Electromagnetism

A refresher

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The CERN Accelerator School

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Electric force

- Common life examples
  - A kid sliding on a plastic surface

Electric force

- Common life examples
  - Polystyrene on a cat

Electric force

- The force that
  - repels the hairs
  - attracts polystyrene to the cat's fur

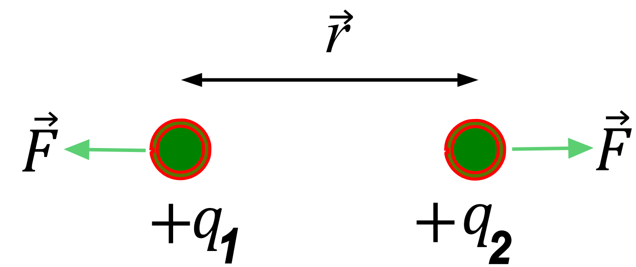
  is due to **electric charge**

- If electric charges are at rest then we call it

  electrostatic force

Coulomb law

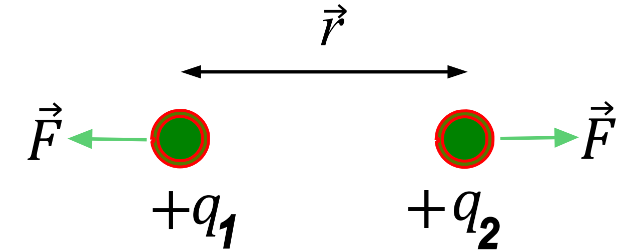
$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$



- Electrostatic force between point-like objects is
  - Proportional to electric charge of each of the two interacting objects

Coulomb law

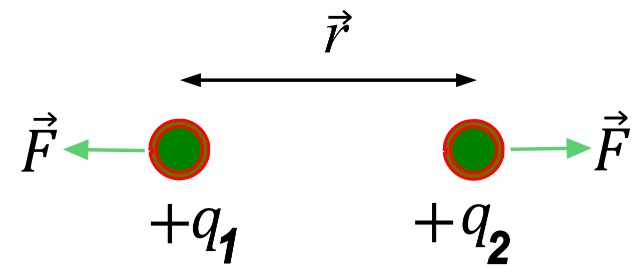
$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$



- Electrostatic force between point-like objects is
  – Proportional to electric charge of each of the two interacting objects
  – Inversely proportional to square of the distance

Coulomb law

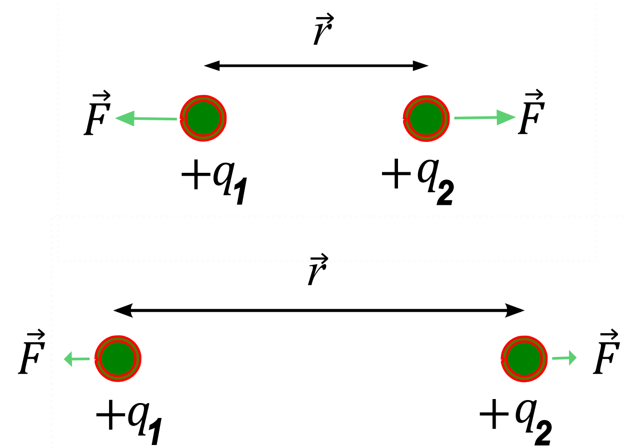
$$F_E = \boldsymbol{K} \cdot \frac{q_1 \cdot q_2}{r^2}$$



- Electrostatic force between point-like objects is
  - Proportional to electric charge of each of the two interacting objects
  - Inversely proportional to square of the distance
  - Proportional to Coulomb constant K
    - Which depends on medium type (vacuum, air, water)

Distance

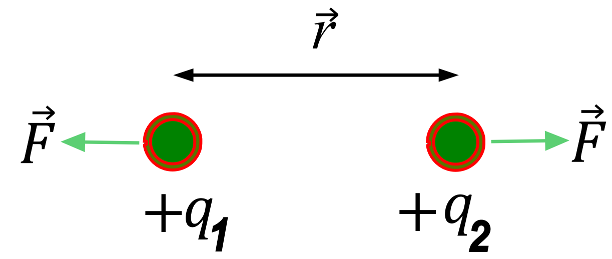
$$F_E = \boldsymbol{K} \cdot \frac{q_1 \cdot q_2}{r^2}$$



- Electrostatic force between point-like objects is
  - Inversely proportional to square of the distance
  - If we increase the distance 2x then the force is 4x smaller
  - If we increase the distance 10x then the force is 100x smaller

Electric charge

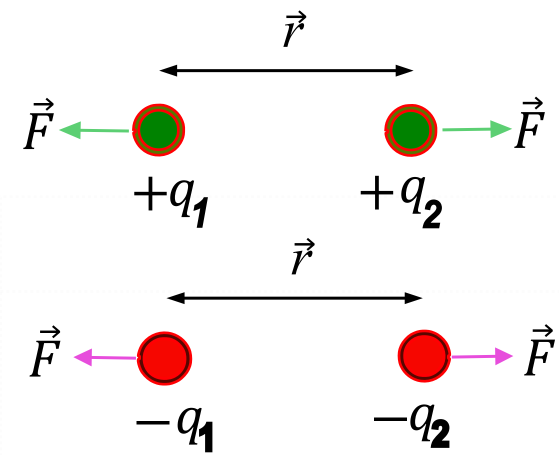
$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$



- Electrostatic force between point-like objects is
  - Proportional to electric charge of each interacting objects
  - **It means that if one of the objects has 2x more charge then the force is 2x stronger**

Electric charge

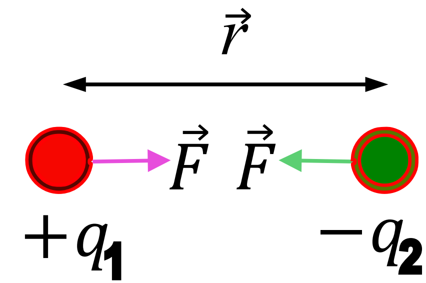
$$F_E = \boldsymbol{K} \cdot \frac{q_1 \cdot q_2}{r^2}$$



- Electric charge can be negative or positive
- **If charge of both objects is the same then the force is repelling**

Electric charge

$$F_E = K \cdot - \frac{q_1 \cdot q_2}{r^2}$$



- Electric charge can be negative or positive
- **If charge of both objects is opposite then the force is attracting**

What is electric charge?

- It is a fundamental property of some elementary particles

- It has unit of Coulomb [C]

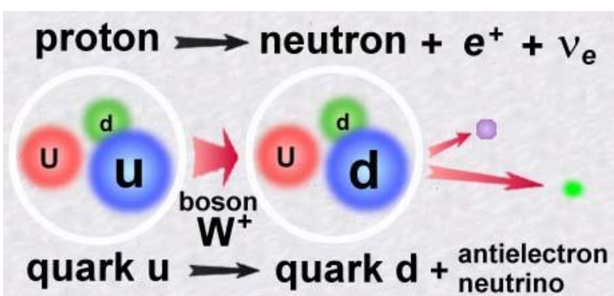
What is electric charge?

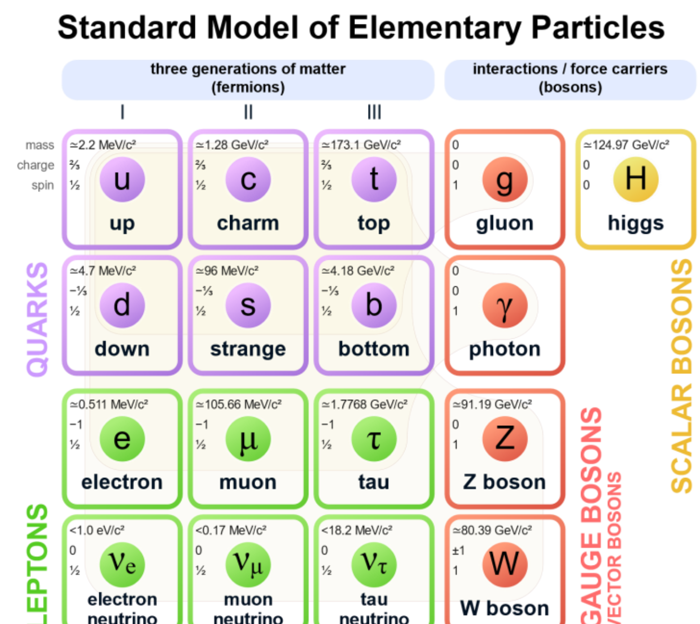
- Charge can be negative, for example for electron, or positive, for example for proton
- Electron charge, $e = -1.602 \cdot 10^{-19}$ C, is exactly opposite of proton charge
  - Why? Because proton can decay to positron, i.e. anti-electron, plus neutral stuff

$$^{11}_{6}C \rightarrow \, ^{11}_{5}B + e^{+} + \nu_{e} + 0.96 \text{ MeV}$$

Electric charge

- All free particles have charge that is multiple of *e*
- Quarks have charge of 2/3 or -1/3 of *e*
  - But they are bound to exist only in triplets such that the total charge is 0, e, 2e
  - N.B. beta decay is in fact a decay of up quark to down quark

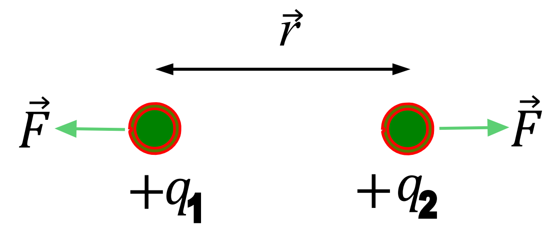




Electrostatic force

- The force acts in direction of the 2 objects

$$F_E = K \cdot \frac{q_1 \cdot q_2}{\boldsymbol{r}^2}$$

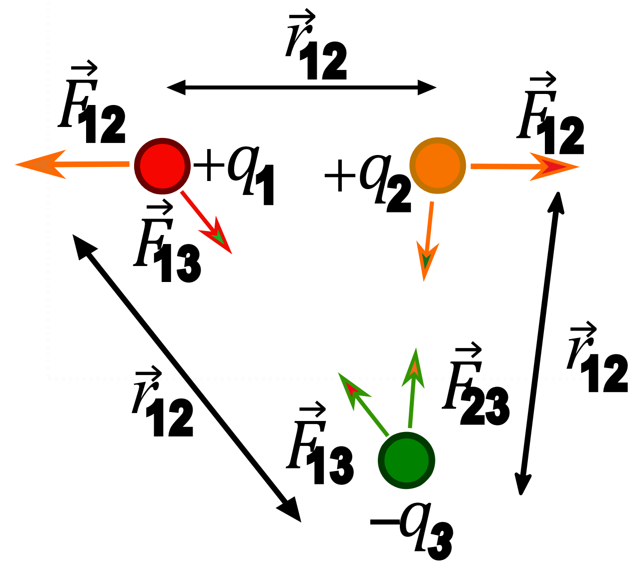


- In vector notation the equation is written

$$\overrightarrow{F_E} = K \cdot \frac{q_1 \cdot q_2 \cdot \vec{\boldsymbol{r}}}{\|\boldsymbol{r}\|^3}$$

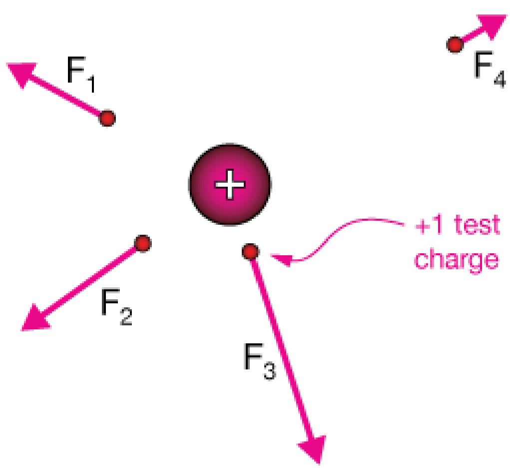
Multibody interaction

- If there is more than 2 charges interacting then we can calculate force of each pair and add the resulting forces as vectors: **principle of superposition**



Electric field

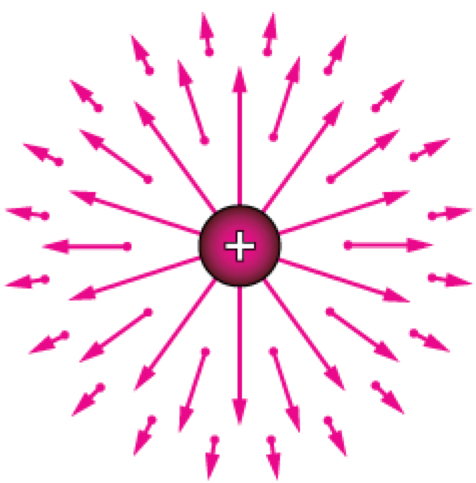
- It is much easier to do multibody calculations if we introduce **electric field**
- To every point in space we assign a **vector**
- It corresponds to force that the charged object creating the field would exert on a 1 Coulomb point like charge



Electric field of a positively charged sphere
e.g., a proton

$$\vec{E} = K \cdot \frac{q}{\|\boldsymbol{r}\|^3} \vec{r}$$

$$E = K \cdot \frac{q}{r^2}$$



Electric field

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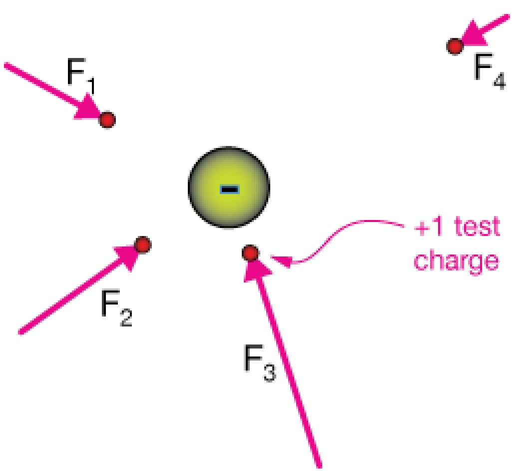
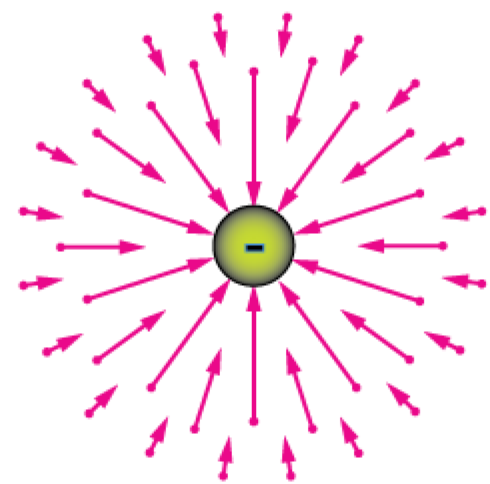


Electric field of a negatively charged sphere
e.g., an electron

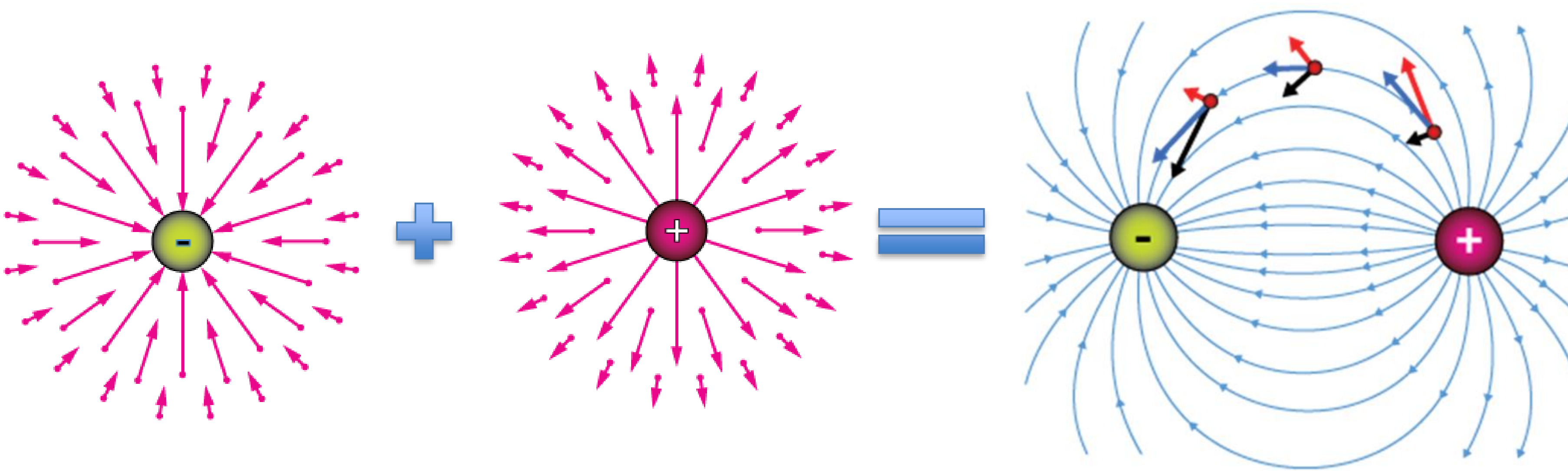
$$\vec{E} = -K \cdot \frac{q}{\|\boldsymbol{r}\|^3} \vec{r}$$

$$E = -K \cdot \frac{q}{r^2}$$



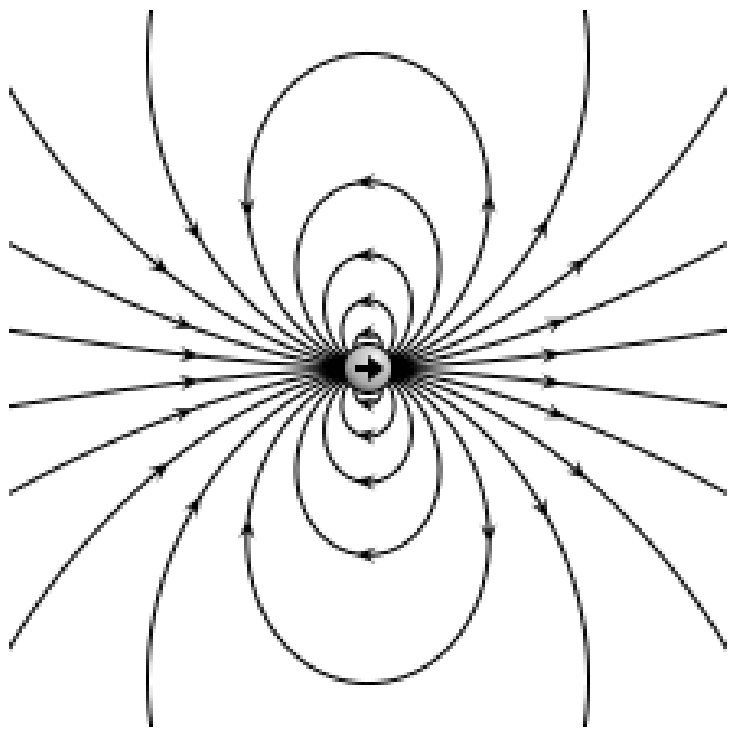
Superposition of electric fields

- The fields can be simply added
- Having electric field $\vec{E}$ we can calculate force $\vec{F} = q\vec{E}$



