Nonlinear Beam Dynamics

Lecture 2

Contents

Hamiltonian Dynamics

In the previous lecture we saw how nonlinear elements such as sextupoles can perturb the beam motion, excite resonances and potentially cause beam loss or restrict the dynamic aperture.

In general the equations of motion including nonlinear elements cannot be solved analytically; rather they are solved numerically through particle tracking.

However, in order to understand and control the impact that nonlinear elements can have, it is also enlightening to see how the tools of Hamiltonian Mechanics and perturbation theory can be used.

In the first part of this lecture we will provide a brief introduction to this topic in order to illustrate how the tools can be applied. There are many references that provide a more in-depth treatment in the context of particle accelerators, see for example [1-6]. For a more general introduction to Hamiltonian mechanics, see [7].

Hamiltonian Dynamics

The starting point in the analysis is to construct the Hamiltonian of the system; an expression that contains information describing the interactions of the dynamical quantities. The Hamiltonian is expressed in terms of a set of canonical coordinates $r_i = (q_i, p_i)$, where q_i is the generalised canonical coordinate and p_i is the generalised momentum. i.e.

$$H = H(\mathbf{q}, \mathbf{p}; t)$$

The time evolution of the system is then uniquely defined by Hamilton's Equations

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}$$
$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$

Hamiltonian Dynamics

In the case of a charged particle moving in an electromagnetic field, the Hamiltonian is given by [8]

$$H = e\Phi + \sqrt{(c\mathbf{p} - e\mathbf{A})^2 + m^2c^4}$$

This is not in a particularly convenient form for application to particle accelerators. As such, it is usual to make a number of canonical transformations to move to a different coordinate frame, i.e.:

- Move from a Cartesian to a co-moving curvilinear reference frame
- Choose the Coulomb gauge and ignored static electric fields (setting $\Phi = 0$)
- Convert to using s as the independent variable (rather than t)
- Normalise the transverse momenta to the reference momentum
- Assume the magnetic fields are purely transverse, $A_x = A_y = 0$ (hard-edged magnet model with no end-effects)
- Take the small angle approximation ($p_x \ll p_0$; $p_y \ll p_0$)
- Assume the curvature is small in the dipoles (ρ large)

In addition, the multipole expansion of the magnetic field is typically employed, namely

$$\frac{eA_s}{p_0} = \frac{1}{\rho}\left(x + \frac{x^2}{2\rho}\right) - Re\sum_{n=1}^{M}\frac{(k_n + ij_n)}{n!}(x+iy)^n; \qquad k_n = \frac{1}{B_0\rho_0}\frac{d^nB_y}{dx^n}; \quad j_n = \frac{1}{B_0\rho_0}\frac{d^nB_x}{dx^n}$$

After these canonical transformations have been completed, we are left with a Hamiltonian in the form [9]:

$$H\left(x, p_x, y, p_y; s\right) = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{x\delta}{\rho} - \frac{x^2}{2\rho^2} + \mathrm{Re}\left[\sum_{n=1}^{M} \frac{(k_n + i j_n)}{(n+1)!}(x+iy)^{n+1}\right]$$

As a quick check, we see that if we keep only the terms up to quadrupole ($n = 1$), and assume on-energy particles ($\delta = 0$), we have

$$H\left(x, p_x, y, p_y; s\right) = \frac{p_x^2 + p_y^2}{2} - \frac{x^2}{2\rho^2} + \frac{k_1}{2}(x^2 - y^2)$$

Applying Hamilton's equations, and noting that $x' = p_x/(1+\delta)$ and $y' = p_y/(1+\delta)$ we end up with Hill's equations:

$$x'' + \left(\frac{1}{\rho^2} - k_1\right)x = 0$$

$$y'' + k_1 y = 0$$

Hamiltonian for Accelerator Components

Now that the general Hamiltonian for an accelerator has been established, we can apply it to the individual components. There are typically four basic building blocks: the drift space, bending magnet, quadrupole and sextupole.

For the drift space, we have $k_n = 0, j_n = 0$ and $\rho \to \infty$, leaving:

$$H_{drift} = \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

After applying Hamilton's Equations, the horizontal particle coordinates after passing through the drift space are (similar in y):

$$x_1 = x_0 + \frac{p_x}{1+\delta}\Delta s; \qquad p_{x1} = p_{x_0}$$

For a quadrupole we have:

$$H_{quad} = \frac{p_x^2+p_y^2}{2(1+\delta)} + \frac{k_1}{2}(x^2 - y^2)$$

$$x_1 = x_0 \cos(k\Delta s) + \frac{p_{x_0}}{k(1+\delta)}\sin(k\Delta s)$$

$$p_{x_1} = p_{x0}\cos(k\Delta s) - k(1+\delta)x_0 \sin(k\Delta s)$$

$$k = \sqrt{k_1/1+\delta}$$

Hamiltonian Analysis for Nonlinear Motion

Rather than analysing the particle motion element-by-element, we can also define an effective Hamiltonian for the ring as a whole by concatenating the individual maps from each element.

The Hamiltonian for nonlinear motion is analysed using perturbation theory by splitting the Hamiltonian into two parts:

$$H(x, p_x, y, p_y; s) = H_0(x, p_x, y, p_y; s) + V(x, p_x, y, p_y; s)$$

H_0 is the linear, solvable part and V is a perturbation due to the nonlinear elements:

$$H_0 = \frac{p_x^2 + p_y^2}{2(1+\delta)} - \frac{\mathrm{x}\delta}{\rho} - \frac{x^2}{2\rho^2} + \frac{k_1}{2}(x^2 - y^2)$$

$$V = \mathrm{Re}\left[\sum_{n=2}^{M} \frac{(k_n + i j_n)}{(n+1)!}(x + iy)^{n+1}\right]$$

We thus defined the Hamiltonian order-by-order, with the n-th order Hamiltonian denoted by h_n.

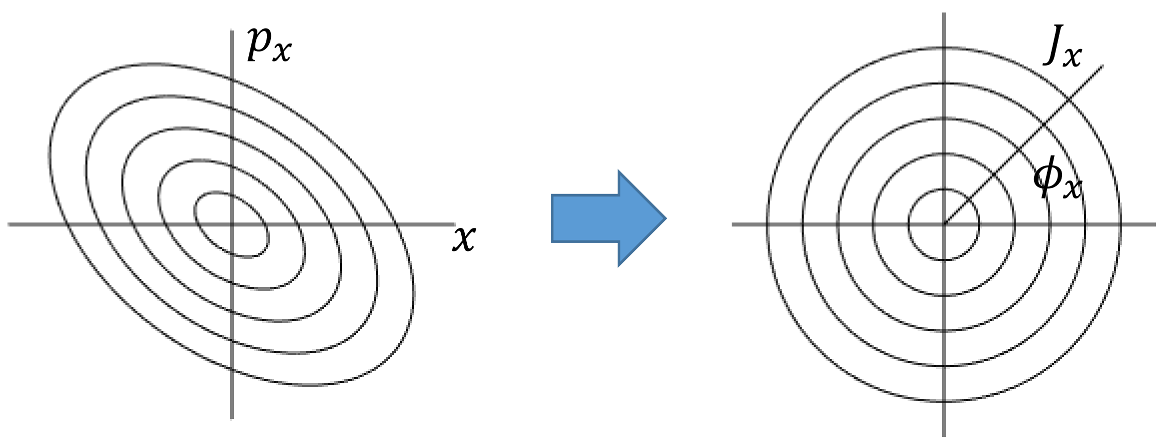
Hamiltonian Analysis for Nonlinear Motion

To simplify the analysis and isolate the nonlinear components as far as possible, it is common to move to a resonance basis via the action-angle coordinates.

$$\hat{x}_{\pm} = \sqrt{2J_x}e^{\pm\phi_x} = x \mp ip_x \qquad J_x = \gamma x^2 + 2\alpha_x x p_x + \beta_x p_x^2$$

$$\hat{y}_{\pm} = \sqrt{2J_y}e^{\pm\phi_y} = y \mp ip_y \qquad \phi_x = \frac{1}{Q_x}\int \frac{1}{\beta_x} ds$$

Here, J_x and J_y are the action (amplitude) terms and ϕ_x and ϕ_y are the angle terms. By making this transformation, the (linear) particle motion is reduced to a simple (circular) rotation in phase space.



