \frac{1}{\sqrt{6}}\alpha_1 - \frac{1}{\sqrt{2}}\alpha_2, \ \beta_3 = -\frac{2}{\sqrt{6}}\alpha_1.$$

We can again consider the plane (x, y) as

$$x = r\cosh\Phi,$$
$$y = r\sinh\Phi,$$

where a new hyperbolic angle Φ is defined by

$$\Phi = \int d\tilde{t}\sqrt{\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}^2}.$$

We have reduced a four-dimensional problem to the old two-dimensional one, on using the fact that the variables α_1, α_2 and ϕ enter into the equation of motion for the scale factor $\tilde{a}$ only through the squares of their time derivatives.

The behaviour of the scale factor before and after the crossing of the singularity can be matched by using the transition to the new coordinates x and y, which mix geometrical and scalar field variables in a particular way.

To describe the behaviour of the anisotropic factors it is enough to fix the constants β_{i0}.

Duality between static spherically or hyperbolically symmetric solutions and cosmological solutions in scalar-tensor gravity

- A duality between spherically symmetric static solutions in the presence of a massless scalar field and the Kantowski-Sachs cosmological models which instead possess hyperbolic symmetry was found.
- The spherically symmetric Kantowski-Sachs universes are connected by a duality transformation to the static solutions possessing hyperbolic symmetry.

- The main ingredient of this duality is the exchange of roles between the radial coordinate and the temporal coordinate combined with the exchange between the spherical two-dimensional geometry and the hyperbolical two-dimensional geometry.
- We have found exact solutions for static spherically and hyperbolically symmetric geometries in the presence of a massless scalar field conformally coupled to gravity and their respective Kantowski-Sachs cosmologies.

$$ds^2 = \frac{a_0^2\left(A_0\left(\tanh\frac{t}{2}\right)^{2\sqrt{\frac{1-\gamma^2}{3}}}+1\right)^2 \sinh^2 t}{4A_0\left(\tanh\frac{t}{2}\right)^{2\gamma+2\sqrt{\frac{1-\gamma^2}{3}}}}$$

$$\times(dt^2 - d\chi^2 - \sinh^2\chi d\phi^2)$$

$$-\frac{b_0^2\left(A_0\left(\tanh\frac{t}{2}\right)^{2\sqrt{\frac{1-\gamma^2}{3}}}+1\right)^2\left(\tanh\frac{t}{2}\right)^{2\gamma-2\sqrt{\frac{1-\gamma^2}{3}}}}{4A_0}dr^2.$$

For $\gamma = 1/2$

$$ds^2 = \frac{a_0^2\left(A_0 \tan\frac{t}{2} + 1\right)^2 \cos^4\frac{t}{2}}{A_0}(dt^2 - d\theta^2 - \sin^2\theta d\phi^2)$$
$$-\frac{b_0^2\left(A_0 \tan\frac{t}{2} + 1\right)^2}{4A_0}dr^2.$$

This metric is regular at $t = 0$ and has singularities at $t = \pm\pi$ and at $t = t_0 = -2\arctan\frac{1}{A_0}$.
At $t \to \pm\pi$ the scale factor $a \to 0$ while $b \to \infty$. At $t \to t_0$ both scale factors vanish.
At $t < 0$, we find ourselves in the region with antigravity because $U_c < 0$.
The expression contains only integer powers of the trigonometrical functions and one can describe the crossing of the singularities in a unique way.
Thus, we can imagine an infinite periodic evolution of the universe.

In the vicinity of the moment $t \to \pi$, the asymptotic expressions for the metric coefficients are

$$ds^2 = dT^2 - c_1^2 T d\theta^2 - c_2^2 T \sin^2\theta d\phi^2 - c_3 \frac{1}{T} dr^2.$$

This form has a structure similar to that of the Kasner solution for a Bianchi-I universe, where the Kasner indices have the values

$$p_1 = \frac{1}{2},\ p_2 = \frac{1}{2},\ p_3 = -\frac{1}{2}.$$

These indices do not satisfy the standard Kasner relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1,$$

they satisfy the generalized relation

$$\sum_{i=1}^{3} p_i^2 = 2\sum_{i=1}^{3} p_i - \left(\sum_{i=1}^{3} p_i\right)^2.$$

In the vicinity of the singularity at $t = t_0$:

$$ds^2 = dT^2 - c_1^2 T d\theta^2 - c_2^2 T \sin^2\theta d\phi^2 - c_3 T dr^2.$$

This behavior is isotropic.