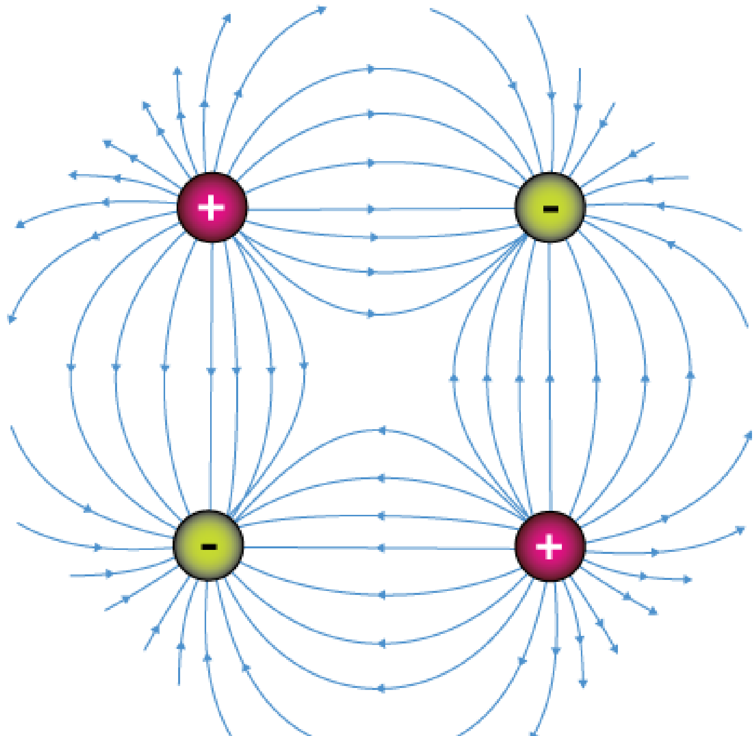
Electric fields

- Field of an electric dipole



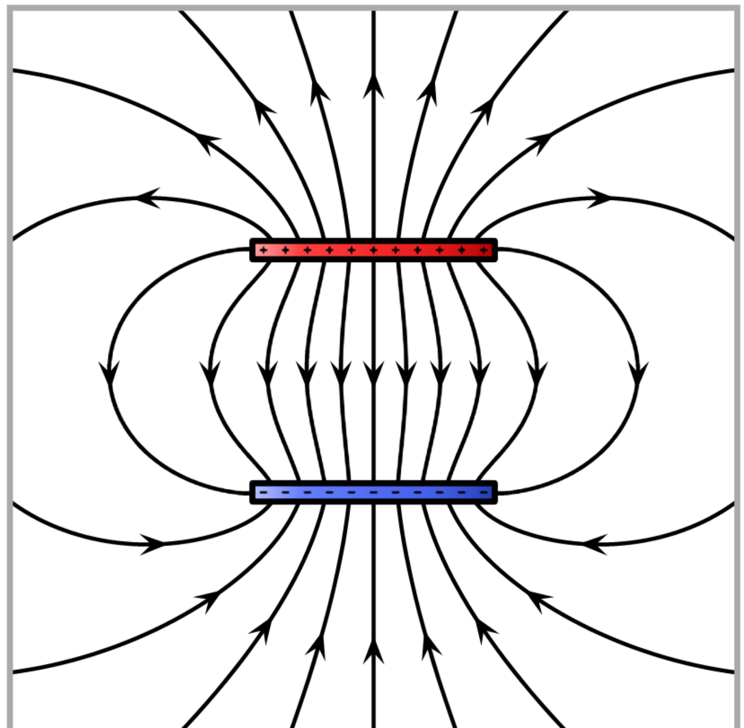
Electric fields

- Field of an electric quadrupole



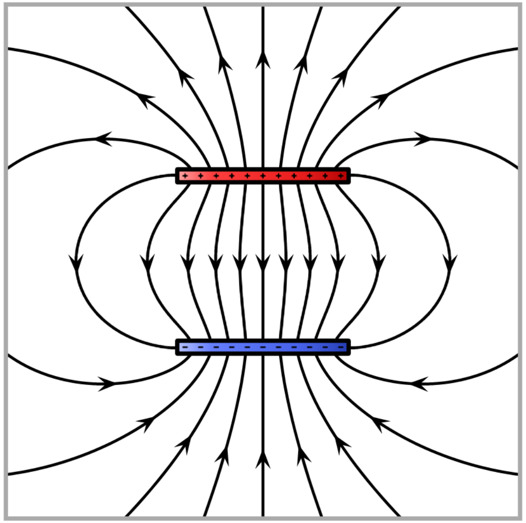
Electric fields

- Field between charged plates



Electrostatic forces and object shapes

- The electric field distribution depends on the shape of the charged objects
- The same way the electrostatic force between arbitrary objects depends on their shape



Influence of medium

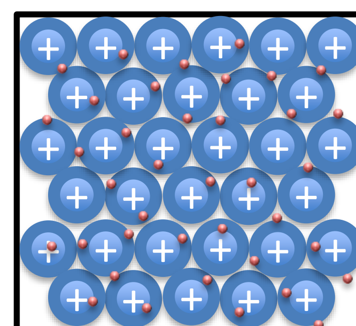
$$F_E = \boldsymbol{K} \cdot \frac{q_1 \cdot q_2}{r^2}$$

- The force changes depending on the medium that the objects are immersed in
  - *Why? Electric charge stays the same …*
  - Because the medium is made of charged particles

Influence of medium

$$F_E = \boldsymbol{K} \cdot \frac{q_1 \cdot q_2}{r^2}$$

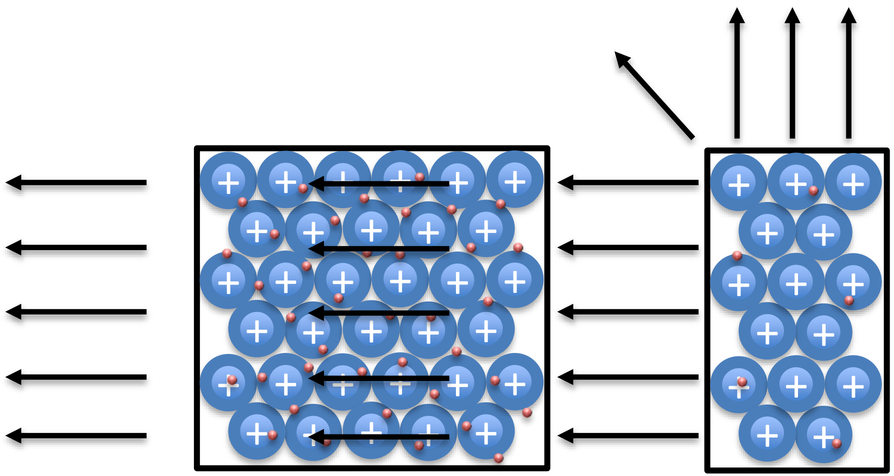
- In metals electrons can freely move within volume



Influence of medium

$$F_E = \boldsymbol{K} \cdot \frac{q_1 \cdot q_2}{r^2}$$

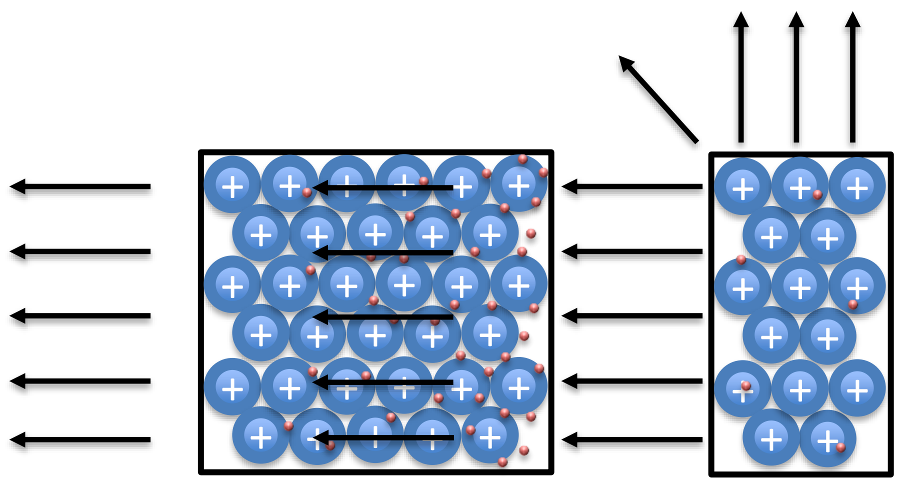
- In metals electrons can freely move within volume
- External charge exerts force on the electrons and protons



Influence of medium

$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$

- In metals electrons can freely move within volume
- External charge exerts force on the electrons and protons
  - Electrons are attracted towards a positive charge and are repelled from a negative one
  - Their displacement creates uneven distribution within the volume



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- In metals electrons can freely move within volume
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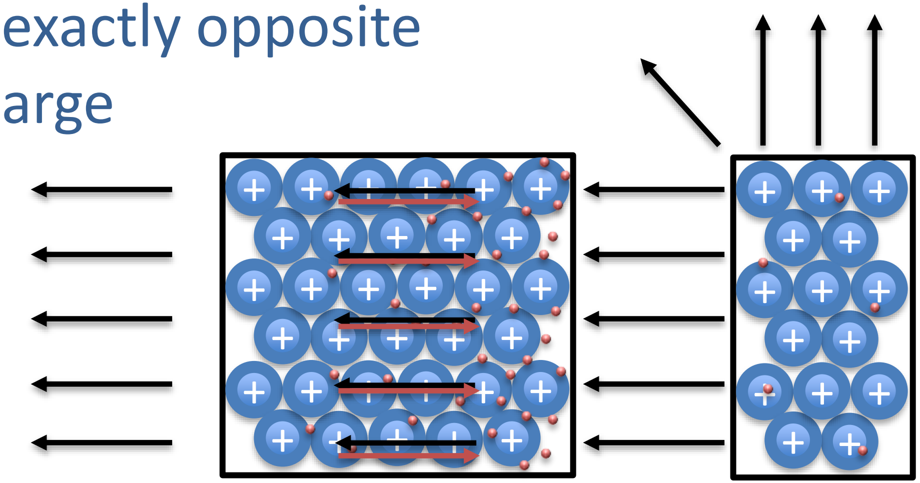
  - The resulting electric field is exactly opposite to the one of the external charge



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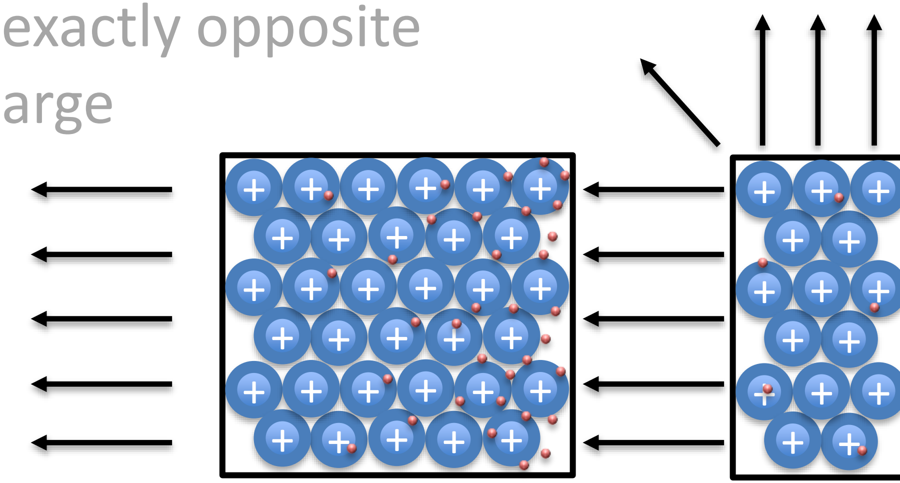
- In metals electrons can freely move within volume
- External charge exerts force on the electrons and protons
  - Electrons are attracted towards positive charge and are repelled from a negative one
  - Their displacement creates uneven distribution within the volume
  - The resulting electric field is exactly opposite to the one of the external charge
  - The electron motion continues until there is no electric field in the volume



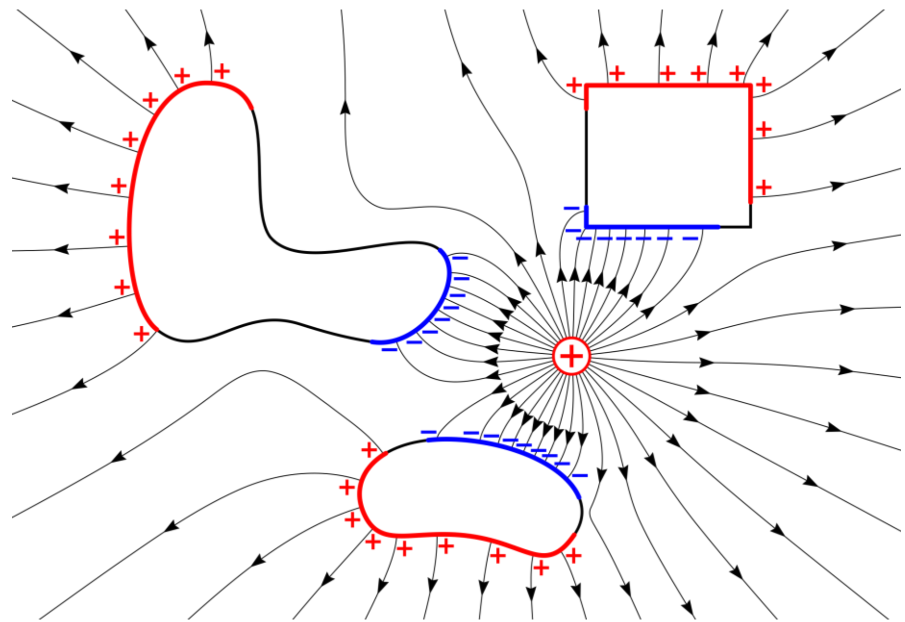
Influence of medium

$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$

- In metals electrons can freely move within volume

  → **Electric fields cannot penetrate metallic volumes**

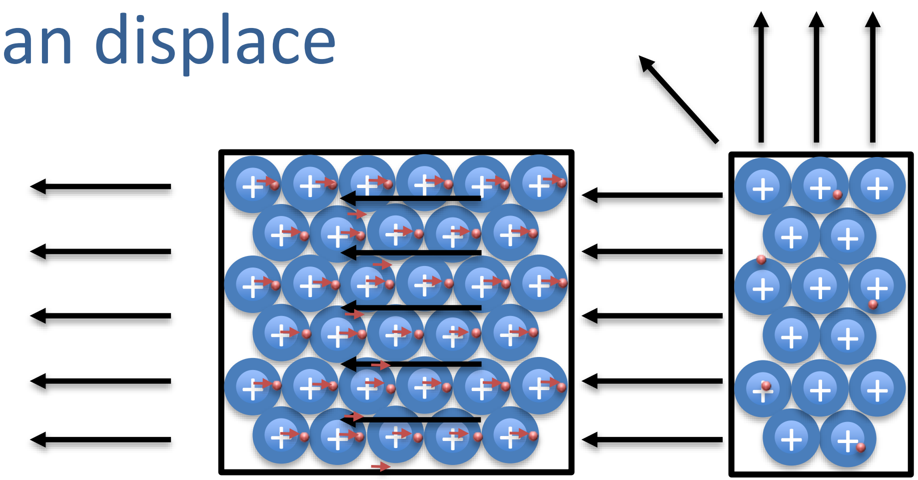
  → **Field lines are perpendicular to metallic surfaces**



Influence of medium

$$F_E = K \cdot \frac{q_1 \cdot q_2}{r^2}$$

- In non-metallic materials electrons cannot freely move
- Still upon external electric field electrons displace within their molecules and the material becomes polarized
- Induced electric field reduces the external field by the amount that depends on how much the electrons can displace



