Transverse Linear Beam Dynamics

Wolfgang Hillert

The CERN Accelerator School

Introduction to Accelerator Physics
22 September – 05 October 2024
Santa Susanna, Spain

The CERN Accelerator School

Copyright statement and speaker's release for video publishing

Table of Contents

Lecture 1

Lecture 2

Table of Contents

Table of Contents

1. Introduction

Picture taken from: https://www.cambridge.org/elt/blog/2023/07/27/5-quick-and-effective-warm-up-activities-for-the-english-classroom/

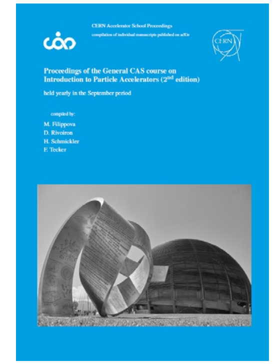
Warm-up:

- **Literature**
- **Why Magnets?**

Literature

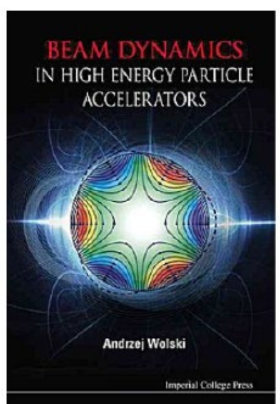
Highly recommended and directly related to the lecture:

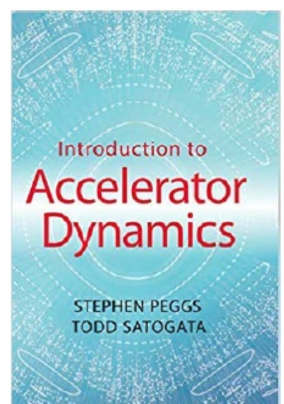
- W. Hillert: ***Transverse Linear Beam Dynamics***,
  2nd version, uploaded to the program @ the course indico page
  part of CAS proceedings available under
  https://cernbox.cern.ch/s/8GTKGkebEDf8FSY



Further recommended reading:

- A. Wolski: ***Beam Dynamics in high energy particle accelerators***,
  Imperial College Press, ISBN 978-1-78326-277-9
- S. Peggs, T. Satogata: ***Introduction to Accelerator Dynamics***,
  Cambridge University Press, ISBN 978-1-10713-284-9





Additional Literature

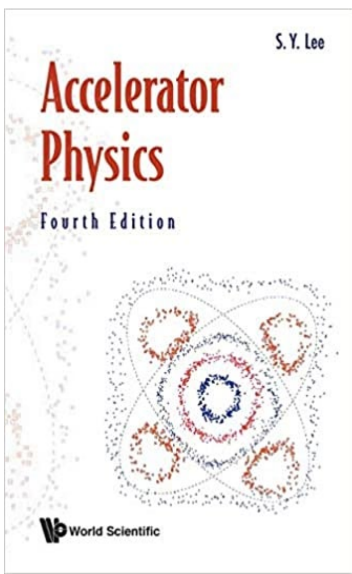
English Textbooks:

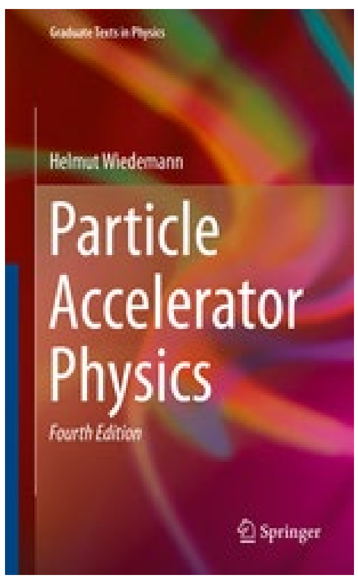
- S.Y. Lee: ***Accelerator Physics*,**
  4th edition, World Scientific, New Yersey 2018, ISBN 978-981-4374-94-1
- Bryant/Johnson: ***The Principles of Circular Accelerators and Storage Rings***,
  Cambridge University Press, Cambridge 2005, ISBN 978-0-521-61969-1
- Edwards/Syphers: ***An Introduction to the Physics of High Energy Accelerators***,
  John Wiley & Sons, New York 1992, ISBN 978-0-471-55163-8
- K. Wille: ***The physics of particle accelerators***,
  Oxford Univ. Press 2005, Oxford, ISBN 0-19-850550-7
- H. Wiedemann: ***Particle Accelerator Physics***,
  4th edition, Springer 2015, Berlin, ISBN 978-3-319-18316-9
- Chao / Tigner: ***Handbook of Accelerator Physics and Engineering*,**
  2nd edition, World Scientific, Singapore 2013, ISBN 987-4417-17-4

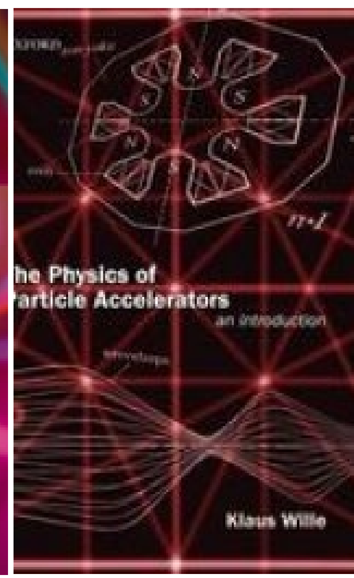
Additional Literature

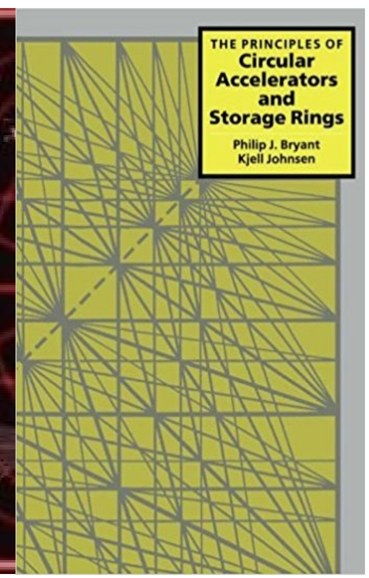
German Textbooks:

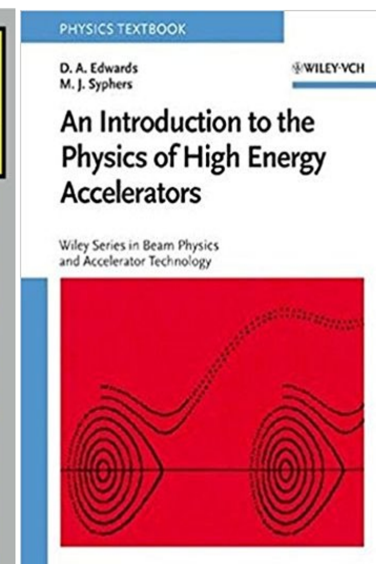
- F. Hinterberger: ***Physik der Teilchenbeschleuniger und Ionenoptik***, 2. Ausgabe, Springer 2008, Berlin, ISBN 978-3-540-75281-3
- K. Wille: ***Physik der Teilchenbeschleuniger und Synchrotronstrahlungsquellen***, 2. überarb. und erw. Ausgabe, Teubner 1996, Stuttgart, ISBN 978-3-519-13087-1











Why Magnets?

