Recap 2nd Lecture

Paraxial Optics: trajectory described by offsets (x, x', y, y') from design orbit,
important approximation: **displacements $|x|, |y| \ll \rho$**

Geometric Optics: each element i is represented by a transfer matrix $\mathbf{M}_i$,
→ treatment of **linear elements only**

Matrices (simple 2x2 approx.): dipole and drift $\mathbf{M}_D = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$, quadrupole $\mathbf{M}_Q = \begin{pmatrix} 1 & 0 \\ -1/f_{x,y} & 1 \end{pmatrix}$

Matrix Formalism: **trajectory (single particle's orbit)** from $\vec{x} = \prod_{i=1}^{n} \mathbf{M}_i \cdot \vec{x}_0$

Equations of Motion:

$$x''(s) + \left\{ \frac{1}{\rho^2(s)} - k(s) \right\} x(s) = \frac{1}{\rho(s)} \frac{\Delta p}{p_0}$$ ← **linearization of all terms**

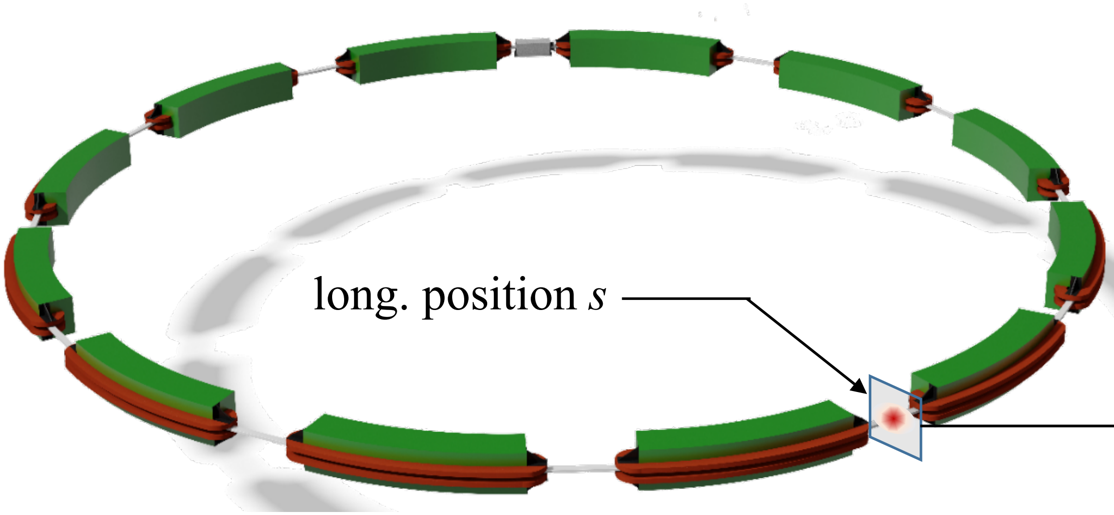
$$y''(s) + k(s)\, y(s) = 0$$ ← **"flat" accelerator**

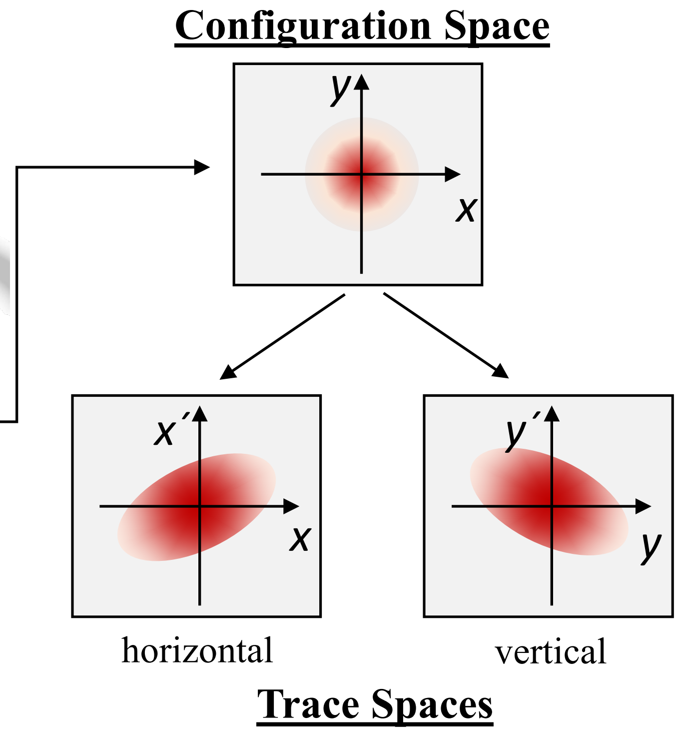
Matrices from EQM: build from solution of EQM for individual elements
→ piecewise solution of EQM by applying the matrix formalism