Influence of medium $F_E = \boldsymbol{K} \cdot \frac{q_1 \cdot q_2}{r^2}$

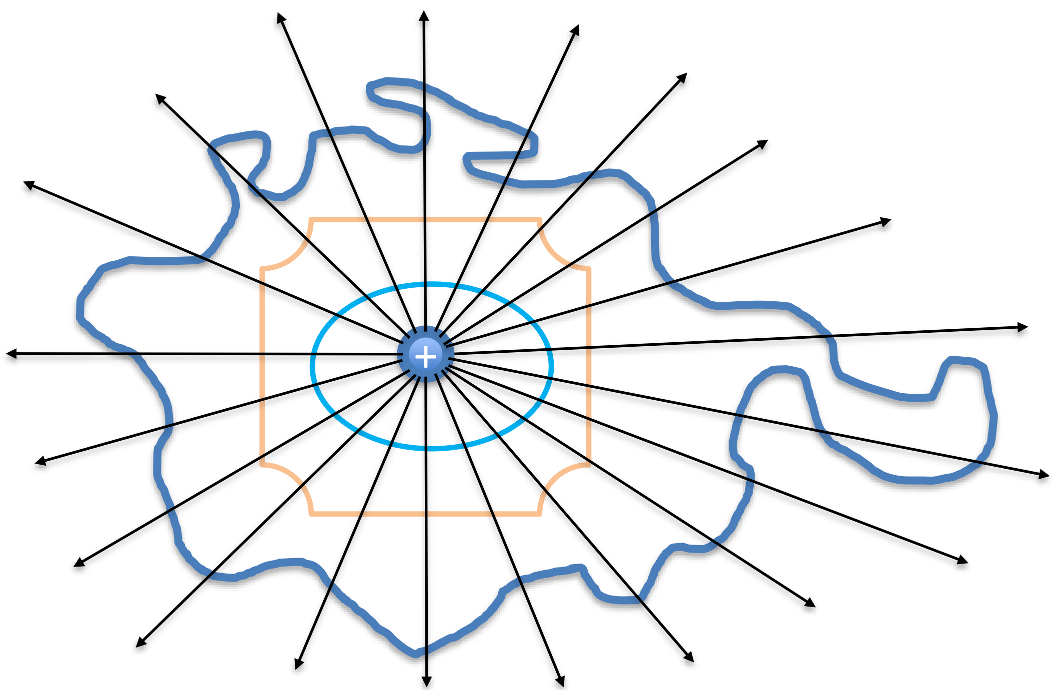
- Coulomb constant depends on the medium
- For vacuum $K = \frac{1}{4\pi\varepsilon_0} = 9 \cdot 10^9 \frac{C}{N \cdot m}$
  - where ε_0 is the vacuum permittivity
- For dielectrics $K = \frac{1}{4\pi\varepsilon}$, where ε is the material permittivity
- $\varepsilon = \varepsilon_r \varepsilon_0 = (1 + \chi)\varepsilon_0$, where
  - ε_r is the relative permittivity of the material
  - χ is susceptibility of the material

Influence of medium $F_E = \boldsymbol{K} \cdot \frac{q_1 \cdot q_2}{r^2}$

- The material permittivity $\varepsilon = \varepsilon_r \varepsilon_0 = (1 + \chi)\varepsilon_0$, where
  - ε_r is the relative permittivity of the material
  - χ is the susceptibility of the material
- Material permittivity in general depends on many factors
  - Temperature, pressure, if external electric field is time varying then on its frequency, …
  - One needs to take into account multiple phenomena to calculate correctly the electric field in dielectric
    - Sound waves, heat waves, ….

Gauss Law

Field flux Φ_E out of an arbitrary closed surface is proportional to the charge enclosed by the surface irrespective of how that charge is distributed



$$\Phi_E = \frac{q}{\varepsilon_0}$$

or

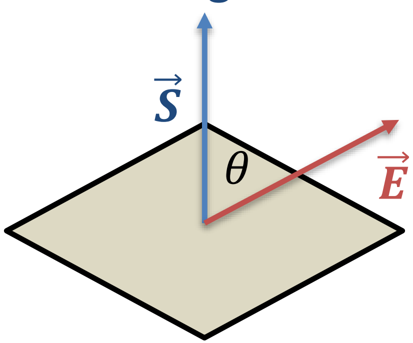
$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

Gauss Law

Field flux Φ_E out of an arbitrary closed surface is proportional to the charge enclosed by the surface irrespective of how that charge is distributed

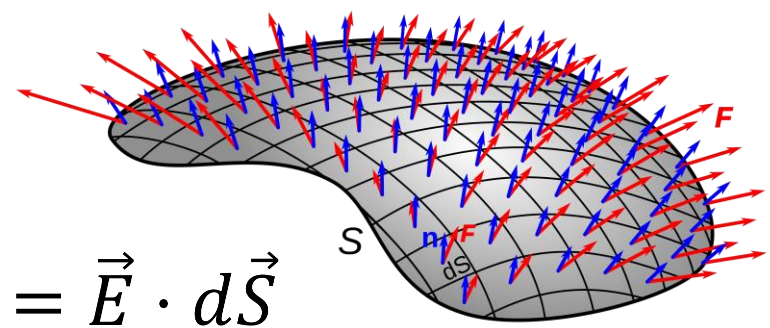
How much field $\vec{E}$ crosses area $\vec{S}$

S is a vector sticking out of a surface.
The vector length is the area of the surface



$$\Phi_E = \mathbf{E} \cdot \mathbf{S} = ES\cos\theta,$$

For and infinitesimally small area we get a differential equation

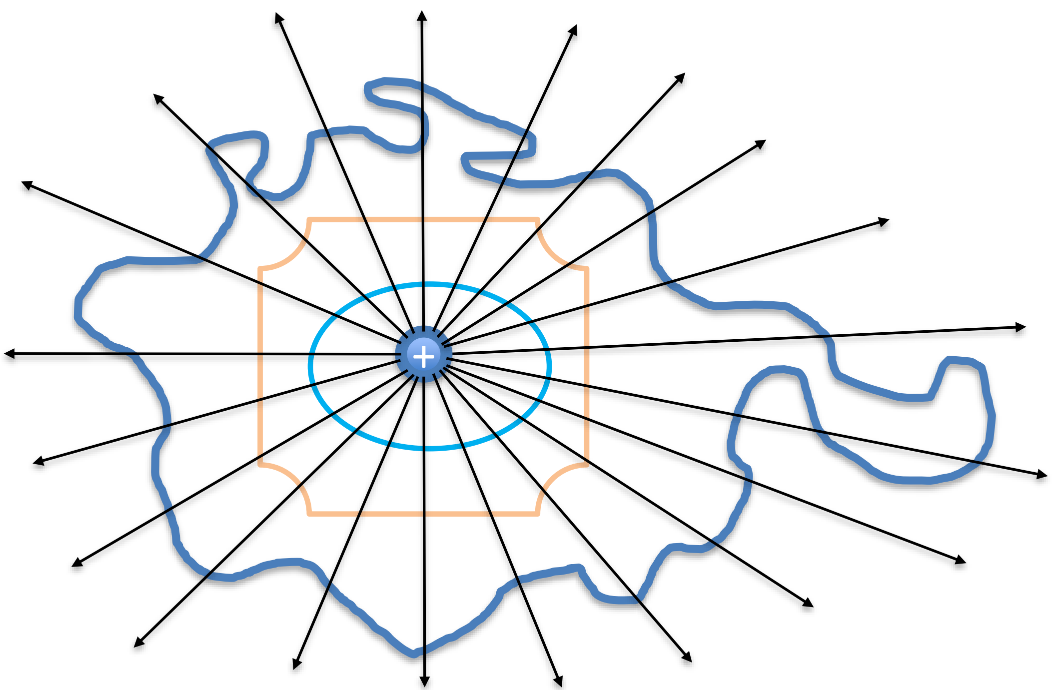


$$d\Phi_E = \vec{E} \cdot d\vec{S}$$

$$\Phi_E = \oiint_S \vec{E} \cdot d\vec{S}$$

Gauss Law

Field flux out of **an arbitrary closed surface**
is proportional to the charge enclosed by the surface
irrespective of how that charge is distributed



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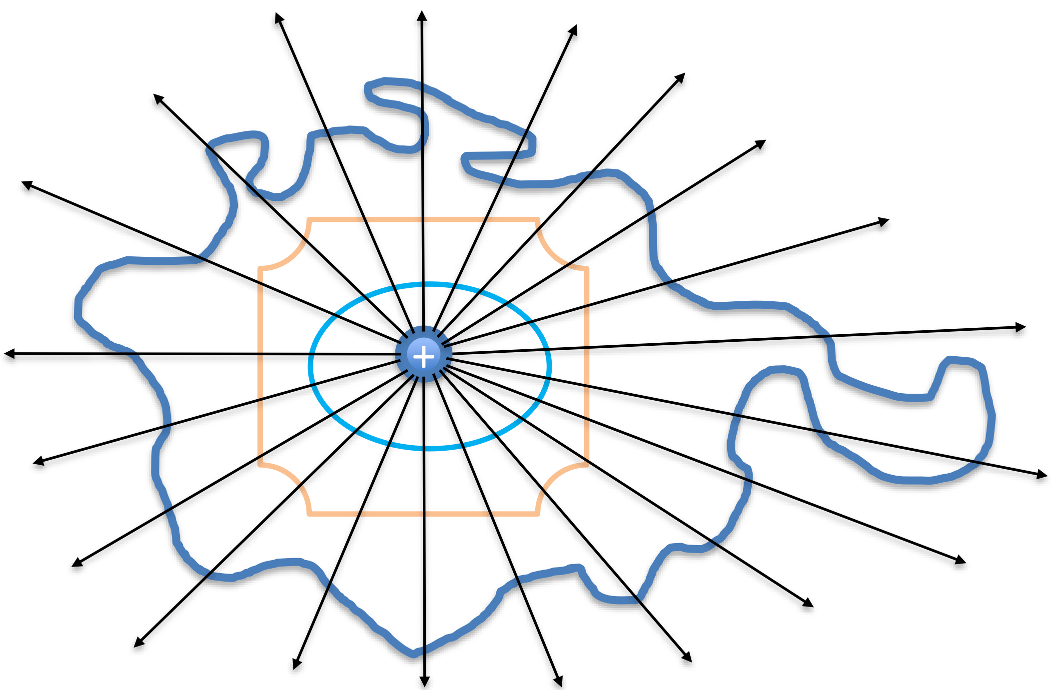
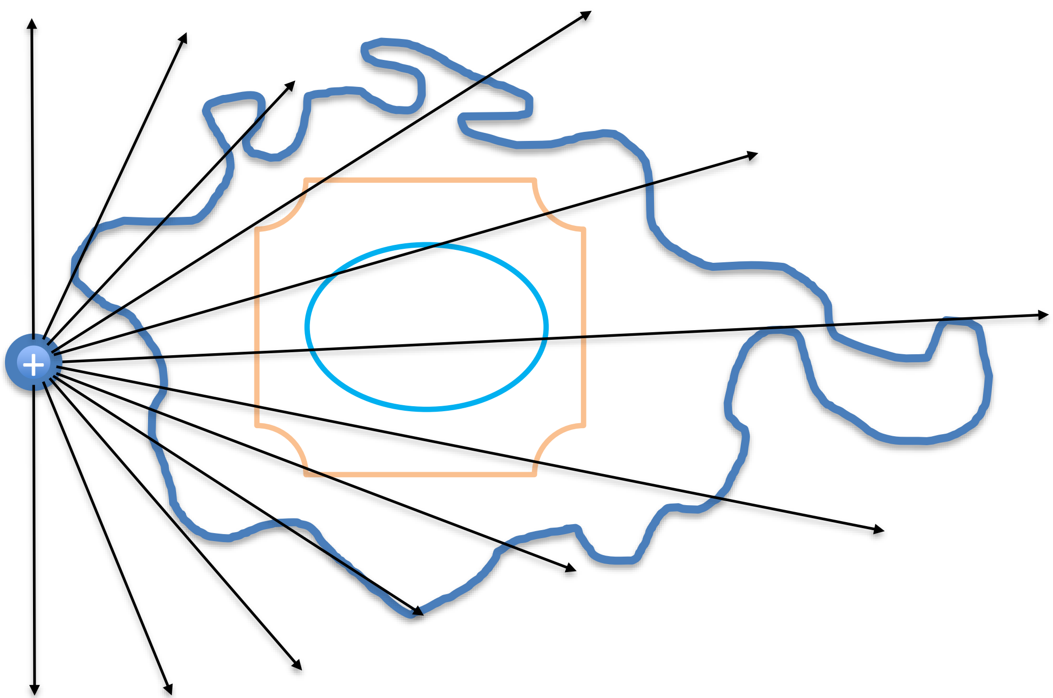


Only electric charges can create field lines

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

Gauss Law

If there is no charge inside the volume, then the total flux is zero, because the same amount of field enters the volume as it leaves it



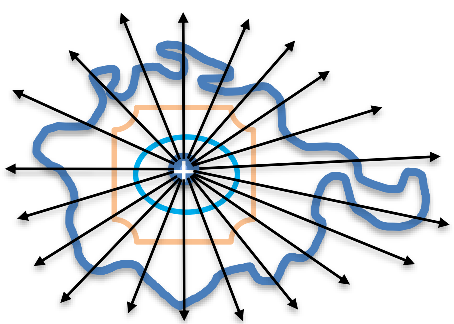
Gauss Law

- Gauss Law has another form using divergence operator

$$\nabla \cdot \vec{E} = \frac{\partial \vec{E}_x}{\partial x} + \frac{\partial \vec{E}_y}{\partial y} + \frac{\partial \vec{E}_z}{\partial z} = \frac{\rho}{\varepsilon_0}$$

  - Where ρ is the volume charge density
- Divergence tells how much field is created at a given point
- Only electric charges can create electric field lines

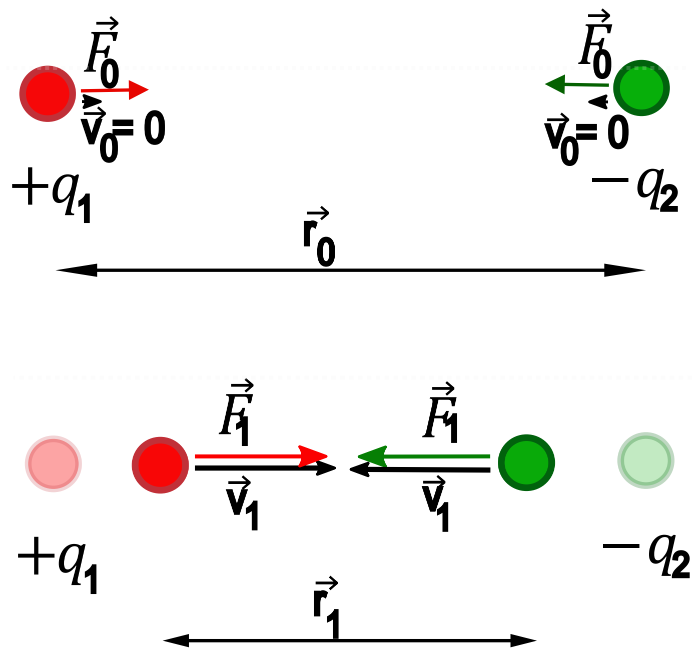
$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$



Field flux out of an arbitrary closed surface is proportional to the charge enclosed by the surface irrespective of how that charge is distributed

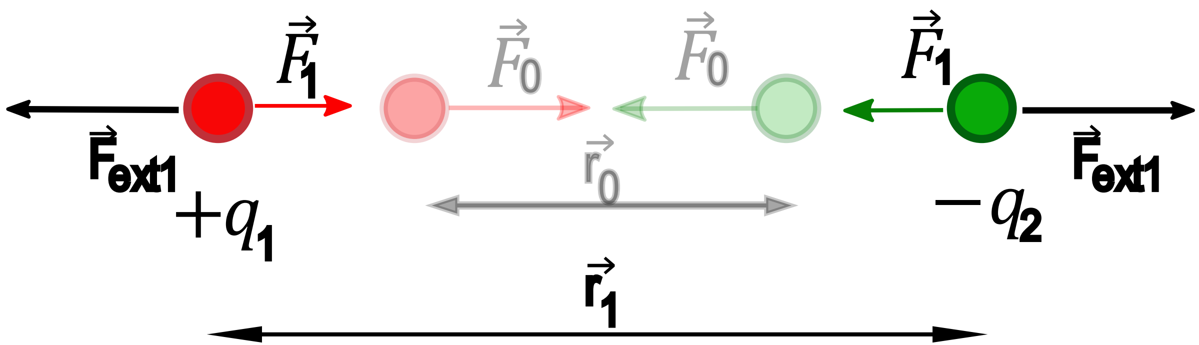
Electrostatic potential energy

- If we let the charges move upon electrostatic force, then they start accelerating → they gain kinetic energy



Electrostatic potential energy

- If we want to separate opposite sign charges, then we need to put work into it



Electrostatic potential energy

- Work needed to bring 2 point-like charges to a distance r

$$W = \int_{\infty}^{r} \vec{F} \cdot d\vec{r} = q_1 \int_{\infty}^{r} \vec{E} \cdot d\vec{r} = K q_1 q_2 \int_{\infty}^{r} \frac{dr}{r^2} = K q_1 \, q_2 \frac{1}{r}$$

Electrostatic potential energy

- If we let the charges move upon electrostatic force, then they start accelerating → they gain kinetic energy
- If we want to separate opposite sign charges, then we need to put work into it
- Electric field has potential energy
  - For example, potential energy of 2 point-like charges of 1 C brought together to a distance of 1 cm is

$$U_E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 \cdot q_2}{r} = 9 \cdot 10^9 \cdot \frac{1 \cdot 1}{0.01} = 9 \cdot 10^{11}\, J$$

Electrostatic potential energy

- Potential energy of electric field
  - For example, potential energy of 2 point-like charges of +1 C brought together at distance of 1cm is

$$U_E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 \cdot q_2}{r} = 9 \cdot 10^9 \cdot \frac{1 \cdot 1}{0.01} = 9 \cdot 10^7 \, J$$

  - If one of the charges has mass of 1 kg and we let it go, then all the potential energy will be converted to kinetic energy

$$U_E = E_K = \frac{mv^2}{2} \Rightarrow$$

$$v = \sqrt{2U_E/m} = 1'341.6 \, km/s = 4'829'907 \, km/h$$

  - 1 Coulomb it is a lot of charge!