The impact of the nonlinear elements then appears as a perturbation on top of the phase-space rotation.

Hamiltonian Analysis for Nonlinear Motion

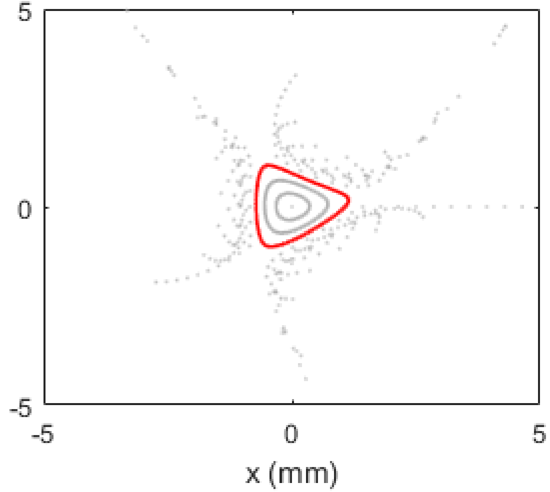
Using this basis, the n-th order Hamiltonian can be expanded as:

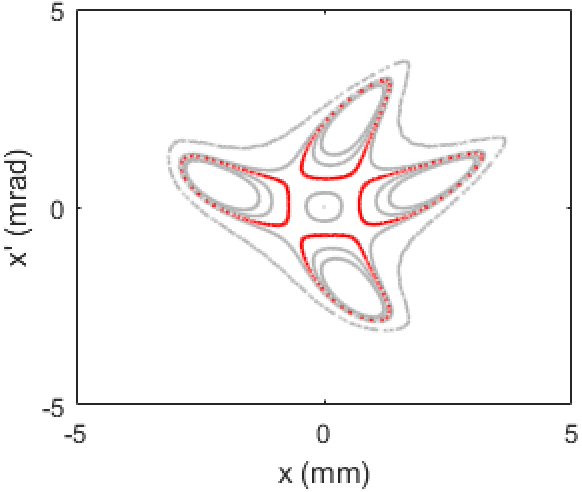
$$h_n = \sum_{j,k,l,m,p>0}^{j+k+l+m+p-1=n} h_{jklmp} \hat{x}_+^j \hat{x}_-^k \hat{y}_+^l \hat{y}_-^m \delta^p$$

The individual h_{jklmp} are known as the resonance driving terms (RDTs). They generate angle (phase) dependent terms in the Hamiltonian that can lead to resonant motion of the particles, such as motion on a chain of islands or on a separatrix. On the islands, the betatron tunes satisfy a resonance of the type:

$$P = NQ_x + MQ_y$$

where $N = j - k$ and $M = l - m$.





Hamiltonian Analysis for Nonlinear Motion

The second order Hamiltonian h_2 describing the impact of the sextupoles will also contain terms due to the quadrupoles via δ for the off-energy particles:

$$\frac{eA_s}{p} = \frac{e}{p_0}\frac{A_s}{1+\delta} \quad \Rightarrow \quad k_n(\delta) = \frac{k_n}{1+\delta}$$

Note that second order here refers to the particle coordinates; the Hamiltonian is first order in sextupole strength!

The general expression for the resonance driving terms for the h_2 Hamiltonian is [9]:

$$h_{jklmp} \propto \sum_{i=1}^{N_{sext}} (k_2L)_i \, \beta_{xi}^{\frac{j+k}{2}} \beta_{yi}^{\frac{l+m}{2}} \eta_i^p e^{i\{(j-k)\phi_{xi}+(l-m)\phi_{yi}\}} + \left[-\sum_{i=1}^{N_{quad}} (k_1L)_i \, \beta_{xi}^{\frac{j+k}{2}} \beta_{yi}^{\frac{l+m}{2}} e^{i\{(j-k)\phi_{xi}+(l-m)\phi_{yi}\}}\right]_{\delta\neq 0}$$

The summations are carried out over all quadrupoles and sextupoles, and depend upon the Twiss and dispersion parameters at the magnet locations.

Hamiltonian Analysis for Nonlinear Motion

There are thus 10 resonance driving terms that are first order in sextupole strength; five chromatic and five geometric. The first two are chromatic and are independent of the phase advance:

$$h_{11001} = -\frac{1}{4}\left[\sum_{i=1}^{N_{sext}} 2(k_2L)_i\,\eta_{xi} - \sum_{i=1}^{N_{quad}} (k_1L)_i\right]\beta_{xi}$$

$$h_{00111} = +\frac{1}{4}\left[\sum_{i=1}^{N_{sext}} 2(k_2L)_i\,\eta_{xi} - \sum_{i=1}^{N_{quad}} (k_1L)_i\right]\beta_{yi}$$

These terms drive the linear chromaticity; the reason for introducing sextupoles in the first place. The remaining eight RDTs vary with the phase advance and appear as complex conjugate pairs. The three remaining chromatic RDTs are:

$$h_{20001} = -\frac{1}{8}\left[\sum_{i=1}^{N_{sext}} 2(k_2L)_i\,\eta_{xi} - \sum_{i=1}^{N_{quad}} (k_1L)_i\right]\beta_{xi}e^{i2\phi_{xi}}$$

$$h_{00201} = +\frac{1}{8}\left[\sum_{i=1}^{N_{sext}} 2(k_2L)_i\,\eta_{xi} - \sum_{i=1}^{N_{quad}} (k_1L)_i\right]\beta_{yi}e^{i2\phi_{yi}}$$

$$h_{10002} = -\frac{1}{2}\left[\sum_{i=1}^{N_{sext}} 2(k_2L)_i\,\eta_{xi} - \sum_{i=1}^{N_{quad}} (k_1L)_i\right]\eta_{xi}\sqrt{\beta_{xi}}e^{i\phi_{xi}}$$

The first two of these generate momentum dependence for the beta functions and drive synchro-betatron resonances. The third drives second order dispersion.

Hamiltonian Analysis for Nonlinear Motion

The remaining five RDTs for the h_2 Hamiltonian are purely geometric and do not depend on the dispersion function or quadrupole strengths. They are:

$$h_{21000} = -\frac{1}{8}\sum_{i=1}^{N_{sext}}(k_2L)_i\,\beta_{xi}^{3/2}e^{i\phi_{xi}}$$
$$h_{10110} = \frac{1}{4}\sum_{i=1}^{N_{sext}}(k_2L)_i\,\beta_{xi}^{1/2}e^{i\phi_{xi}}$$
$$h_{30000} = -\frac{1}{24}\sum_{i=1}^{N_{sext}}(k_2L)_i\,\beta_{xi}^{3/2}e^{i3\phi_{xi}}$$
$$h_{10020} = \frac{1}{8}\sum_{i=1}^{N_{sext}}(k_2L)_i\,\beta_{xi}^{1/2}\beta_{yi}e^{i(\phi_{xi}-2\phi_{yi})}$$
$$h_{10200} = \frac{1}{8}\sum_{i=1}^{N_{sext}}(k_2L)_i\,\beta_{xi}^{1/2}\beta_{yi}e^{i(\phi_{xi}+2\phi_{yi})}$$

Of the geometric terms, the first two drive integer resonances of type Q_x, the third drives the $3Q_x$ resonance, and the final two drive the sextupole coupling resonances $Q_x \pm 2Q_y$.

