Recap 1st Lecture

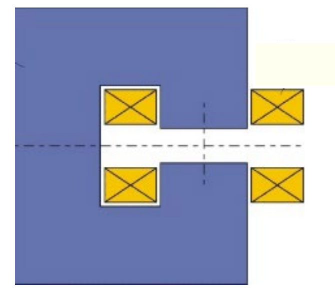
Magnetic Rigidity $B\rho$:

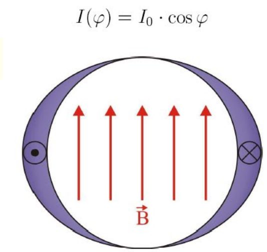
corresponding beam momentum $p = q\left(B\rho\right)$ defines "required" $B\rho$

Beam Guidance:

dipole magnets, 2 flat poles, $B_y = \text{const.}$

dipole strength $\kappa = \dfrac{1}{\rho} = \dfrac{q}{p} B_0, \quad \left[\kappa\right] = \text{m}^{-1}$ (curvature)



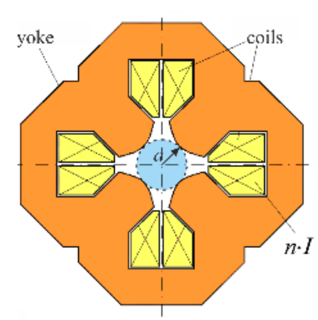


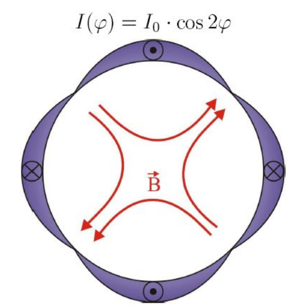
Beam Focusing:

quadrupole magnets, 4 hyperbolic poles, $B_y \sim x$

quadrupole strength $k = -\dfrac{q}{p} \dfrac{\partial B_y}{\partial x}, \quad \left[k\right] = \text{m}^{-2}$

focal length (thin magnets) $1/f_{x,y} = \mp kL$





Magnetic Multipoles:

2n poles, **"normal"** and "skew", multipole strength $s_n = \dfrac{q}{p} \cdot \dfrac{\partial^{n-1} B_y}{\partial x^{n-1}}, \quad \left[s_n\right] = \text{m}^{-n}$

rotational symmetry $2\pi/n$

(additional minus sign for $n = 2$)

3. Linear Beam Optics

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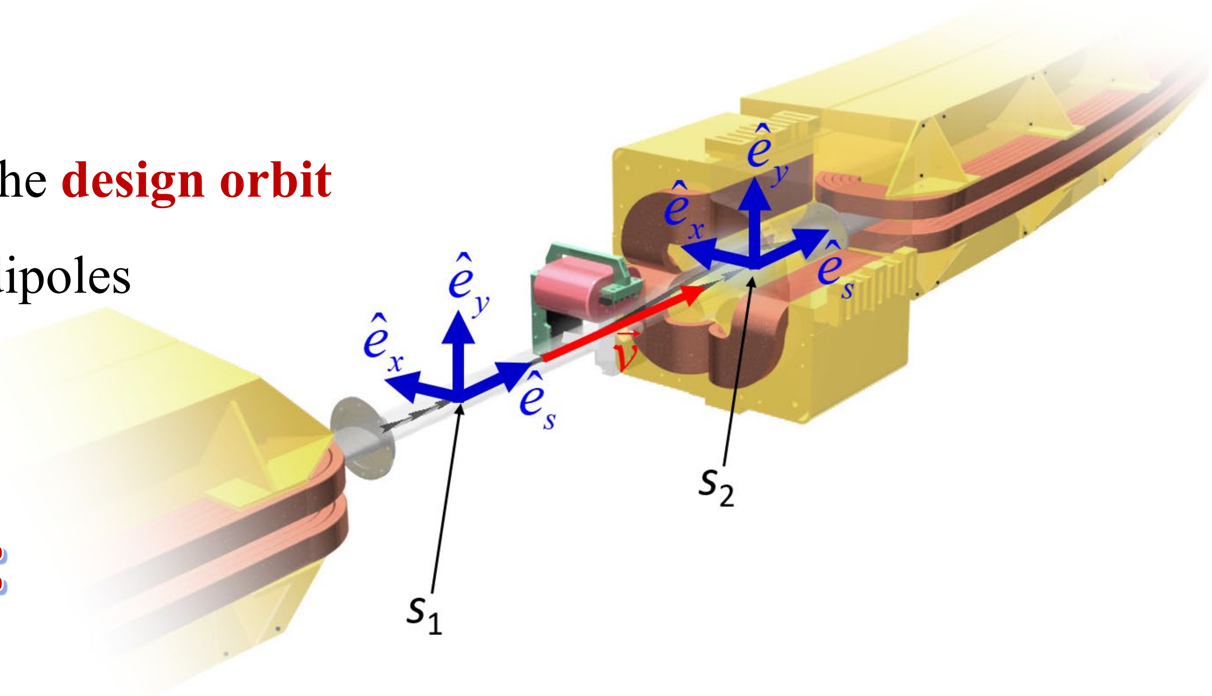
- Geometric Optics
- Equations of Motion
- Matrix Formalism
- Beams and Trace Space

Coordinate System

Reference path:

Path of a particle moving on the **design orbit**

- has nominal curvature in dipoles
- is centered in all magnets

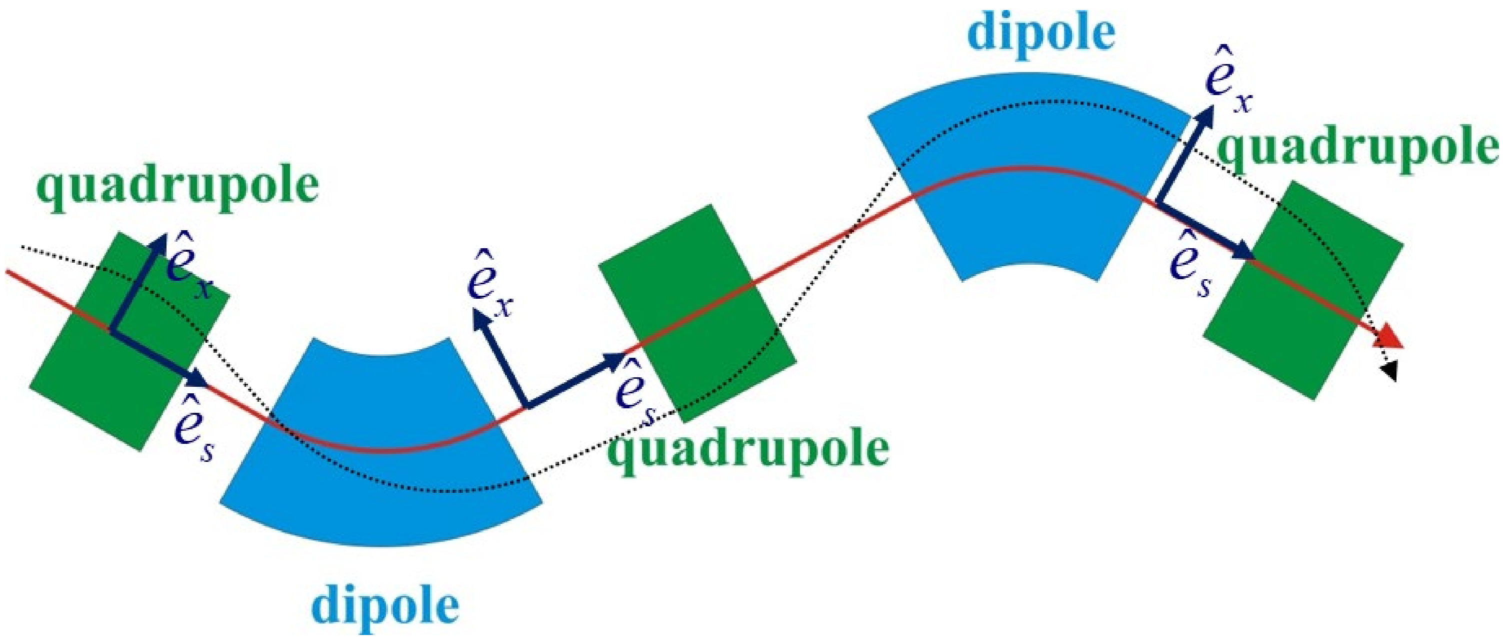


"Flat" Machines:

Use right-handed orthogonal coordinate system with:

- origin is fixed to reference orbit
- $\hat{e}_s$ is tangential to reference orbit and pointing in particle's speed direction
- $\hat{e}_y$ is perpendicular on plane defined by reference orbit and points vertically
- $\hat{e}_x = \hat{e}_y$ x $\hat{e}_s$ is in the plane of the reference orbit

Coordinate System

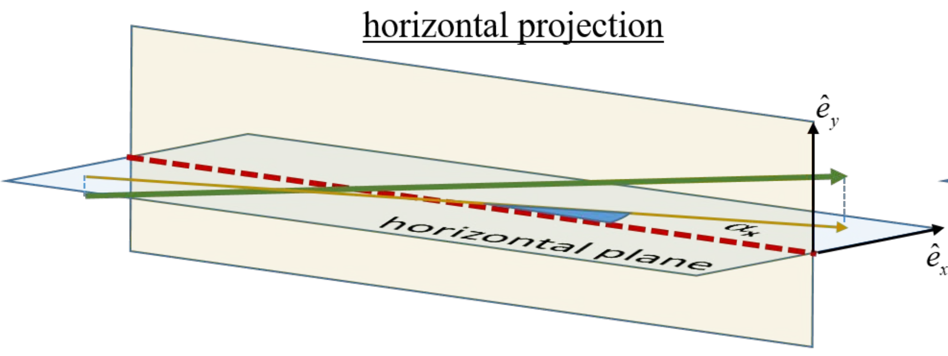


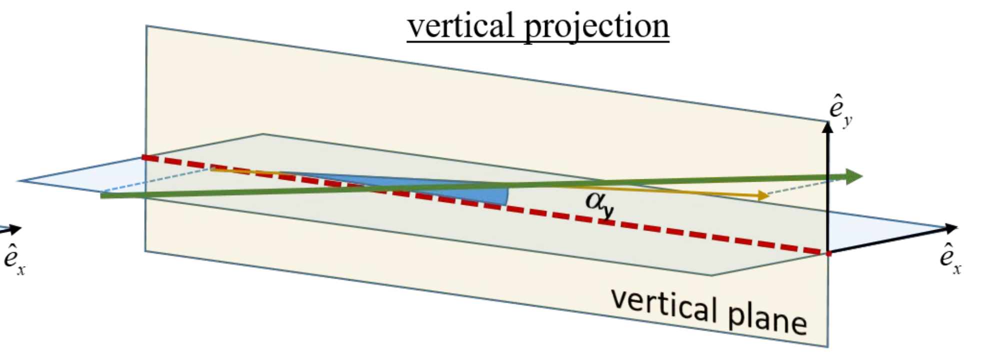
Coordinate system is “following” the design orbit:

- is **not** co-moving with reference particle (↔ Lorentz transformation!)
- is rotating in dipole magnets due to curved design orbit
- s-position is measured along the reference orbit from start position s_0

Transverse Displacements

Project particle's trajectory on the transverse planes:





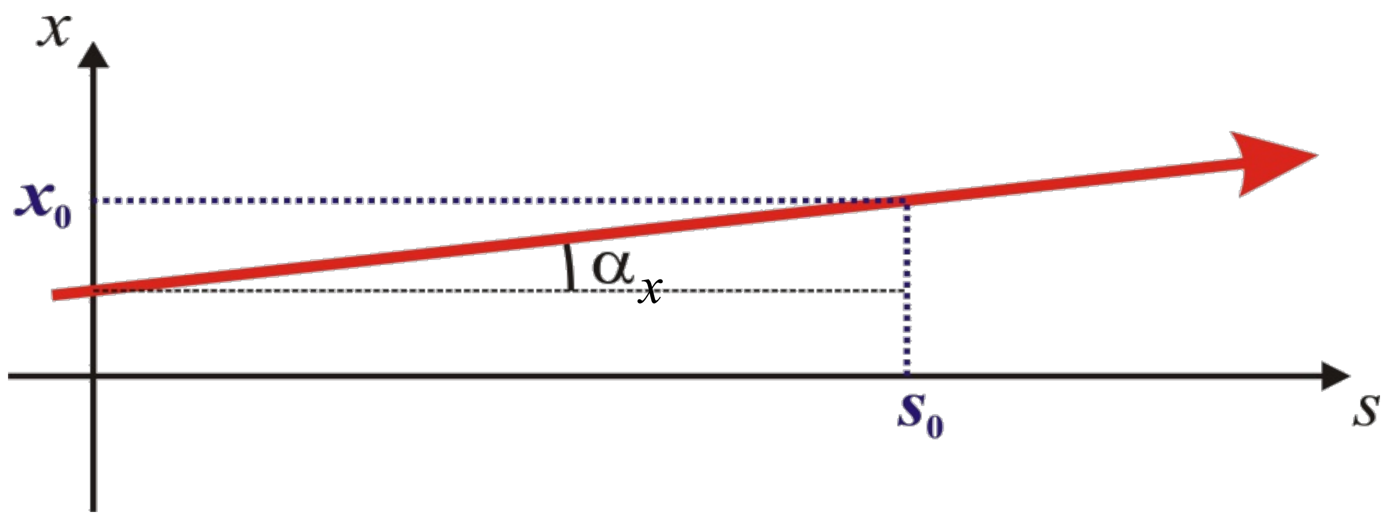
Definition of transverse displacements in our coordinate system "following" the design orbit:

- transverse position displacements x, y
- transverse angular displacements $x' = \tan\alpha_x$, $y' = \tan\alpha_y$

Characterization of particles by vectors $\vec{x}(s) = \begin{pmatrix} x \\ x' \end{pmatrix}$, $\vec{y}(s) = \begin{pmatrix} y \\ y' \end{pmatrix}$

Paraxial Optics

Paraxial "rays" ↔ small displacements $x \ll \rho$, $x' = \tan\alpha_x \approx \alpha_x$



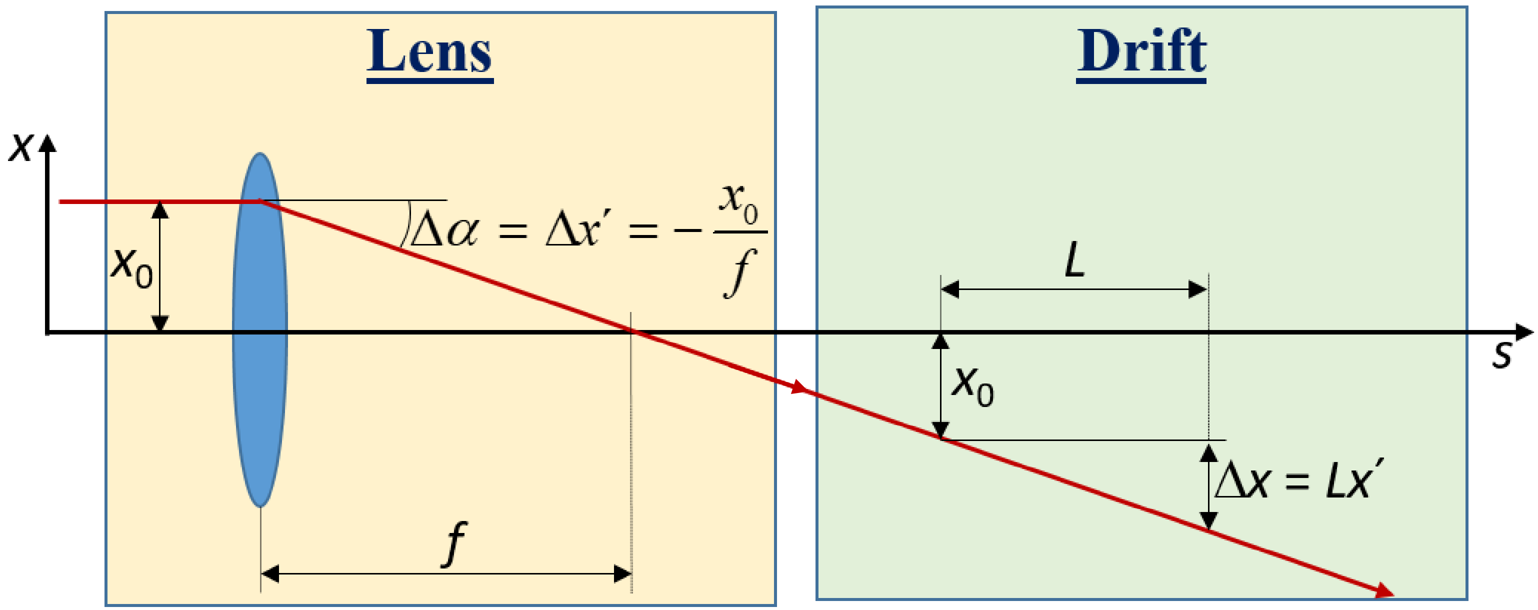
Impact of magnets in a very rough approximation:

dipole magnet: drift of length L_D

quadrupole magnet: thin lens with focal length $f_x = -\frac{1}{kL_Q}$, $f_y = \frac{1}{kL_Q}$

Geometric Optics

Change of particle displacements in horizontal / vertical plane



$$x = x_0$$
$$x' = x_0' - x_0 / f$$

$$x = x_0 + L \cdot x_0'$$
$$x' = x_0'$$

Linear transformation → can be described by matrices!