Quantum cosmology and singularities

Speaking about quantum cosmology and singularities people mean two different things:
Modification of the Friedmann equation.

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \rho_{\rm matter} + \rho_{\rm quantum\ corrections}.$$

Vanishing of the quantum state of the universe.

$$\Psi({\rm geometry} + {\rm matter})_{\rm geometry\ is\ singular} = 0.$$

Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = 0.$$

Where is the time ?
What is the probability ?

A time can be defined as a certain function of geometrical variables.
After that the wavefunction describing matter variables satisfies an effective Schrödinger equation.
The singularity is associated with such values of the matter variables when this singularity arises in the classical theory.
Our analysis of some simple models tells that the probability of the arising of soft singularities is not suppressed by the wave function of the universe, while the probability of Big Bang – Big Crunch singularity tends to zero.

The suppression of the Big Bang – Big Crunch singularity follows from the requirement of the normalizability of the wave function of the Universe

$$\int d\phi \bar{\Psi}(\phi)\Psi(\phi) < \infty.$$

When $|\phi| \to \infty$, the probability density $\bar{\Psi}\Psi$ should tend to zero rapidly.

If $|\phi| \to \infty$ corresponds to Big Bang – Big Crunch singularity, when this singularity is suppressed.

Particles, fields and singularities

What happens with particles (in quantum field theoretical sense) when the universes passes through the cosmological singularity ?
The scalar field in the flat Friedmann universe satisfies the Klein-Gordon equation:

$$\Box\phi + V'(\phi) = 0.$$

One can consider a spatially homogeneous solution of this equation ϕ_0, depending only on time t as a classical background.
A small deviation from this background solution can be represented as a sum of Fourier harmonics satisfying linearized equations

$$\ddot{\phi}(\vec{k},t) + 3\frac{\dot{a}}{a}\dot{\phi}(\vec{k},t) + \frac{\vec{k}^2}{a^2}\phi(\vec{k},(t)) + V''(\phi_0(t))\phi(\vec{k},(t)) = 0.$$

The corresponding quantized field is

$$\hat{\phi}(\vec{x}, t) = \int d^3\vec{k}(\hat{a}(\vec{k})u(k, t)e^{i\vec{k}\cdot\vec{x}} + \hat{a}^+(\vec{k})u^*(k, t)e^{-i\vec{k}\cdot\vec{x}}),$$

where the creation and the annihilation operators satisfy the standard commutation relations:

$$[\hat{a}(\vec{k}), \hat{a}^+(\vec{k}')] = \delta(\vec{k} - \vec{k}').$$

The basis functions should be normalized so that the canonical commutation relations between the field ϕ and its canonically conjugate momentum $\hat{\mathcal{P}}$ were satisfied

$$[\hat{\phi}(\vec{x}, t), \hat{\mathcal{P}}(\vec{y}, t')] = i\delta(\vec{x} - \vec{y}).$$

$$u(k,t)\dot{u}^*(k,t) - u^*(k,t)\dot{u}(k,t) = \frac{i}{(2\pi)^3 a^3(t)}.$$

The linearized Klein-Gordon equation has two independent solutions.
To define a particle it is necessary to have two independent non-singular solutions.
It is a non-trivial requirement in the situations when a singularity or other kind of irregularity of the spacetime geometry occurs.
It is convenient also to construct explicitly the vacuum state for quantum particles as a Gaussian function of the corresponding variable. Let is introduce an operator

$$\hat{f}(\vec{k},t) = (2\pi)^3(\hat{a}(\vec{k})u(k,t) + \hat{a}^+(-\vec{k})u^*(k,t)).$$

Its canonically conjugate momentum is

$$\hat{p}(\vec{k},t) = a^3(t)(2\pi)^3(\hat{a}(\vec{k})\dot{u}(k,t) + \hat{a}^+(-\vec{k})\dot{u}^*(k,t)).$$

We can express the annihilation operator as

$$\hat{a}(\vec{k}) = i\hat{p}(\vec{k},t)u^*(k,t) - ia^3(t)\hat{f}(\vec{k},t)\dot{u}^+(k,t).$$

Representing the operators $\hat{f}$ and $\hat{p}$ as

$$\hat{f} \to f, \quad \hat{p} \to -i\frac{d}{df},$$

one can write down the equation for the corresponding vacuum state in the following form:

$$\left(u^*\frac{d}{df} - ia^3\dot{u}^* f\right)\Psi_0(f) = 0.$$

$$\Psi_0(f) = \frac{1}{\sqrt{|u(k,t)|}}\exp\left(\frac{ia^3(t)\dot{u}^*(k,t)f^2}{2u^*(k,t)}\right).$$

In the case of the Big Bang - Big Crunch singularity, one of the basis functions in the vicinity of the singularity becomes singular and it is impossible to construct a Fock space.