Hamiltonian Analysis for Nonlinear Motion

$$h_{jklmp} \propto \sum_{i=1}^{N_{sext}} (k_2 L)_i \, \beta_{xi}^{\frac{j+k}{2}} \beta_{yi}^{\frac{l+m}{2}} \eta_i^p e^{i\{(j-k)\phi_{xi}+(l-m)\phi_{yi}\}} + \left[-\sum_{i=1}^{N_{quad}} (k_1 L)_i \, \beta_{xi}^{\frac{j+k}{2}} \beta_{yi}^{\frac{l+m}{2}} e^{i\{(j-k)\phi_{xi}+(l-m)\phi_{yi}\}} \right]_{\delta \neq 0}$$

The analysis of the sextupole Hamiltonian therefore enables deep insight into the causes of nonlinear particle motion.

During the design-phase of a storage ring a huge amount of effort is spent on trying to control the nonlinear beam dynamics and keep the particle motion as linear and regular as possible. This in turn enables a large dynamic aperture and hence long lifetime and good injection efficiency.

Through careful adjustment of the beta-functions and dispersion the amplitude of the RDTs can be reduced, or by controlling the phase advance between the sextupoles the RDT from one magnet can be cancelled by the next.

Transverse Resonance Island Buckets

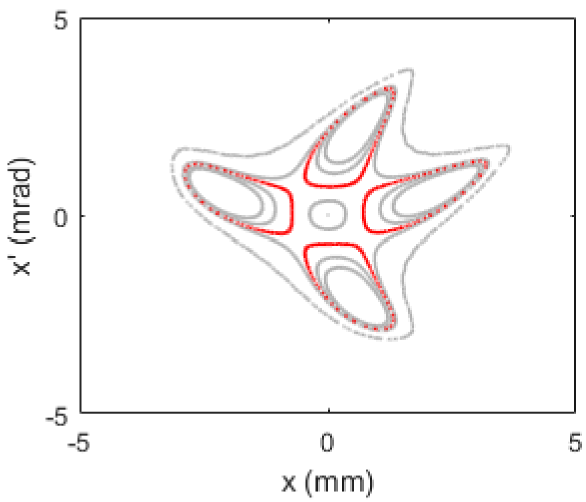
Although efforts are generally made to supress the nonlinear perturbations, there are also examples of how they can be exploited for specific applications. One example of this is the use of transverse resonance island buckets at the MLS and BESSY-II electron storage rings.

As we have seen, if a machine is operated with a tune close to a resonance, then the tune-shift with amplitude experienced by the particles leads to the creation of additional stable fixed points of motion in the transverse plane.

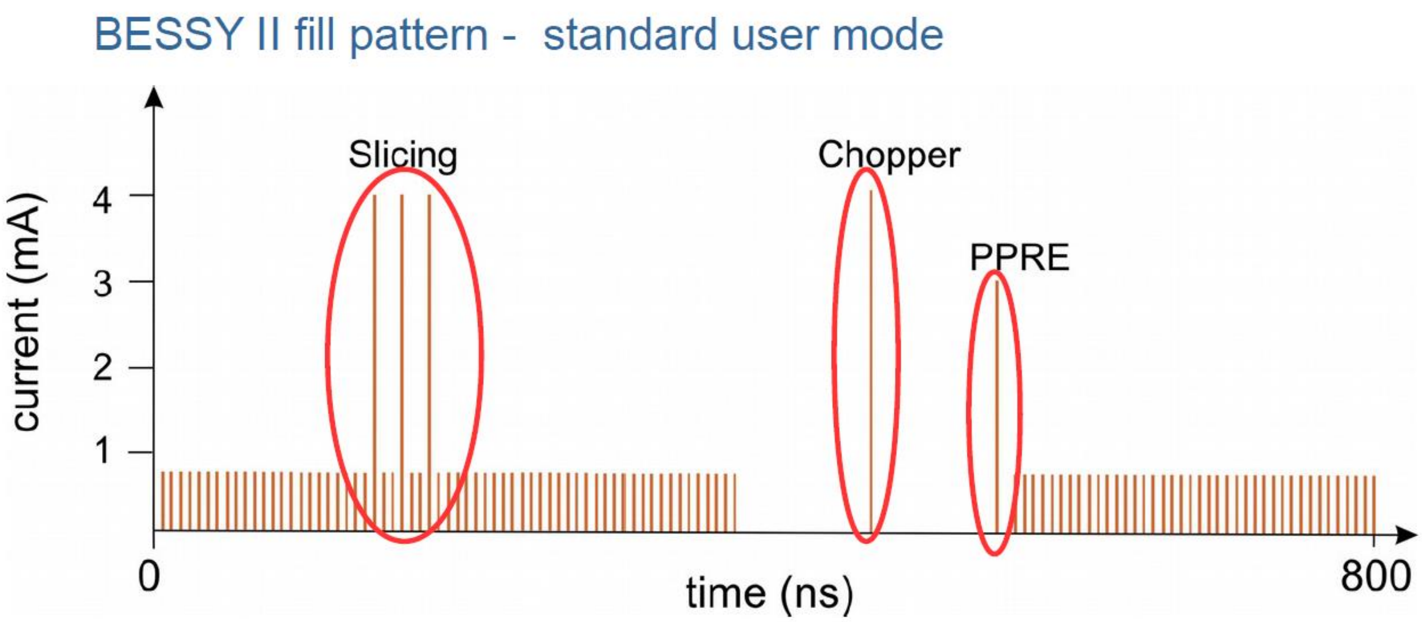
The radiation emitted by particles in the island buckets has two interesting features:

- Spatial separation from the core beam
- Sub-revolution frequency temporal emission

These properties can be exploited for experimental purposes.



Transverse Resonance Island Buckets



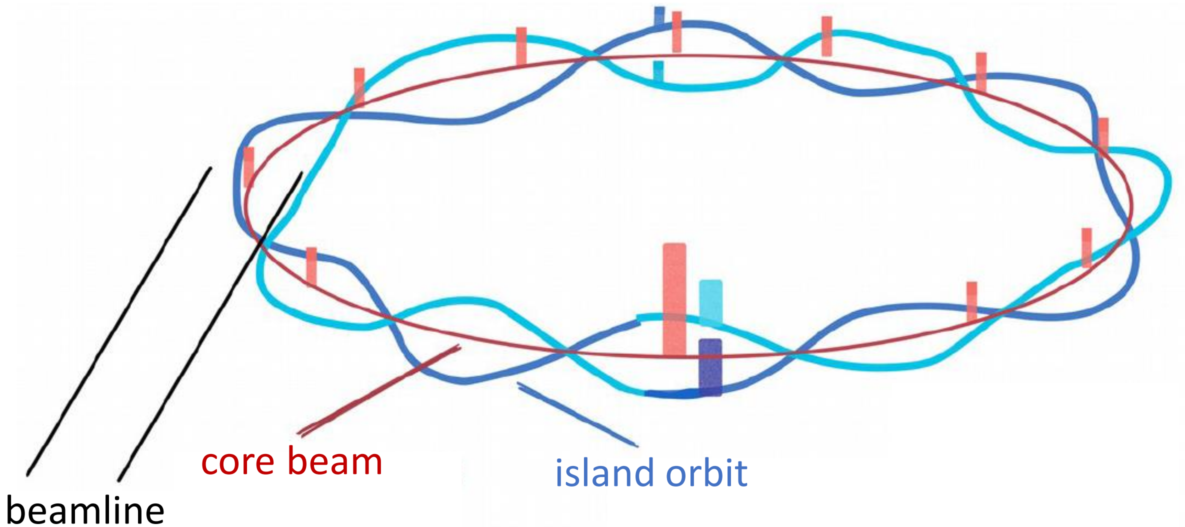
Motivation:

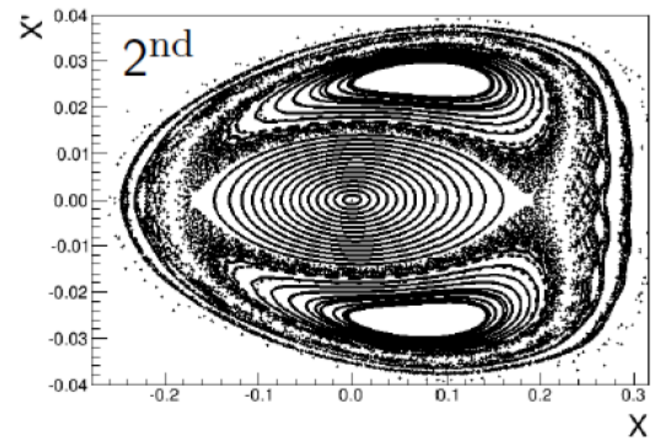
- Standard fill pattern at BESSY-II has to satisfy many users simultaneously
- Multi-bunch train gives high average brightness for majority of beamlines
- Single bunch for time-resolved experiments
- By populating island buckets, individual bunches can be isolated

P. Goslawski et al., XXIII ESLS Workshop, Villigen, Switzerland, (2015)

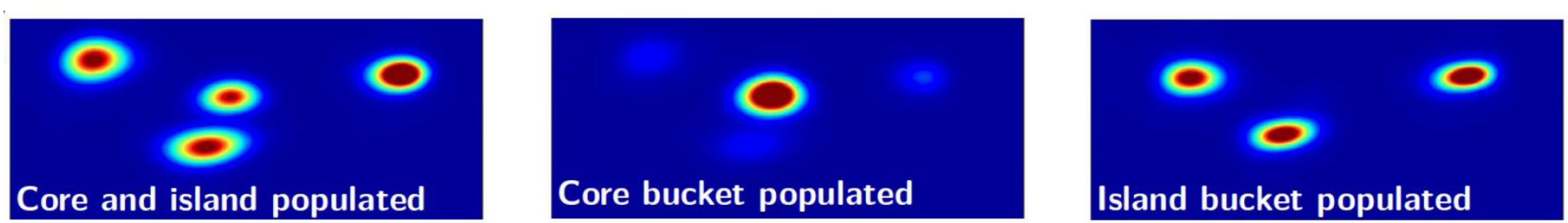
Transverse Resonance Island Buckets

- Operation close to $Q_x = 1/2$ leads to two islands being formed
- Orbit distortion for island bucket closes after two turns





- Close to $Q_x = 1/3$, three islands are formed
- Can control position of island buckets by varying skew quads and /or normal multipoles
- Vary amount of charge in each bucket with transverse excitation (multi-bunch feedback)



P. Goslawski et al., XXIII ESLS Workshop, Villigen, Switzerland, (2015)

Nonlinear Motion in the Longitudinal Plane

Similar tricks can be played in the longitudinal plane. The equations of motion in this case are:

$$\frac{d\phi}{dt} = \omega_{RF}\alpha_c\delta$$

$$\frac{d\delta}{dt} = \frac{eV}{T_0 E_0}[\sin(\phi) - \sin(\phi_s)]$$

where ω_{RF} is the angular frequency of the RF, ϕ is the phase of the particle and α_c is the momentum compaction factor. The momentum compaction factor is defined as the path lengthening with momentum deviation:

$$\alpha = \frac{\Delta L}{\delta}$$

and can itself be expanded as a function of δ:

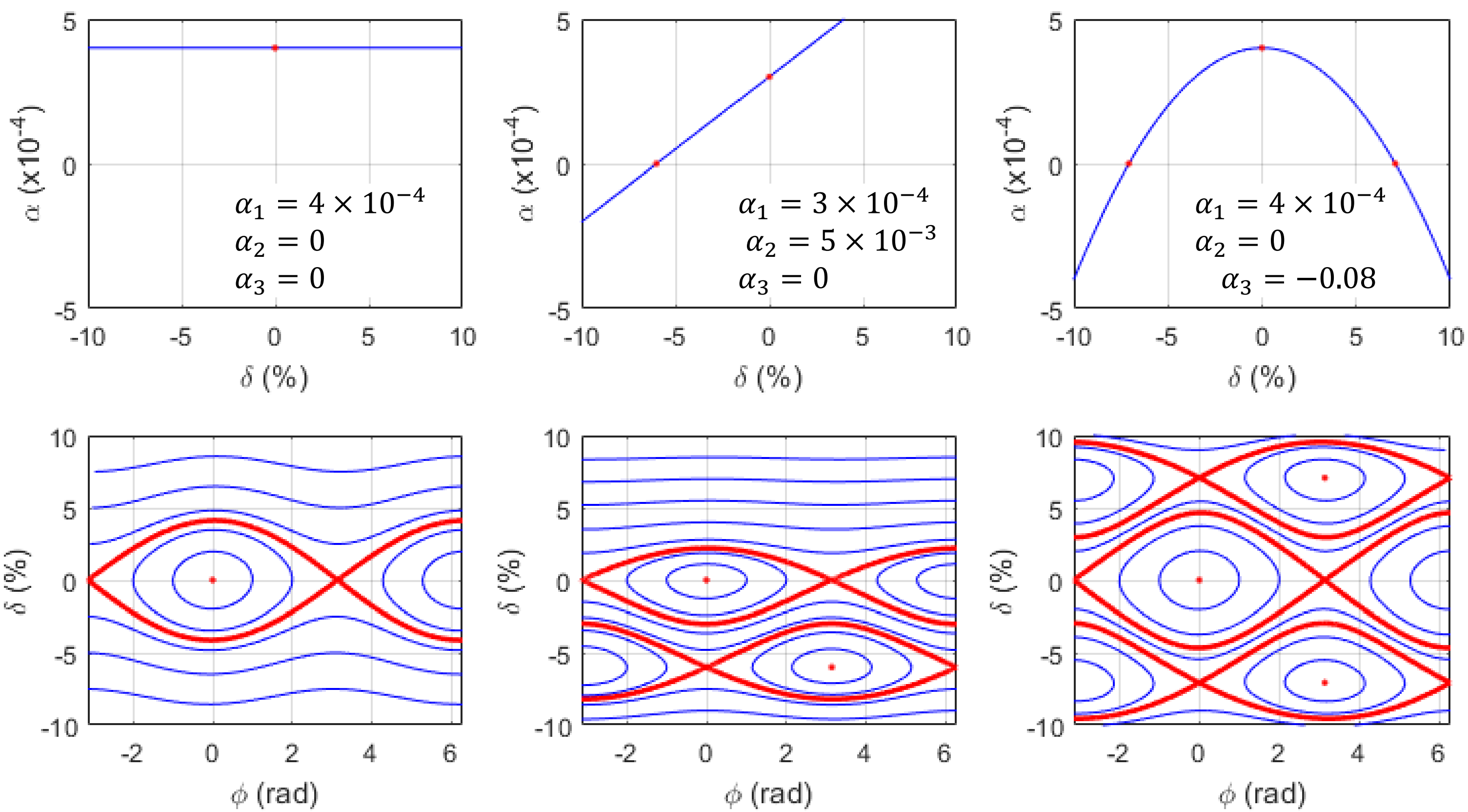
$$\alpha_c = \alpha_1 + \alpha_2\delta + \alpha_3\delta^2 + \cdots$$

Each of this terms can be adjusted with an appropriate order of multipole, i.e., α_1 can be controlled with quadrupoles, α_2 with sextupoles and so on.