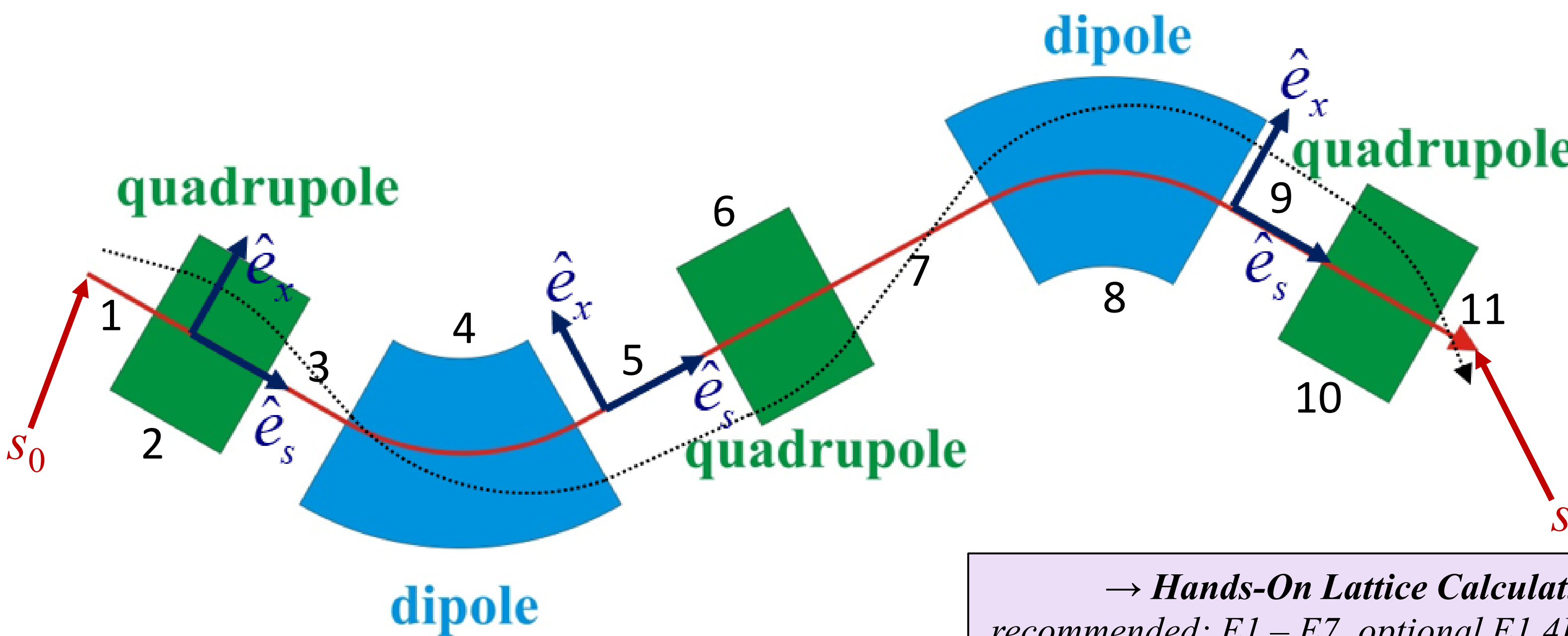
Geometric Optics

Planes (x, x') and (y, y') are called transverse trace spaces

Trace space	Position	Drift	Dipole	Quadrupole
horizontal	$\vec{x}(s) = \begin{pmatrix} x \\ x' \end{pmatrix}$	$\mathbf{M}_d = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$	$\mathbf{M}_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$	$\mathbf{M}_Q = \begin{pmatrix} 1 & 0 \\ -1/f_x & 1 \end{pmatrix}$
Trace Space	**Position**	**Drift**	**Dipole**	**Quadrupole**
vertical	$\vec{y}(s) = \begin{pmatrix} y \\ y' \end{pmatrix}$	$\mathbf{M}_d = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$	$\mathbf{M}_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$	$\mathbf{M}_Q = \begin{pmatrix} 1 & 0 \\ -1/f_y & 1 \end{pmatrix}$

Quadrupole: $f_x = -f_y$, Convention: $f > 0 \rightarrow$ focusing

Particle Trajectory

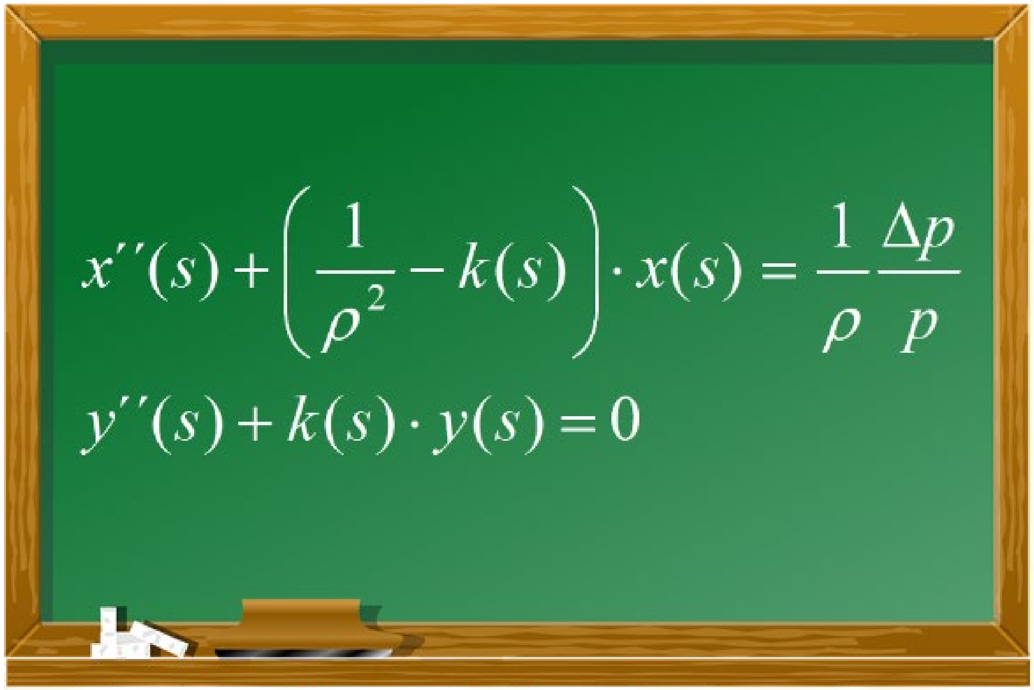


→ ***Hands-On Lattice Calculations***
recommended: E1 – E7, optional E1.4Py – E1.5Py

Calculation of single particle trajectories by matrix multiplication, e.g.:

$$\vec{x} = \underbrace{\mathbf{M}_{d11} \cdot \mathbf{M}_{Q10} \cdot \mathbf{M}_{d9} \cdot \mathbf{M}_{D8} \cdot \mathbf{M}_{d7} \cdot \mathbf{M}_{Q6} \cdot \mathbf{M}_{d5} \cdot \mathbf{M}_{D4} \cdot \mathbf{M}_{d3} \cdot \mathbf{M}_{Q2} \cdot \mathbf{M}_{d1}}_{= \text{Transfer Matrix } \mathbf{M}} \cdot \vec{x}_0$$

More precise description → matrices have to be derived from equations of motion!



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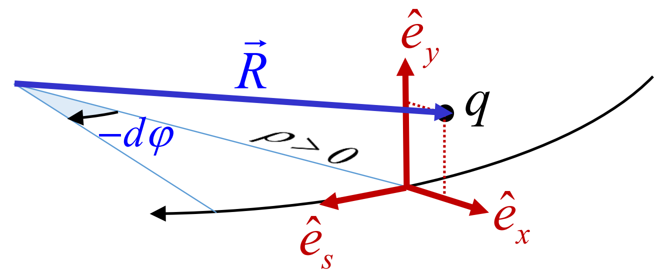
Taken from https:/www.wikipedia.org/wiki/Datei:Lämpel.jpg

“Derivation” of the equations of motion!

Equations of Motion

Charge q moving on a curved orbit

Important restrictions/approximations:



- constant longitudinal velocity $\rightarrow \dot{s} = v_s = \text{const.}$
- "slowly" varying curvature $\rightarrow \rho \approx \text{const.}$
- design orbit is in (s, x) plane $\rightarrow \hat{e}_y = \text{const.}$
- paraxial optics ($|x| << |\rho|$) $\rightarrow$ no coupling to long. motion ($R_s = R_s' = 0$)

Use polar coordinates $\hat{e}_x = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$, $\hat{e}_\varphi = -\hat{e}_s = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$ and $\mathrm{d}\varphi = -\frac{1}{\rho}\mathrm{d}s$

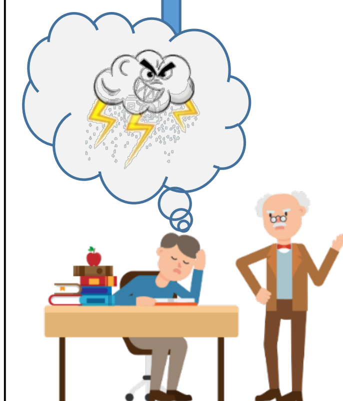
Transverse position vector of moving charge q:

$$\rightarrow \quad \vec{R}(s) = [\rho + x(s)] \cdot \hat{e}_x(s) + y(s) \cdot \hat{e}_y(s) \quad \text{and} \quad \hat{e}_x' = +\frac{1}{\rho}\hat{e}_s, \quad \hat{e}_s' = -\frac{1}{\rho}\hat{e}_x$$

Equations of Motion

Continue with $\gamma_r m_0 \ddot{\vec{R}} = q\left(\vec{v} \times \vec{B}\right)$ using following approximations:

- small displacements $x, y \ll \rho$, $v_s = \text{const.}$ **(paraxial optics)**
- only dipole and quadrupole magnets **(linear field changes)**
- design orbit lies in a plane **(flat accelerator)**
- no coupling of motion in hor. and vert. plane **(upright magnets)**
- small momentum deviations **(quasi monochromatic beam)**
- in general: **no non-linear terms (linear beam optics)**



Proceedings!