Deflection of charged particles:

Can use either components $E_\perp$ or $B_\perp$ perpendicular on particle's velocity v:

Zentripetal Force → $\gamma_r m_0 \frac{\mathrm{v}^2}{\rho} = \frac{p\mathrm{v}}{\rho} = qE_\perp + q\mathrm{v}B_\perp$ ← Lorentz Force

Lecture of Irina Shreyber: ***Electromagnetic Theory***

Definition of the **stiffness:**

- electric stiffness: $p\mathrm{v} = q\rho E_\perp$ ($\rho E_\perp$ is called the electric stiffness)
- magnetic stiffness: $p = q\rho B_\perp$ ($\rho B_\perp$ is called the magnetic stiffness)

Required stiffness to deflect ultra-relativistic particles (v ≈ c):

„easy"

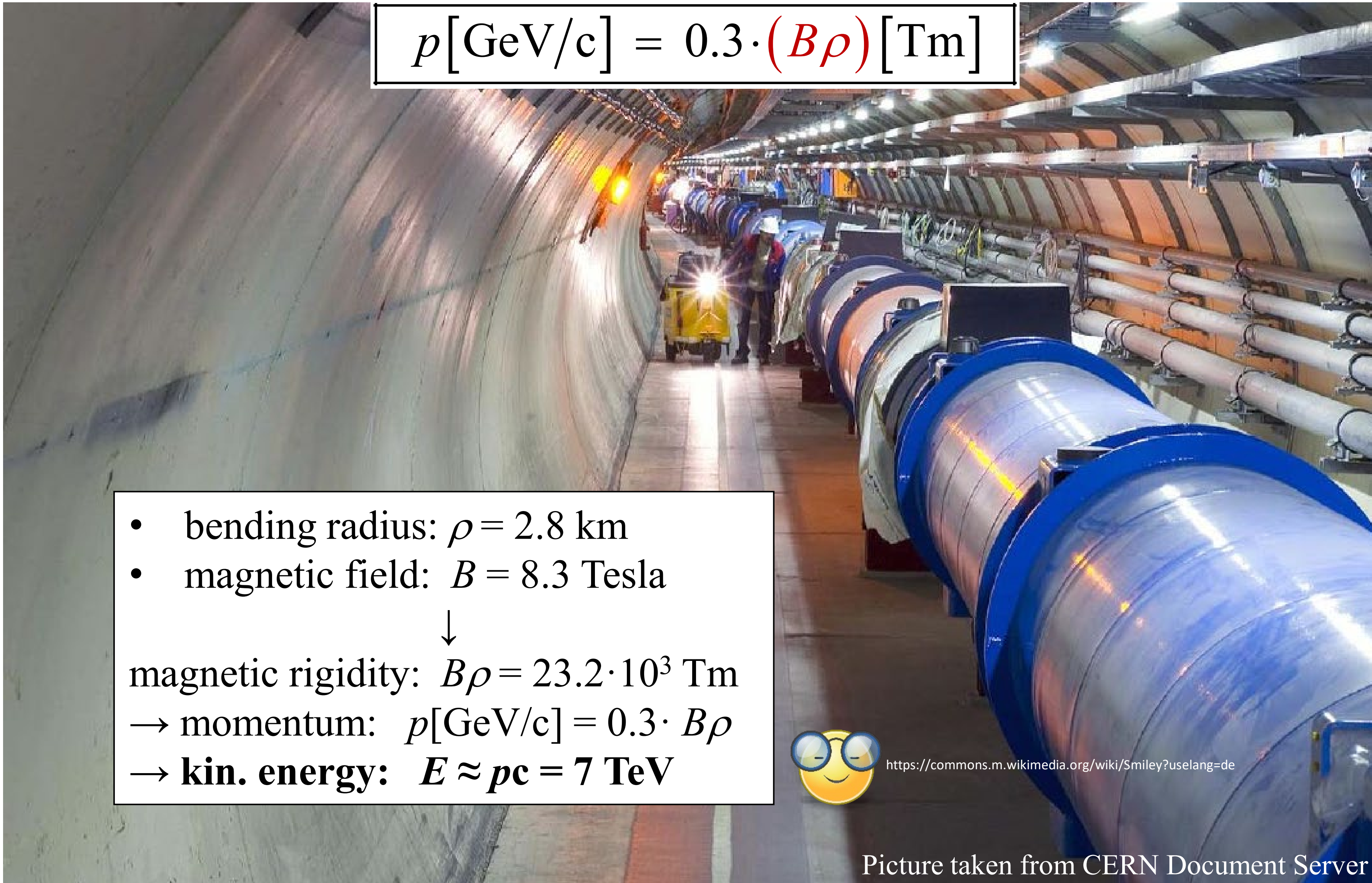
$$\rho B_\perp = 1\,\mathrm{Tm} \quad \overset{c\cdot\rho B_\perp \hat{=} \rho E_\perp}{\leftrightarrow} \quad \rho E_\perp = 300\,\mathrm{MV}$$

„impossible"

→ Only magnetic fields are used for deflection of ultra-relativistic beams!

$B\rho$ is often called the **magnetic rigidity** of a beam!

Example LHC



$$p[\mathrm{GeV/c}] = 0.3 \cdot (B\rho)[\mathrm{Tm}]$$

- bending radius: $\rho = 2.8$ km
- magnetic field: $B = 8.3$ Tesla

$\downarrow$

magnetic rigidity: $B\rho = 23.2 \cdot 10^3$ Tm

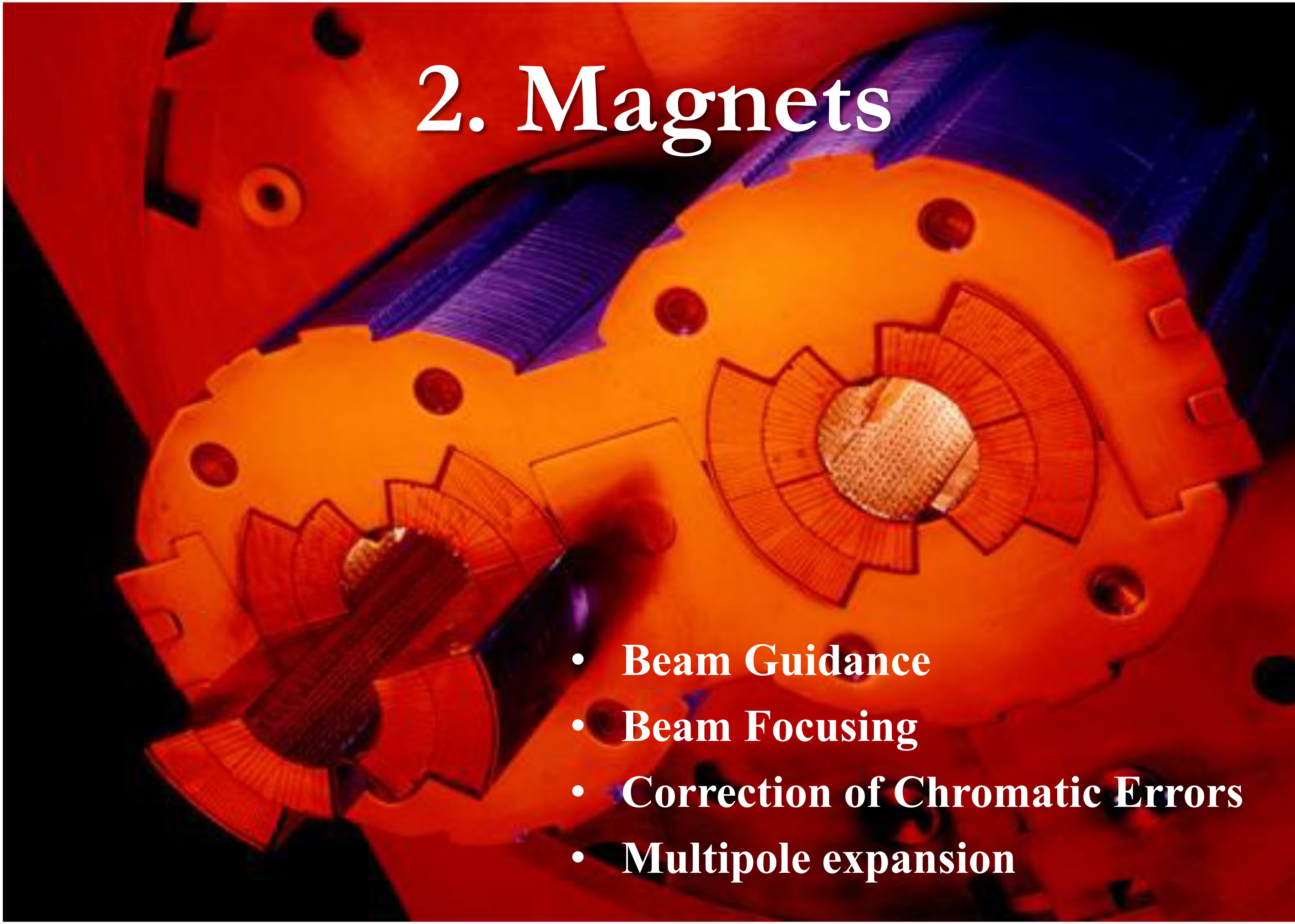
$\rightarrow$ momentum: $p[\mathrm{GeV/c}] = 0.3 \cdot B\rho$