Nonlinear Motion in the Longitudinal Plane

The longitudinal motion has stable or unstable fixed-point whenever $d\phi/dt = d\delta/dt = 0$. For the phase, fixed points occur at $\phi = \phi_s$ and $\phi = \pi - \phi_s$. For the energy, this occurs at either $\delta = 0$ or $\alpha_c = 0$

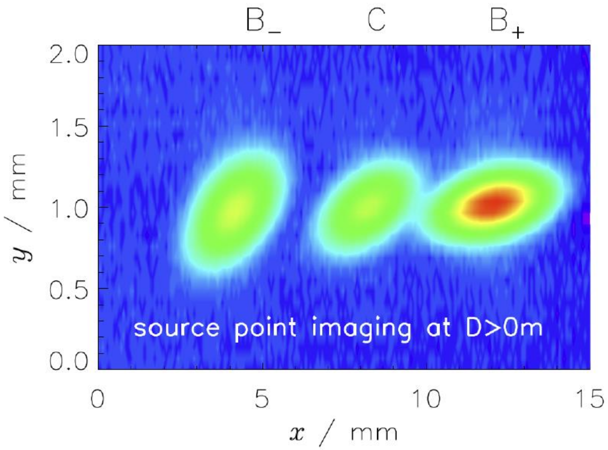
$$\frac{d\phi}{dt} = \omega_{RF}\alpha_c\delta; \qquad \frac{d\delta}{dt} = \frac{eV}{T_0E_0}[\sin(\phi) - \sin(\phi_s)]; \qquad \alpha_c = \alpha_1 + \alpha_2\delta + \alpha_3\delta^2 + \cdots$$

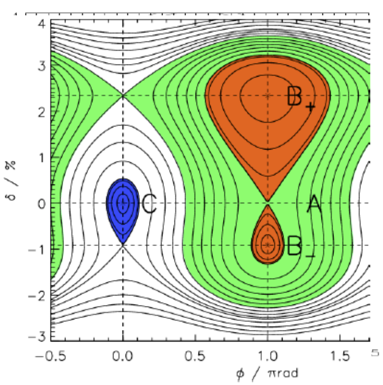


Nonlinear Motion in the Longitudinal Plane

Many examples of storage rings storing multiple beams simultaneously in different RF buckets, usually in the context of 'low-alpha' (short bunch length) studies. In other words, higher order terms in the momentum compaction factor only become significant as the magnitude of the leading order term (α_1) is reduced.

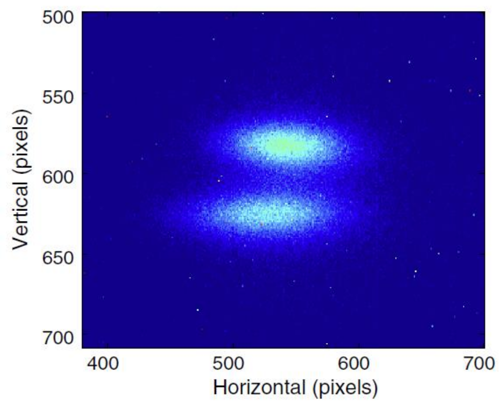
MLS (Germany)

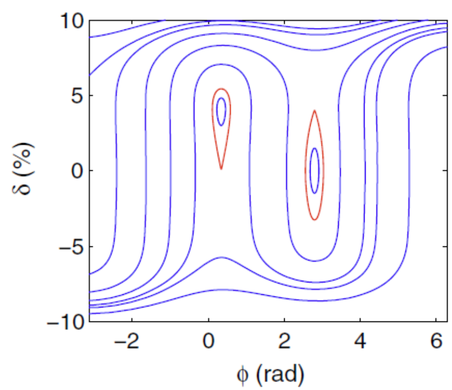




M. Reis et al., Low-ε-Rings, Heraklion, Greece, (2011)

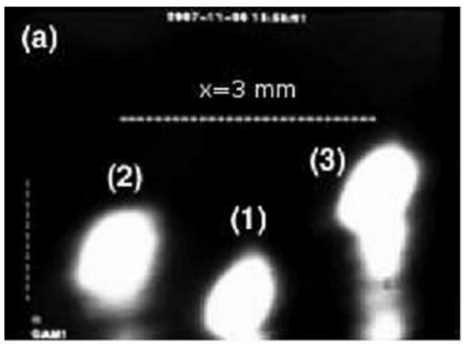
Diamond (UK)

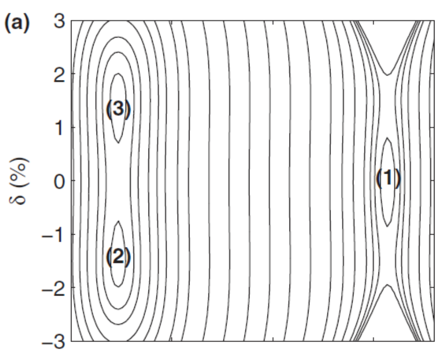




I. Martin et al., PRSTAB 14, 040705 (2011)

SOLEIL (France)





M. Attal et al., PRSTAB 16, 054001 (2013)

Nonlinear Motion in Linacs

The final part of this lecture will concentrate on how nonlinear effects can affect the performance of linear-accelerator based facilities.

As we will see in the lecture on free-electron lasers, in order to maximise the FEL gain the electron density must be high in order benefit from the coherent emission of synchrotron radiation. Similarly, a high density is desirable for linear colliders in order to maximise the luminosity. In both cases, the electron bunch is compressed by applying a correlated energy chirp along the bunch length, then passing it through a magnetic chicane.

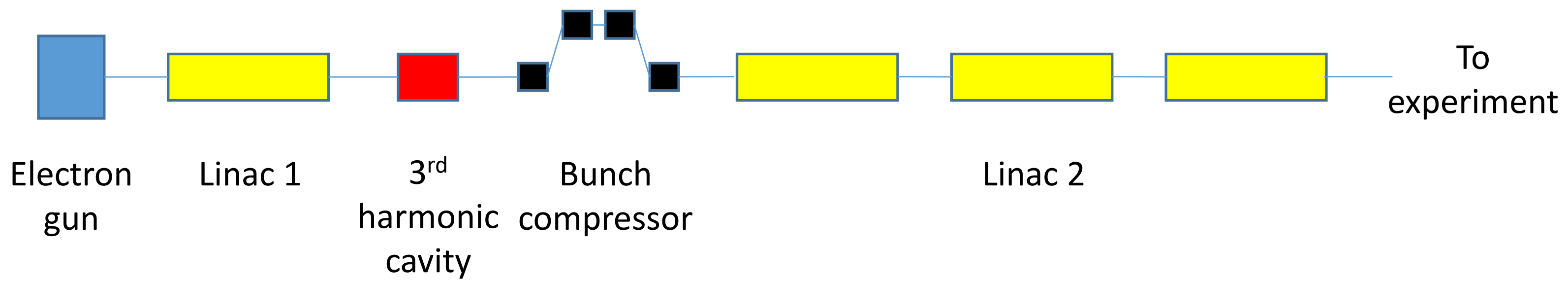
There are two main sources of nonlinear motion in linacs:

1) Curvature from the finite bunch length and the accelerating RF field
2) Nonlinearities introduced by the magnetic chicane

We will start by looking at the layout of a generic linear accelerator.