EQM:

$$x''(s) + \left(\frac{1}{\rho^2(s)} - k(s)\right) \cdot x(s) = \frac{1}{\rho(s)} \frac{\Delta p}{p_0}$$

$$y''(s) + k(s) \cdot y(s) = 0$$

Geometric Focusing

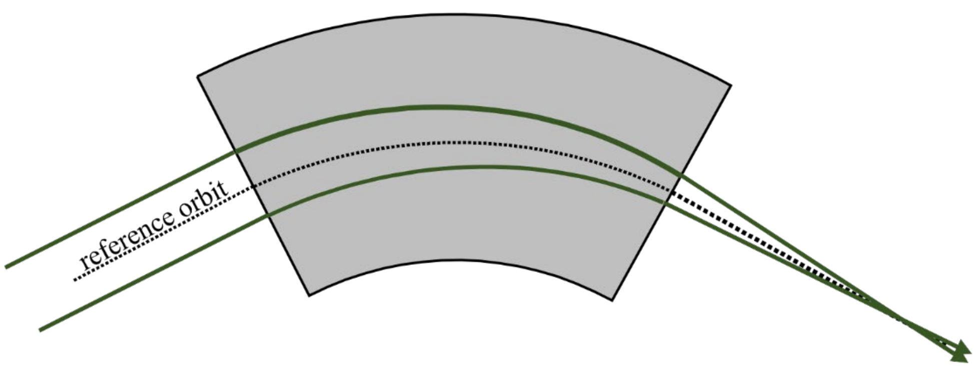
$$x''(s) + \left(\frac{1}{\rho^2(s)} - k(s)\right)\cdot x(s) = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}$$

$$y''(s) + k(s)\cdot y(s) = 0$$

geometric focusing

quadrupole focusing

→ **Additional focusing of a sector dipole magnet:**



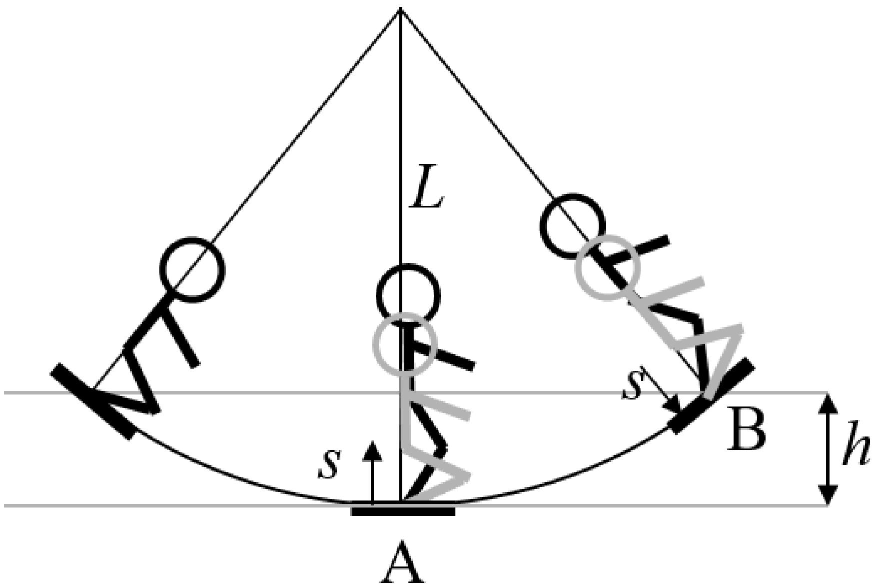
Transversely displaced beam is travelling on a longer/shorter path in the dipole and thus bent more/less!

Geometric focusing (= weak!) in the horizontal plane!

Optical Resonances

Compare:

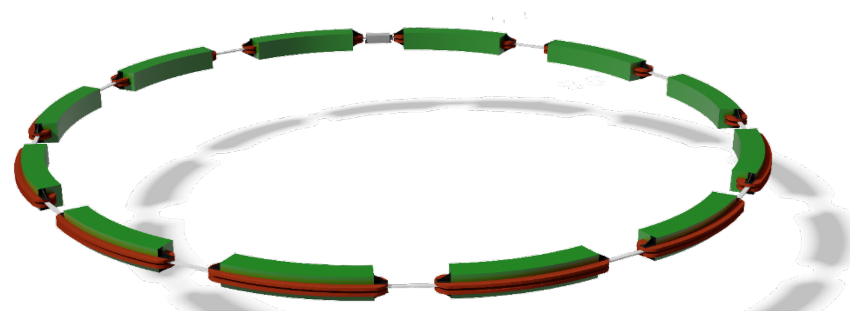
Children's swing



$$\ddot{\varphi}(t) + \omega^2(t) \cdot \varphi(t) = 0$$

$$\omega(t) = \sqrt{\frac{g}{L(t)}}$$

Circular accelerator



$$y''(s) + k(s) \cdot y(s) = 0$$

$$\Omega(s) = \sqrt{k(s)}$$

periodic excitation possible!

→ more later (circular accelerators) …

General Solution Approach

For every individual linear element the equations of motions are linear differential equations with constant coefficients!

$$x''(s) + \left(\frac{1}{\rho^2} - k\right) \cdot x(s) = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

$$y''(s) + k \cdot y(s) = 0$$

a) Drift: $1/\rho(\mathrm{s}) = k(\mathrm{s}) = 0$

b) Dipole: $\rho(\mathrm{s}) = \rho = \text{const.},\ k(\mathrm{s}) = 0$

c) Quadrupole: $1/\rho(\mathrm{s}) = 0,\ k(\mathrm{s}) = k = \text{const.}$

→ **Stepwise solution for individual multipoles.**

Concentrate first on mono-energetic beam with $\Delta p = 0$!

