

Transverse Linear Beam Dynamics

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Introduction to Accelerator Physics
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Table of Contents

1. *Introduction*

1.1. Literature	7
1.2. Magnetic rigidity	10

2. *Magnets*

2.1. Deflection and focusing	13
2.2. Taylor expansion	14
2.3. Beam guidance (dipoles)	16
2.4. Beam focusing (quadrupoles)	18
2.5. Correction of chromatic errors (sextupoles)	23
2.6. Multipole expansion	26
2.7. Effective field length	29

3. *Linear Beam Optics*

3.1. Coordinate system	33
3.2. Geometric optics	36
3.3. Linearized equations of motion	40
3.4. Matrix formalism	48
3.4.1. Drift, dipoles and quadrupoles	50
3.4.2. Particle orbits	56

Lecture 1

Lecture 2

Table of Contents

3.5. Particle beams and trace space	60
3.5.1. Beam emittance	63
3.5.2. Twiss parameters	65
3.5.3. Beta functions	69
3.5.4. Transformation in trace space	75
<hr/>	
4. Circular Accelerators	85
4.1. Weak focusing	86
4.2. Strong focusing	89
4.3. Periodic focusing systems	91
4.2.1. Floquet's theorem	91
4.2.2. Stability criterion	94
4.3.1. General FODO lattice	96
4.3.2. Periodic beta functions	100
4.4. Transverse beam dynamics	103
4.4.1. Closed orbit	103
4.4.2. Betatron tune	104
4.4.3. Filamentation	105
<hr/>	

Lecture 3

Lecture 4

Table of Contents

4.5. Effect of magnet errors	109
4.5.1. Dipole errors	111
4.5.2. Gradient errors	116
4.5.3. Optical resonances	123
4.6. Adiabatic damping	127
<hr/>	
5. <i>Dynamics with Off Momentum Particles</i>	130
5.1. Dispersion and dispersion functions	133
5.2. Dispersion in circular accelerators	139
5.3. Chromaticity	141
5.4. Dynamic Aperture	145

Lecture 5

Lecture 6

1. Introduction



Warm-up:

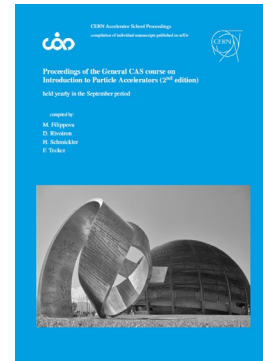
- Literature
- Why Magnets?

Picture taken from:
<https://www.cambridge.org/elt/blog/2023/07/27/5-quick-and-effective-warm-up-activities-for-the-english-classroom/>

Literature

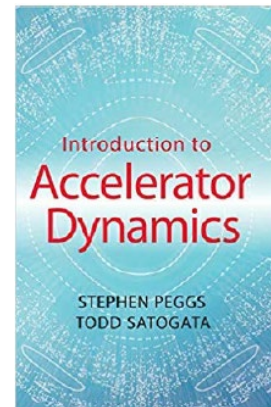
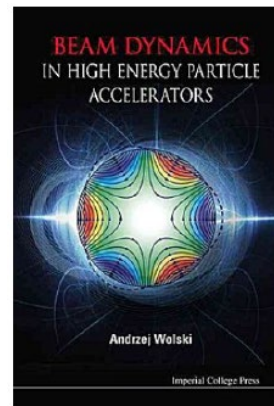
Highly recommended and directly related to the lecture:

- W. Hillert: *Transverse Linear Beam Dynamics*, 2nd version, uploaded to the program @ the course indico page part of CAS proceedings available under <https://cernbox.cern.ch/s/8GTKGkebEDf8FSY>



Further recommended reading:

- A. Wolski: *Beam Dynamics in high energy particle accelerators*, Imperial College Press, ISBN 978-1-78326-277-9
- S. Peggs, T. Satogata: *Introduction to Accelerator Dynamics*, Cambridge University Press, ISBN 978-1-10713-284-9



Additional Literature

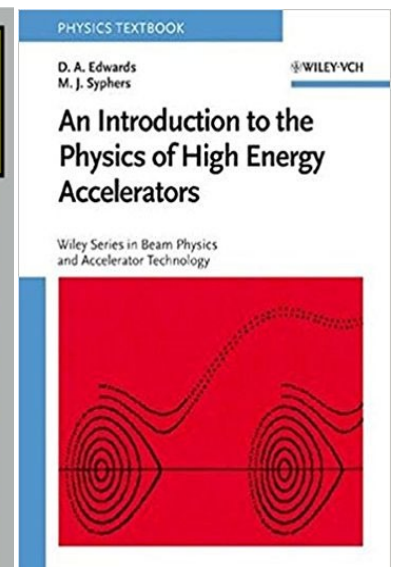
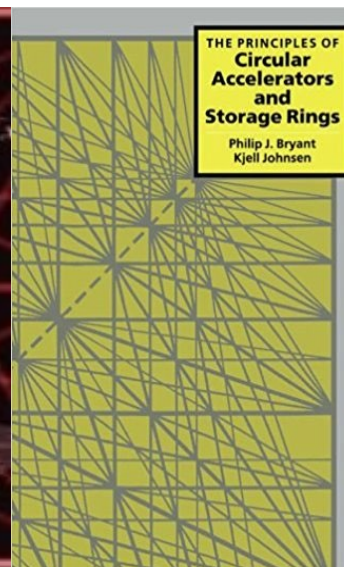
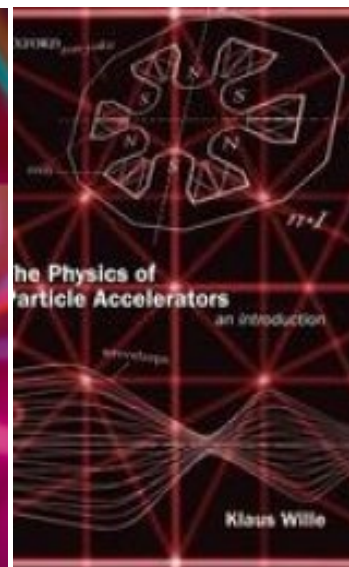
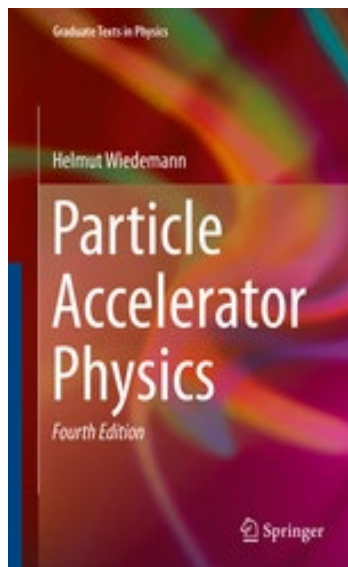
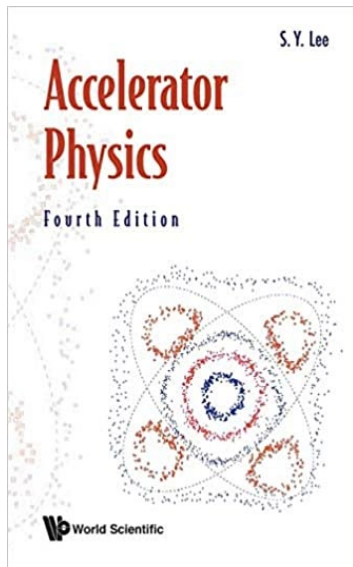
English Textbooks:

- S.Y. Lee: *Accelerator Physics*, 4th edition, World Scientific, New Jersey 2018, ISBN 978-981-4374-94-1
- Bryant/Johnson: *The Principles of Circular Accelerators and Storage Rings*, Cambridge University Press, Cambridge 2005, ISBN 978-0-521-61969-1
- Edwards/Syphers: *An Introduction to the Physics of High Energy Accelerators*, John Wiley & Sons, New York 1992, ISBN 978-0-471-55163-8
- K. Wille: *The physics of particle accelerators*, Oxford Univ. Press 2005, Oxford, ISBN 0-19-850550-7
- H. Wiedemann: *Particle Accelerator Physics*, 4th edition, Springer 2015, Berlin, ISBN 978-3-319-18316-9
- Chao / Tigner: *Handbook of Accelerator Physics and Engineering*, 2nd edition, World Scientific, Singapore 2013, ISBN 987-4417-17-4

Additional Literature

German Textbooks:

- F. Hinterberger: *Physik der Teilchenbeschleuniger und Ionenoptik*, 2. Ausgabe, Springer 2008, Berlin, ISBN 978-3-540-75281-3
- K. Wille: *Physik der Teilchenbeschleuniger und Synchrotronstrahlungsquellen*, 2. überarb. und erw. Ausgabe, Teubner 1996, Stuttgart, ISBN 978-3-519-13087-1



Why Magnets?

Deflection of charged particles:

Can use either components E_{\perp} or B_{\perp} perpendicular on particle's velocity v :

$$\boxed{\text{Zentripetal Force}} \rightarrow \gamma_r m_0 \frac{v^2}{\rho} = \frac{pv}{\rho} = qE_{\perp} + qvB_{\perp} \leftarrow \boxed{\text{Lorentz Force}}$$

Definition of the **stiffness**:

Lecture of Irina Shreyber: *Electromagnetic Theory*

- electric stiffness: $pv = q\rho E_{\perp}$ (ρE_{\perp} is called the **electric stiffness**)
- magnetic stiffness: $p = q\rho B_{\perp}$ (ρB_{\perp} is called the **magnetic stiffness**)

Required stiffness to deflect ultra-relativistic particles ($v \approx c$):

„easy“ $\rho B_{\perp} = 1 \text{ Tm}$ $\overset{c \cdot \rho B_{\perp} \hat{=} \rho E_{\perp}}{\leftrightarrow}$ $\rho E_{\perp} = 300 \text{ MV}$ „impossible“

→ **Only magnetic fields are used for deflection of ultra-relativistic beams!**

$B\rho$ is often called the **magnetic rigidity** of a beam!

Example LHC

$$p[\text{GeV}/c] = 0.3 \cdot (B\rho) [\text{Tm}]$$

- bending radius: $\rho = 2.8 \text{ km}$
 - magnetic field: $B = 8.3 \text{ Tesla}$
- ↓
- magnetic rigidity: $B\rho = 23.2 \cdot 10^3 \text{ Tm}$
→ momentum: $p[\text{GeV}/c] = 0.3 \cdot B\rho$
→ **kin. energy:** $E \approx pc = 7 \text{ TeV}$



<https://commons.m.wikimedia.org/wiki/Smiley?uselang=de>

Picture taken from CERN Document Server

2. Magnets



- **Beam Guidance**
- **Beam Focusing**
- **Correction of Chromatic Errors**
- **Multipole expansion**

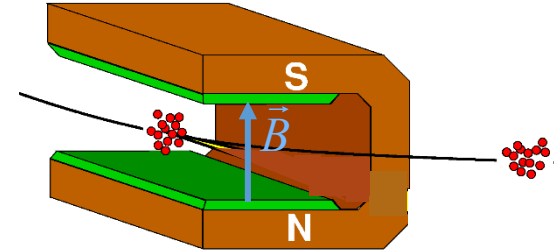
Picture taken from https://www.lhc-closer.es/taking_a_closer_look_at_lhc/0.magnetic_dipoles

Deflection and Focusing

1) Beam deflection:

need homogenous (constant) vertical magnetic field

→ dipole magnet, $B_y = B_0 = \text{const.}$



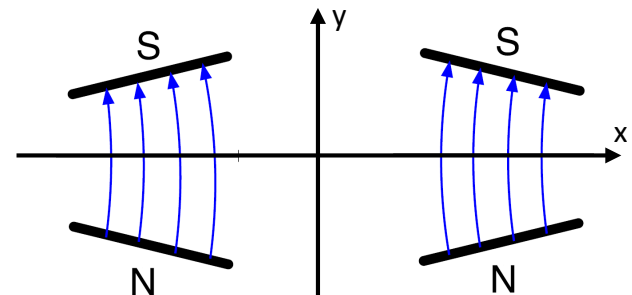
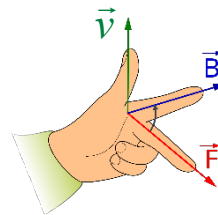
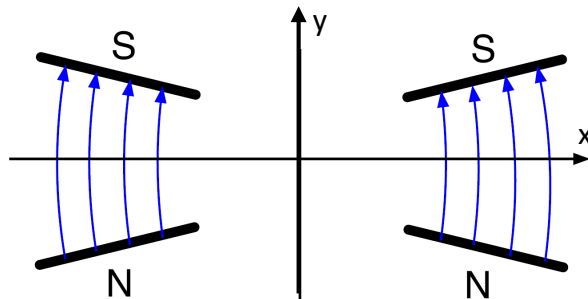
2) Beam focusing:

need magnetic field which is linearly increasing with distance from center

2 options: $g = -\frac{\partial B_y}{\partial x} = \text{const.} > 0$

or

$g = -\frac{\partial B_y}{\partial x} = \text{const.} < 0$



horizontal defocusing
vertical focusing

quadrupole magnet

horizontal focusing
vertical defocusing



Taylor Expansion

More general: Taylor expansion of the vertical magnetic field

$$B_y(x, y) = \underbrace{B_y(0, y)}_{\text{Dipoles}} + \underbrace{x \cdot \frac{\partial B_y}{\partial x}(0, y)}_{\text{Quadrupoles}} + \underbrace{x^2 \cdot \frac{\partial^2 B_y}{\partial x^2}(0, y)}_{\text{Sextupoles}} + \dots$$

Scaling defines types of “normal” magnets:

- **Dipole** magnets $\leftrightarrow B_y = \text{const.}$ \rightarrow *Beam deflection*
- **Quadrupole** magnets $\leftrightarrow B_y \sim x$ \rightarrow *Beam focusing*
- **Sextupole** magnets $\leftrightarrow B_y \sim x^2$ \rightarrow *Chromatic correction*
- **Octupole** magnets $\leftrightarrow B_y \sim x^3$ \rightarrow ...?