Typical layout of linear accelerator

A linear accelerator typically has the following main components:



The energy at which the bunch is compressed is chosen in order to minimise the impact of various effects, such as:

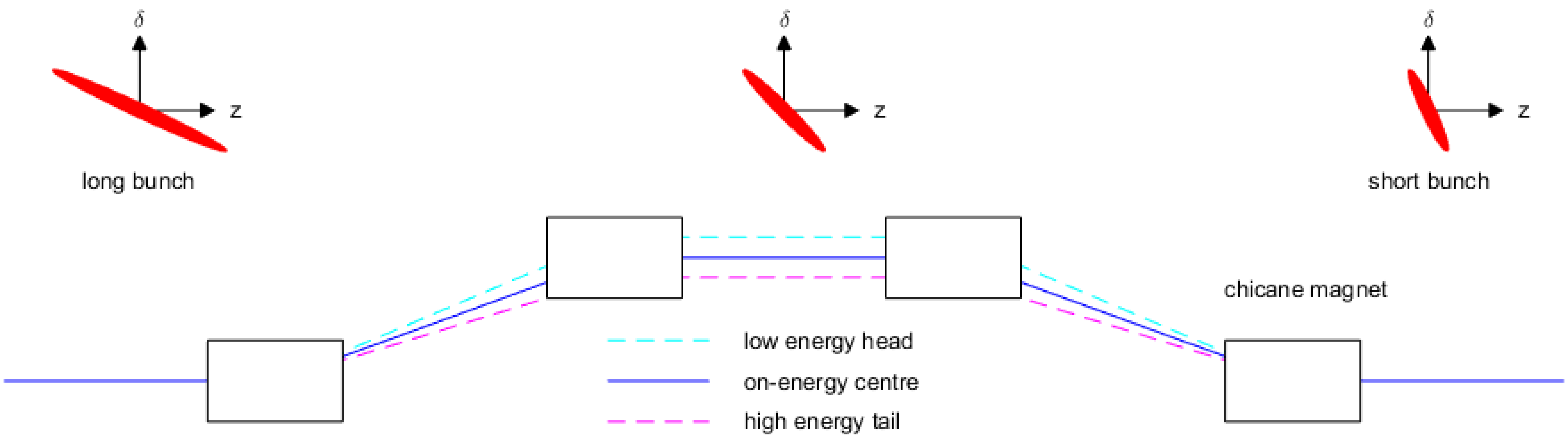
- space charge forces (dominant at low energies, $\propto 1/\gamma^2$)
- coherent and incoherent synchrotron radiation emission
- curvature from the RF wave (better to shorten the bunch early on)

Frequently, several bunch compressors are included at different energies in order to establish an effective trade-off between the competing effects.

Principles of longitudinal bunch compression

The bunch compression has the following stages:

1) Low energy electron bunch is generated by the electron gun
2) The bunch is accelerated off-crest to apply an energy ‘chirp’ (correlated energy variation)
3) The low energy head is delayed with respect to the tail by passing the bunch through a magnetic chicane



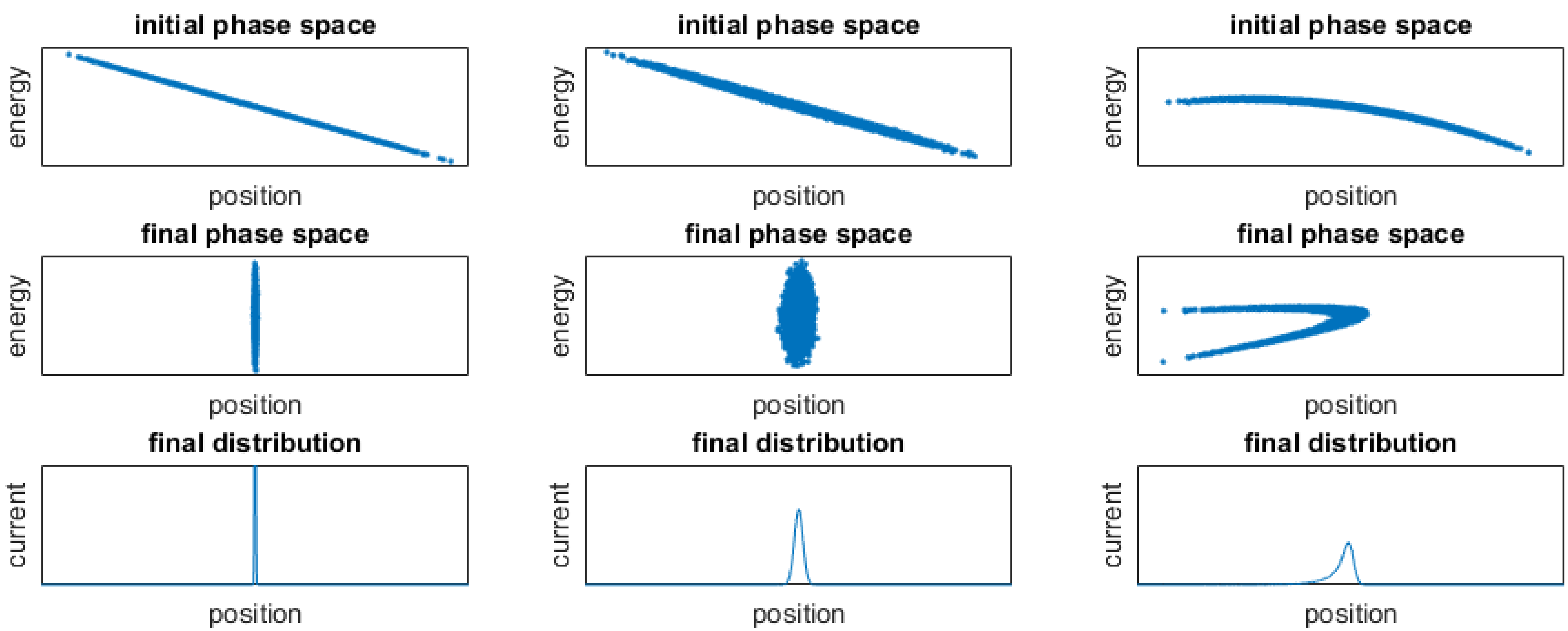
Principles of longitudinal bunch compression

The degree to which a bunch can be compressed depends to a large extent on the details of the longitudinal phase space of the bunch as it enters the chicane.

Ideal bunch:
- zero energy spread
- linear energy chirp

Ideal RF:
- finite energy spread
- linear energy chirp

Real accelerator:
- finite energy spread
- nonlinear chirp



<u>RF Curvature</u>

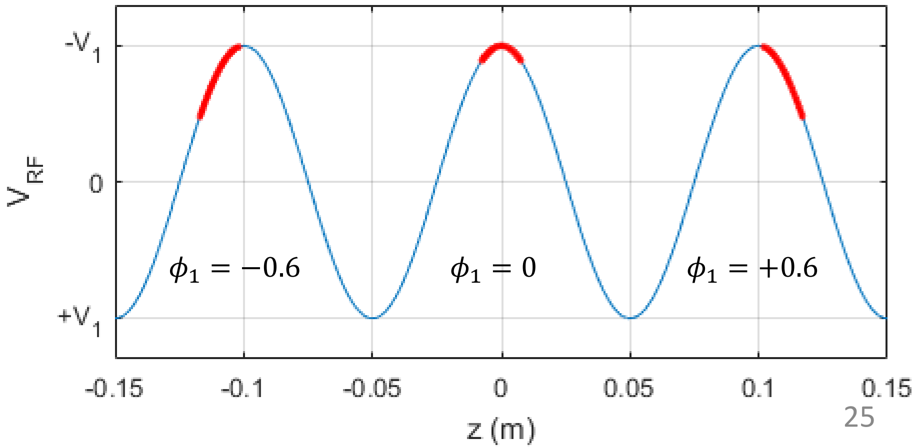
First we investigate the impact of the RF cavity. Defining the centre of the electron bunch as $s = 0$, the energy of the particles at the entrance to the bunch compressor will be

$$E(s) = eV_1 \cos(k_1 s + \phi_1)$$

where V_1, ϕ_1 and $k_1 = 2\pi/\lambda_1$ are the amplitude, phase and wavenumber of the main linac structure relative to the centre of the electron bunch.