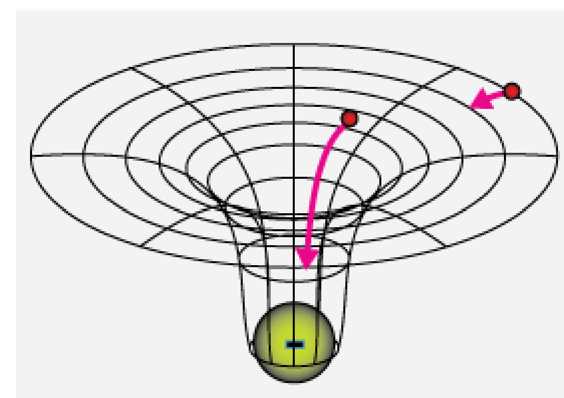
Electric potential

- Potential energy per 1 Coulomb is called **potential**

$$V = \frac{U_E}{q}$$

- It corresponds to the energy needed to bring 1 C charge from infinity to a given point
- Unit is called Volt [V]
- For point-like charges

$$V = K\frac{q}{r}$$



Electric potential

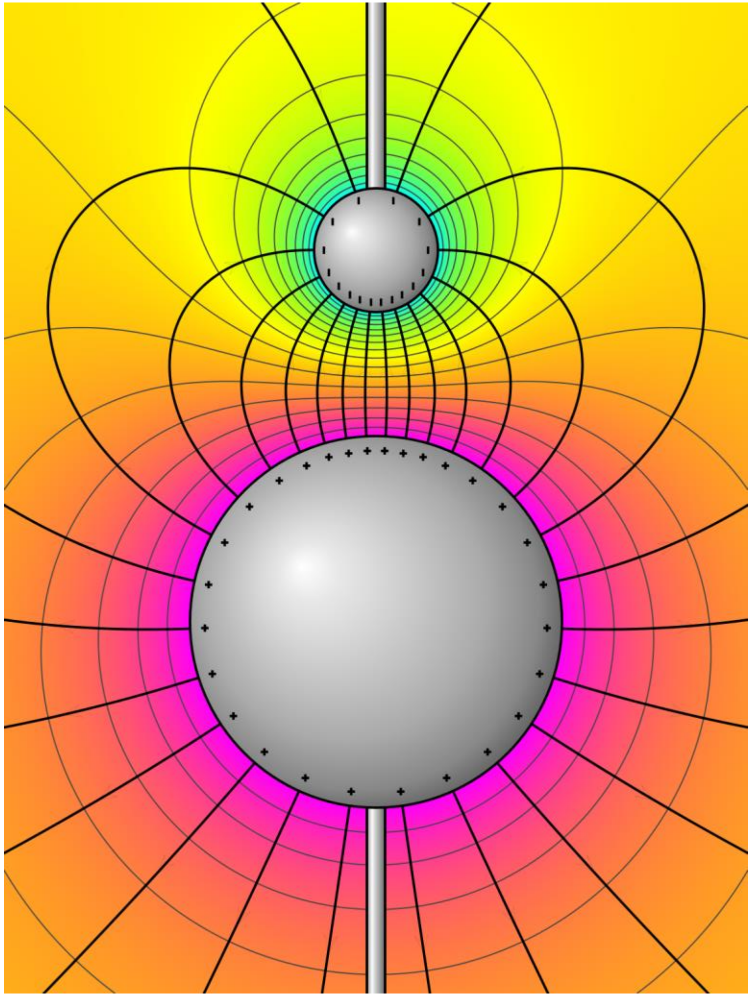
- Usually, it is much easier to solve equations using potentials than using fields and forces
  - Potential is a scalar, single value at each point in space
  - Field is a vector, it has 3 values for each point in space, so normally 3 equations are needed
  - Field can be easily obtained from potential, namely, field is equal to gradient of potential:

$$\vec{E} = \left(E_x, E_y, E_z\right) = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right) = \nabla \mathrm{V}$$

Field and potential

$$\vec{E} = (\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z})$$

- Thin lines = equipotential lines
- Thick lines = electric field lines
- Electric field lines are always perpendicular to equipotential lines



Capacitance

- It's the ratio between charge and produced voltage

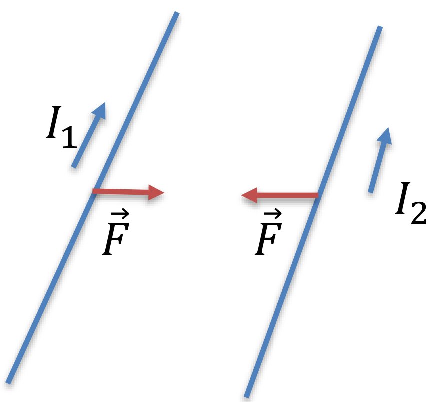
$$C = \frac{q}{V}$$

- Unit is Farad [F]

MAGNETISM

Magnetic Force

- Real life examples
  - Compass
  - Magnets
  - Attracted pair of wires

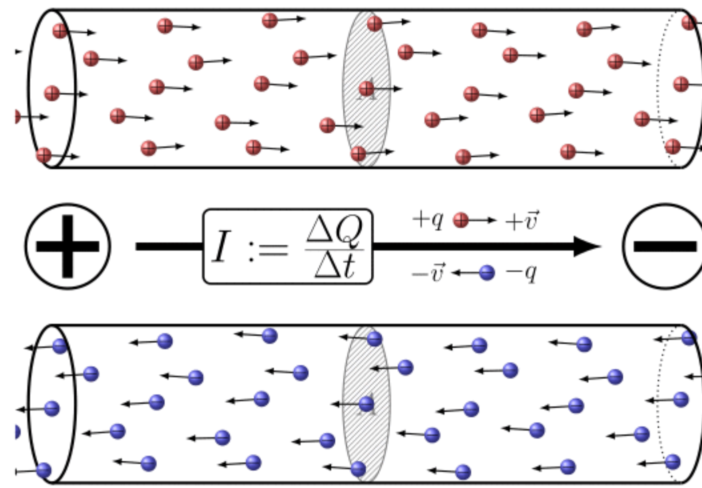


Electric current

- Magnetic force is due to moving electric charges
- Flow of charges is called electric current

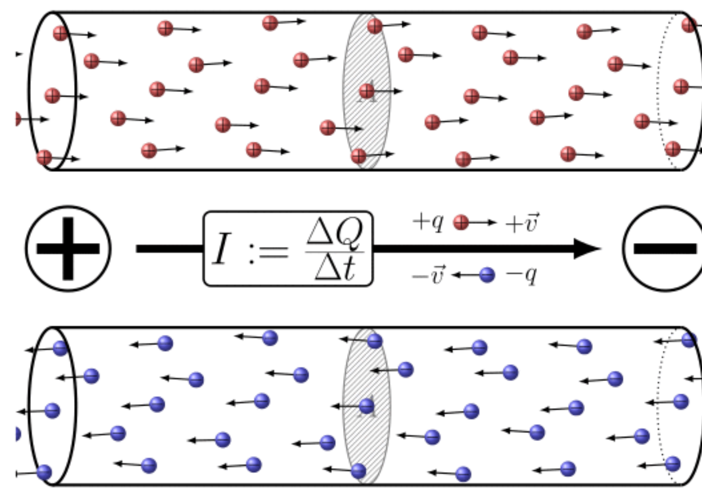
$$I = \frac{dq}{dt}$$

- It measures how much charge flows through a surface in a unit of time
- Unit is Ampere [A]



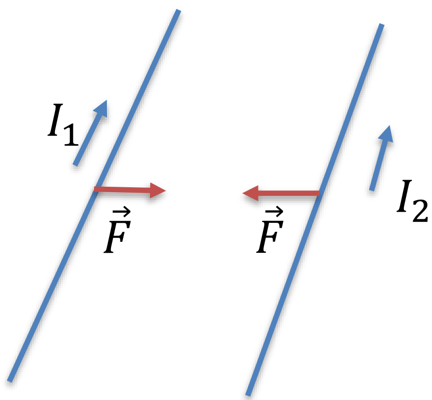
Electric current

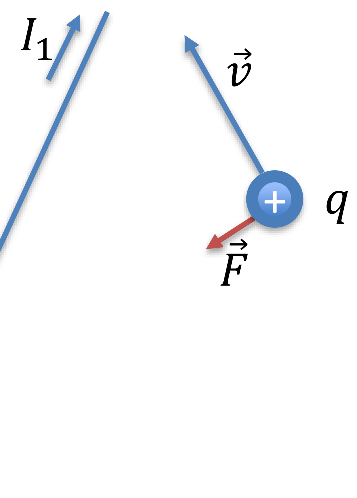
- Positive current when
  - Positive charge moves towards positive direction
  - Negative charge moves towards negative direction



Magnetic Force

- Magnetic force occurs only when both charges are moving

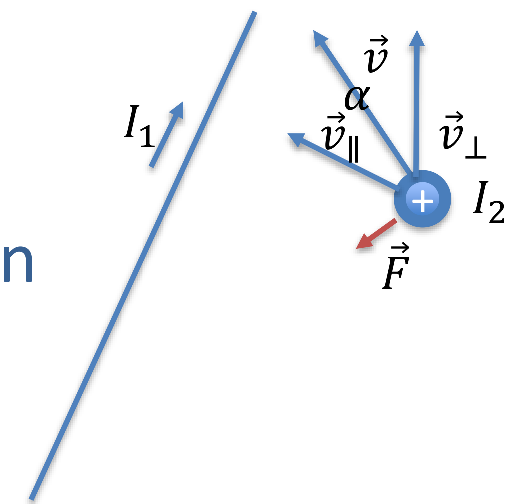




Magnetic Force charge and wire

- Only velocity component in plane with the wire and charge is important

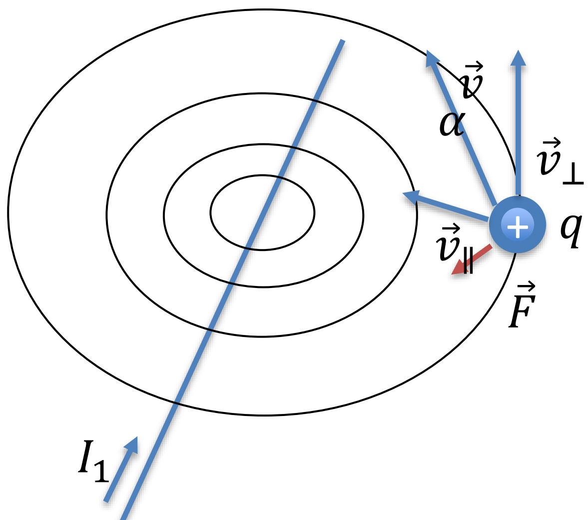
$$\vec{F} = \frac{\mu_0}{2\pi r} q \vec{v}_{\parallel} I_1$$



- If current and charge are positive, then the force is always towards the wire
- As closer to the wire as stronger the force
- Is proportional to charge and current
- μ_0 is the magnetic vacuum permeability (physical const.)

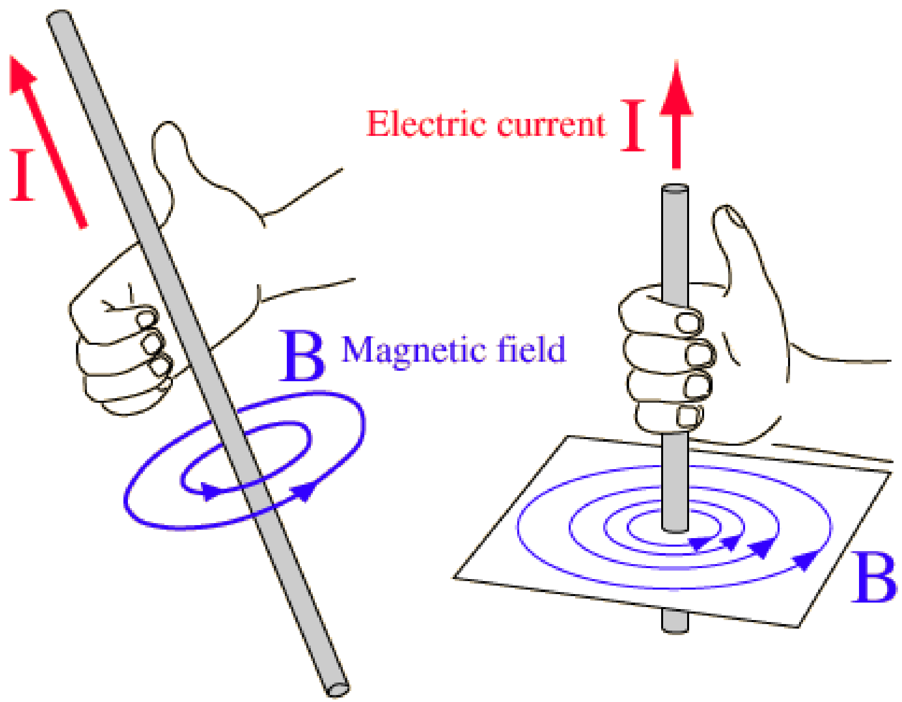
Magnetic Field

- The force is the same for the same r: $\vec{F} = \frac{\mu_0}{2\pi r} q \vec{v}_{\parallel} I_1$
- Field of magnetic force creates circles around the wire
- Strength of magnetic field from a wire is $B = \frac{\mu_0}{2\pi r} I_1$



Direction of magnetic field

- For positive current direction of magnetic field is determined with rule of right hand

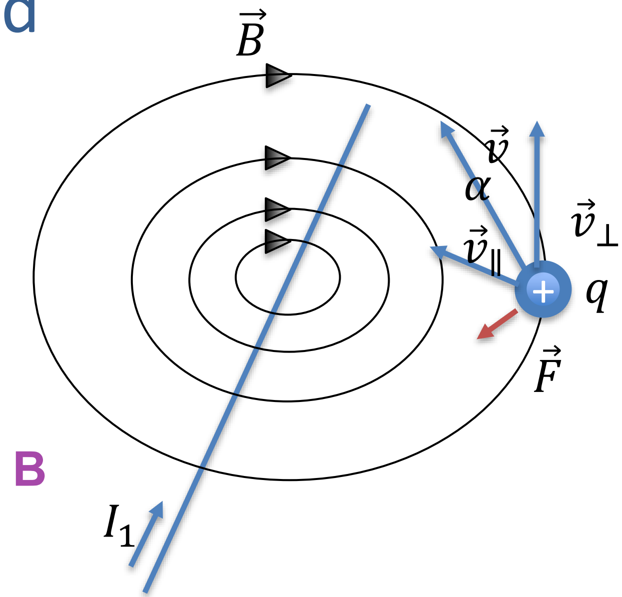


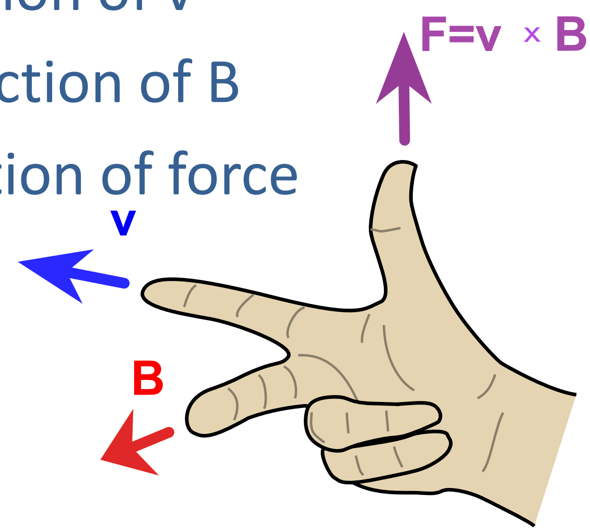
Magnetic Force

- Equation for force from magnetic field

$$\vec{F} = q \cdot \vec{v} \times \vec{B}$$

- Direction of force determined with rule of right hand
  - Index finger: direction of v
  - Middle finger: direction of B
  - Thumb gives direction of force