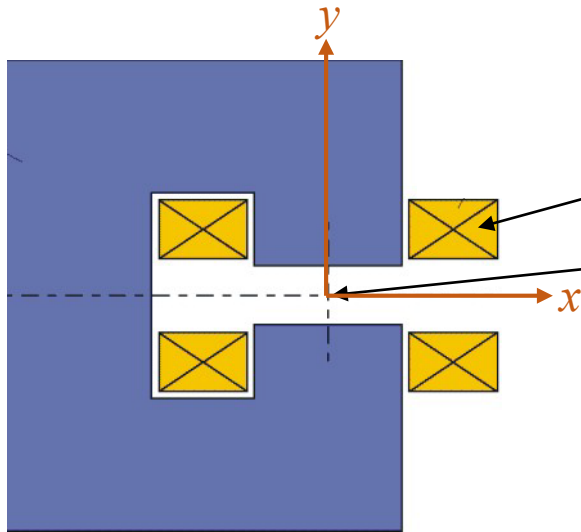
• ...  *non-linear* \rightarrow *no detailed treatment here* 

What must they look like???

Calculation Strategy

↔ iron dominated magnets!

Let's consider e.g. a deflecting dipole magnet:



Maxwell's equations:

$$\vec{\nabla} \times \vec{H} = \vec{j} \quad (\text{coils})$$

$$\vec{\nabla} \times \vec{H} = 0 \quad (\text{gap}) \rightarrow \vec{H} = -\vec{\nabla} \Phi$$

n.c. Magnets (from $B \perp \Phi = \text{const.}$):

$\Phi = \text{const.}$ defines the pole's contour!

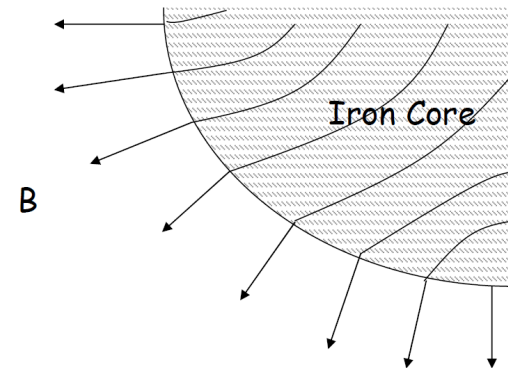
Magn. induction from $\vec{B} = \mu_0 \mu_r \vec{H}$

Or more simple and obvious:

Use of soft steel = ferromagnetic material:

→ B is perpendicular to the surface (poles)

→ B is “shaped” by the pole's contour

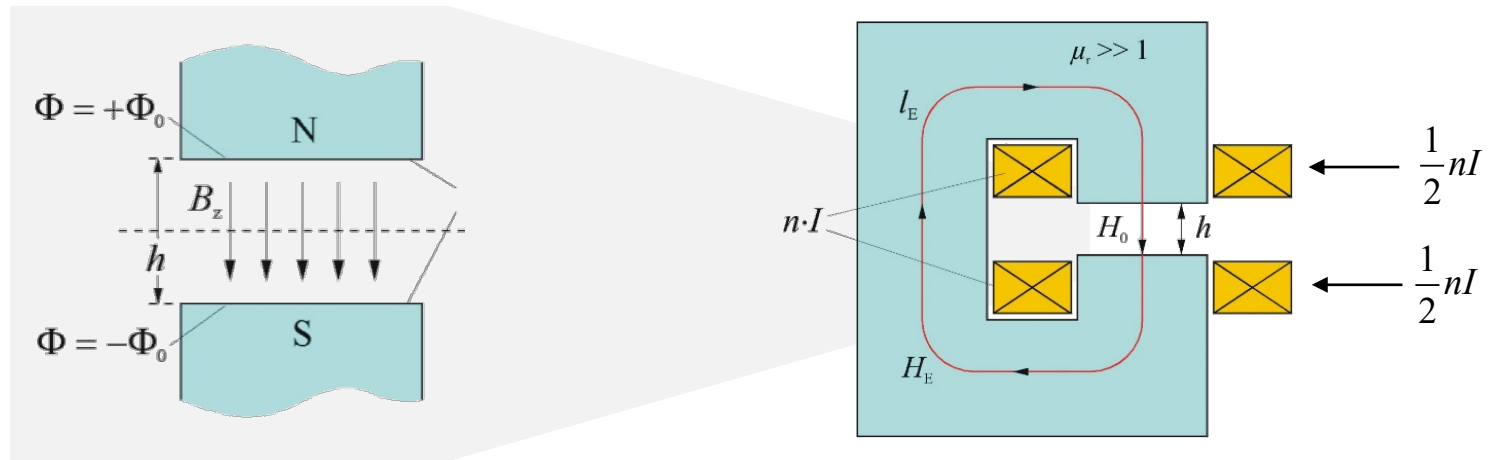


Dipole Strength

Deflection of particles \rightarrow homogenous field: $\vec{B} = B_0 \cdot \hat{e}_y = \text{const.}$

Corresponding magnetic potential: $\Phi(x, y) = -B_0 \cdot y$

\rightarrow Defining the pole's profile to be flat and parallel: **Dipole Magnets!**



Required **Ampère windings $n \cdot I$** from Ampère's law:

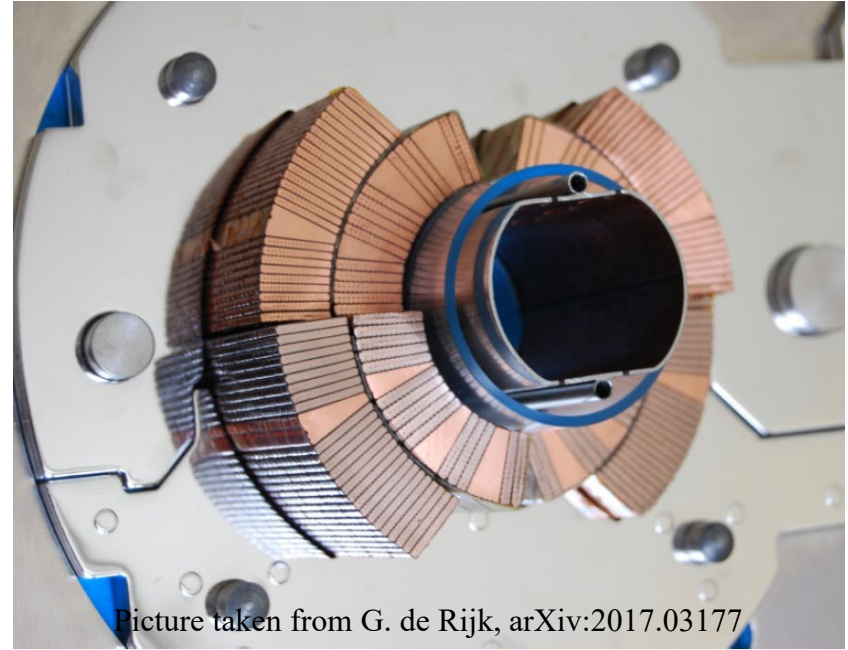
$$n \cdot I = \oint \vec{H} \cdot d\vec{s} = \int_{\text{gap}} \vec{H}_0 \cdot d\vec{s} + \int_{\text{yoke}} \vec{H}_E \cdot d\vec{s} \quad \mu_r \cdot |H_E| = |H_0| \Rightarrow |H_0| \gg |H_E|$$

$$B_0 = \mu_0 \frac{n \cdot I}{h} \quad \text{Dipole strength: } \kappa = \frac{q}{p} B_0 = \frac{q \mu_0 n \cdot I}{p h} = \frac{1}{\rho}, \quad [\kappa] = \text{m}^{-1}$$