In the case of the Big Rip singularity, when in finite interval of time the universe achieves an infinite volume and infinite time derivative of the scale factor, the Fock space can be constructed for a spectator scalar field, but it does not exist for the phantom scalar field driving the expansion.

In the case of the model with tachyon field, presented above, we have considered three situations.

The non-singular transformation of the tachyon into pseudo-tachyon. In this case both basis functions are regular and hence the operators of creation and annihilation are well defined.
However, at the moment of the transformation the dispersion of the Gaussian wave function of the vacuum becomes infinite and then becomes finite again.

In the vicinity of the Big Brake singularity it is impossible to define a Fock vacuum.

However, if we add to the universe dust, the character of the soft singularity is slightly changed and then the presence of the Fock vacuum is restored.

Covariant approach to singularities

The crossing of the Big Bang - Big Crunch singularities looks rather counterintuitive.
However, it can be sometimes described by using the reparametrization of fields, including the metric.
One can say that to do this, it is necessary to resort to one of two ideas, or a combination thereof.
One of these ideas is to employ a reparameterization of the field variables which makes the singular geometrical invariant non-singular.

Another idea is to find such a parameterization of the fields, including, naturally, the metric, that gives enough information to describe consistently the crossing of the singularity even if some of the curvature invariants diverge.
The application of these ideas looks in a way as a craftsman work.
Our goal is to develop a general formalism to distinguish "dangerous" and "non-dangerous" singularities, considering the field variable space of the model under consideration.
When the spacetime singularities can be removed by a reparametrization of the field variables?

Our hypothesis: when the geometry of the space of the field variables is non-singular.
The field space $\mathcal{S}$ was developed in order to treat on the same (geometrical) footing both changes of coordinates in the spacetime $\mathcal{M}$ and field redefinitions in the functional approach to quantum field theory.
This approach requires introducing a local metric G in field space $\mathcal{S}$ and computing the associated geometric scalars by defining a covariant derivative which is compatible with G.
G is actually determined by the kinetic part of the action and its dimension depends on the field content of the latter.

After some cumbersome calculations in the functional space, we have shown that the Kretschmann scalar

$$\mathcal{K} = \mathcal{R}_{ABCD}\,\mathcal{R}^{ABCD}$$

is finite in every theory of pure gravity

$$\mathcal{K} = \frac{n}{8}\left(\frac{n^3}{4} + \frac{3n^2}{4} - 1\right),$$

where n is the spacetime dimension.
It can be interpreted as a statement that all the singularities in empty universe can be crossed.

Another hypothesis: quantum effective action and to homotopy group

Let us introduce the functional

$$\psi[\varphi] = e^{i\,\Gamma[\varphi]},$$

where $\Gamma[\varphi]$ is the effective action. We shall call $\psi[\varphi]$ the functional order parameter because ψ plays the analogous role of an order parameter in the theory of phase transitions in ordered media or cosmology.

The field space $\mathcal{M}$ can be thought of as the ordered medium itself, whereas functional singularities correspond to topological defects.

The functional order parameter ψ defines the map

$$\psi : \mathcal{M} \to \mathbb{S}^1,$$

from the field space to the unit circle, the latter playing the role of the order parameter space.

The singularities can be characterized by the fundamental group (first homotopy group).
If this group is trivial the singularity can be removed.
We have checked on the example of some simple systems with removable singularity that the corresponding homotopy group is indeed trivial.

Bianchi-I cosmologies, magnetic fields and singularities

We have studied a Bianchi-I universe filled with a magnetic field oriented along one of the spatial axes.
In the vicinity of the initial singularity, such a universe is described by the Kasner solution.
The Kasner indices satisfy the standard Kasner relations for an empty Bianchi-I universe.
At the "end of the evolution" i.e. when the volume of the universe tends to infinity, the solution again becomes Kasner one.
Again the Kasner indices satisfy the standard relations.
The transition between these two Kasner regimes coincides with that arising in an empty Bianchi-II universe.

The mechanisms of the description of the crossing of the singularities in an empty Bianchi-I universe and in a Bianchi-I universe filled with a scalar field work well in this case too.

Conclusions and discussion

- General relativity contains many surprises concerning relations between matter and geometry. It is enough to take it seriously.
- There is no need to be afraid of singularities!