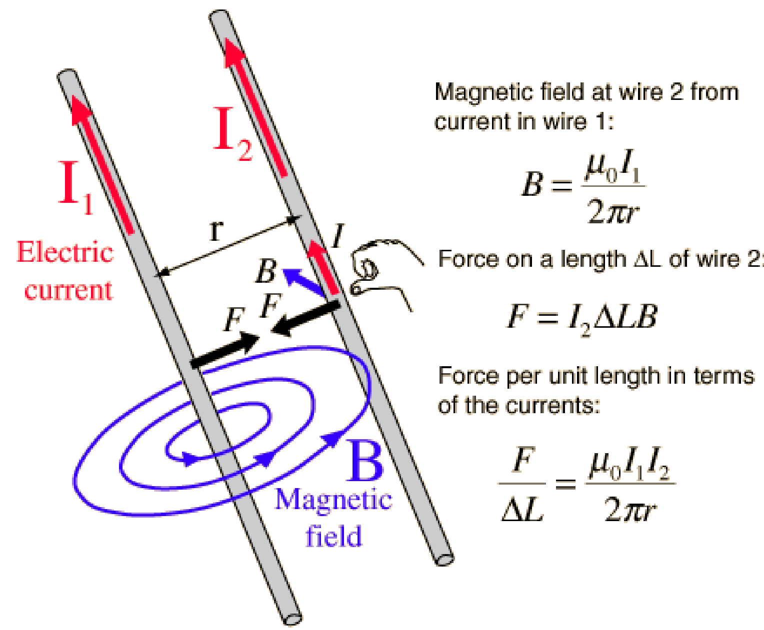


Magnetic Force

- Equation for force from magnetic field

$$\vec{F} = q \cdot \vec{v} \times \vec{B}$$

- Direction of force determined with rule of right hand
  - Index finger: direction of I
  - Middle finger: direction of B
  - Thumb gives direction of force



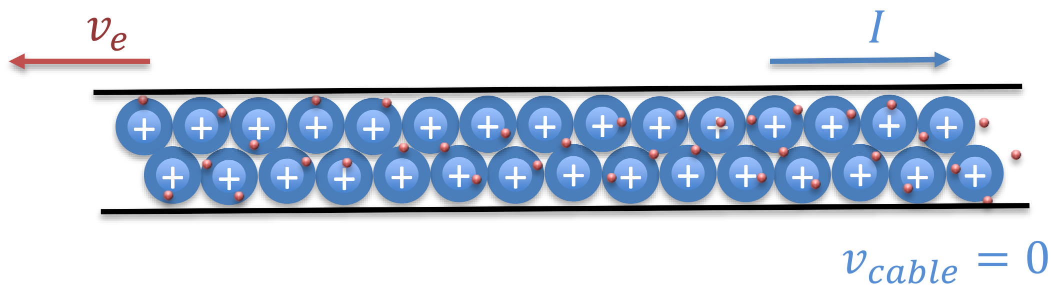
ELECTROMAGNETISM

Electromagnetism

- Magnetic force is electrostatic force transformed by relativistic motion of charges

Electromagnetism

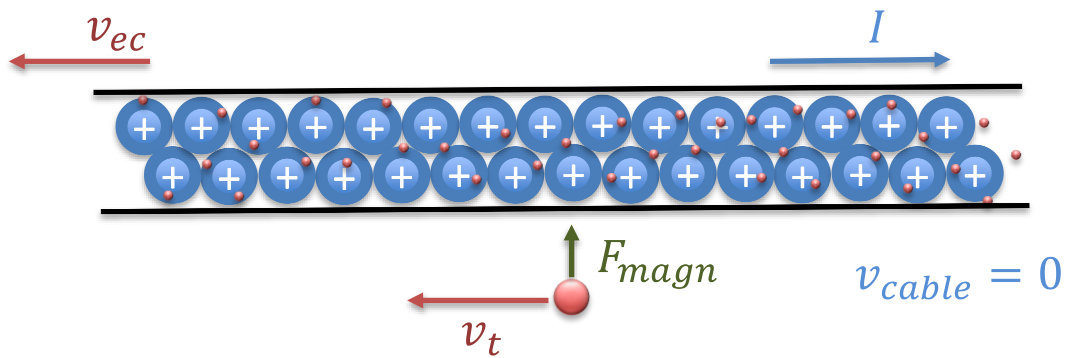
- Magnetic force is electrostatic force transformed by relativistic motion of charges



- Let's take a cable with flowing electric current
  - Positive current to the right
  - Electrons are flowing to the left
- The cable is electrically neutral
- Atoms of the metal are at rest

Electromagnetism

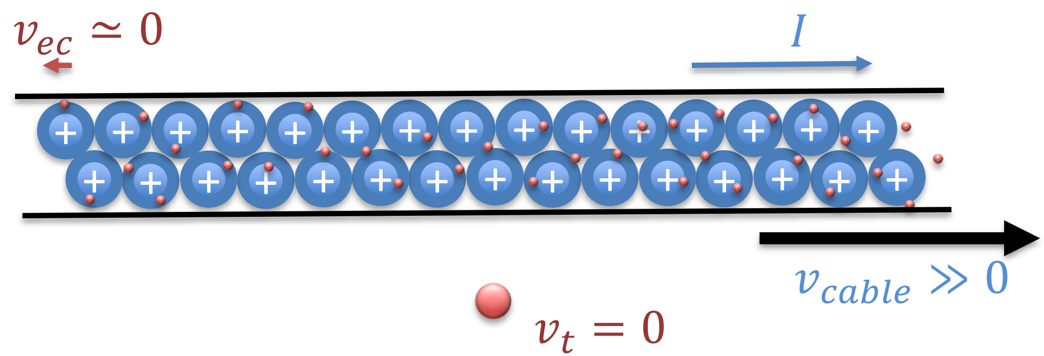
- Magnetic force is electrostatic force transformed by relativistic motion of charges



- A test negative charge moves along the cable to the left
- Magnetic force attracts the test charge to the cable

Electromagnetism

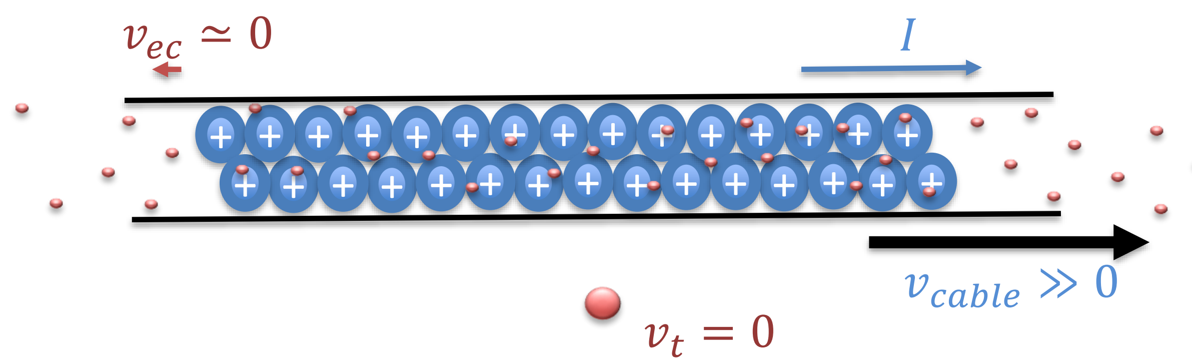
- Magnetic force is electrostatic force transformed by relativistic motion of charges



- Lets move to the rest frame of the test charge
- The test charge now is at rest
- Electrons in the cable are almost at rest ($v_t \simeq v_{ec}$)
- Cable atoms move to the right with $-v_t$

Electromagnetism

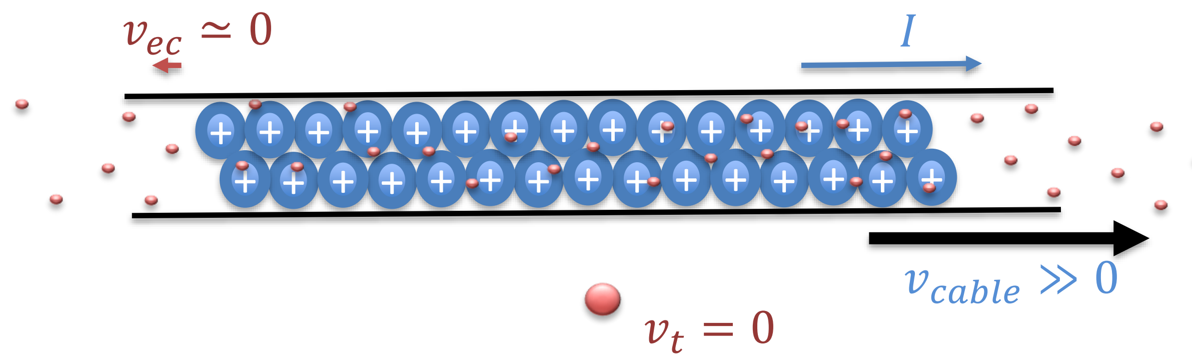
- Magnetic force is electrostatic force transformed by relativistic motion of charges



- Due to relativistic Lorentz contraction the cable gets shorter
- Distance between atoms (positive charges) gets smaller
- Density of positive charges gets larger

Electromagnetism

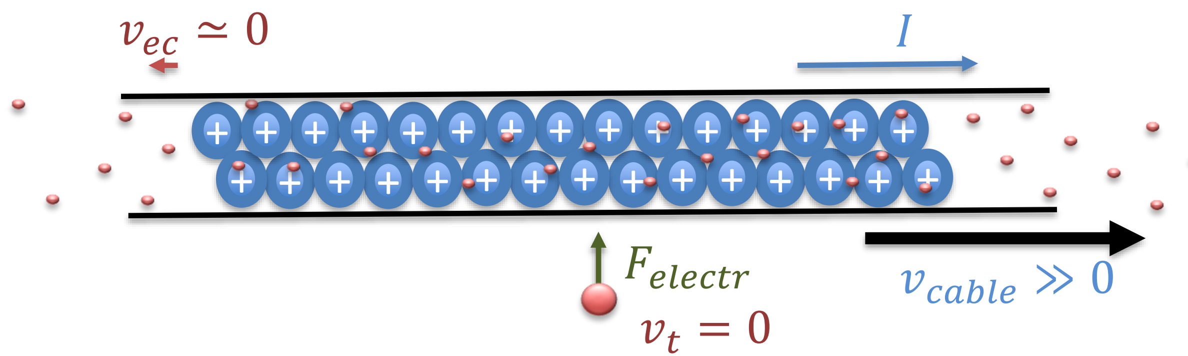
- Magnetic force is electrostatic force transformed by relativistic motion of charges



- Density of positive charges gets larger
- Density of electrons gets smaller
- Therefore, the cable is positively charged

Electromagnetism

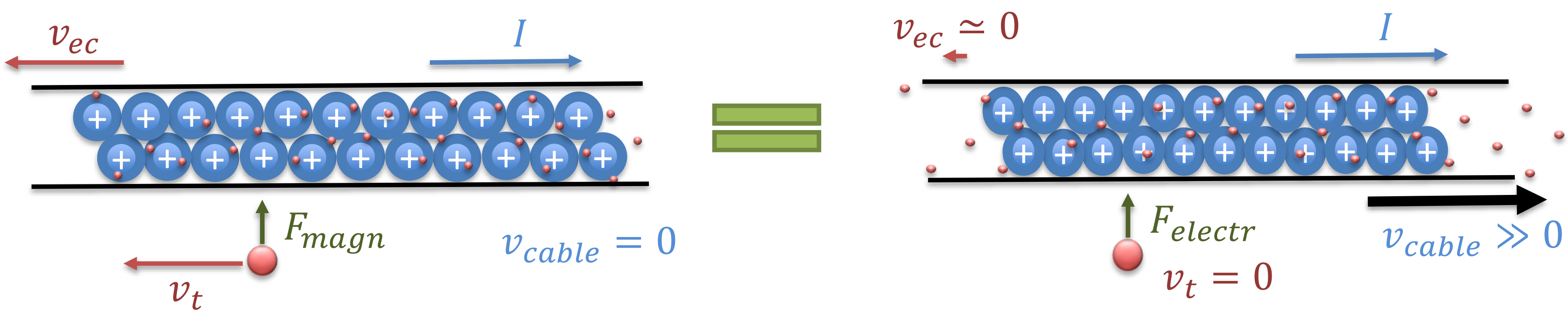
- Magnetic force is electrostatic force transformed by relativistic motion of charges



- Cable is positively charged
- Electrostatic force attracts the test charge to the cable

Electromagnetism

- Magnetic force is electrostatic force transformed by relativistic motion of charges



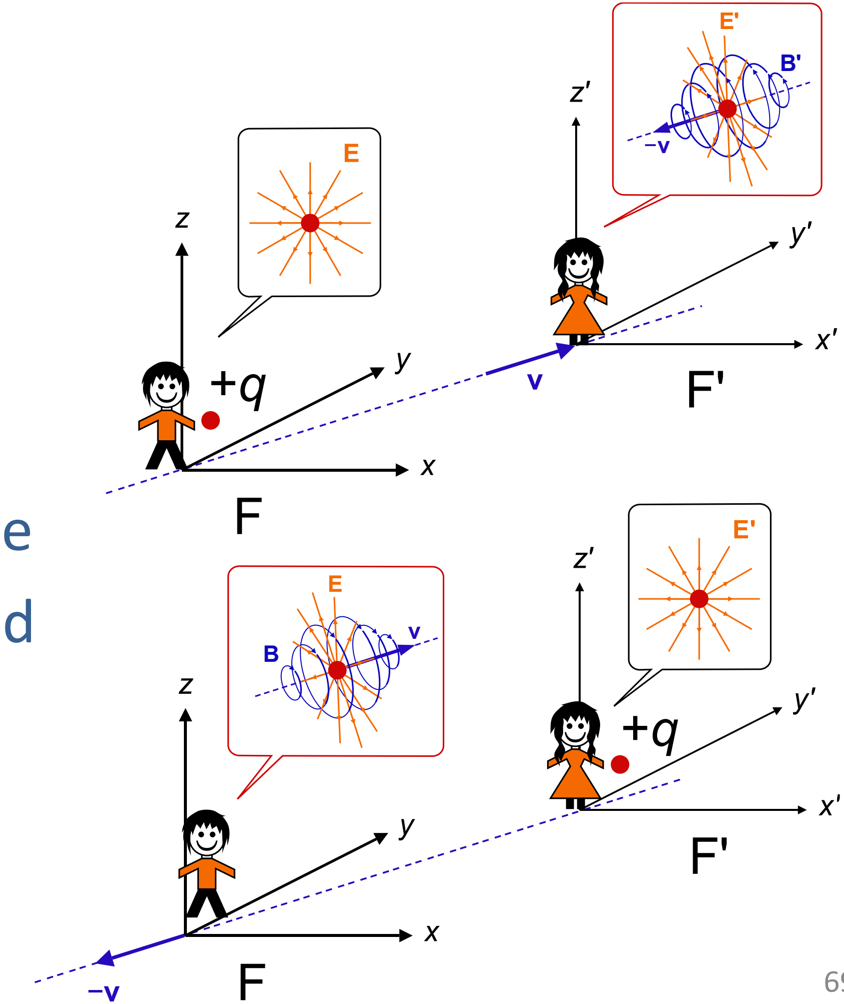
- Magnetic force when cable is at rest is the same as electric force when test charge is at rest
- With reference frame change one field changes to another one

Electromagnetism

- With reference frame change one field changes to another one

$$\mathbf{E}_{\parallel}{}' = \mathbf{E}_{\parallel}$$
$$\mathbf{B}_{\parallel}{}' = \mathbf{B}_{\parallel}$$
$$\mathbf{E}_{\perp}{}' = \gamma\left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}\right)$$
$$\mathbf{B}_{\perp}{}' = \gamma\left(\mathbf{B}_{\perp} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right)$$

- Only transverse components change
- Electric field gets weaker with speed
- Magnetic field gets stronger
- The reason we use magnetic fields at high particle energies



Force strength comparison

- As an example, let's compare "easily achievable" electric and magnetic fields
  - Electric: 1 MV/m (10 MV voltage over 10 cm gap)
  - Magnetic: 1 T

$$\frac{F_{magn}}{F_{elec}} = \frac{qvB}{qE} = \frac{c\beta_{rel}B}{E} = \frac{c\beta_{rel}B}{E} = \beta_{rel}\frac{3\cdot 10^8\cdot 1}{10^6} = 300\cdot\beta_{rel}$$

- If β_{rel} is smaller than 1/300 then the electric force is stronger
  - At CERN only behind the particle sources and in ELENA

LORENTZ FORCE

Lorentz force

- The electromagnetic force is called Lorentz force

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right)$$

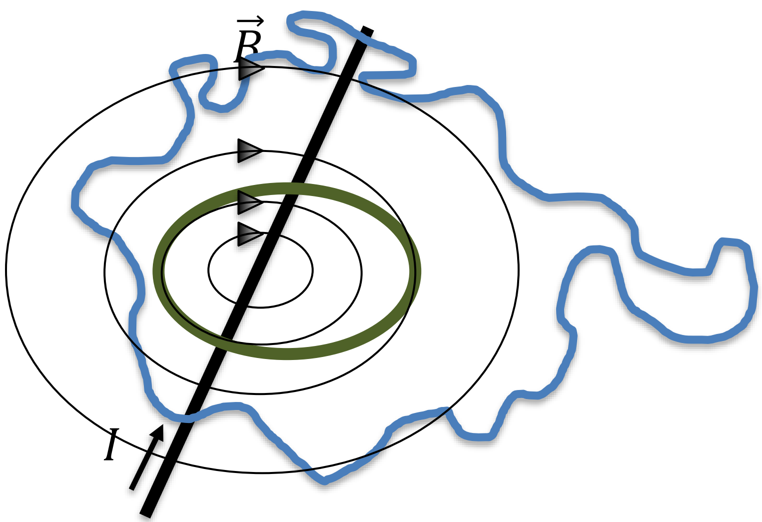
- It is the sum of forces due to electric and magnetic fields

AMPERE’S LAW

Ampere's law

The field integrated around any closed loop
is proportional to the current enclosed by the loop
irrespective of how that current is distributed

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$$

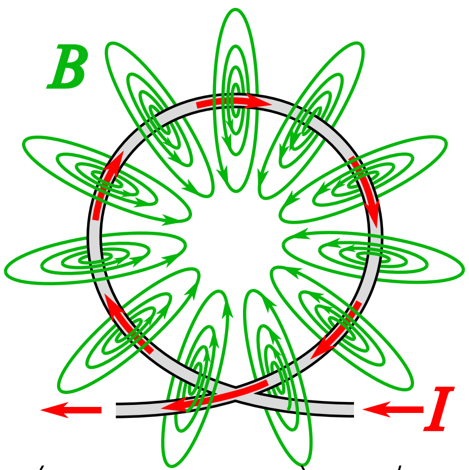


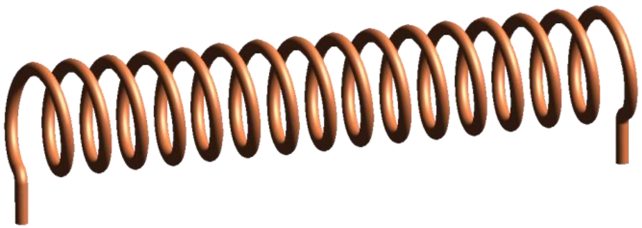
Magnetostatic case!