Types of Dipole Magnets



Picture taken from CERN Document Server



Picture taken from G. de Rijk, arXiv:2017.03177

Iron dominated:

field determined by
geometry of poles
→ 2 flat poles

Superconducting:

field determined by
geometry of coils
→ $j(\phi) \sim \cos \phi$

Lectures of Gijs de Rijk: *Warm Magnets, Cold Magnets*

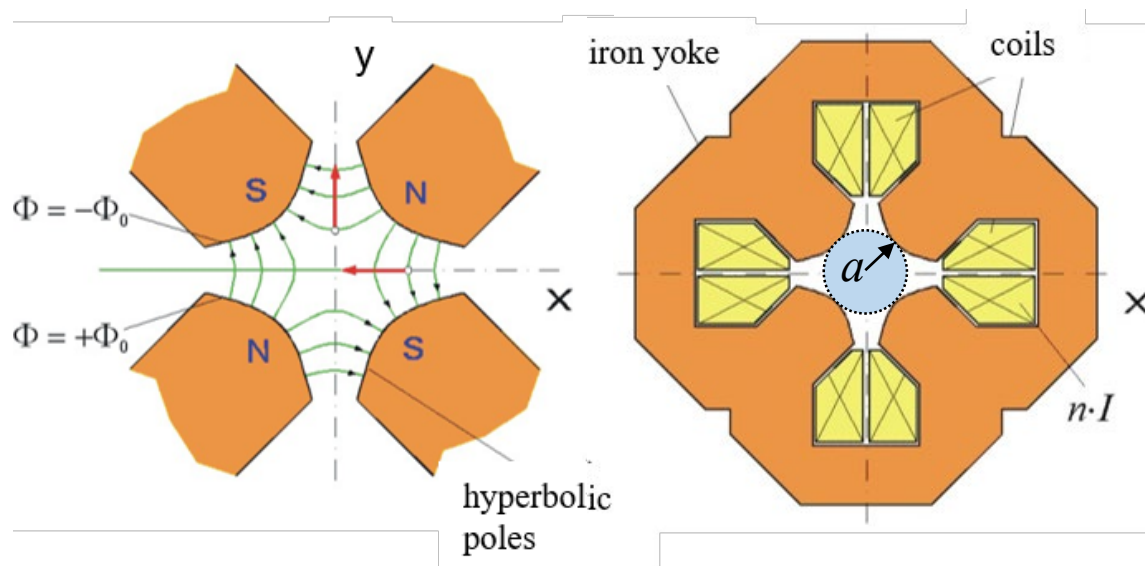
Beam Focusing

Restoring force, linearly increasing with increasing distance from the axis:

$$\boxed{B_y = -g \cdot x, \quad B_x = -g \cdot y} \quad \text{with} \quad g = -\frac{\partial B_y}{\partial x} = -\frac{\partial B_x}{\partial y} = \text{const.} \quad \leftarrow \textit{Proceedings}$$

Corresponding potential: $\Phi(x, y) = g \cdot x \cdot y$ solves $\vec{\nabla} \cdot \vec{B} = -\Delta\Phi = 0$

$\Phi = \text{const.}$ defines the pole's profile to 4 hyperbolic poles: **Quadrupole Magnets**



Pole's contour:
4 poles, profile

$$y(x) = \pm \frac{\Phi_0}{g \cdot x} = \pm \frac{a^2}{2x}$$

$$a = \sqrt{2\Phi_0/g} \quad \uparrow$$

Quadrupole Strength

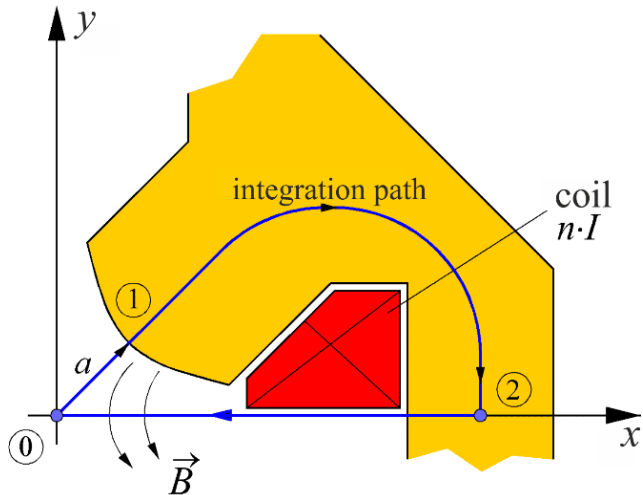
Restoring force on the particle:

$$\vec{F} = q \cdot (\vec{v} \times \vec{B}) = qvg \cdot (x \hat{e}_x - y \hat{e}_y)$$

A quadrupole magnet is therefore focusing only in one plane and defocusing in the other, depending on the sign of g !

The g parameter may be related to the current of the coils by evaluating the

closed loop integral $n \cdot I = \oint \vec{H} \cdot d\vec{s} = \int_0^1 \vec{H}_0 \cdot d\vec{s} + \int_1^2 \vec{H}_E \cdot d\vec{s} + \int_2^0 \vec{H}_0 \cdot d\vec{s} \approx \int_0^1 \vec{H}_0 \cdot d\vec{s}$