$\rightarrow$ **kin. energy: $E \approx pc = 7$ TeV**

https://commons.m.wikimedia.org/wiki/Smiley?uselang=de

Picture taken from CERN Document Server

2. Magnets



- **Beam Guidance**
- **Beam Focusing**
- **Correction of Chromatic Errors**
- **Multipole expansion**

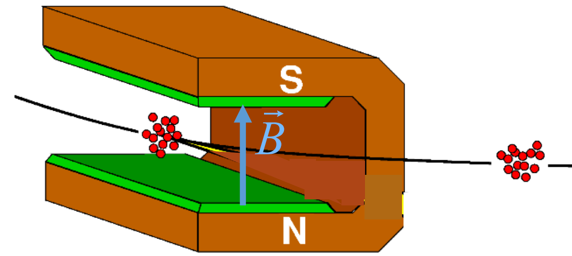
Picture taken from https://www.lhc-closer.es/taking_a_closer_look_at_lhc/0.magnetic_dipoles

Deflection and Focusing

1) Beam deflection:

need <u>homogenous</u> (constant) vertical magnetic field

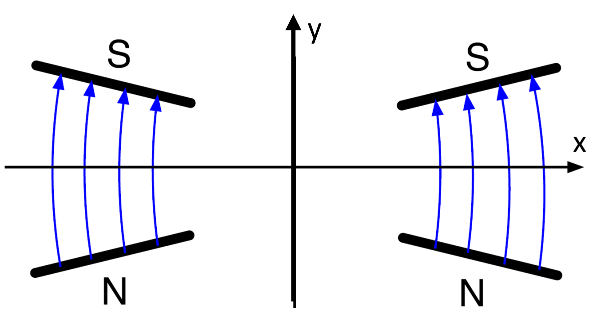
→ dipole magnet, $\boxed{B_y = B_0 = \text{const.}}$



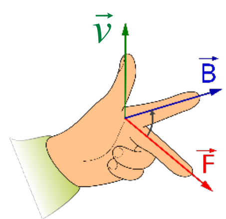
2) Beam focusing:

need magnetic field which is <u>linearly increasing</u> with distance from center

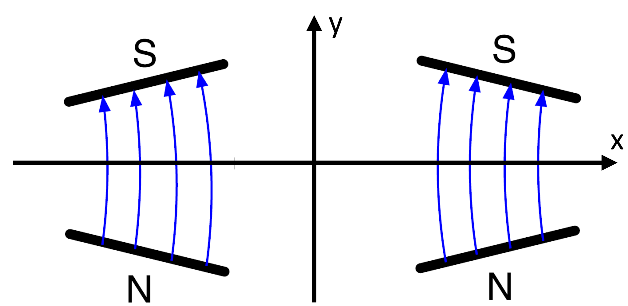
2 options: $\boxed{g = -\frac{\partial B_y}{\partial x} = \text{const.} > 0}$ or $\boxed{g = -\frac{\partial B_y}{\partial x} = \text{const.} < 0}$



horizontal defocusing
vertical focusing



quadrupole magnet



horizontal focusing
vertical defocusing

Taylor Expansion

More general: Taylor expansion of the vertical magnetic field

$$B_y(x,y) = \underbrace{B_y(0,y)}_{\text{Dipoles}} + \underbrace{x \cdot \frac{\partial B_y}{\partial x}(0,y)}_{\text{Quadrupoles}} + \underbrace{x^2 \cdot \frac{\partial^2 B_y}{\partial x^2}(0,y)}_{\text{Sextupoles}} + \ldots$$

Scaling defines **types of "normal" magnets**:

- **Dipole** magnets ↔ $B_y = \text{const.}$ → *Beam deflection*
- **Quadrupole** magnets ↔ $B_y \sim x$ → *Beam focusing*
- **Sextupole** magnets ↔ $B_y \sim x^2$ → *Chromatic correction*
- **Qctupole** magnets ↔ $B_y \sim x^3$ → *…?*
- …

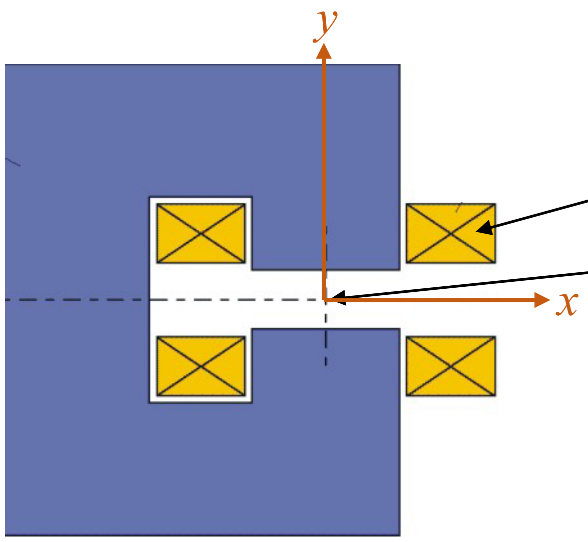
non-linear → no detailed treatment here

What must they look like???

Calculation Strategy

↔ iron dominated magnets!

Let's consider e.g. a deflecting dipole magnet:



Maxwell's equations:

$\vec{\nabla} \times \vec{H} = \vec{j}$ (coils)

$\vec{\nabla} \times \vec{H} = 0$ (gap) $\rightarrow$ $\vec{H} = -\vec{\nabla}\Phi$

n.c. Magnets (from $B \perp \Phi = \text{const.}$):

Φ = const. defines the pole's contour!