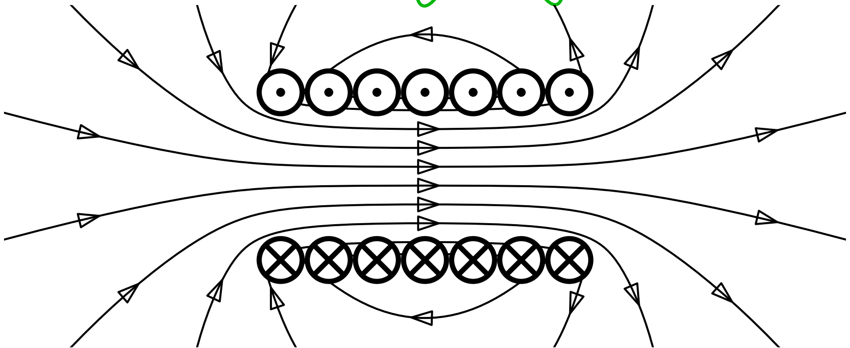
When fields are time-varying additional term needs to be added on R.H.S.

Magnetic coil

- Ampere's law allows to calculate magnetic fields from given distribution of electric currents
  - Or shapes of the wire
- For example, of a
  - Loop
  - Solenoid coil







Solenoid coil

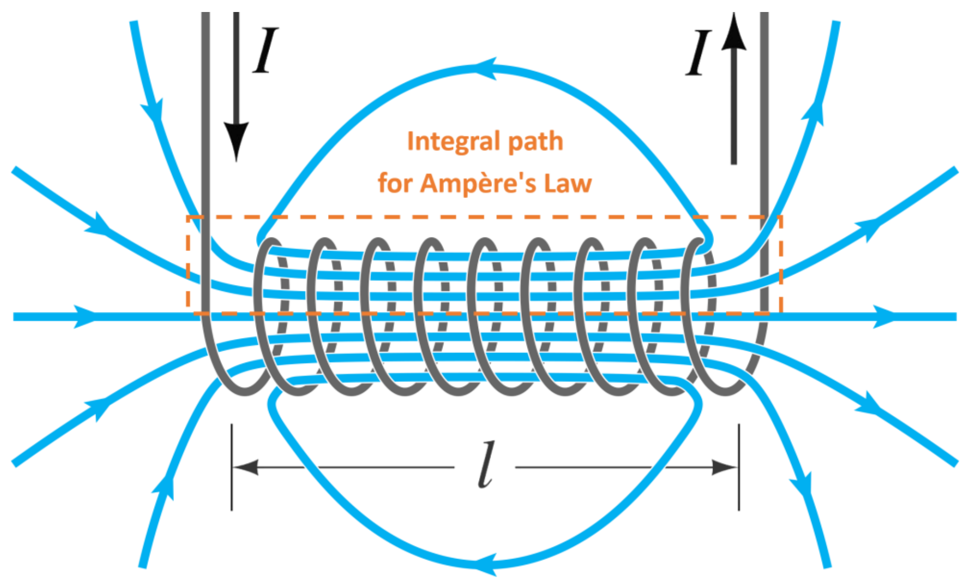
- Selecting contour like the orange box:

  only the line inside solenoid counts and it has constant B field

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$Bl = \mu_0 N I,$$

$$B = \mu_0 \frac{NI}{l}.$$



N: number of windings

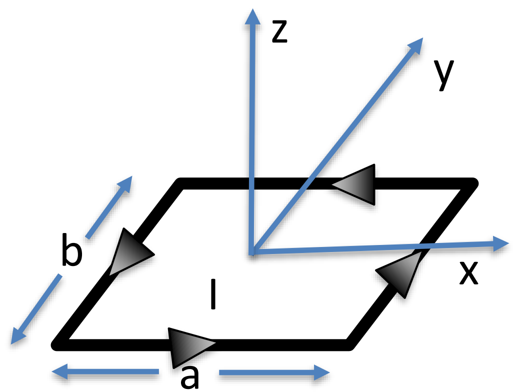
A loop with current

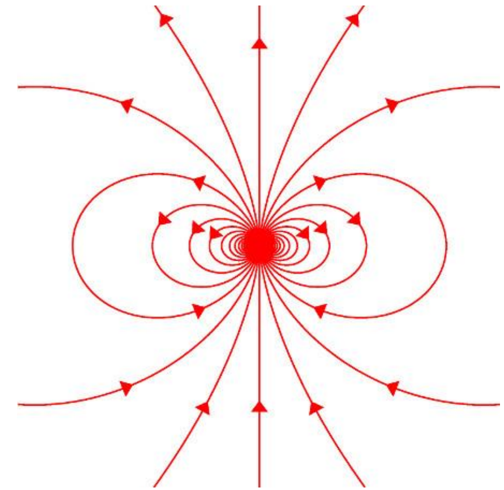
- In many problems it is conceptually useful to split a source of magnetic field into very small loops with current

$$B_x = \frac{Iab}{4\pi\varepsilon_0 c^2}\frac{3xz}{R^5}$$

$$B_y = \frac{Iab}{4\pi\varepsilon_0 c^2}\frac{3yz}{R^5}$$

$$B_x = \frac{Iab}{4\pi\varepsilon_0 c^2}(\frac{1}{R^3} - \frac{3z^2}{R^5})$$

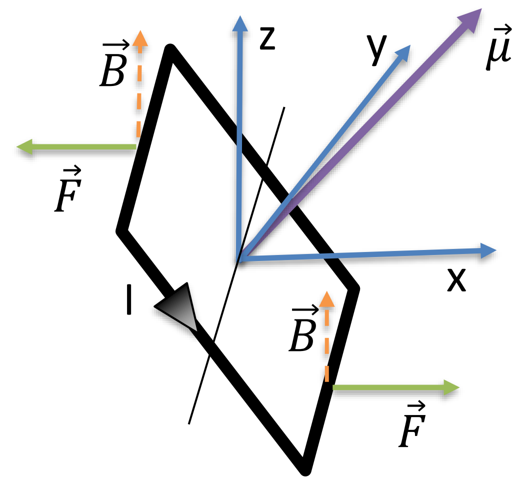




- Iab or $I\vec{A}$ is called magnetic dipole moment vector or simply magnetic moment $\vec{\mu}$

Forces acting on loop with current

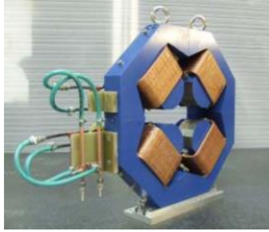
- Put a loop with current in uniform magnetic field and it will rotate such that μ is in direction of the field
- Torque $\vec{\tau} = \vec{\mu} \times \vec{B}$
- Energy $U = -\vec{\mu} \cdot \vec{B}$



Magnetic moments of particles and atoms

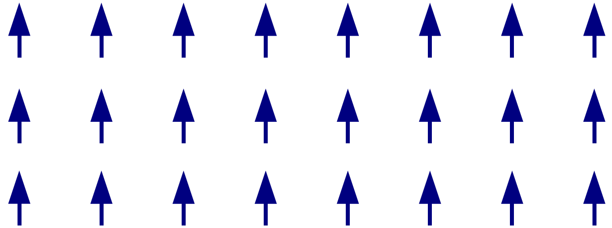
- Charged elementary particles have magnetic moments
- They act like very small loops with current
- One can think of it as charge rotating due to spin
- Usually, magnetic moments are distributed randomly
- But some materials can have moments of their atoms aligned: they are magnets

Magnets

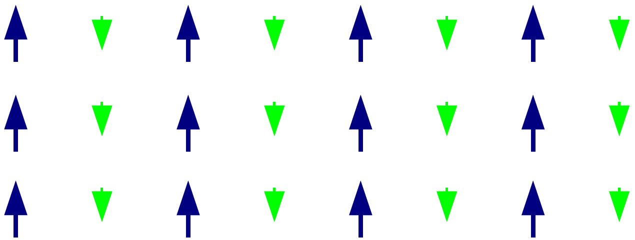


- Magnetic materials can have their moments aligned, they can be magnetized by external magnetic field
- Magnetization can stay forever: permanent magnets
- Or only when external field is present: electromagnets

Ferromagnetic



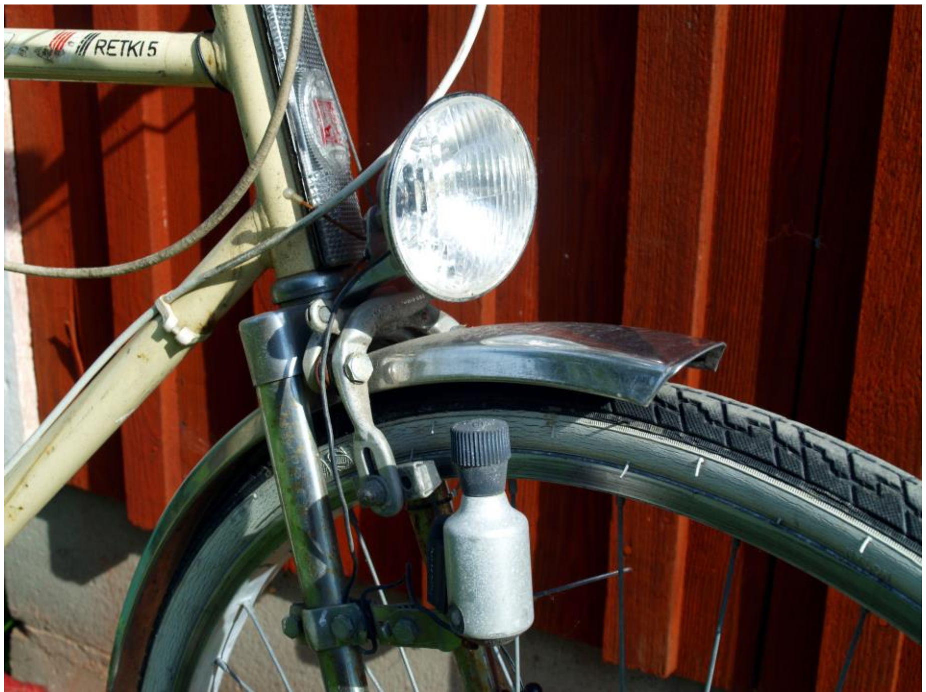
Ferrimagnetic



MAGNETIC INDUCTION

Magnetic Induction

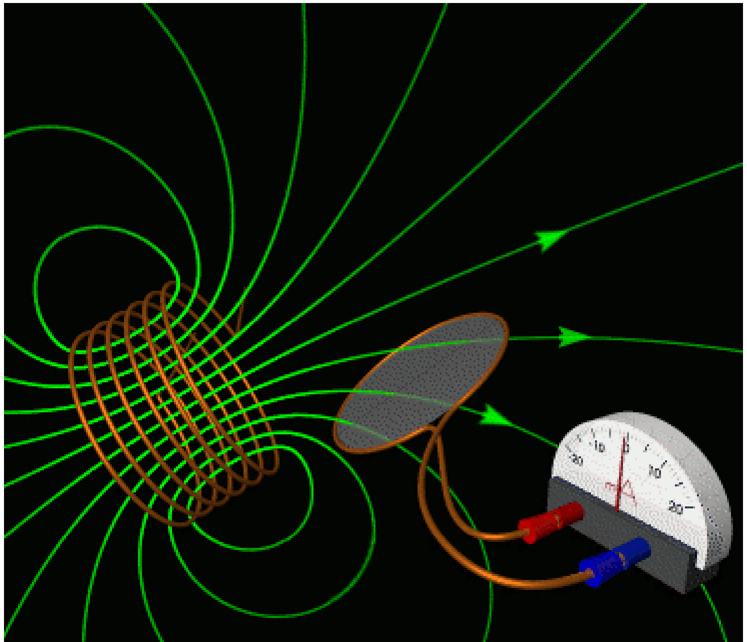
- Real life examples:
  - Electric generator
    - Voltage out of rotating magnet



Induction

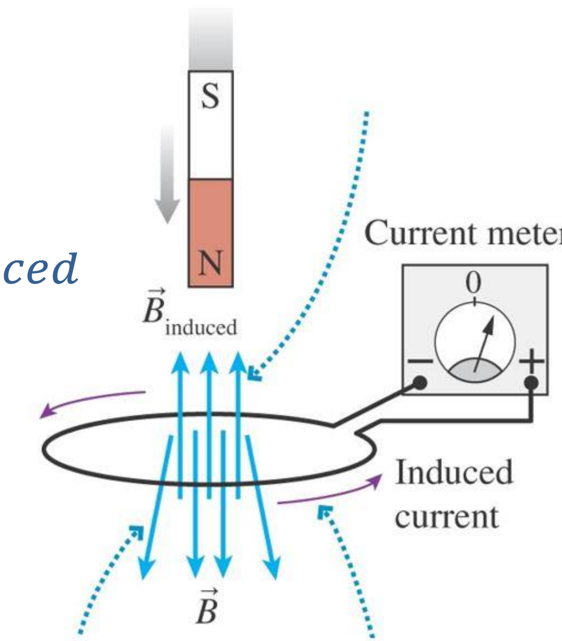
- **Change of magnetic field induces electric field**

- **Change of electric field induces magnetic field**

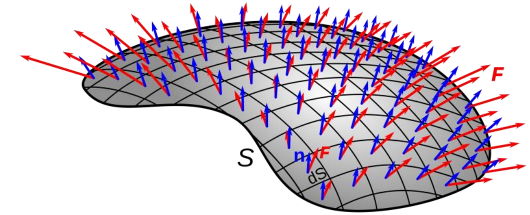


Faraday law

- Change of magnetic field induces electric field
  - When magnet is inserted into the loop then $\vec{B}$ is increasing and electric field is induced along the wire loop
  - Electric field pushes electrons in the wire and generates induced current
  - Induced current creates magnetic field $\vec{B}_{induced}$ such that it is against the external field $\vec{B}$



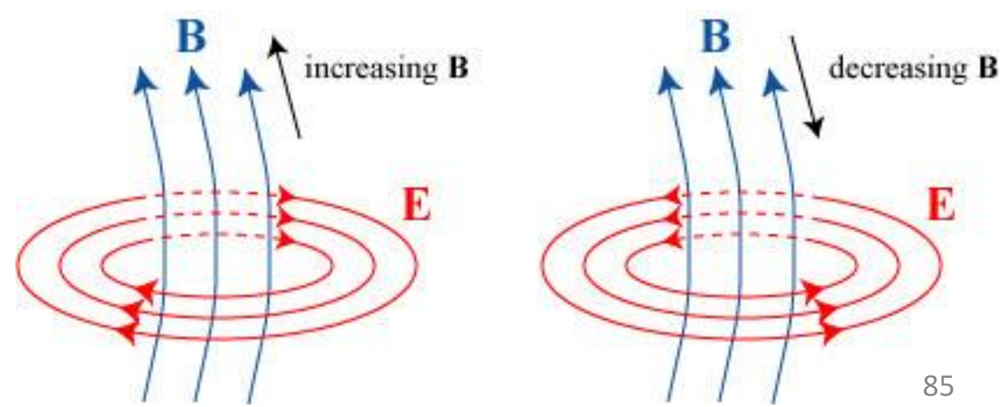
Faraday law



$$\varepsilon_{elecromotive} = \oint_C \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\Phi_B = \oiint_S \vec{B} \cdot d\vec{S}$$

- Electromotive force around a closed loop C is opposite to change of magnetic field flux in time
- Unit is Volt [V]
- The sign of B field is not important
- But what is important it is if B field is increasing or if it is decreasing in time

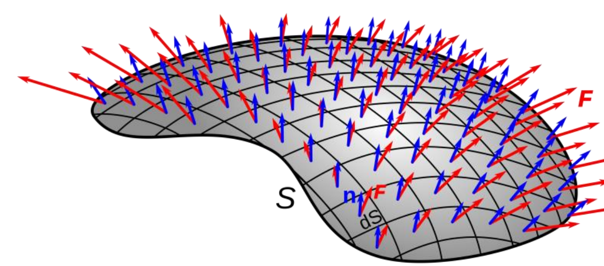


Maxwell's addition to Ampere's law

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

- Magnetic Field integrated around any closed loop is proportional to the current enclosed by the loop plus change of electric field flux in time

$$\Phi_E = \oiint_S \vec{E} \cdot d\vec{S}$$



Putting it all together

MAXWELL EQUATIONS

Maxwell equations

1. Gauss law for electric field

   - Electric charge is the source of electric field

2. Gauss law for magnetic field

   - Magnetic field has no source, there is no “magnetic charges”

3. Faraday law

   - Electric field around a loop (electromotive force) is opposite to change of magnetic field flux through the loop

4. Ampere law

   - Magnetic field around a loop is equal to electric current plus change of electric field flux through the loop

Maxwell equations

- Integral form

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$$\oiint_S \vec{B} \cdot d\vec{S} = 0$$

$$\oint_C \vec{E} \cdot d\vec{l} = -\oiint_S \frac{\overrightarrow{\partial B}}{\partial t} \cdot d\vec{S}$$

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \oiint_S \frac{\overrightarrow{\partial E}}{\partial t} \cdot d\vec{S}$$

Curl

- Curl is, by definition, line integral around infinitesimally small closed loop

$$(\nabla \times \mathbf{F})(p) \cdot \hat{\mathbf{n}} \stackrel{\text{def}}{=} \lim_{A \to 0} \frac{1}{|A|} \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \hat{\boldsymbol{\imath}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \hat{\boldsymbol{k}} = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}$$

- It measures how much the field is curling, or circulating

Maxwell equations

- Integral form

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

$$\oiint_S \vec{B} \cdot d\vec{S} = 0$$

$$\oint_C \vec{E} \cdot d\vec{l} = -\oiint_S \frac{\overrightarrow{\partial B}}{\partial t} \cdot d\vec{S}$$

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \oiint_S \frac{\overrightarrow{\partial E}}{\partial t} \cdot d\vec{S}$$

- Differential form

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\overrightarrow{\partial B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\overrightarrow{\partial E}}{\partial t}$$

ELECTROMAGNETIC WAVES

How the fields propagate ?