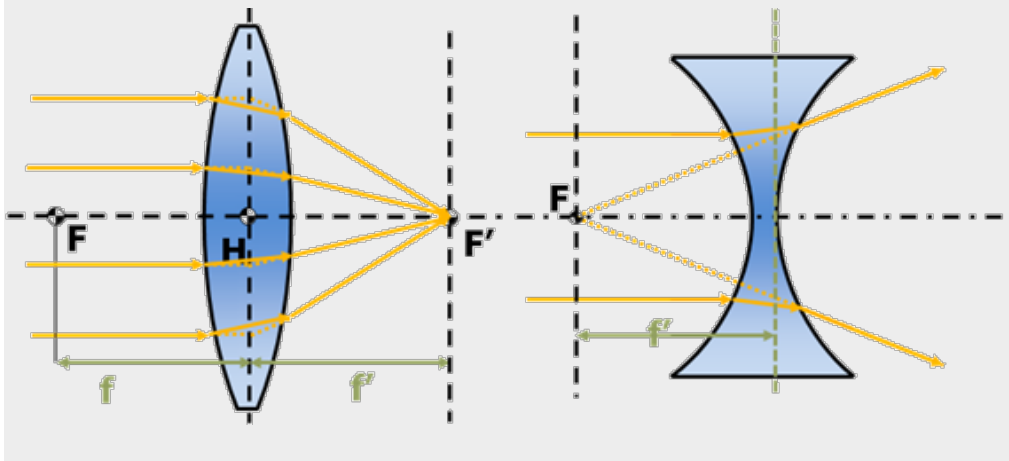
$$\int_0^1 \vec{H}_0 \cdot d\vec{s} = -\int_0^a \frac{g}{\mu_0} r dr = \frac{-g}{2\mu_0} a^2 \rightarrow g = -\frac{2 \cdot \mu_0 \cdot n \cdot I}{a^2}$$

Normalization \rightarrow **quadrupole strength**

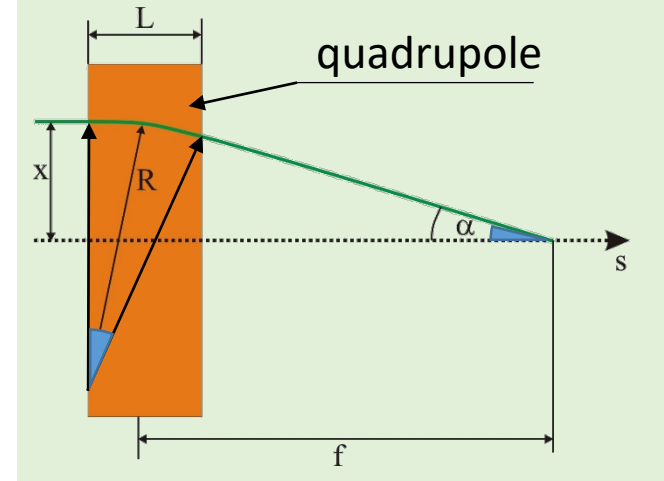
$$k = \frac{q}{p} g = -\frac{2q\mu_0}{p} \frac{n \cdot I}{a^2}, \quad [k] = \text{m}^{-2}$$

Thin Lens Approximation

Light optics: refraction



Magnet optics: deflection



Thin lens approximation:

$L \ll R \leftrightarrow$ transverse offset of beam remains unchanged in magnet

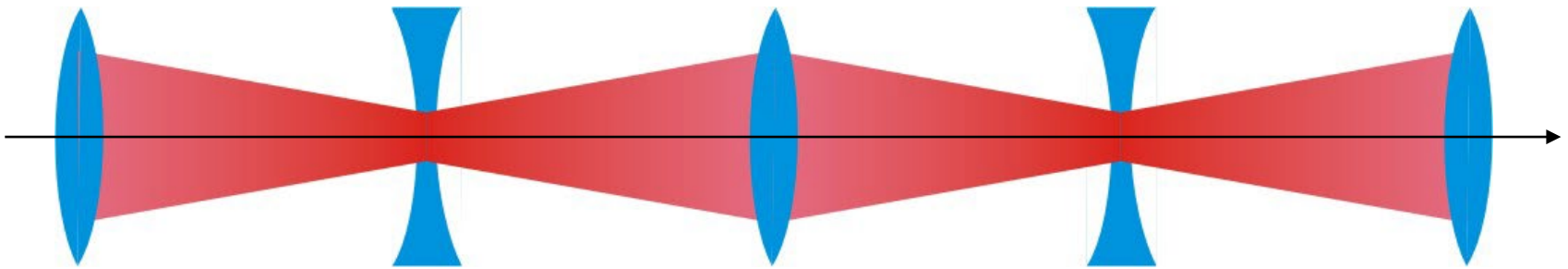
Comparing triangles: $\tan \alpha = \frac{x}{|f|} = \frac{L}{R} = L \cdot \left| \frac{q}{p} B_y \right| = \left| \frac{q}{p} g x \right| L = |x k| L$

Focusing length of a thin quadrupole: $\frac{1}{f_{x,y}} = \mp k \cdot L$ **Def.:** $\begin{cases} k > 0: & \text{focusing in } y \\ k < 0: & \text{focusing in } x \end{cases}$

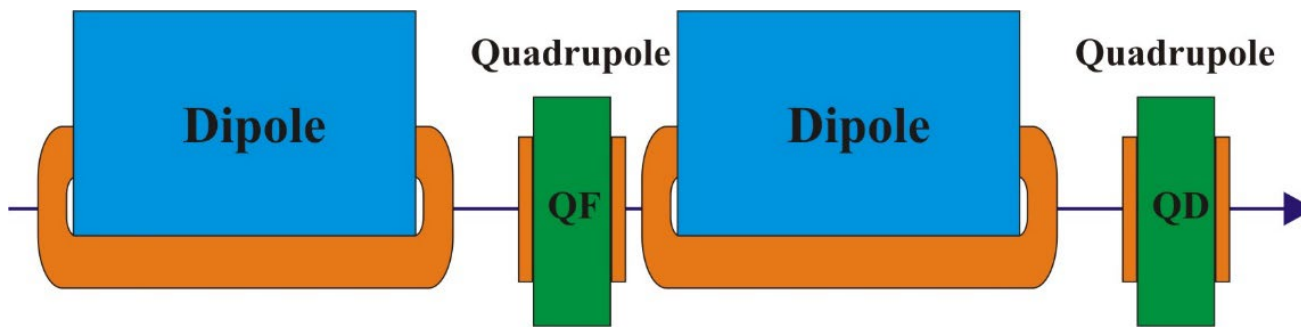
Overall 2D Focusing

Combining focusing and defocusing elements can still lead to overall focusing:

Light optics:



Magnet optics:

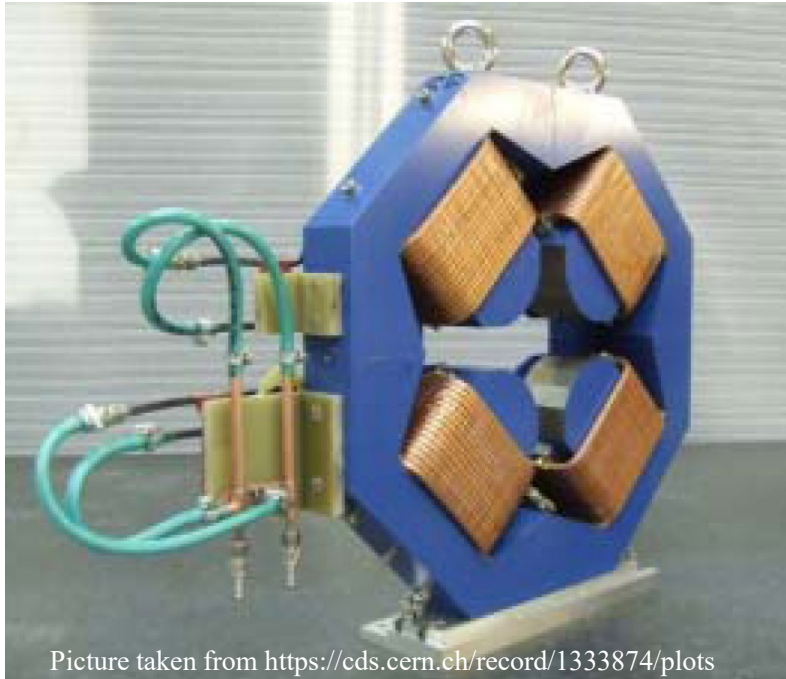


**Strong focusing
or
AG focusing**

Simplest way:
FODO lattice

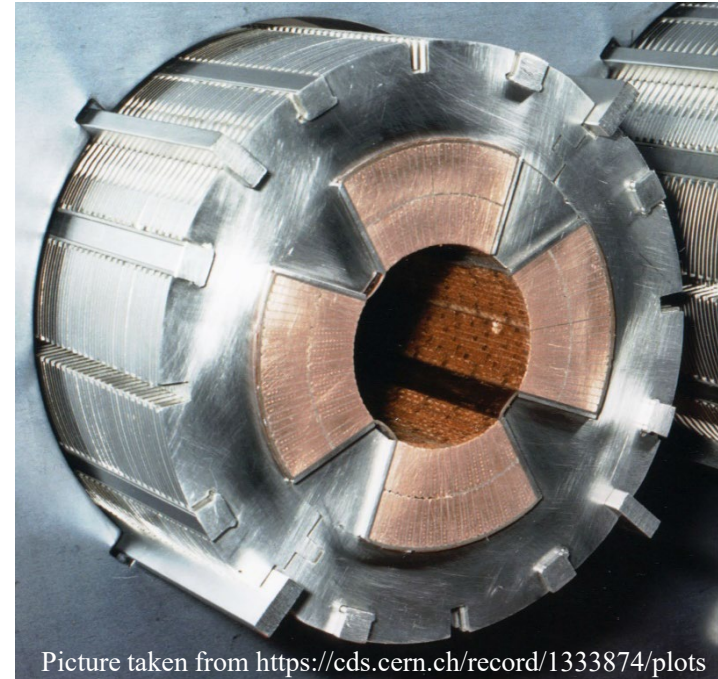
→ Detailed discussion later!

Types of Quadrupole Magnets



Picture taken from <https://cds.cern.ch/record/1333874/plots>

Iron dominated:
field determined by
geometry of poles
→ 4 hyperbolic poles



Picture taken from <https://cds.cern.ch/record/1333874/plots>

Superconducting:
field determined by
geometry of coils
→ $j(\phi) \sim \cos(2\phi)$

Lectures of Gijs de Rijk: *Warm Magnets, Cold Magnets*

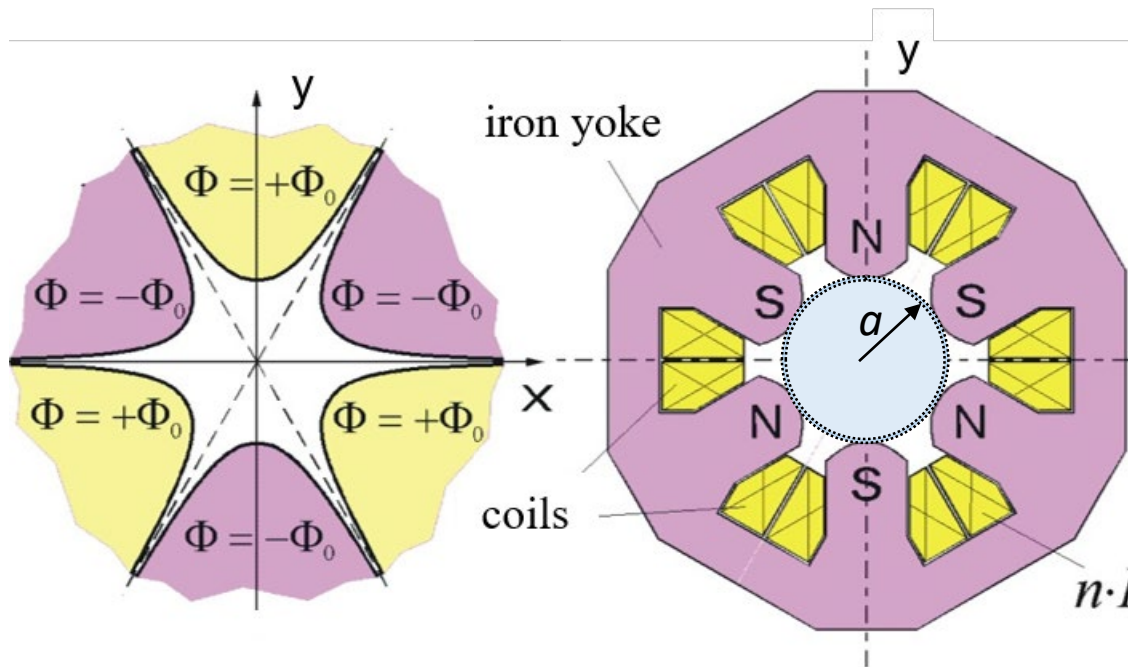
Sextupole Magnets

Quadratic increase of magnetic fields with increasing distance from axis:

$$B_y = \frac{1}{2} g' \cdot (x^2 - y^2) \quad \text{with} \quad g' = \frac{\partial^2 B_y}{\partial x^2} = \text{const.}$$

← *Proceedings*

Corresponding potential: $\Phi(x, y) = \frac{1}{6} g' (y^3 - 3x^2 y)$, solves $\vec{\nabla} \cdot \vec{B} = -\Delta\Phi = 0$



Pole's contour:

6 poles, profile

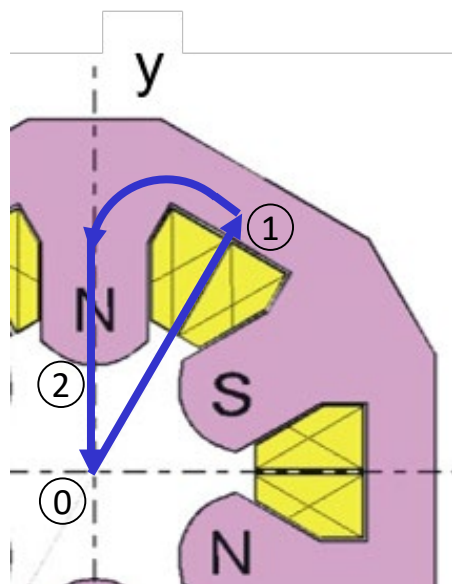
$$x(y) = \pm \sqrt{\frac{y^2}{3} \pm \frac{2\Phi_0}{g' y}}$$

or using the aperture

$$a = \sqrt[3]{6\Phi_0/g'}$$

$$\rightarrow x(y) = \pm \sqrt{\frac{y^2}{3} \pm \frac{a^3}{3y}}$$

Sextupole Strength

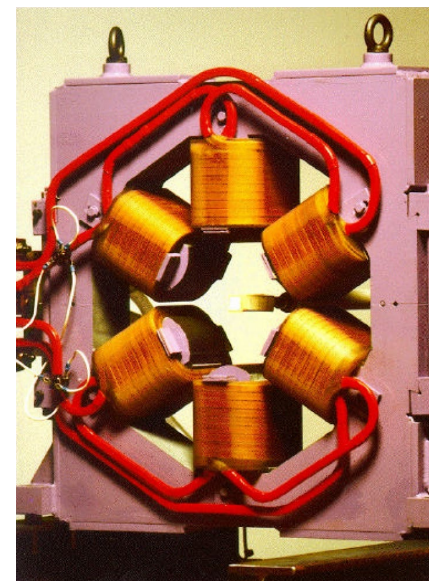


Again, the g parameter may be related to the current of the coils by evaluating the closed loop integral

$$n \cdot I = \oint \vec{H} \cdot d\vec{s} \approx - \int_0^2 \vec{H}_0 \cdot d\vec{s} = \frac{1}{6} g' a^3$$

revealing

$$g' = \frac{\partial^2 B_y}{\partial x^2} = 6 \mu_0 \frac{nI}{a^3}$$



Normalization \rightarrow **sextupole strength:**

$$m = \frac{q}{p} g' = \frac{6q\mu_0}{p} \frac{nI}{a^3}, \quad [m] = \text{m}^{-3}$$

Transverse magnetic fields \rightarrow coupling of particle's horizontal and vertical motion

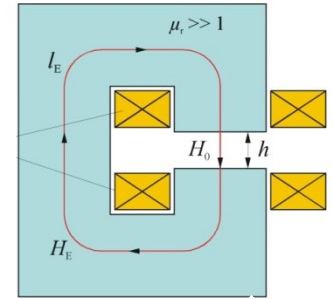
$$B_x(x, y) = -\frac{\partial \Phi}{\partial x} = g' x y \quad \text{and} \quad B_y(x, y) = -\frac{\partial \Phi}{\partial y} = \frac{1}{2} g' (x^2 - y^2)$$

Again: Basic Types of Magnets

Beam guidance:

dipole magnets

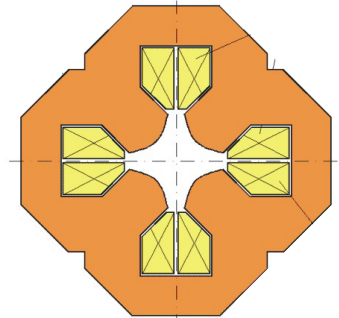
$$\frac{1}{\rho} = \kappa = \frac{q}{p} B_y, \quad B_y = \text{const.}$$



Beam focusing:

quadrupole magnets

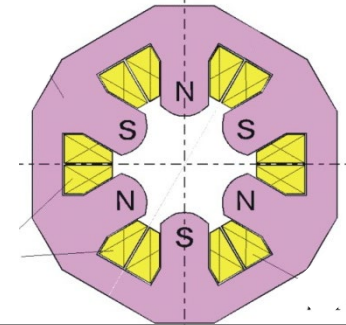
$$\frac{1}{f_x} = -kL, \quad B_x = -\frac{p}{q} ky, \quad B_y = -\frac{p}{q} kx$$



Chromatic correction:

sextupole magnets

$$B_x = \frac{p}{q} mxy, \quad B_y = \frac{p}{2q} m(x^2 - y^2)$$



Properties defined by pole profiles in iron dominated magnets



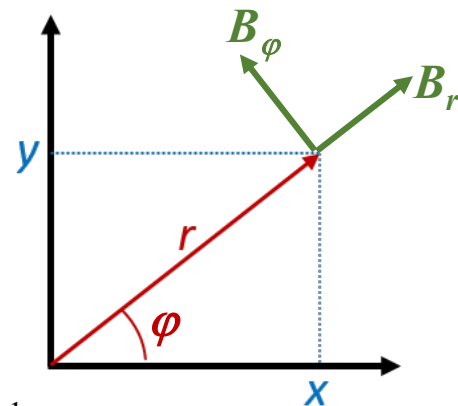
Multipole Expansion (r, φ)



Proceedings!

General treatment by multipole expansion, e.g. in polar coordinates:

$$B_r(r, \varphi) = B_0 \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} \cdot (b_n \sin(n\varphi) - a_n \cos(n\varphi))$$
$$B_\varphi(r, \varphi) = B_0 \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{n-1} \cdot (a_n \sin(n\varphi) + b_n \cos(n\varphi))$$



Contribution of multipole n : $|B|_n = \sqrt{B_{r,n}^2 + B_{\varphi,n}^2} = B_0 \left(\frac{r}{r_0} \right)^{n-1} \cdot \sqrt{a_n^2 + b_n^2}$

Generally: $2n$ pole has symmetry $2\pi/n$, $|B|_n$ scales with r^{n-1}

$n = 1$: dipole magnet

$n = 2$: quadrupole magnet

$n = 3$: sextupole magnet

$n = 4$: octupole magnet

$n = 5$: decapole magnet

Classification:

$b_n \neq 0$: "upright" magnets

$a_n \neq 0$: "skew" magnets, rotated by $\pi/2n$

Field Pattern of Different Multipoles

Normal or upright magnets

no vert. deflection in the horz. mid-plane!

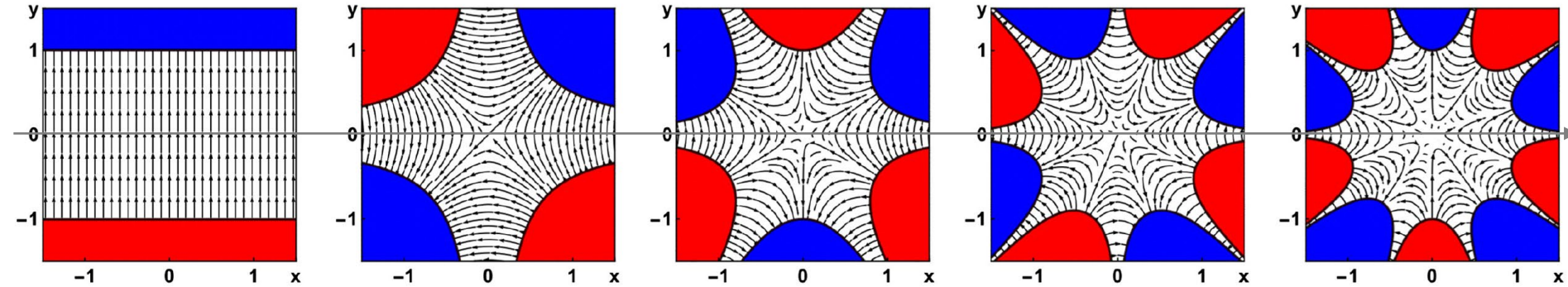
Normal R-Dipole

Normal R-Quadrupole

Normal R-Sextupole

Normal R-Octupole

Normal R-Decapole



Skew or rotated magnets

always vert. deflection in the horz. mid-plane!