Dipole Focusing: horizontal **geometric focusing** in sector dipole magnets
vertical **edge focusing** in rectangular dipole magnets

weak!

Configuration and Trace Space





Trace Space (x, x') ↔ Phase Space (x, p_x)

$$p_x = \beta_r \gamma_r m_0 c \cdot x'$$

Famous theorem of Liouville:

The phase space distribution function describing the density of possible states around a phase space point is invariant under conservative forces"!

→ The phase space area covered by the beam remains constant!

Beam and Trace Space

Beam = statistical set of points in trace space!

→ ***Hands-On Lattice Calculations***
recommended: E8 – E11

Consider 2D trace space, u is representing x or y:

→ each particle i is represented by a point (u_i, u_i') in trace space

Choose origin of the coordinate system (u, u') at the barycenter of the points:

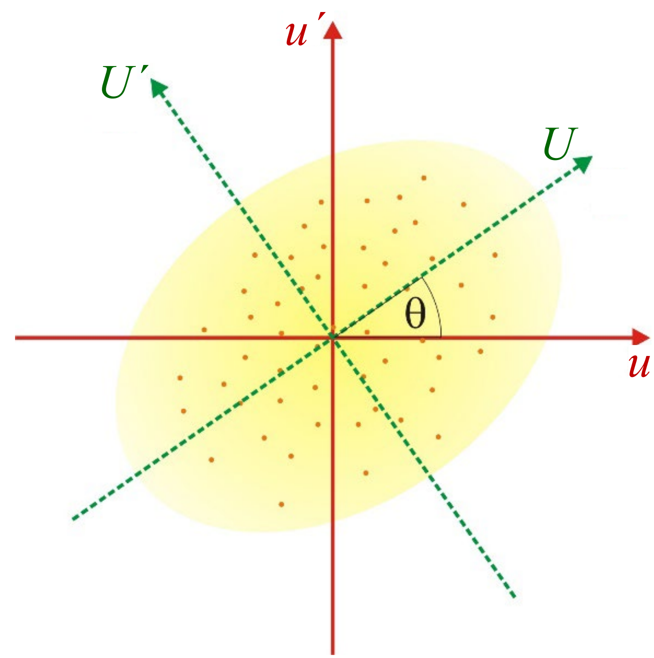
$$\bar{u} = \frac{1}{N}\sum_{i=1}^{N} u_i = 0, \qquad \overline{u'} = \frac{1}{N}\sum_{i=1}^{N} u_i' = 0$$

Interested in variances (rms spread)

$$\sigma_u^{\,2} = \frac{1}{N}\sum_{i=1}^{N} u_i^{\,2}, \qquad \sigma_{u'}^{\,2} = \frac{1}{N}\sum_{i=1}^{N} u_i'^{\,2}$$

System (U, U') which is rotated by θ:

$$U_i = u_i \cdot \cos\theta + u_i' \cdot \sin\theta$$
$$U_i' = -u_i \cdot \sin\theta + u_i' \cdot \cos\theta$$



Beam and Trace Space

Proceedings!

Variances in rotated system (U, U'):

$$\sigma_U^{\,2} = \frac{1}{N}\sum_{i=1}^{N} U_i^{\,2} = \overline{u^2}\cos^2\theta + \overline{u'^2}\sin^2\theta + \overline{uu'}\sin 2\theta$$

$$\sigma_{U'}^{\,2} = \frac{1}{N}\sum_{i=1}^{N} U_i'^{\,2} = \overline{u^2}\sin^2\theta + \overline{u'^2}\cos^2\theta - \overline{uu'}\sin 2\theta$$