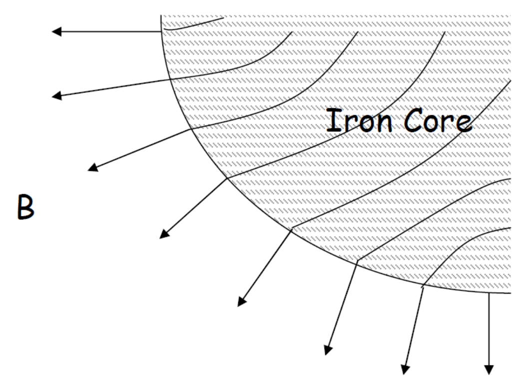
Magn. induction from $\vec{B} = \mu_0 \, \mu_r \, \vec{H}$

Or more simple and obvious:

Use of soft steel = **ferromagnetic material**:

→ B is perpendicular to the surface (poles)

→ B is "shaped" by the pole's contour

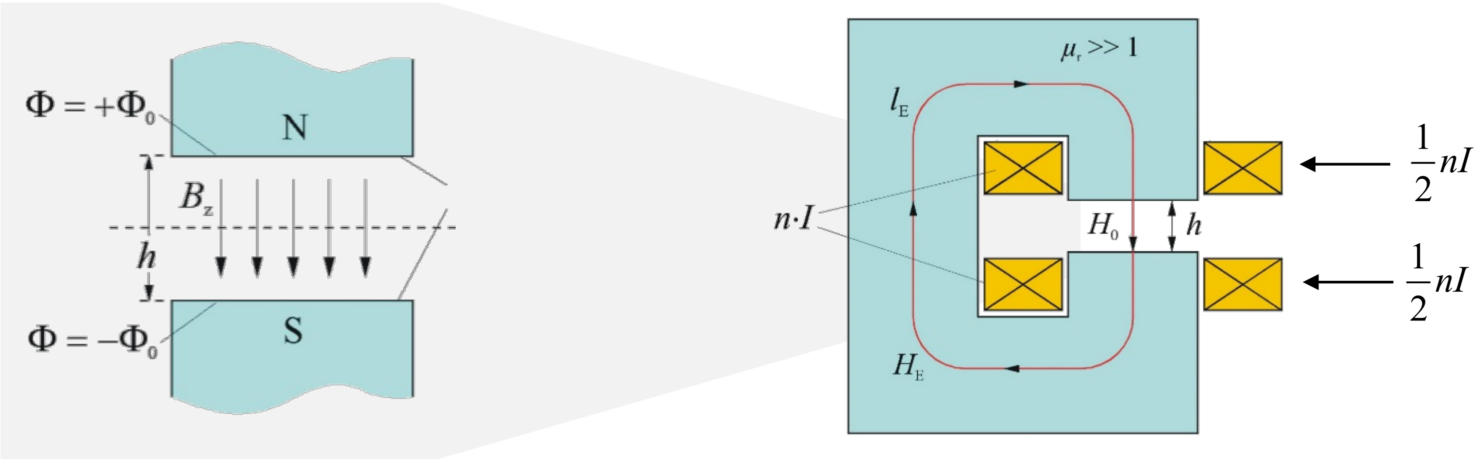


Dipole Strength

Deflection of particles $\rightarrow$ homogenous field: $\boxed{\vec{B} = B_0 \cdot \hat{e}_y = \text{const.}}$

Corresponding magnetic potential: $\Phi(x,y) = -B_0 \cdot y$

$\rightarrow$ **Defining the pole's profile to be flat and parallel: Dipole Magnets!**



Required **Ampère windings *n·I*** from Ampère's law:

$$n \cdot I = \oint \vec{H} \cdot d\vec{s} = \int_{gap} \vec{H}_0 \cdot d\vec{s} + \int_{yoke} H_E \qquad \mu_r \cdot |H_E| = |H_0| \;\Rightarrow\; |H_0| \gg |H_E|$$

$$B_0 = \mu_0 \frac{n \cdot I}{h}$$ **Dipole strength:** $$\kappa = \frac{q}{p} B_0 = \frac{q\,\mu_0}{p} \frac{n \cdot I}{h} = \frac{1}{\rho}, \qquad [\kappa] = \text{m}^{-1}$$

Types of Dipole Magnets

Picture taken from CERN Document Server

Picture taken from G. de Rijk, arXiv:2017.03177

Iron dominated:
field determined by
geometry of poles
$\rightarrow$ 2 flat poles

Superconducting:
field determined by
geometry of coils
$\rightarrow j(\phi) \sim \cos\phi$

Lectures of Gijs de Rijk: ***Warm Magnets***, ***Cold Magnets***

Beam Focusing

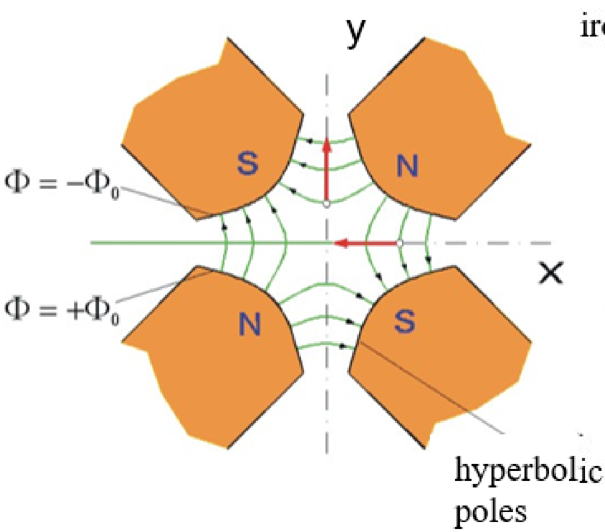
Restoring force, linearly increasing with increasing distance from the axis:

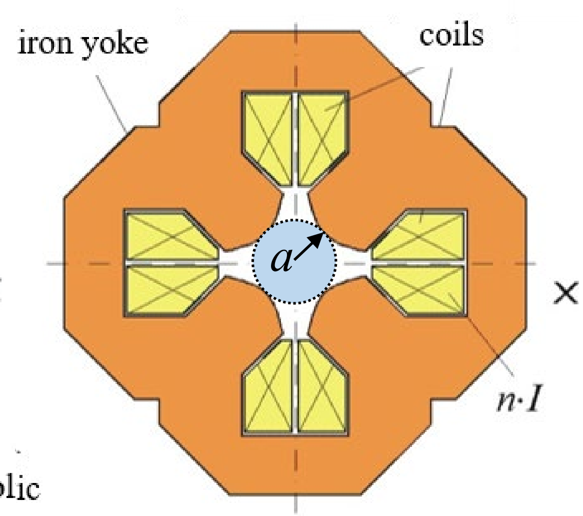
$$\boxed{B_y = -g \cdot x, \quad B_x = -g \cdot y} \quad \text{with} \quad g = -\frac{\partial B_y}{\partial x} = -\frac{\partial B_x}{\partial y} = \text{const.}$$

← *Proceedings*

Corresponding potential: $\Phi(x, y) = g \cdot x \cdot y$ solves $\vec{\nabla} \cdot \vec{B} = -\Delta\Phi = 0$

Φ = const. defines the pole's profile to 4 hyperbolic poles: **Quadrupole Magnets**





Pole's contour:

4 poles, profile

$$y(x) = \pm\frac{\Phi_0}{g \cdot x} = \pm\frac{a^2}{2x}$$

$$a = \sqrt{2\Phi_0 / g}$$

Quadrupole Strength

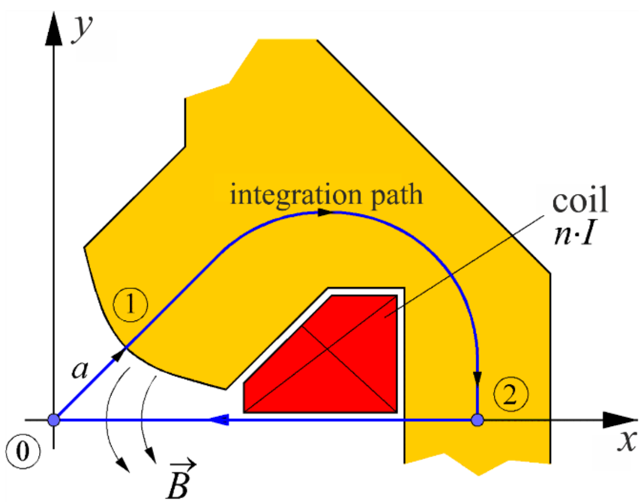
Restoring force on the particle:

$$\vec{F} = q \cdot \left(\vec{\mathrm{v}} \times \vec{B} \right) = q\mathrm{v}g \cdot \left(x\hat{e}_x - y\hat{e}_y \right)$$

A quadrupole magnet is therefore focusing only in one plane and defocusing in the other, depending on the sign of *g*!