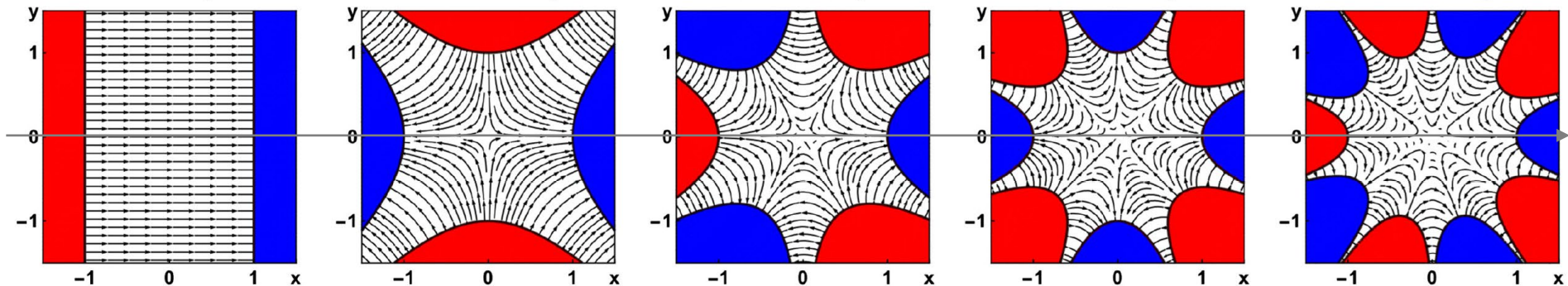
Skew R-Dipole

Skew R-Quadrupole

Skew R-Sextupole

Skew R-Octupole

Skew R-Decapole



Taken from Zolkin, Timofey, Phys.Rev.Accel.Beams **20** (2017) no.4, 043501



Multipole Expansion: (x, y)



Magnetic Potential:

Proceedings!

Dipole	$-\frac{q}{p} \cdot \Phi_1 = -\underline{\kappa}x + \kappa y$
Quadrupole	$-\frac{q}{p} \cdot \Phi_2 = 1/2 \underline{k}(x^2 - y^2) - kxy$
Sextupole	$-\frac{q}{p} \cdot \Phi_3 = -1/6 \underline{m}(x^3 - 3xy^2) + 1/6 m(3x^2y - y^3)$
Octupole	$-\frac{q}{p} \cdot \Phi_4 = -1/24 \underline{r}(x^4 - 6x^2y^2 + y^4) + 1/6 r(x^3y - xy^3)$

Upright Magnets:

Dipole	$\frac{q}{p} \vec{B}_1 = \kappa \hat{e}_y$	“constant”
Quadrupole	$\frac{q}{p} \vec{B}_2 = -ky \hat{e}_x - kx \hat{e}_y$	“linear”
Sextupole	$\frac{q}{p} \vec{B}_3 = mxy \hat{e}_x + \frac{1}{2}m(x^2 - y^2) \hat{e}_y$	“cubic”
Octupole	$\frac{q}{p} \vec{B}_4 = \frac{1}{6}r(3x^2y - y^3) \hat{e}_x + \frac{1}{6}r(x^3 - 3xy^2) \hat{e}_y$	

Multipole strengths:

b_n : upright

a_n : skew

$$\begin{aligned} \underline{\kappa} &= \frac{q B_0}{p} a_1, & \kappa &= \frac{q B_0}{p} b_1 \\ \underline{k} &= -\frac{q B_0}{p r_0} a_2, & k &= -\frac{q B_0}{p r_0} b_2 \\ \underline{m} &= \frac{q B_0}{p r_0^2} a_3, & m &= \frac{q B_0}{p r_0^2} b_3 \\ \underline{r} &= \frac{q B_0}{p r_0^3} a_4, & r &= \frac{q B_0}{p r_0^3} b_4 \end{aligned}$$

$$s_n = \frac{q}{p} \cdot \frac{\partial^{n-1} B_y(0,0)}{\partial x^{n-1}}$$

add. minus sign for $n = 2!$

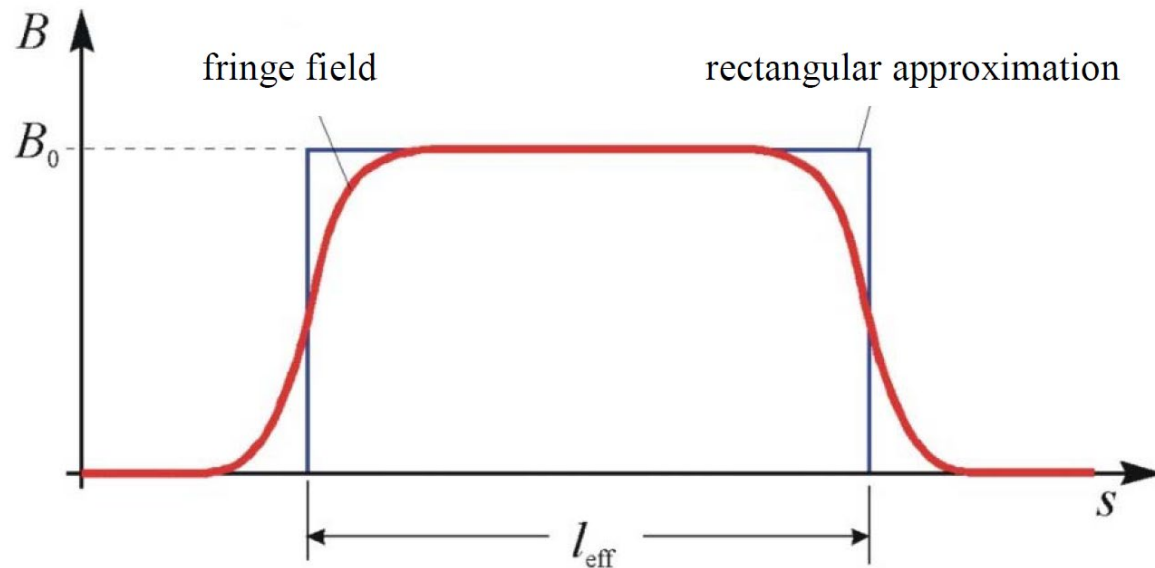
Effective Field Length

So far: assumption of constant field distribution along \hat{e}_s !

In reality: fringe fields at the end of the magnets

→ Definition of an effective field length l_{eff} via

$$\int_{-\infty}^{\infty} \vec{B} \cdot d\vec{s} = \vec{B}_0 \cdot l_{eff}$$



End of 1st Lecture!



Questions?