From this, we see the absolute energy variation within the bunch is larger for long bunches relative to the RF wavelengths and for large voltages. Clearly, if we wish to minimise the nonlinear effects we should shorten the electron bunch early and limit the extent relative to the RF wave.

In order to compress the electron bunch we need to apply an energy chirp. This is controlled by the phase of the main linac, ϕ_1, i.e. by running 'off-crest'.



Linearisation of the energy chirp

We would like to apply a linear energy chirp to the electron bunch, and set the higher order terms to zero. To do this, we need to add in a linearising RF cavity operating at the 3rd harmonic ($V_{3HC}(s) = V_3 \cos(3k_1 s + \phi_3)$) and use this to cancel the derivatives of the energy distribution, i.e.:

$$
\begin{aligned}
E(s=0) &= eV_1 \cos(\phi_1) + eV_3 \cos(\phi_3) \\
E'(s=0) &= -k_1 eV_1 \sin(\phi_1) - 3k_1 eV_3 \sin(\phi_3) \\
E''(s=0) &= -k_1^2 eV_1 \cos(\phi_1) - 9k_1^2 eV_3 \cos(\phi_3) \\
E'''(s=0) &= k_1^3 eV_1 \sin(\phi_1) + 27k_1^3 eV_3 \sin(\phi_3)
\end{aligned}
$$

The first derivative is the linear energy chirp which we would like to keep, allowing us to choose V_1 and ϕ_1. In order to cancel the quadratic term, we need to set:

$$V_3 = -\frac{V_1}{9}\frac{\cos(\phi_1)}{\cos(\phi_3)}$$

If we wish to also cancel the cubic term, we need to set

$$V_3 = -\frac{V_1}{27}\frac{\sin(\phi_1)}{\sin(\phi_3)}$$

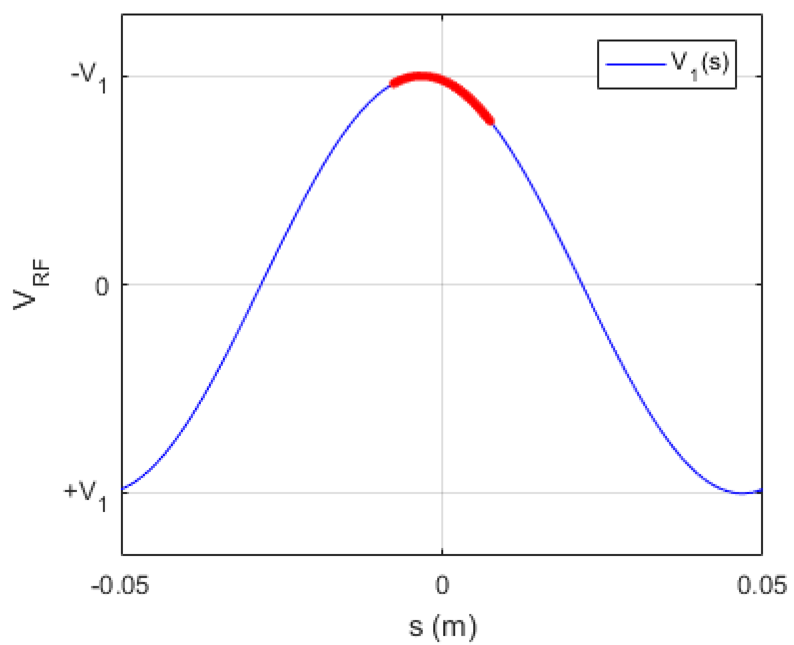
This gives two equations with two unknowns, defining V_3 and ϕ_3 uniquely.

Linearisation of the energy chirp

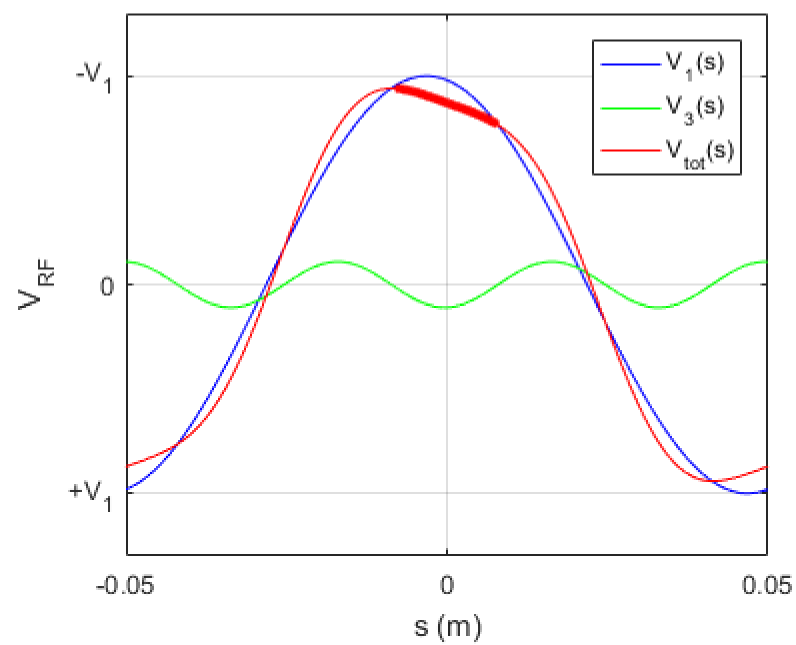
Example:

main linac voltage	= V_1
main linac phase	= 0.2 rad
⇒ 3HC phase	= 0.0675 rad
⇒ 3HC voltage	= $V_1/8.841$

Main RF only

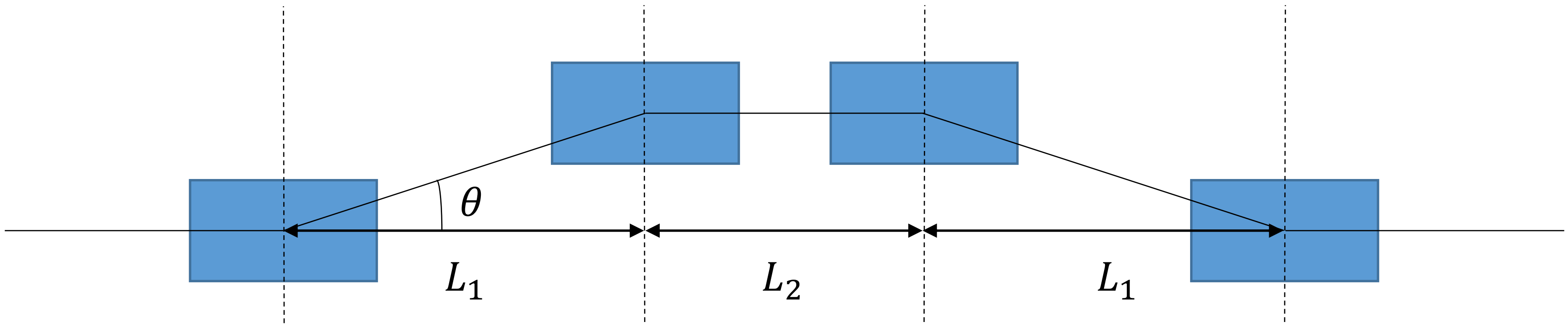


Main RF + 3HC



Nonlinearities from the bunch compressor

The second source of nonlinear motion in our example linear accelerator comes from the magnetic chicane. If we assume the energy of the particles are constant when passing through the chicane (i.e. neglect the effect of synchrotron radiation), then only the path-length will have a dependence on the energy deviation.



In this case, the total distance travelled by the particles when passing through the chicane is:

$$L_{tot} = \frac{2L_1}{\cos\theta} + L_2$$

where the bend angle θ depends upon the energy deviation, i.e. $\theta(\delta) = \theta_0/(1+\delta) \approx \theta_0(1-\delta)$ (high energy will be deflected by a smaller angle and vice versa).