using $\sin 2\theta = 2\sin\theta\cos\theta$

are minimized / maximized with respect to the angle θ when

$$\frac{\partial \sigma_U^{\,2}}{\partial\theta} = \frac{\partial \sigma_{U'}^{\,2}}{\partial\theta} = 0 \quad \rightarrow \quad \tan 2\theta = \frac{2\overline{uu'}}{\overline{u^2} - \overline{u'^2}}$$

and using again $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

which yields (from $\sigma_U^{\,2} + \sigma_{U'}^{\,2} = \overline{u^2} + \overline{u'^2}$ and $\sigma_U^{\,2} - \sigma_{U'}^{\,2} = \dfrac{2\overline{uu'}}{\sin 2\theta}$):

$$\sigma_U^{\,2} = \frac{1}{2}\left(\overline{u^2} + \overline{u'^2} + \frac{2\overline{uu'}}{\sin 2\theta}\right)$$

$$\sigma_{U'}^{\,2} = \frac{1}{2}\left(\overline{u^2} + \overline{u'^2} - \frac{2\overline{uu'}}{\sin 2\theta}\right)$$

and with $\dfrac{1}{\sin^2\alpha} = 1 + \dfrac{1}{\tan^2\alpha}$ $\rightarrow\rightarrow\rightarrow$

Beam Emittance

Emittance ε $\leftrightarrow$ defined by spread of the distribution

$$\varepsilon_u = \sigma_U \cdot \sigma_{U'} = \sqrt{\overline{u^2} \cdot \overline{u'^2} - \left(\overline{uu'}\right)^2} \qquad [\varepsilon_u] = \mathrm{m} \cdot \mathrm{rad}$$

It is important to note that this is a statistical definition of ε!

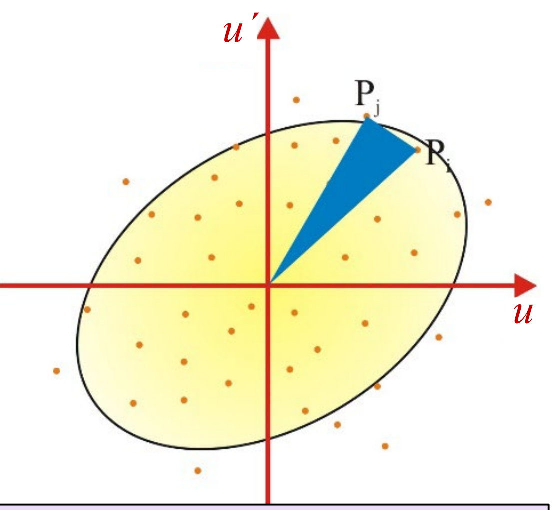
More general, ε_u will be defined over the area $\varepsilon_u = \iint \mathrm{d}u'\mathrm{d}u$!

The emittance can be considered as a statistical mean area:

$$\varepsilon_u = \frac{1}{N}\sqrt{\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(u_i u_j' - u_j u_i'\right)^2} = \frac{1}{N}\sqrt{2\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}^2}$$

(remember $2A_\Delta = \left|\vec{a} \times \vec{b}\right| \overset{a_3=b_3=0}{=} \left|a_1 b_2 - a_2 b_1\right|$)

where A_{ij} is the area of the triangle $0P_iP_j$ and ε is a measure of the spread of the points around their barycenter.



→ ***Hands-On Lattice Calculations***
optional: E2.2Ph

Beam Emittance

The area A of the envelope ellipse is just π times the emittance ε

$$A = \pi a b = \pi \sigma_U \sigma_{U'} = \pi \varepsilon_u$$

and its equation with respect to the rotated coordinates X and X'

$$\frac{U^2}{\sigma_U{}^2} + \frac{U'^2}{\sigma_{U'}{}^2} = \frac{U^2}{a^2} + \frac{U'^2}{b^2} = 1 \qquad \rightarrow \qquad U^2 \sigma_{U'}{}^2 + U'^2 \sigma_U{}^2 = \varepsilon_u^2$$

where a and b are the two semi-axes of the envelope ellipse.