The g parameter may be related to the current of the coils by evaluating the closed loop integral

$$n \cdot I = \oint \vec{H} \cdot \mathrm{d}\vec{s} = \int_{\boxed{0}}^{\boxed{1}} \vec{H}_0 \cdot \mathrm{d}\vec{s} + \int_{\boxed{1}}^{\boxed{2}} \vec{H}_E \cdot \mathrm{d}\vec{s} + \int_{\boxed{2}}^{\boxed{0}} \vec{H}_0 \cdot \mathrm{d}\vec{s} \approx \int_{\boxed{0}}^{\boxed{1}} \vec{H}_0 \cdot \mathrm{d}\vec{s}$$



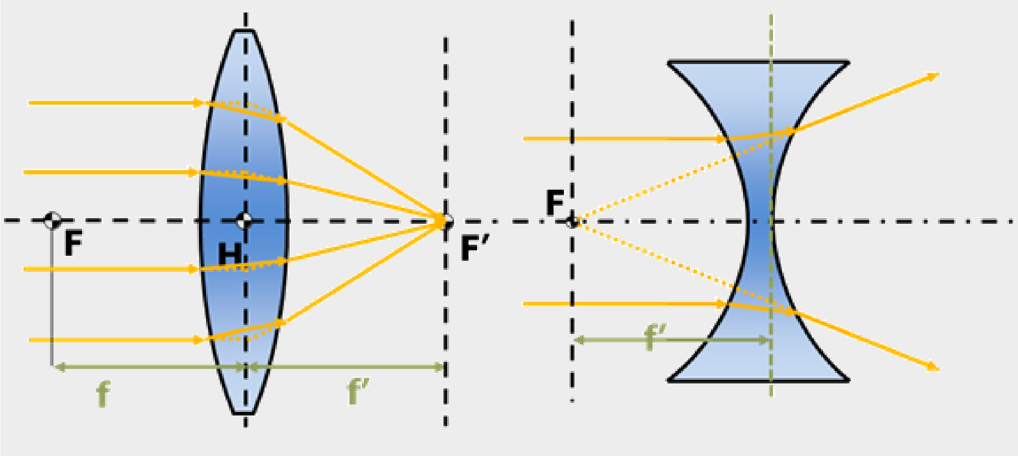
$$\int_{\boxed{0}}^{\boxed{1}} \vec{H}_0 \cdot \mathrm{d}\vec{s} = -\int_0^a \frac{g}{\mu_0} r\,\mathrm{d}r = \frac{-g}{2\mu_0} a^2 \quad \rightarrow \quad g = -\frac{2 \cdot \mu_0 \cdot n \cdot I}{a^2}$$

Normalization → **quadrupole strength**

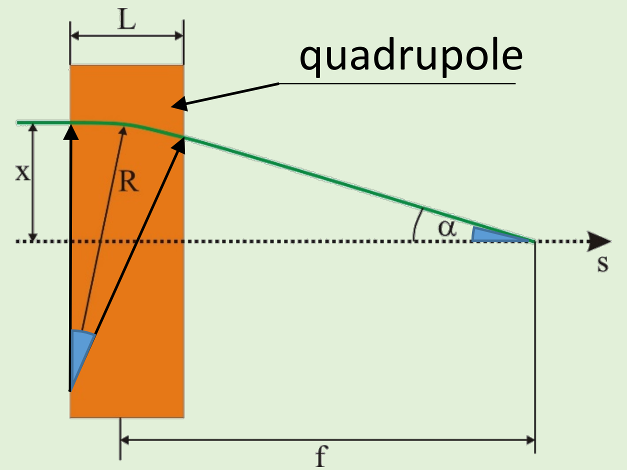
$$k = \frac{q}{p} g = -\frac{2q\mu_0}{p} \frac{n \cdot I}{a^2}, \quad [k] = \mathrm{m}^{-2}$$

Thin Lens Approximation

Light optics: refraction



Magnet optics: deflection



<u>Thin lens approximation:</u>

$L << R$ $\leftrightarrow$ transverse offset of beam remains unchanged in magnet

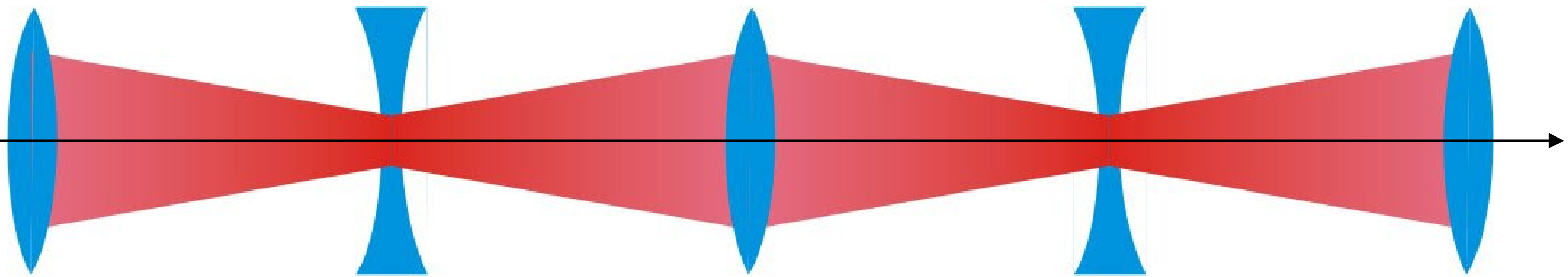
Comparing triangles: $\tan\alpha = \frac{x}{|f|} = \frac{L}{R} = L \cdot \left|\frac{q}{p} B_y\right| = \left|\frac{q}{p} g\, x\right| L = |x\, k| L$

Focusing length of a thin quadrupole: $\frac{1}{f_{x,y}} = \mp\, k \cdot L$ **Def.:** $\begin{cases} k > 0: & \text{focusing in } y \\ k < 0: & \text{focusing in } x \end{cases}$

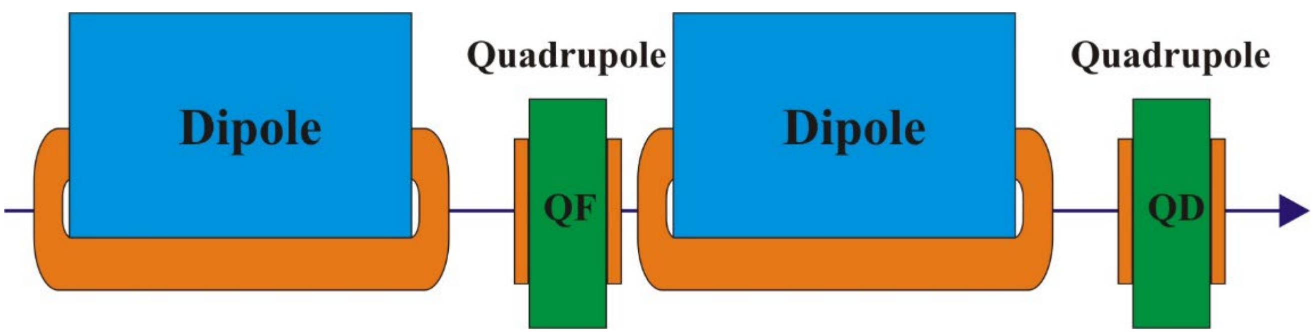
Overall 2D Focusing

Combining focusing and defocusing elements can still lead to overall focusing:

Light optics:



Magnet optics:

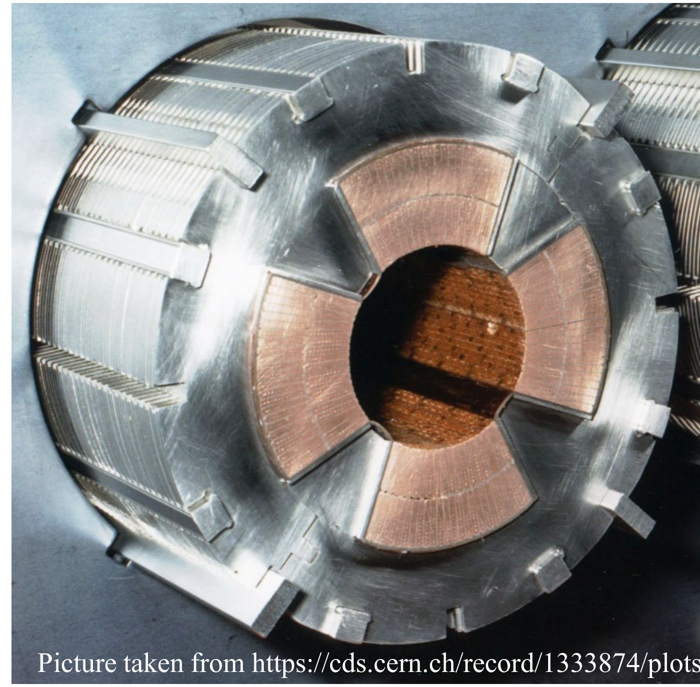


Strong focusing
or
AG focusing

Simplest way:
FODO lattice

→ Detailed discussion later!

Types of Quadrupole Magnets



Iron dominated:
field determined by
geometry of poles
→ 4 hyperbolic poles

Superconducting:
field determined by
geometry of coils
→ $j(\phi) \sim \cos(2\phi)$

Lectures of Gijs de Rijk: ***Warm Magnets***, ***Cold Magnets***

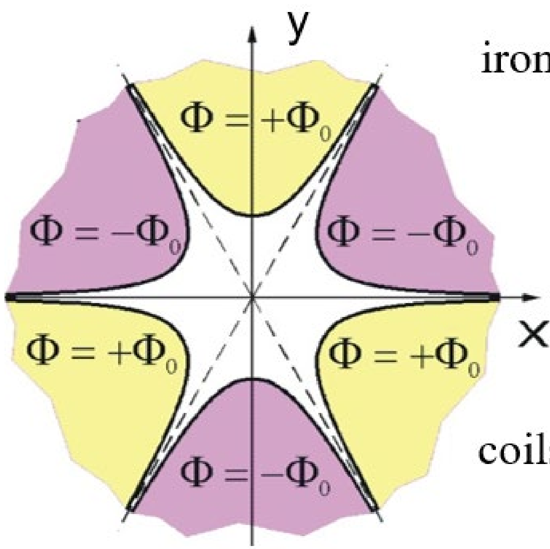
Sextupole Magnets

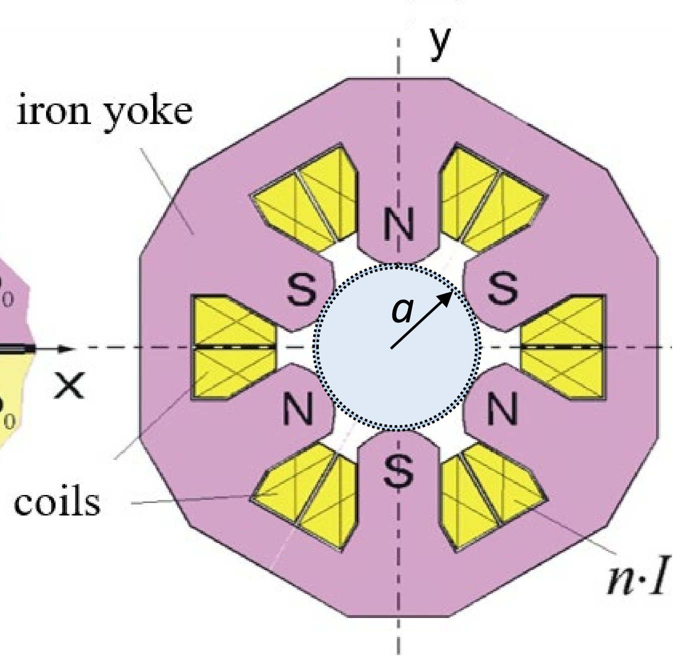
Quadratic increase of magnetic fields with increasing distance from axis:

$$B_y = \frac{1}{2} g' \cdot \left(x^2 - y^2\right) \quad \text{with} \quad g' = \frac{\partial^2 B_y}{\partial x^2} = \text{const.}$$

← *Proceedings*

Corresponding potential: $\Phi(x,y) = \frac{1}{6} g'\left(y^3 - 3x^2 y\right)$, solves $\vec{\nabla} \cdot \vec{B} = -\Delta\Phi = 0$





Pole's contour:

6 poles, profile

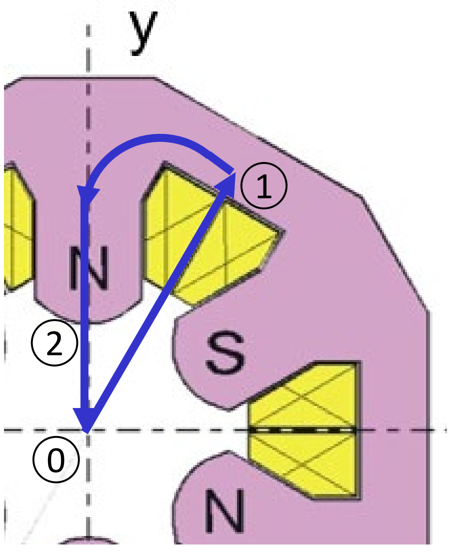
$$x(y) = \pm\sqrt{\frac{y^2}{3} \pm \frac{2\Phi_0}{g' y}}$$

or using the aperture

$$a = \sqrt[3]{6\Phi_0 / g'}$$

$$\rightarrow \quad x(y) = \pm\sqrt{\frac{y^2}{3} \pm \frac{a^3}{3y}}$$

Sextupole Strength

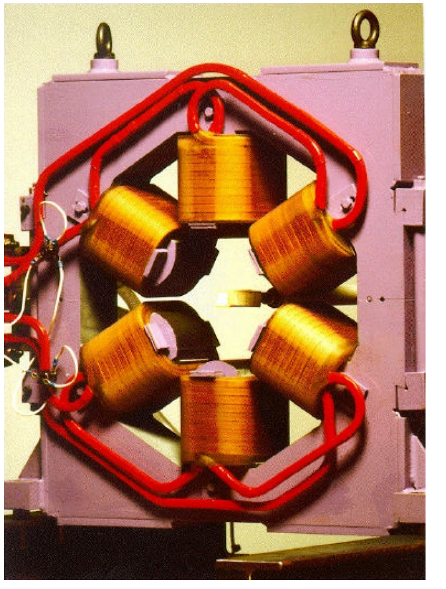


Again, the g parameter may be related to the current of the coils by evaluating the closed loop integral

$$n \cdot I = \oint \vec{H} \cdot d\,\vec{s} \approx -\int_{\boxed{0}}^{\boxed{2}} \vec{H}_0 \cdot d\,\vec{s} = \frac{1}{6} g' a^3$$

revealing

$$g' = \frac{\partial^2 B_y}{\partial x^2} = 6\,\mu_0 \frac{n\,I}{a^3}$$



Normalization → **sextupole strength:** $\boxed{m = \frac{q}{p} g' = \frac{6\,q\,\mu_0}{p} \frac{n\,I}{a^3}}$, $[m] = \mathrm{m}^{-3}$