- Solve Maxwell equations for different situations and conditions

EM field propagation in vacuum: Wave equations

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

- Derivations in Appendix
- These are vector equations
  - Vectors have direction
    - We can always choose a reference frame
      we chose it so the wave moves along x axis
  - This allows to reduce them to a scalar equation

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

Solution of wave eq. in vacuum: Plane waves

- General solution:

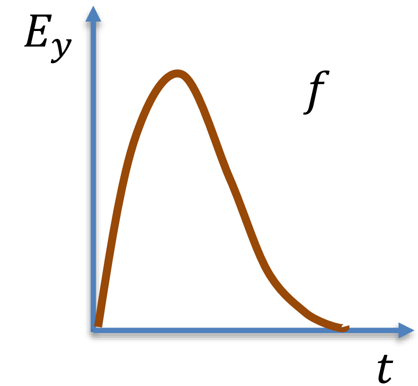
$$E_y = \mathrm{f}(x - t/\sqrt{\mu_0 \varepsilon_0}) + g(x - t/\sqrt{\mu_0 \varepsilon_0})$$

  - x is direction of wave propagation
- How did we find it?
  Someone guessed it and it fits
- **f and g are arbitrary functions, they are defined by the source of the wave**

Plane wave

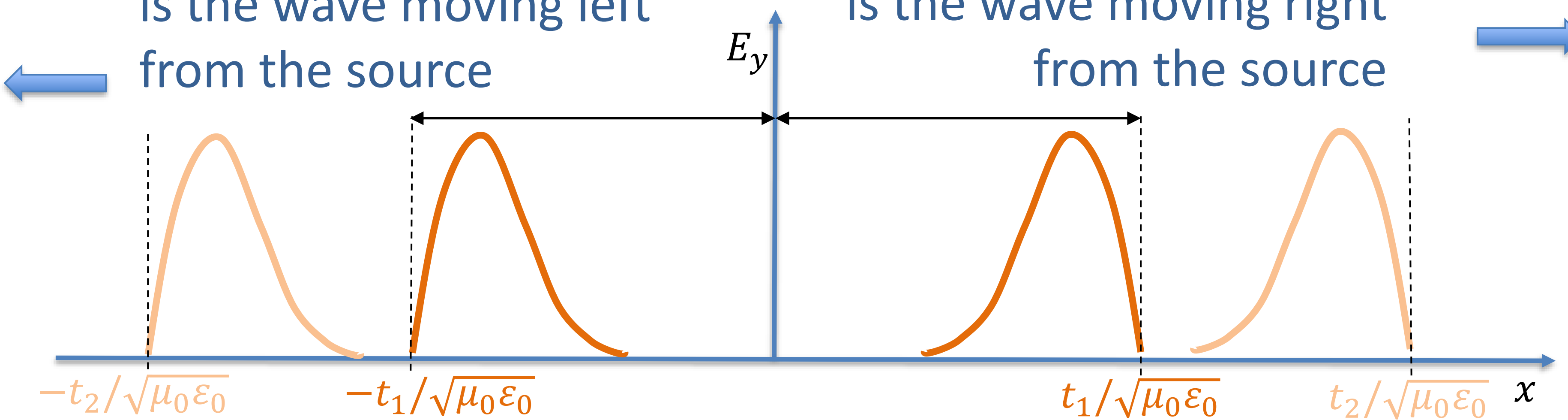
$$E_y = f(x - ct) + f(x + ct)$$

- f can be an arbitrary function
  - At t=0 source at x=0 generates E_y pulse
  - Pulse in space at time t_1 and t_2:



$f(x + ct)$ is the wave moving left from the source

$f(x - ct)$ is the wave moving right from the source



Sinusoidal plane waves

- If the source is resonating with frequency ω, and the source amplitude is sinusoidal $A_s = A_{0s} \sin \omega t$, then the wave accordingly has sinusoidal form

$$E_y = E_{0y} \sin \omega(t - x/c)$$

- To facilitate calculations and equation, very often complex exponents are employed

$$E_y = E_{0y} e^{i\omega(t-x/c)}$$

  – But only the real part has a physical meaning

A function of (t-x/c) is also function of (x-ct)

$$F\left(t - \frac{x}{c}\right) = F\left(-\frac{x-ct}{c}\right) = f(x - ct)$$

Wavenumber

- If plane wave propagates in an arbitrary direction then equation becomes

$$E_y = E_{0y} e^{i\omega\left(t - \frac{x}{c} - \frac{y}{c} - \frac{z}{c}\right)}$$

- To simplify, we put k = ω/c

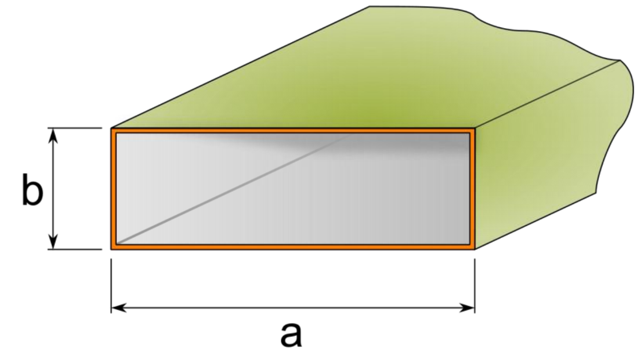
$$E_y = E_{0y} e^{i(\omega t - kx - ky - kz)}$$

- k is called angular wavenumber
  - It describes how phase changes per unit length
  - The unit is radian/m

RECTANGULAR WAVEGUIDE

Rectangular waveguide

- Metallic (ideally conducting) tube with a rectangular profile
- Vacuum inside the tube
  - No charges, no currents
- Wave equations still hold
- **Boundary conditions need to be respected**
  - **No electric fields along the metallic surfaces**
  - **No magnetic fields perpendicular to metallic surfaces**



$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Rectangular waveguide

- Independent 2 equations that lead to independent 2 solutions

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right) E_z = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right) B_z = 0$$

  – The final solution is sum of these 2 solutions

TE and TM modes

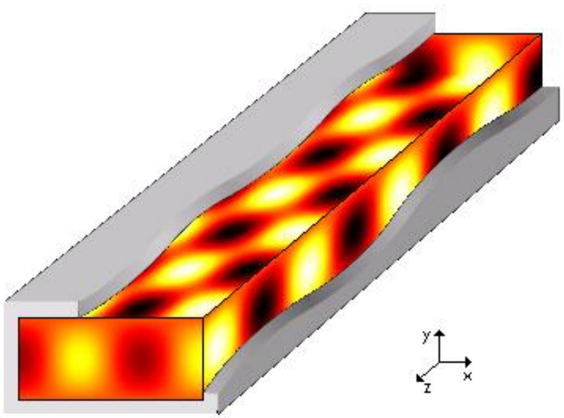
- If one takes $E_z = 0$ and $B_z \neq 0$, then only transverse E_x and E_y remain: **TE** (**T**ransverse **E**lectric) wave

- If one takes $E_z \neq 0$ and $B_z = 0$, then only transverse B_x and B_y remain: **TM** (**T**ransverse **M**agnetic) wave

Rectangular waveguide modes

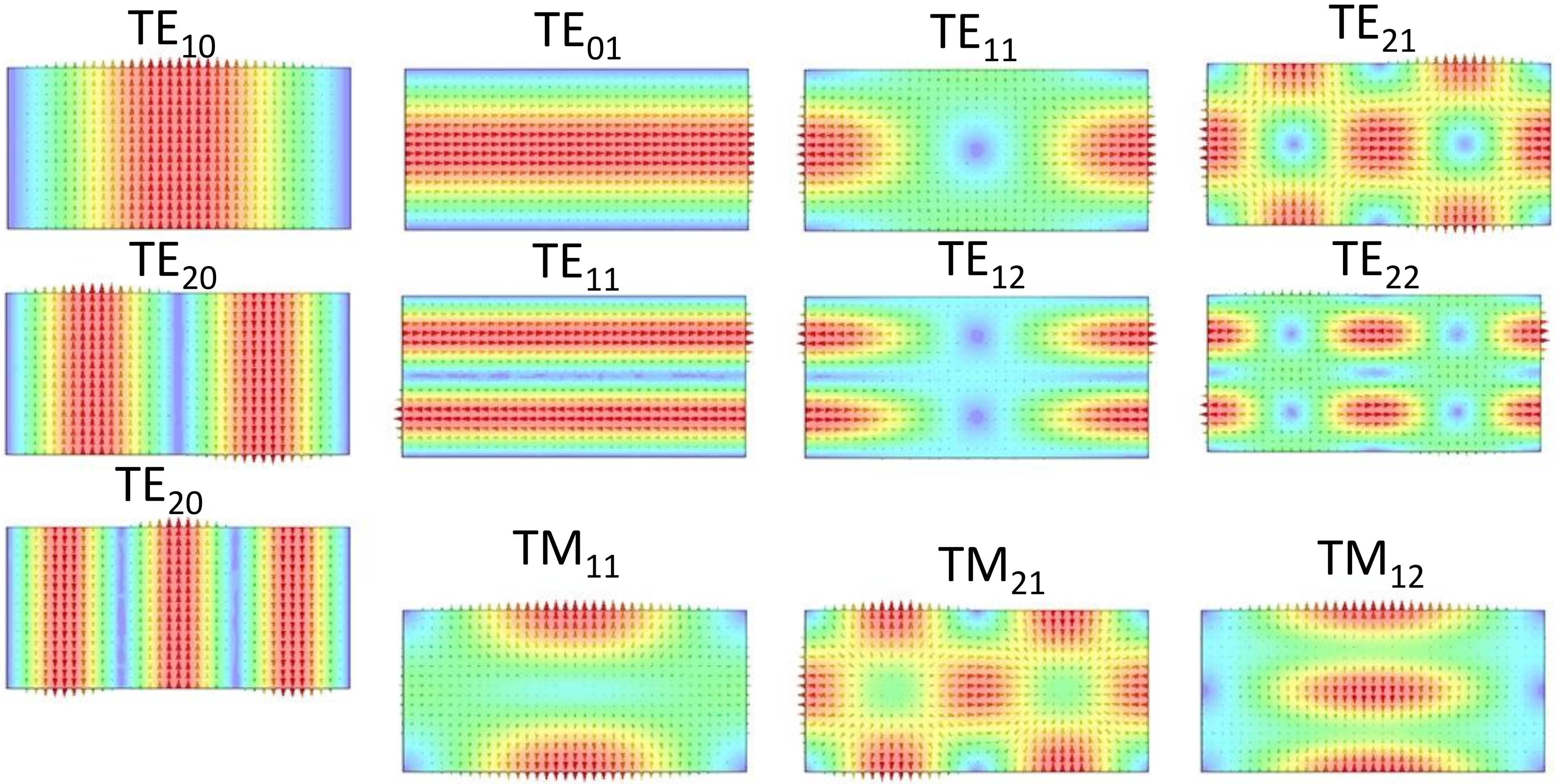
$$X\left(x\right) = A\sin\left(k_x x\right) + B\cos\left(k_x x\right)$$

- $\frac{\partial B_z}{\partial x} = 0$ for x = 0 and x = a ()
  so, $k_x = m\pi/a$
- The same applies for Y
- Finally $$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos(\omega(t - z/c))$$



Electric field Ex component of the TE_{31} mode inside a hollow metallic waveguide.

Electric field for different modes



https://cds.cern.ch/record/1416619/

Frequencies of the modes

$$-k_x^2 - k_y^2 + \frac{\omega^2}{c^2} - k^2 = 0 \qquad B_z = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

- Wavenumber $k = \sqrt{(\omega/c)^2 - \pi^2 \left[(m/a)^2 + (n/b)^2\right]}$
- It must be positive, what gives condition for minimum frequency that can propagate, **the cutoff frequency**: $\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$
- The lowest ω_{mn} is for mode TE_{10} (m=1 and n=0): $\omega_{10} = \frac{c\pi}{a}$
- Waves with lower frequencies cannot propagate in waveguide having width a (assuming b<a)

Credits

- Certain figures were copied from
  - Wikipedia.org
  - xaktly.com by Dr. Jeff Cruzan
    -
  - Erk Jensen's lecture on CAS 2010 https://cds.cern.ch/record/1416619/

Rectangular waveguide

- The wave needs to propagate along the waveguide so the x dependent component needs to be $E_x = e^{i(\omega t - kx)}$

$$E(x, y, z, t) = E_T(y, z)e^{i(\omega t - kx)}$$

$$B(x, y, z, t) = B_T(y, z)e^{i(\omega t - kx)}$$

- Putting these to Faraday **and** Ampere laws

$$\frac{\partial E_z}{\partial y} + \frac{\partial E_y}{\partial z} = i\omega B_x$$

$$\frac{\partial E_x}{\partial z} + \frac{\partial E_z}{\partial x} = i\ \omega B_y$$

$$\frac{\partial E_y}{\partial x} + \frac{\partial E_z}{\partial y} = i\omega B_z$$

APPENDIXES

SOLVING MAXWELL EQUATIONS IN VACUUM: WAVE EQUATION

Solving Maxwell equation in vacuum

- The trick to quickly solve the equations:

$$\nabla \times \nabla \times \vec{E} = \nabla \times (-\frac{\partial \vec{B}}{\partial t})$$

Apply curl operator on both sides on Faraday Law

$$\nabla \times \nabla \times \vec{E} \equiv \boxed{\nabla(\nabla \cdot} \vec{E}) - \nabla^2 \vec{E} = -\boxed{\frac{\partial}{\partial t} \nabla \times} \vec{B}$$

Apply Gauss and Ampere laws

$$\boxed{\nabla \cdot \vec{E}} = \frac{\rho}{\varepsilon_0}$$

$$\boxed{\nabla \times \vec{B}} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla(\frac{\rho}{\varepsilon_0}) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial \vec{J}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

How the fields propagate in vacuum?

$$\nabla\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2 \vec{E} = -\mu_0 \frac{\partial \vec{J}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

- In vacuum there is no charges and no currents

$\rho = 0$
$\vec{J} = 0$

$$\nabla^2 \vec{E} \equiv \frac{\partial^2 E_x}{\partial t^2} + \frac{\partial^2 E_y}{\partial t^2} + \frac{\partial^2 E_z}{\partial t^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

How the fields propagate in vacuum?

- The same trick with curl applied on both sides of Ampere's Law leads to

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla^2 \vec{B} \equiv \frac{\partial^2 B_x}{\partial t^2} + \frac{\partial^2 B_y}{\partial t^2} + \frac{\partial^2 B_z}{\partial t^2} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

How the fields propagate in vacuum?

- Wave equations

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

- These are vector equations
  - Vectors have direction
    - We can always choose a reference frame
      we chose it so the wave moves along x axis
  - We induce that the amplitudes do not change in transverse
    i.e. all derivatives w.r.t. y and z are zero

How the fields propagate in vacuum?

- Gauss Law $\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=0 \Rightarrow \frac{\partial E_x}{\partial x}+0+0=0 \Rightarrow$

  $\frac{\partial E_x}{\partial x}=0 \Rightarrow$ E_x constant along x

  not interesting, choosing special case $E_x = 0$

- Faraday Law, developing curl operator

$$(\nabla\times\vec{E})_x=-\frac{\partial B_x}{\partial t}\Rightarrow\frac{\partial E_z}{\partial y}+\frac{\partial E_y}{\partial z}=-\frac{\partial B_x}{\partial t}\Rightarrow\frac{\partial B_x}{\partial t}=0$$

B_x constant in time

$$(\nabla\times\vec{E})_y=-\frac{\partial B_y}{\partial t}\Rightarrow\frac{\partial E_x}{\partial z}+\frac{\partial E_z}{\partial x}=-\frac{\partial B_y}{\partial t}\Rightarrow\frac{\partial B_y}{\partial t}=0$$

B_y constant in time

$$(\nabla\times\vec{E})_z=-\frac{\partial B_z}{\partial t}\Rightarrow\frac{\partial E_y}{\partial x}+\frac{\partial E_x}{\partial y}=\frac{\partial B_z}{\partial t}\Rightarrow\frac{\partial E_y}{\partial x}=-\frac{\partial B_z}{\partial t}$$

- Only B_z is changing in time
  putting B_x=0 and B_y=0 (constant fields are not interesting)

How the fields propagate in vacuum?