By an inverse rotation of angle $-\theta$ in trace space we obtain the ellipse equation in trace space coordinates u and u':

$$\varepsilon_u^2 = u^2 \cdot \sigma_{u'}^2 - 2uu' \cdot \overline{uu'} + u'^2 \cdot \sigma_u^2 = u^2 \cdot \sigma_{u'}^2 - 2uu' \cdot r\,\sigma_u\,\sigma_{u'} + u'^2 \cdot \sigma_u^2$$

where we have defined the correlation coefficient $r = \dfrac{\overline{uu'}}{\sqrt{\overline{u^2} \cdot \overline{u'^2}}}$

Optical Functions

Liouville's theorem ↔ density of states around a phase space point = const.

→ particles occupy the same area in phase space at different *s*

mono-energetic beam: $p_u = \gamma_r m_0 v_u = \beta_r \gamma_r m_0 c \cdot u' \quad \leftrightarrow \quad u' \sim p_u$

→ Liouville's theorem holds as well for the trace space (if $\beta_r \gamma_r$ = const.!)

The emittance ε_u remains constant under conservative forces!
It characterizes the beam's "internal" properties!

→ Normalization of the beam parameters by the emittance:
Separation of the impact of **magnet optics** and the **beam's internal properties**!

Definition of α, β, γ:

$$\sigma_u^2(s) = \overline{u^2(s)} = \varepsilon_u \cdot \beta_u(s)$$

$$\sigma_{u'}^2(s) = \overline{u'^2(s)} = \varepsilon_u \cdot \gamma_u(s)$$

$$r\sigma_u \sigma_{u'} = \overline{uu'} = -\varepsilon_u \cdot \alpha_u(s)$$

Optical Functions

We call the newly defined $\alpha_u, \beta_u, \gamma_u$ **optical functions** (Twiss parameters)!

Using them, the equation of the envelope ellipse reads in the „conventional“ form:

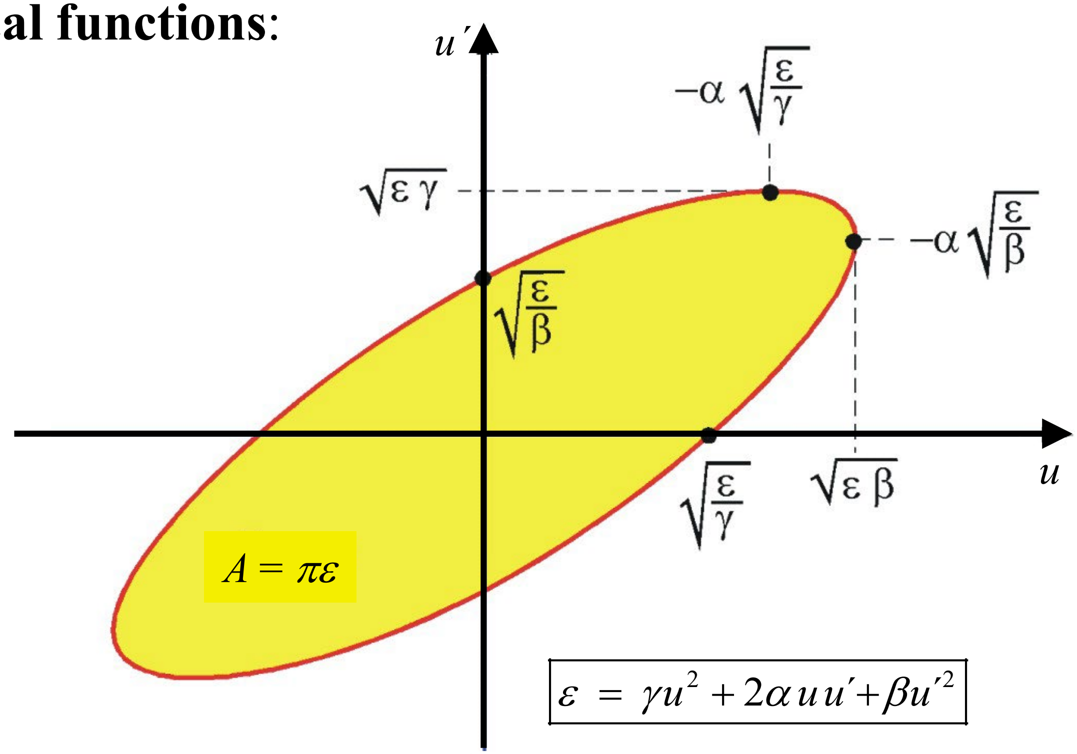
$$\varepsilon_u = \gamma_u u^2 + 2\alpha_u uu' + \beta_u u'^2$$

All the above derived equations appear in identical form for the horizontal and vertical plane. In the following, we will skip the index u for reason of simplicity. Please note, that this doesn't imply that emittances and corresponding Twiss parameters are equal in both planes – they are not!