Transverse magnetic fields → coupling of particle's horizontal and vertical motion

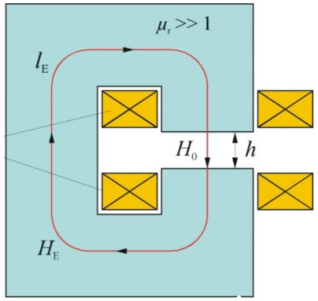
$$\boxed{B_x(x,y) = -\frac{\partial \Phi}{\partial x} = g'\,x\,y \quad \text{and} \quad B_y(x,y) = -\frac{\partial \Phi}{\partial y} = \frac{1}{2} g' \left(x^2 - y^2\right)}$$

Again: Basic Types of Magnets

Beam guidance:

dipole magnets

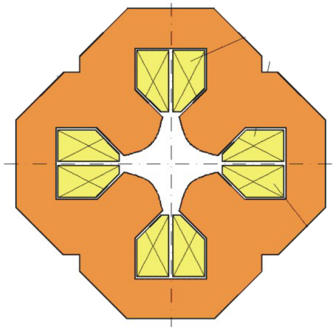
$$\frac{1}{\rho} = \kappa = \frac{q}{p} B_y, \quad B_y = \text{const.}$$



Beam focusing:

quadrupole magnets

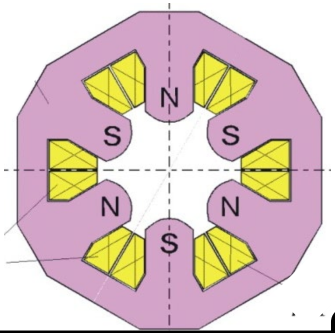
$$\frac{1}{f_x} = -kL, \quad B_x = -\frac{p}{q} k y, \quad B_y = -\frac{p}{q} k x$$



Chromatic correction:

sextupole magnets

$$B_x = \frac{p}{q} m x y, \quad B_y = \frac{p}{2q} m \left(x^2 - y^2 \right)$$



Properties defined by pole profiles in iron dominated magnets

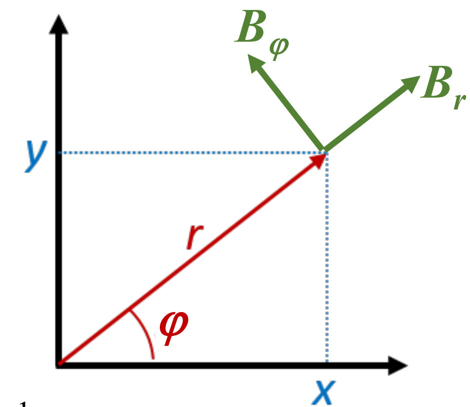
Multipole Expansion (r,φ)

Proceedings!

General treatment by multipole expansion, e.g. in polar coordinates:

$$B_r(r,\varphi) = B_0 \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \cdot \left(b_n \sin(n\varphi) - a_n \cos(n\varphi)\right)$$

$$B_\varphi(r,\varphi) = B_0 \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} \cdot \left(a_n \sin(n\varphi) + b_n \cos(n\varphi)\right)$$



Contribution of multipole n: $\left|B\right|_n = \sqrt{B_{r,n}^2 + B_{\varphi,n}^2} = B_0 \left(\frac{r}{r_0}\right)^{n-1} \cdot \sqrt{a_n^2 + b_n^2}$

Generally: 2*n* pole has symmetry $2\pi/n$, $|B|_n$ scales with r^{n-1}

$n = 1$: dipole magnet
$n = 2$: quadrupole magnet
$n = 3$: sextupole magnet
$n = 4$: octupole magnet
$n = 5$: decapole magnet

Classification:

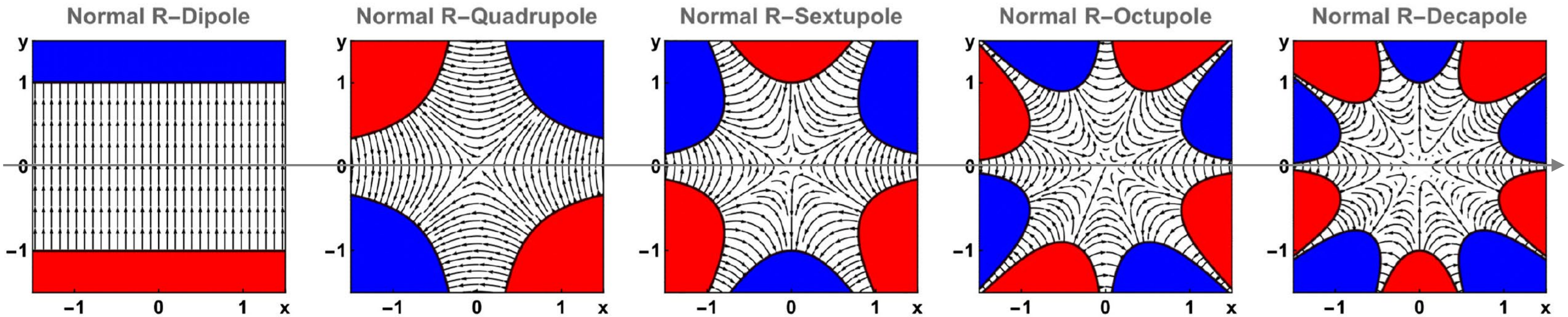
$b_n \neq 0$: "upright" magnets

$a_n \neq 0$: "skew" magnets, rotated by $\pi/2n$

Field Pattern of Different Multipoles

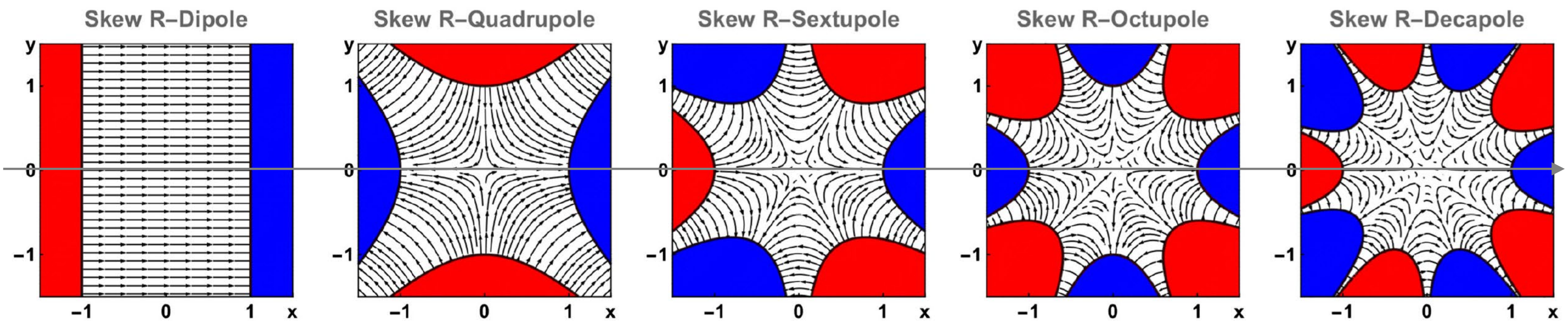
Normal or *upright* magnets

no vert. deflection in the horz. mid-plane!