General Solution Approach

Universal notation of the equation of motion

$$u''(s) + K(s)\cdot u(s) = 0$$

with:

- magnet parameter $K(s) = \begin{cases} 1/\rho^2(s) - k(s) & \text{motion in the horizontal plane} \\ k(s) & \text{motion in the vertical plane} \end{cases}$

- position displacement $u(s) = \begin{cases} x(s) & \text{motion in the horizontal plane} \\ y(s) & \text{motion in the vertical plane} \end{cases}$

General solution for constant *K* and initial values u_0 and u_0':

$$u(s) = u_0 \cdot \cos\left(\sqrt{K}\cdot s\right) + \frac{u_0'}{\sqrt{K}} \cdot \sin\left(\sqrt{K}\cdot s\right)$$

$$u'(s) = -u_0\sqrt{K}\cdot \sin\left(\sqrt{K}\cdot s\right) + u_0' \cdot \cos\left(\sqrt{K}\cdot s\right)$$

$$u_0 = u(s=0)$$

$$u_0' = u'(s=0)$$

General Solution Approach

Implication of a negative sign of *K* (*K* < 0):

$$K < 0 \quad \rightarrow \quad \sqrt{K} = i\sqrt{|K|}$$

and with $\cos(i\alpha) = \cosh\alpha, \quad \sin(i\alpha) = i \cdot \sinh\alpha$

$$u(s) = u_0 \cdot \cosh\left(\sqrt{|K|} \cdot s\right) + \frac{u_0'}{\sqrt{|K|}} \cdot \sinh\left(\sqrt{|K|} \cdot s\right)$$

$$u'(s) = u_0 \sqrt{|K|} \cdot \sinh\left(\sqrt{|K|} \cdot s\right) + u_0' \cdot \cosh\left(\sqrt{|K|} \cdot s\right)$$

We can use functions *C*(s) and *S*(s) and generally write

$$\begin{aligned} u(s) &= C(s) \cdot u_0 + S(s) \cdot u_0' \\ u'(s) &= C'(s) \cdot u_0 + S'(s) \cdot u_0' \end{aligned} \quad \rightarrow \quad \begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$

Matrix Formalism

→ Definition of a transfer matrix **M**

$$\mathbf{M}(s) = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \mathbf{M}(s) \cdot \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$

For a sequence (K_1, K_2, K_3, … K_n) of elements the procedure yields

$$\begin{pmatrix} u_n \\ u_n' \end{pmatrix} = \mathbf{M}_n \cdot \mathbf{M}_{n-1} \cdot \mathbf{M}_{n-2} \cdots \mathbf{M}_2 \cdot \mathbf{M}_1 \cdot \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$

$$\mathbf{M}_{s_0 \to s_n} = \prod_i \mathbf{M}_i$$

→ *Hands-On Lattice Calculations*
optional: E1.1Py – E1.3Py

The transfer Matrix $\mathbf{M}_i$ characterizes the individual element with its specific K_i!

Important feature of the transfer matrix:

→ *Hands-On Lattice Calculations*
recommended: E13

$$\det(\mathbf{M}) = \det \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = CS' - SC' = 1$$

← *Proceedings*

4x4 Matrix Formalism

Characterize particle's state by a vector built from transverse displacements:

$$\vec{X} = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} \text{horizontal displacement} \\ \text{horizontal angular displacement} \\ \text{vertical displacement} \\ \text{vertical angular displacement} \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{hor. trace space} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{vert. trace space} \end{matrix}$$

Use the matrix formalism to describe particles trajectories: $\vec{X} = \mathbf{M} \cdot \vec{X}_0$

In case of upright magnets there will be no coupling of the transverse planes:

$$\mathbf{M} = \begin{pmatrix} r_{11} & r_{12} & 0 & 0 \\ r_{21} & r_{22} & 0 & 0 \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & r_{43} & r_{44} \end{pmatrix} = \begin{pmatrix} \langle x|x_0\rangle & \langle x|x_0'\rangle & 0 & 0 \\ \langle x'|x_0\rangle & \langle x'|x_0'\rangle & 0 & 0 \\ 0 & 0 & \langle y|y_0\rangle & \langle y|y_0'\rangle \\ 0 & 0 & \langle y'|y_0\rangle & \langle y'|y_0'\rangle \end{pmatrix}$$

Next, we have to derive the matrices for drift, dipole and quadrupole magnets.

Drift Space

Equations of motion for a free drift with length L:

$$K(s) = 0 \quad \rightarrow \quad \begin{aligned} x''(s) &= 0 \\ y''(s) &= 0 \end{aligned}$$

We thus get for known starting conditions x_0, x_0', y_0, y_0':

$$\begin{aligned} x &= x_0 + L \cdot x_0' \\ y &= y_0 + L \cdot y_0' \end{aligned} \quad \text{and} \quad \begin{aligned} x' &= x_0' \\ y &= y_0' \end{aligned}$$

yielding the transfer matrix of a drift with length L:

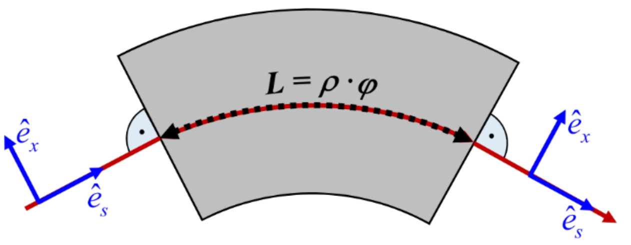
$$\mathbf{M}_{drift} = \begin{pmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Sector Dipole Magnet

Equations of motion for a dipole magnet with length L:

$$K_x(s) = \frac{1}{\rho^2} \quad \rightarrow \quad x(L) = x_0 \cdot \cos\left(\frac{L}{\rho}\right) + \frac{x_0'}{\rho} \cdot \sin\left(\frac{L}{\rho}\right)$$

$$K_y(s) = 0 \qquad y(L) = y_0 + y_0' \cdot L$$



yielding the transfer matrix of a dipole magnet with length L:

$$\mathbf{M}_{dipole} = \begin{pmatrix} \cos\varphi & \rho\sin\varphi & 0 & 0 \\ -1/\rho \cdot \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & \rho\varphi \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

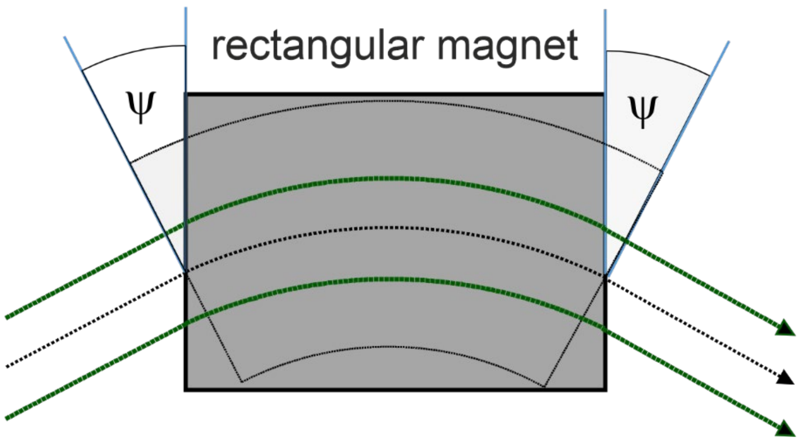
with $\varphi = \frac{L}{\rho}$

drift!

geometric focusing!

Rectangular Dipole Magnet

Beam is additionally affected by the fringe fields (entrance/exit)!



→ **Proceedings:**

- additional weak horizontal defocusing!
- additional weak vertical focusing!