Optical Functions

Meaning of the **optical functions**:

Remark:
According to the statistical definition, the ellipse does NOT enclose all points (= particle positions) in trace space!



- $\sqrt{\beta}$ represents the r.m.s. beam envelope per unit emittance
- $\sqrt{\gamma}$ represents the r.m.s. beam divergence per unit emittance
- α is proportional to the correlation between u and u'

Beam Matrix

Beam matrix = covariance matrix of the particle distribution:

$$\boldsymbol{\Sigma}_{\text{beam}} = \begin{pmatrix} \overline{u^2} & \overline{uu'} \\ \overline{uu'} & \overline{u'^2} \end{pmatrix} = \begin{pmatrix} \sigma_u^2 & \sigma_{uu'} \\ \sigma_{uu'} & \sigma_{u'}^2 \end{pmatrix} = \varepsilon_u \begin{pmatrix} \beta_u & -\alpha_u \\ -\alpha_u & \gamma_u \end{pmatrix}$$

Emittance ↔ beam matrix:

$$\varepsilon_u = \sqrt{\sigma_u^2 \sigma_{u'}^2 - \sigma_{uu'}^2} = \sqrt{\det\left(\Sigma_{\text{beam}}\right)}$$

→ Important relation between the optical functions:

$$\varepsilon = \sqrt{\underbrace{\beta\varepsilon}_{\overline{u^2}} \cdot \underbrace{\gamma\varepsilon}_{\overline{u'^2}} - \underbrace{\alpha^2\varepsilon^2}_{\overline{-uu'}^2}} \quad \rightarrow \quad \boxed{\beta\gamma - \alpha^2 = 1}$$

Envelope ellipse in trace space:

$$\varepsilon_u = \frac{1}{\varepsilon_u}\left(u^2\sigma_{u'}^2 - 2\sigma_{uu'}\,uu' + u'^2\sigma_u^2\right) = \gamma u^2 + 2\alpha uu' + \beta u'^2 = \varepsilon_u \cdot \vec{u}^T \cdot \left(\boldsymbol{\Sigma}_{\text{beam}}\right)^{-1} \cdot \vec{u}$$

Solving the EQM

Equation of motion in general form:

$$\boxed{u''(s) + K(s)\cdot u(s) \;=\; 0}$$

Ansatz for a solution of the equation of motion:

$$u(s) = A\cdot w(s)\cdot\cos\big(\mu(s)+\varphi_0\big) \qquad \big(A,\varphi_0\big) \text{ defined by individual particle}$$

<u>Discussion of the chosen parametrization:</u>

- Phase advance $\mu(s)$ is positive and monotonously increasing

$$\mu(s) > 0, \quad \mu'(s) > 0, \quad \mu(0) = \mu_0 = 0$$

- Amplitude function $w(s) > 0$ and constant $A > 0$ are defined except for a scaling factor since only the product $A\cdot w$(s) enters. We choose

$$w_0 = w(0) = \sqrt{\frac{1}{\mu_0{}'}} \qquad \rightarrow \qquad w_0^2\,\mu_0{}' = 1$$

Decoupled Equations

Inserting the Ansatz in the equation of motion yields

$$\underbrace{\left[w''-w\cdot\mu'^2+K\cdot w\right]}_{=0}\cdot\cos\left(\mu+\varphi_0\right)\;-\;\underbrace{\left[2\cdot w'\cdot\mu'+w\cdot\mu''\right]}_{=0}\sin\left(\mu+\varphi_0\right)\;=\;0$$

Relation is valid for any given phase advance $\mu(s)$ and any given position s

$$w''-w\cdot\mu'^2+K\cdot w\;=\;0$$

$$2\cdot w'\cdot\mu'+w\cdot\mu''\;=\;0$$

Integration of the second equation:

$$\int_0^s\frac{\mu''}{\mu'}\mathrm{d}s=-2\int_0^s\frac{w'}{w}\mathrm{d}s\;\rightarrow\;\frac{\mu'(s)}{\mu'(0)}=\left(\frac{w(0)}{w(s)}\right)^2\;\rightarrow\;\mu(s)=\mu_0{}'w_0^2\int_0^s\frac{\mathrm{d}s}{w^2(s)}$$