Skew or *rotated* magnets

always vert. deflection in the horz. mid-plane!



Taken from Zolkin, Timofey, Phys.Rev.Accel.Beams **20** (2017) no.4, 043501

Multipole Expansion: (x, y)

Magnetic Potential:

Proceedings!

Dipole	$-\frac{q}{p}\cdot\Phi_1 = -\underline{\kappa}\,x + \kappa\,y$
Quadrupole	$-\frac{q}{p}\cdot\Phi_2 = 1/2\,\underline{k}\left(x^2 - y^2\right) - k\,x\,y$
Sextupole	$-\frac{q}{p}\cdot\Phi_3 = -1/6\,\underline{m}\left(x^3 - 3\,x y^2\right) + 1/6\,m\left(3\,x^2 y - y^3\right)$
Octupole	$-\frac{q}{p}\cdot\Phi_4 = -1/24\,\underline{r}\left(x^4 - 6\,x^2 y^2 + y^4\right) + 1/6\,r\left(x^3 y - x\,y^3\right)$

Upright Magnets:

Dipole	$\frac{q}{p}\vec{B}_1 = \kappa\hat{e}_y$	*"constant"*
Quadrupole	$\frac{q}{p}\vec{B}_2 = -k\,y\,\hat{e}_x - k\,x\,\hat{e}_y$	*"linear"*
Sextupole	$\frac{q}{p}\vec{B}_3 = m\,x\,y\,\hat{e}_x + \frac{1}{2}m\left(x^2 - y^2\right)\hat{e}_y$	*"cubic"*
Octupole	$\frac{q}{p}\vec{B}_4 = \frac{1}{6}r\left(3x^2 y - y^3\right)\hat{e}_x + \frac{1}{6}r\left(x^3 - 3\,x\,y^2\right)\hat{e}_y$	

Multipole strengths:

b_n: upright

a_n: skew

$$\underline{\kappa} = \frac{q\,B_0}{p}a_1, \qquad \kappa = \frac{q\,B_0}{p}b_1$$

$$\underline{k} = -\frac{q\,B_0}{p\,r_0}a_2, \qquad k = -\frac{q\,B_0}{p\,r_0}b_2$$

$$\underline{m} = \frac{q\,B_0}{p\,r_0^2}a_3, \qquad m = \frac{q\,B_0}{p\,r_0^2}b_3$$

$$\underline{r} = \frac{q\,B_0}{p\,r_0^3}a_4, \qquad r = \frac{q\,B_0}{p\,r_0^3}b_4$$

$$s_n = \frac{q}{p}\cdot\frac{\partial^{n-1}B_y(0,0)}{\partial x^{n-1}}$$

add. minus sign for n = 2!

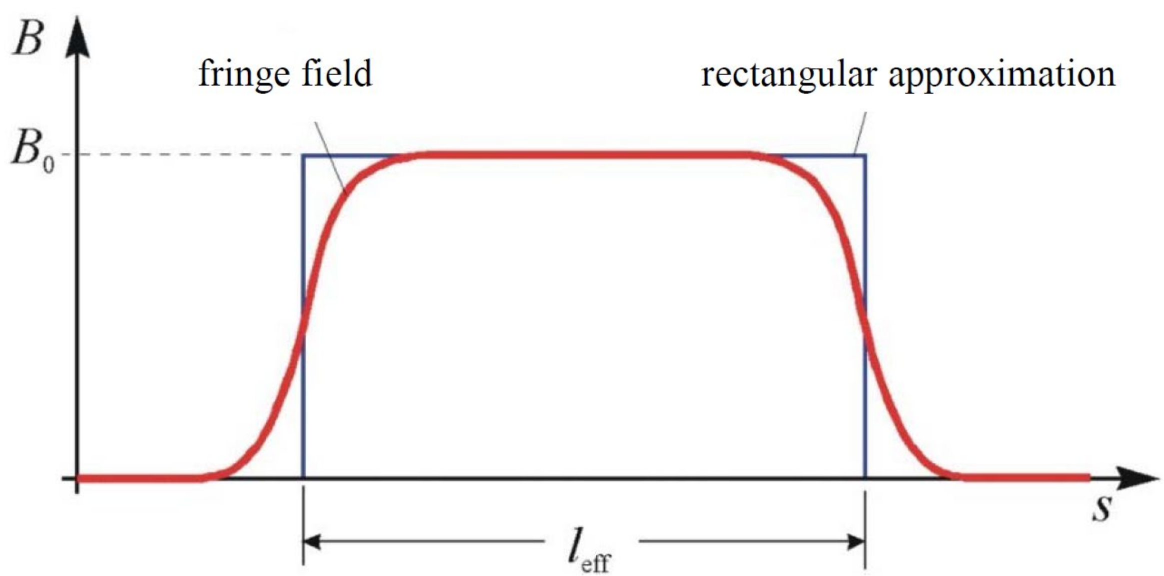
Effective Field Length

So far: assumption of constant field distribution along $\hat{e}_s$!

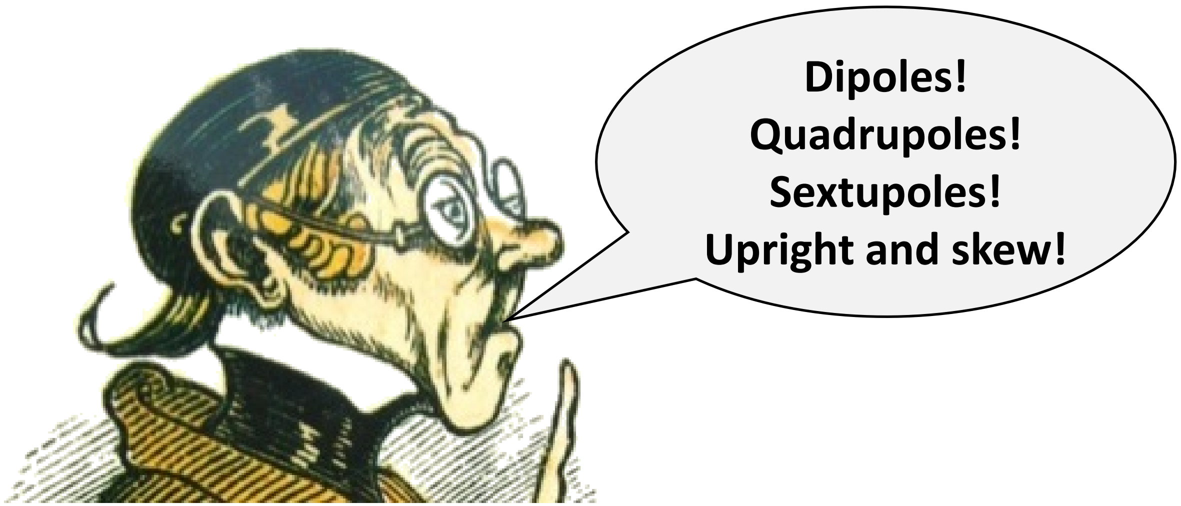
In reality: fringe fields at the end of the magnets

→ Definition of an effective field length l_{eff} via

$$\int_{-\infty}^{\infty} \vec{B} \cdot \mathrm{d}\vec{s} = \vec{B}_0 \cdot l_{eff}$$



End of 1st Lecture!



Questions?