Nonlinearities from the bunch compressor

However, rather than the total distance travelled by the particle, we are interested in the difference in path length with energy deviation compared to the nominal reference particle, $\Delta s = L_{tot}(\delta) - L_{tot}(0)$. In this case we can write:

$$\Delta s = 2L_1\left(\frac{1}{\cos\theta(\delta)} - \frac{1}{\cos\theta_0}\right)$$

Clearly, the deviation in path length with energy is nonlinear with energy deviation. In order to progress, we can expand the cosine terms and re-express the equation as a Taylor series expansion.

As first approximation, we can say:

$$\frac{1}{\cos\theta} \approx 1 + \frac{\theta^2}{2}$$

With the additional approximation that $\theta(\delta) \approx \theta_0(1-\delta)$, we have

$$\Delta s \approx -2L_1\theta_0^2\delta + L_1\theta_0^2\delta^2$$

Aside: Taylor maps for nonlinear elements

In the first lecture, we introduced the linear map of an element that describes the transfer matrix from some initial set of coordinates before the element to a final set, i.e.

$$z^f = \mathbf{R}\, z^i$$

where $\mathbf{R}$ is the transfer matrix and $z^i$and $z^f$are vectors of the initial and final particle coordinates respectively:

$$z = [x, x', y, y', s, \delta]$$

The individual elements of the $\mathbf{R}$ matrix can be denoted with indices that refer to each phase space component. For example, in full matrix form we have

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ s \\ \delta \end{pmatrix}^f = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ s \\ \delta \end{pmatrix}^i$$

i.e., the final longitudinal position coordinate depends on the energy deviation via the R_{56} element.

Aside: Taylor maps for nonlinear elements

In general, we can continue this notation using a Taylor series expansion, such that:

$$z_j^f = \sum_k R_{jk}\, z_k^i + \sum_{kl} T_{jkl}\, z_k^i z_l^i + \sum_{kl} U_{jklm}\, z_k^i z_l^i z_m^i + \cdots$$

In this case, the dependence of the final longitudinal position coordinate depends on the initial energy deviation via the expansion $s^f = s^i + R_{56}\delta + T_{566}\delta^2 + U_{5666}\delta^3 + \cdots$.

Typically a Taylor map is truncated at some order, where the highest order is determined based upon the required accuracy and the strength of the nonlinearities at each order. Clearly, it is desirable to only include the lowest order terms necessary in order to reduce calculation times.

One consequence of truncating the power series is that in general the map will no longer be *symplectic*, that is, the volume of phase space will not be conserved by the transformation. This is in violation of Liouville's Theorem, and can introduce unwanted effects in particle tracking such as artificial damping, excitation or diffusion. Special techniques have to be applied in order to remain a transfer map remains symplectic if it is to be used to study long-term stability during particle tracking.

Nonlinearities from the bunch compressor

Returning to the problem of the magnetic chicane, we have

$$\Delta s = 2L_1\left(\frac{1}{\cos\theta(\delta)} - \frac{1}{\cos\theta_0}\right)$$

or in terms of the second order Taylor map we can write

$$\Delta\, s = R_{56}\delta + T_{566}\delta^2$$

where we have for our approximation:

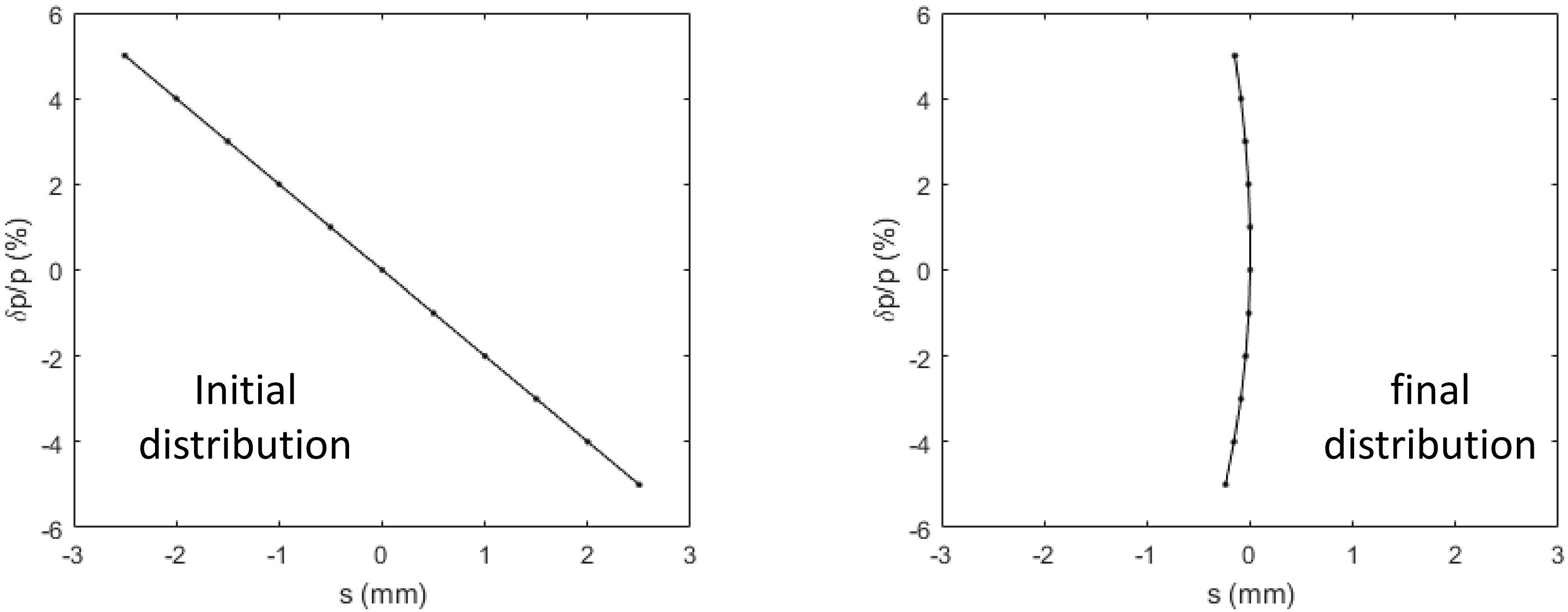
$$R_{56} = -2L_1\theta_0^2, \qquad \mathrm{T}_{566} = +L_1\theta_0^2$$

To investigate the significance of the nonlinearities, let us assume for now that our initial bunch has:

- perfectly linear energy chirp of $d\delta/ds = 20$
- zero energy spread
- initial bunch length of 5 mm.

Nonlinearities from the bunch compressor

If we want total compression, then using the linear approximation we should set $R_{56} = -0.25$. Taking $L_1 = 1\ m$, then the required bend angle is $\theta_0 \approx \sqrt{\frac{R_{56}}{-2L_1}} \approx 20\ deg$.



After passing through the magnetic chicane, the distribution acquires a curvature very similar to the one we found from the RF wave. In practice, the phase and voltage of the harmonic cavity can be set to compensate for both the nonlinearities from the main linac and the chicane. Alternatively, sextupoles can be placed inside the chicane to compensate for the nonlinearities.

Summary

Hamiltonian mechanics can be used to analyse the impact of nonlinear elements in a perturbative form

Resonance Driving Terms can be computed to isolate the impact on different aspects of beam motion, order by order

Nonlinearities can lead to additional stable fixed-points of motion in both the transverse and longitudinal planes. Multiple beams can be stored simultaneously and exploited for user experiments.

Nonlinear effects also appear in linear machines, limiting their performance.

- RF curvature and nonlinear terms from the magnetic chicane can limit the achievable peak current.
- Compensation elements such as higher-harmonic cavities or sextupoles within the chicane can be used

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