- Ampere's Law, developing curl operator

$$(\nabla \times \vec{B})_x = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \Rightarrow \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \Rightarrow \frac{\partial E_x}{\partial t} = 0$$

$$(\nabla \times \vec{B})_y = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \Rightarrow \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \Rightarrow \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

$$(\nabla \times \vec{B})_z = \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t} \Rightarrow \frac{\partial B_z}{\partial x} + \frac{\partial B_z}{\partial y} = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t} \Rightarrow \frac{\partial B_z}{\partial t} = 0$$

- Only E_y is changing in time
  putting E_x=0 and E_z=0 (constant fields are not interesting)

How the fields propagate in vacuum?

$$\frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}$$

- Only E_y and B_z are left
  - Electric field changes only in Y plane
  - Magnetic field changes only in Z plane
  - Magnetic field is perpendicular to electric field

- Putting all together allows to reduce it to a single scalar wave equation

$$\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

Plane waves

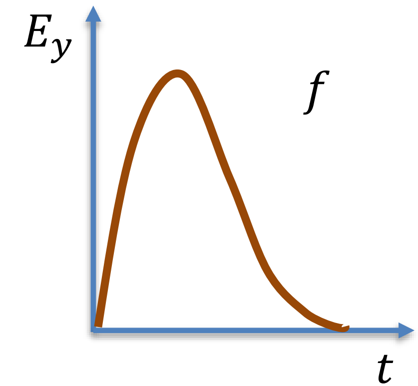
- General solution:

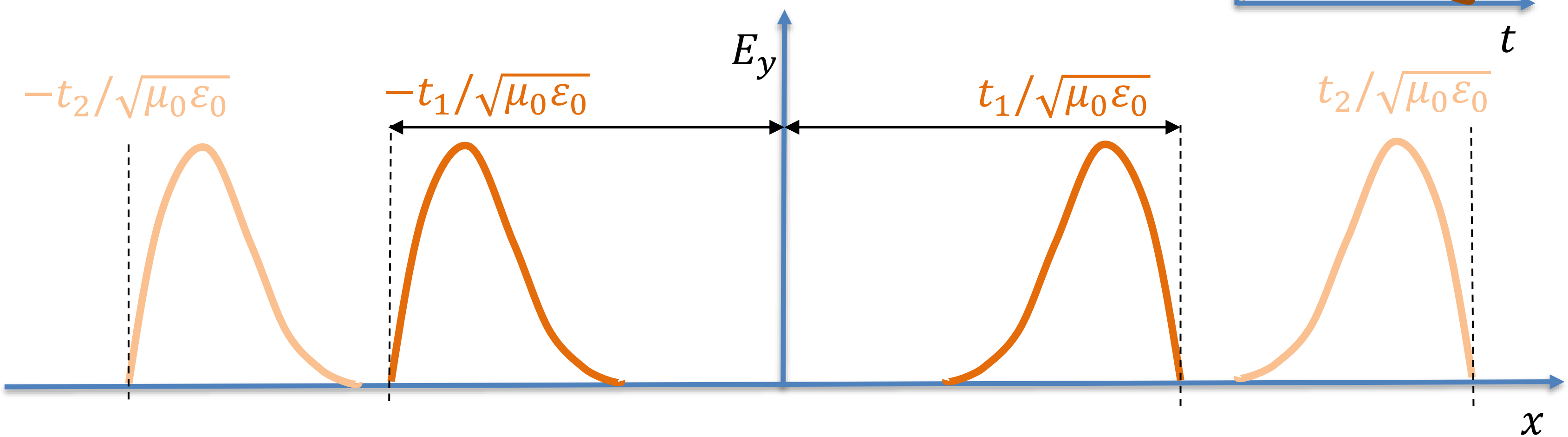
$$E_y = \mathrm{f}(x - t/\sqrt{\mu_0\varepsilon_0}) + g(x - t/\sqrt{\mu_0\varepsilon_0})$$

- How did we find it?
  Someone guessed it and it fits
- f and g are arbitrary functions, they are defined by the source of the wave

Plane wave $E_y = f(x - t/\sqrt{\mu_0\varepsilon_0}) + f(x + t/\sqrt{\mu_0\varepsilon_0})$

- f can be an arbitrary function
  - At t=0 source at x=0 generates E_y pulse
  - Pulse in space at time t_1 and t_2:



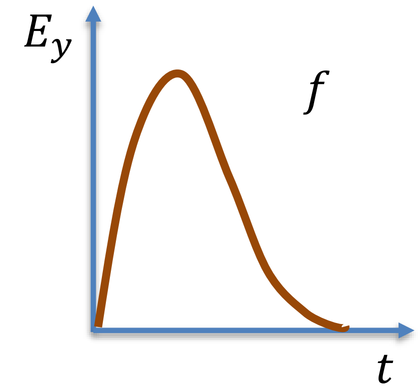


$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = c =$ 299792458 m/s is the speed of propagation

Plane wave

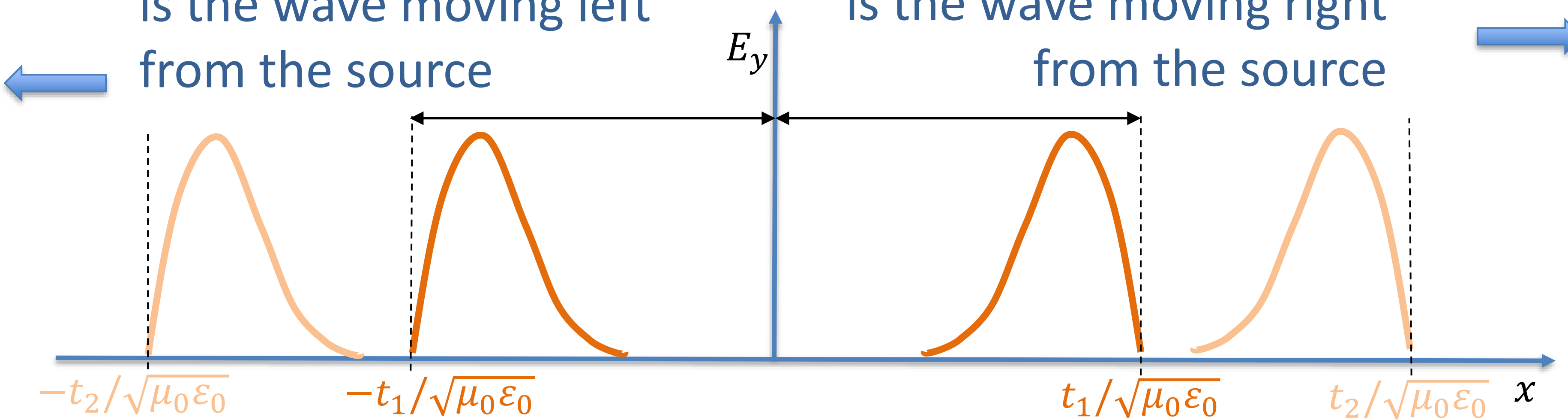
$$E_y = f(x - ct) + f(x + ct)$$

- f can be an arbitrary function
  - At t=0 source at x=0 generates E_y pulse
  - Pulse in space at time t_1 and t_2:



$f(x + ct)$ is the wave moving left from the source

$f(x - ct)$ is the wave moving right from the source



Sinusoidal plane waves

- If the source is resonating with frequency ω, and the source amplitude is sinusoidal $A_s = A_{0s} \sin \omega t$, then the wave accordingly has sinusoidal form

$$E_y = E_{0y} \sin \omega (t - x/c)$$

- To facilitate calculations and equation, very often complex exponents are employed

$$E_y = E_{0y} e^{i\omega(t - x/c)}$$

  - But only the real part has a physical meaning

A function of (t-x/c) is also function of (x-ct)

$$F\left(t - \frac{x}{c}\right) = F\left(-\frac{x - ct}{c}\right) = f(x - ct)$$

Sinusoidal plane wave

- If plane wave propagates in an arbitrary direction then equation becomes

$$E_y = E_{0y} e^{i\omega\left(t - \frac{x}{c} - \frac{y}{c} - \frac{z}{c}\right)}$$

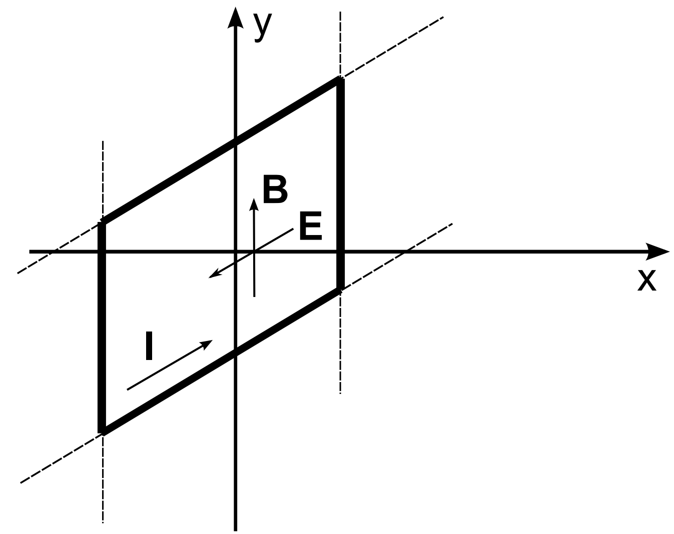
- To simplify, we put k = ω/c

$$E_y = E_{0y} e^{i(\omega t - kx - ky - kz)}$$

- k is called angular wavenumber
  - It describes how phase changes per unit length
  - The unit is radian/m

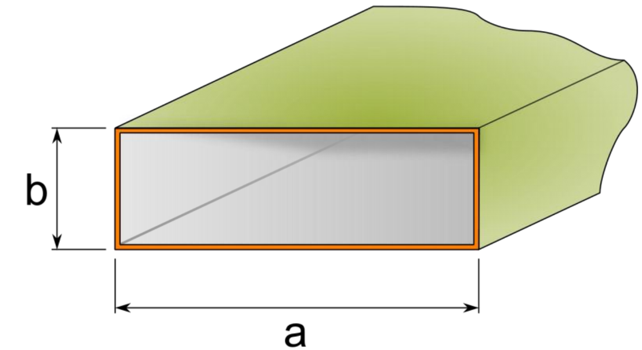
Source of plane waves

- A metallic plate of infinite dimensions
- Not really a realistic concept
- But in many cases
  a very good approximation
  - For example, close to the plate
  - Far away from the source
- It allows to simplify the equations



Rectangular waveguide

- Metallic (ideally conducting) tube with a rectangular profile
- Vacuum inside the tube
  - No charges, no currents
- Wave equations still hold
- **Boundary conditions need to be respected**
  - **No electric fields along the metallic surfaces**
  - **No magnetic fields perpendicular to metallic surfaces**



$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

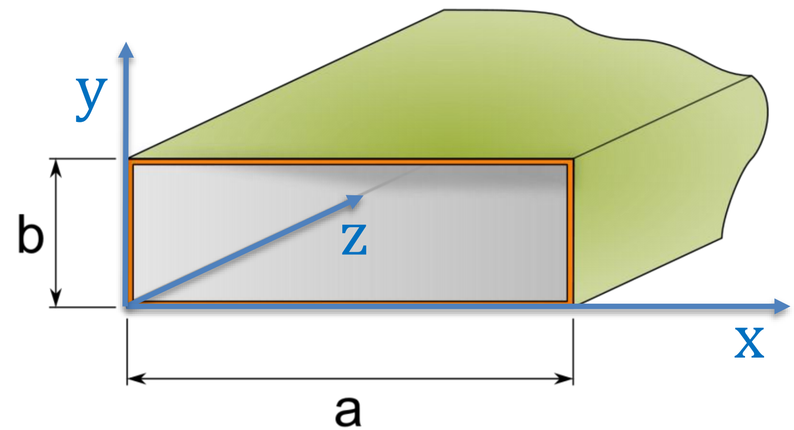
$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Rectangular waveguide

- We assume that source is a resonating one
- The wave propagates along the waveguide, so only z component depends on time t

$$E(x, y, z, t) = E_T(x, y)e^{i(kz-\omega t)}$$
$$B(x, y, z, t) = B_T(x, y)e^{i(kz-\omega t)}$$

Different reference system then used for vacuum!
Now wave propagates towards z



Rectangular waveguide

Step1: Insert these

$$E(x,y,z,t) = E_T(x,y)e^{i(kz-\omega t)}$$
$$B(x,y,z,t) = B_T(x,y)e^{i(kz-\omega t)}$$

into Faraday's and Ampere's Laws

$$\nabla \times \mathbf{E} = -(\partial \mathbf{B}/\partial t),$$
$$\nabla \times \mathbf{B} = c^{-2}(\partial \mathbf{E}/\partial t)$$

It gives

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z$$
$$\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x \qquad \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x$$
$$ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y \qquad ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y$$

Rectangular waveguide

- These equations reduce to

$$E_x = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y} \right)$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right)$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_z}{\partial y} \right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2} \left(k \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_z}{\partial x} \right)$$

- All 4 depend only on the longitudinal fields E_z and B_z

Rectangular waveguide

- Step 2: put these into Gauss Law

$$E(x,y,z,t) = E_T(x,y)e^{i(kz-\omega t)}$$
$$B(x,y,z,t) = B_T(x,y)e^{i(kz-\omega t)}$$

$$\nabla \cdot \mathbf{E} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z \right) e^{i(kz-\omega t)} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0$$

- Use E_x and E_y from previous page

$$E_x = \frac{i}{(\omega/c)^2 - k^2} \left(k\frac{\partial E_z}{\partial x} + \omega \frac{\partial B_z}{\partial y} \right)$$
$$E_y = \frac{i}{(\omega/c)^2 - k^2} \left(k\frac{\partial E_z}{\partial y} - \omega \frac{\partial B_z}{\partial x} \right)$$

$$-i\left(k^2 - \frac{\omega^2}{c^2}\right)^{-1} \left(k\frac{\partial^2 E_z}{\partial x^2} + \omega \frac{\partial^2 B_z}{\partial x \partial y} \right) - i\left(k^2 - \frac{\omega^2}{c^2}\right)^{-1} \left(k\frac{\partial^2 E_z}{\partial y^2} - \omega \frac{\partial^2 B_z}{\partial x \partial y} \right) + ikE_z = 0$$

Rectangular waveguide

- Gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right) E_z = 0$$

- Step 3: Using Gauss Law for magnetic field yields

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right) B_z = 0$$

- Independent 2 equations that lead to independent 2 solutions
  - The final solution is sum of these 2 solutions

Rectangular waveguide

- If one takes $E_z = 0$ and $B_z \neq 0$, then only transverse E_x and E_y remain: **TE** (**T**ransverse **E**lectric) wave

- If one takes $E_z \neq 0$ and $B_z = 0$, then only transverse B_x and B_y remain: **TM** (**T**ransverse **M**agnetic) wave

Rectangular waveguide

- We consider TE wave
- Write $B_z(x,y) = X(x)Y(y)$
- Then $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right) E_z = 0$ is

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \left(\frac{\omega^2}{c^2} - k^2\right) XY = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2 - \frac{\omega^2}{c^2}$$

- $k^2 - \frac{\omega^2}{c^2}$ is a constant, so $\frac{1}{X}\frac{\partial^2 X}{\partial x^2}$ = const and $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$ = const

Rectangular waveguide

- So, we can write $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2$ and $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$

$$-k_x^2 - k_y^2 + \frac{\omega^2}{c^2} - k^2 = 0$$

- General solution of this equation is

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$

- But

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

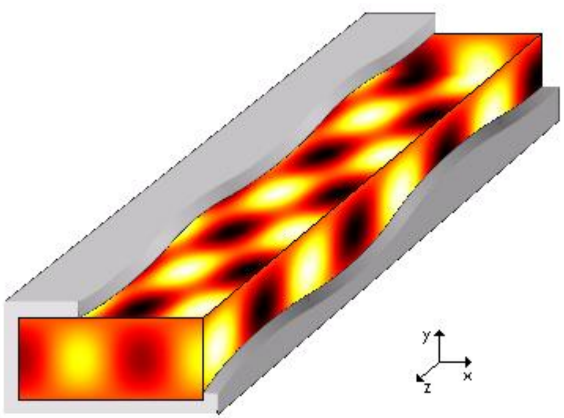
$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

must vanish at the waveguide boundary, so $\frac{\partial B_z}{\partial x} = 0$ for x = 0 and x = a

Rectangular waveguide

$$X(x) = A\sin(k_x x) + B\cos(k_x x)$$

- $\frac{\partial B_z}{\partial x} = 0$ for x = 0 and x = a
  so, $k_x = m\pi/a$
- The same applies for Y
- Finally $$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$



Electric field Ex component of the TE_{31} mode inside a hollow metallic waveguide.

Rectangular waveguide

$$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \qquad -k_x^2 - k_y^2 + \frac{\omega^2}{c^2} - k^2 = 0$$

- Wavenumber $k = \sqrt{(\omega/c)^2 - \pi^2\left[(m/a)^2 + (n/b)^2\right]}$
- It must be positive, what gives condition for minimum frequency that can propagate, **the cutoff frequency**: $\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$
- The lowest ω_{mn} is for mode TE_{10} (m=1 and n=0): $\omega_{10} = \frac{c\pi}{a}$
- Waves with lower frequencies cannot propagate