Impact on orbit for arbitrary entrance/exit angle $\Psi_{1,2}$ is described by additional entrance/exit matrix $\mathbf{M}_{\Psi}$

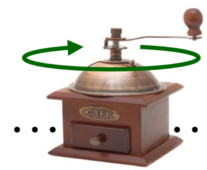
Treatment of beam entrance / exit:

$$\mathbf{M}_{rect} = \mathbf{M}_{\psi_2} \cdot \mathbf{M}_{dipole} \cdot \mathbf{M}_{\psi_1}$$

$$\mathbf{M}_{\psi} = \begin{pmatrix} \begin{matrix} 1 & 0 \\ \frac{\tan\psi}{\rho} & 1 \end{matrix} & \cdots \\ \vdots & \begin{matrix} 1 & 0 \\ -\frac{\tan\psi}{\rho} & 1 \end{matrix} \end{pmatrix}$$

This effect is called edge focusing!

Rectangular Dipole Magnet

For $\Psi_1 = \Psi_2$ (symmetric crossing) we obtain the following transfer matrix …  …

$$\mathbf{M}_{rect} = \begin{pmatrix} 1 & \rho\sin\varphi & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\varphi\tan\dfrac{\varphi}{2} & \rho\varphi \\ 0 & 0 & \dfrac{\varphi}{\rho}\tan^2\dfrac{\varphi}{2} - 2\varphi\tan\dfrac{\varphi}{2} & 1-\varphi\tan\dfrac{\varphi}{2} \end{pmatrix} \approx \begin{pmatrix} 1 & \rho\varphi & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varphi & \rho\varphi \\ 0 & 0 & -\dfrac{\sin^2\varphi}{\rho\varphi} & \cos\varphi \end{pmatrix}$$

drift!

vertical focusing!

A rectangular dipole magnet is therefore focusing in the vertical plane.
It acts like a drift space in the horizontal plane!

Quadrupole Magnet

Equations of motion for a quadrupole magnet with length L:

$$K_x(s) = -k \qquad x(L) = x_0 \cdot \cos\left(\sqrt{-k}L\right) + \frac{x_0'}{\sqrt{-k}} \cdot \sin\left(\sqrt{-k}L\right)$$

$$K_y(s) = +k \quad \rightarrow \quad y(s) = y_0 \cdot \cos\left(\sqrt{k}L\right) + \frac{y_0'}{\sqrt{k}} \cdot \sin\left(\sqrt{k}L\right)$$

According to the definitions made on page 20, we name

- a horizontal focusing quadrupole ($k < 0$) a **focusing quadrupole QF**
- a vertical focusing quadrupole ($k > 0$) a **defocusing quadrupole QD**

We put $\Omega = \sqrt{|k|} \cdot L$ and the focal length $1/f = |k| L$ and get

Quadrupole Magnet

→ ***Hands-On Lattice Calculations:*** *optional E1.1Ph*

QF ($k < 0$):

$$\mathbf{M}_{QF} = \begin{pmatrix} \cos\Omega & \frac{1}{\sqrt{|k|}}\sin\Omega & 0 & 0 \\ -\sqrt{|k|}\sin\Omega & \cos\Omega & 0 & 0 \\ 0 & 0 & \cosh\Omega & \frac{1}{\sqrt{|k|}}\sinh\Omega \\ 0 & 0 & \sqrt{|k|}\sinh\Omega & \cosh\Omega \end{pmatrix} \xrightarrow{L\to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{f} & 1 \end{pmatrix}$$

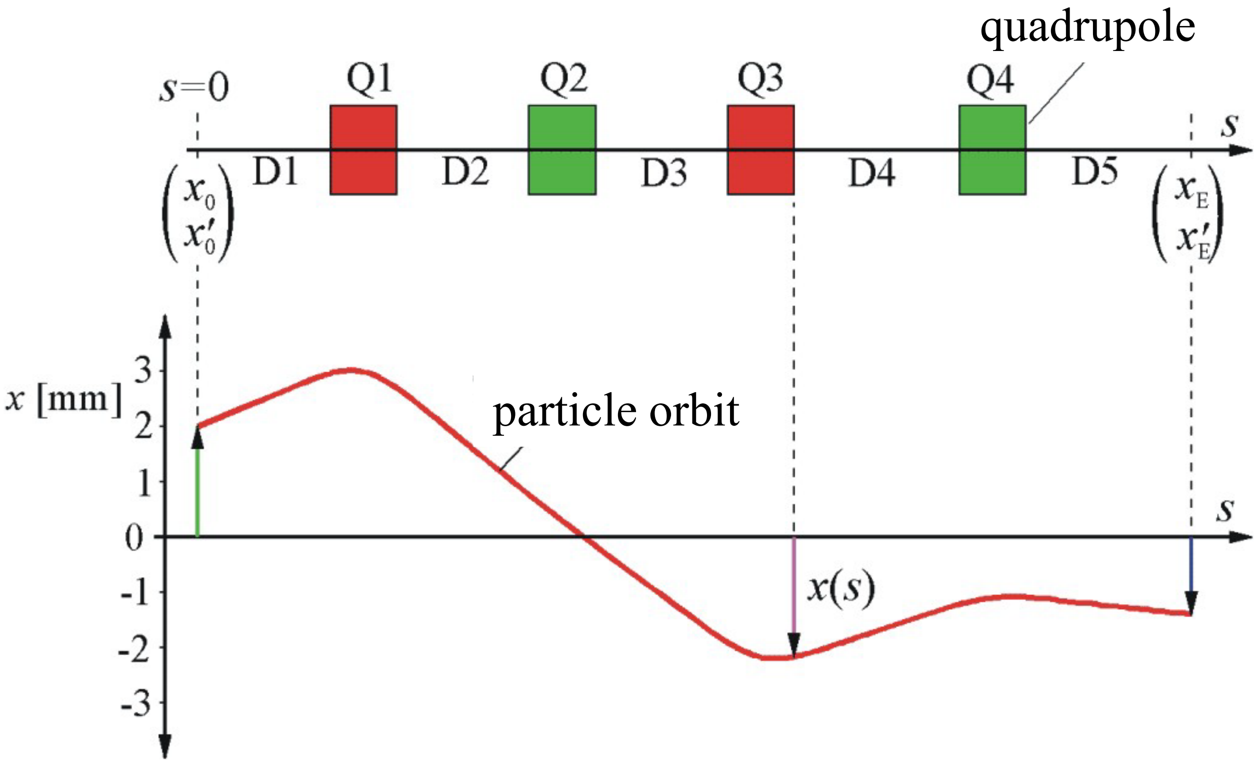
„thin lens“ approximation

QD ($k > 0$):

$$\mathbf{M}_{QD} = \begin{pmatrix} \cosh\Omega & \frac{1}{\sqrt{|k|}}\sinh\Omega & 0 & 0 \\ \sqrt{|k|}\sinh\Omega & \cosh\Omega & 0 & 0 \\ 0 & 0 & \cos\Omega & \frac{1}{\sqrt{|k|}}\sin\Omega \\ 0 & 0 & -\sqrt{|k|}\sin\Omega & \cos\Omega \end{pmatrix} \xrightarrow{L\to 0} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{f} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{f} & 1 \end{pmatrix}$$

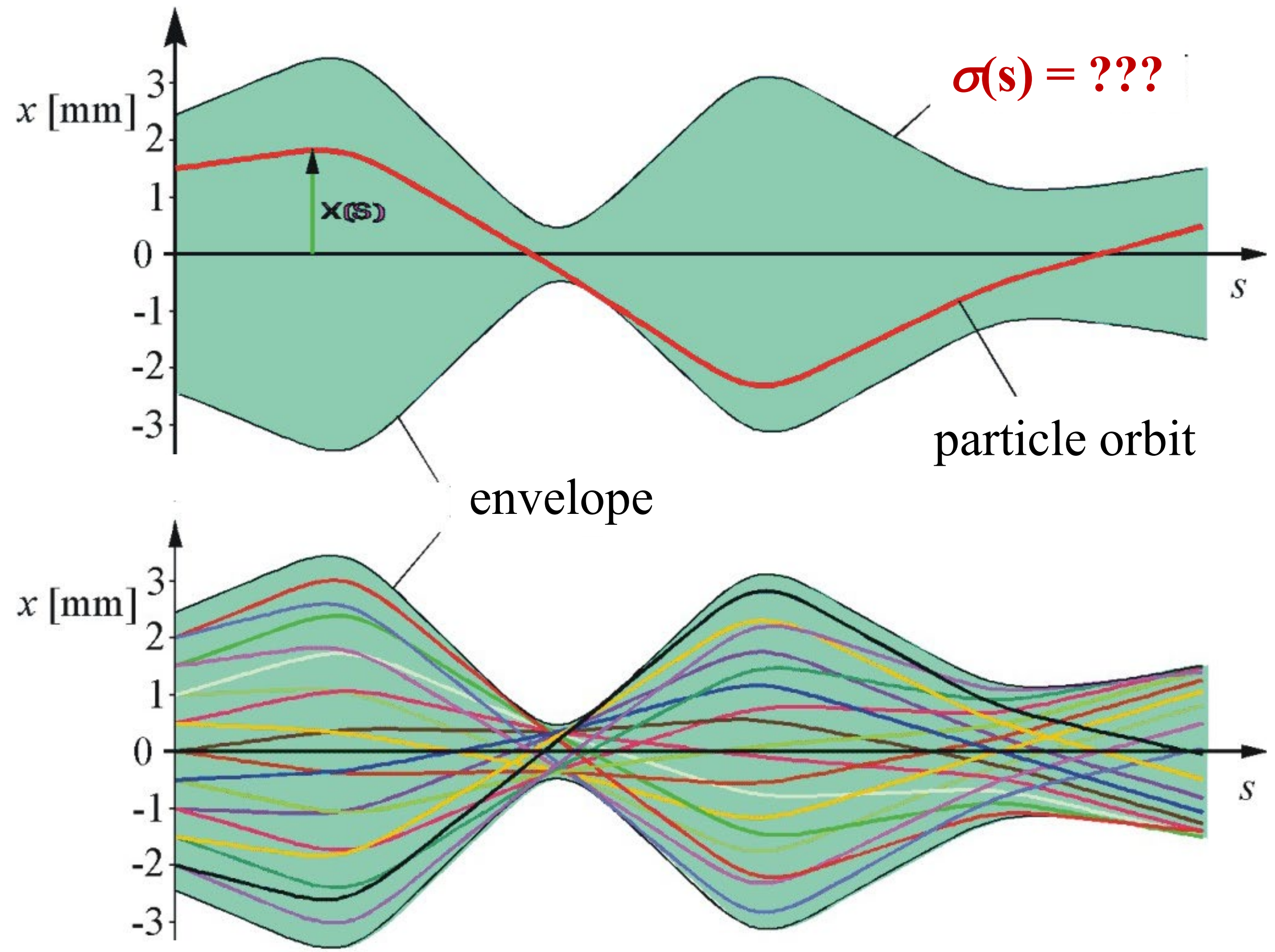
Trajectories

With the derived matrixes particle trajectories may be calculated for any given arbitrary beam transport line by cutting this beam line into smaller uniform pieces so that k = const. and ρ = const. in each of these pieces:



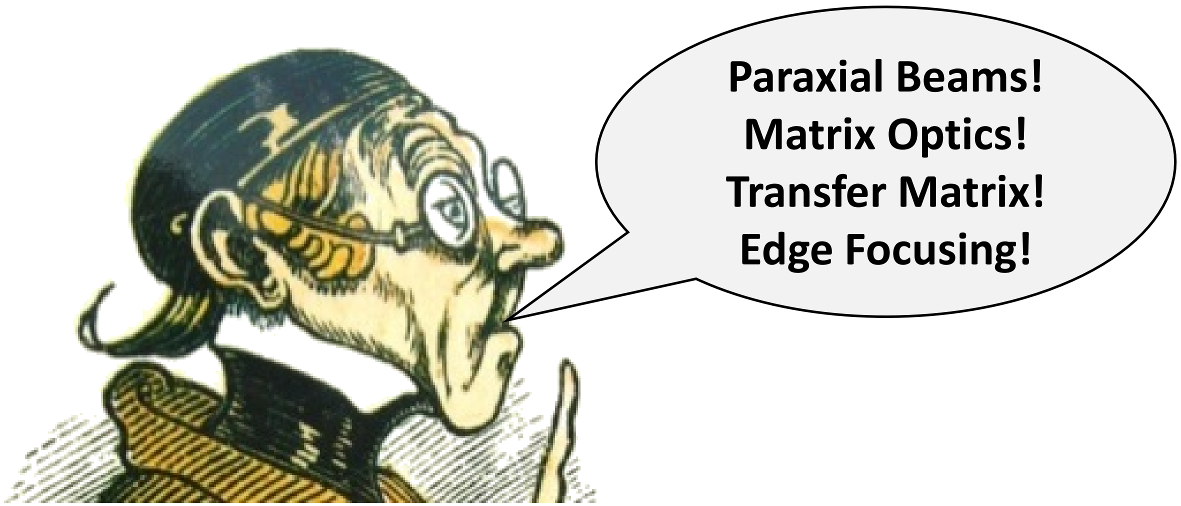
$$\vec{x}_E = \mathbf{M}_{D5} \cdot \mathbf{M}_{Q4} \cdot \mathbf{M}_{D4} \cdot \mathbf{M}_{Q3} \cdot \mathbf{M}_{D3} \cdot \mathbf{M}_{Q2} \cdot \mathbf{M}_{D2} \cdot \mathbf{M}_{Q1} \cdot \mathbf{M}_{D1} \cdot \vec{x}_0$$

Beam Envelope



How can we compute the evolution of the beam's envelope?

End of 2nd Lecture!



Questions?