With our settings ($w_0^2\mu_0{}'=1$) we get:

$$\boxed{\mu(s)=\int_0^s\frac{\mathrm{d}s}{w^2(s)}\qquad\text{and}\qquad w''-\frac{1}{w^3}+K\cdot w=0}$$

Beta Function

Definition of a new function - the beta function:

$$\beta(s) = w^2(s)$$

→ transverse position displacement = oscillation around reference orbit

$$u(s) \;=\; A\cdot\sqrt{\beta(s)}\cdot\cos\left(\mu(s)+\varphi_0\right)$$

$$\mu(s) = \int_0^s \frac{\mathrm{d}\tilde{s}}{\beta(\tilde{s})}$$

Building the first derivative and again defining a new alpha function, we obtain

$$u'(s) = -\frac{A}{\sqrt{\beta(s)}}\left\{\alpha(s)\cdot\cos\left(\mu(s)+\varphi_0\right)+\sin\left(\mu(s)+\varphi_0\right)\right\}$$

with $\alpha(s) = -\dfrac{\beta'(s)}{2}$

Twiss Parameters

The equation for u can be transformed to

$$\cos^2(\mu + \varphi_0) = \frac{u^2}{A^2 \cdot \beta}$$

which can be used in combination with the equation for u' to obtain

$$\sin^2(\mu + \varphi_0) = \left(\frac{\sqrt{\beta}}{A} \cdot u' + \frac{\alpha}{A\sqrt{\beta}} \cdot u \right)^2$$

Using $\cos^2 + \sin^2 = 1$ we derive

$$\frac{u^2}{\beta(s)} + \left(\frac{\alpha(s)}{\sqrt{\beta(s)}} \cdot u + \sqrt{\beta(s)} \cdot u' \right)^2 = A^2$$

Defining a new gamma function by $\gamma = \left(1 + \alpha^2\right) / \beta$ this can be transformed to

$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = A^2$$

… looks perfectly the same like the envelope ellipse equation on slide 66!

Twiss Parameters

Proceedings!

Are the newly defined α, β and γ identical to those defined on slide 65?

Each particle is defined by its individual A_i and $\varphi_{i,0}$!

But all particles are described by the same optical functions α, β, γ, μ

Let's check and calculate the second statistical moments:

$$\sigma_u^2 = \overline{u^2} = \overline{A_i^2 \beta \cos^2\left(\mu + \varphi_{0,i}\right)} = \boxed{\frac{1}{2}\overline{A_i^2}\beta}$$

$$\sigma_{u'}^2 = \overline{u'^2} = \overline{\frac{A_i^2}{\beta}\left\{\alpha^2 \cos^2(\ldots) + \sin^2(\ldots) + 2\alpha\cos(\ldots)\sin(\ldots)\right\}} = \boxed{\frac{1}{2}\overline{A_i^2}\gamma}$$

$$\overline{uu'} = \overline{-A_i^2\left\{\alpha\cos^2\left(\mu + \varphi_{0,i}\right) + \cos\left(\mu + \varphi_{0,i}\right)\sin\left(\mu + \varphi_{0,i}\right)\right\}} = \boxed{-\frac{1}{2}\overline{A_i^2}\alpha}$$

$$\varepsilon^2 = \overline{u^2}\,\overline{u'^2} - \left(\overline{uu'}\right)^2 = \frac{1}{4}\left(\overline{A_i^2}\right)^2\left(\beta\gamma - \alpha^2\right) = \boxed{\left(\frac{1}{2}\overline{A_i^2}\right)^2}$$

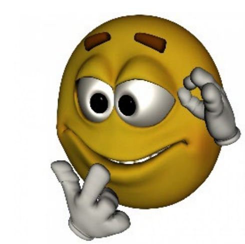
→ indeed – they are:

$$\sigma_u^2 = \varepsilon_u \beta_u$$

$$\sigma_{u'}^2 = \varepsilon_u \gamma_u$$

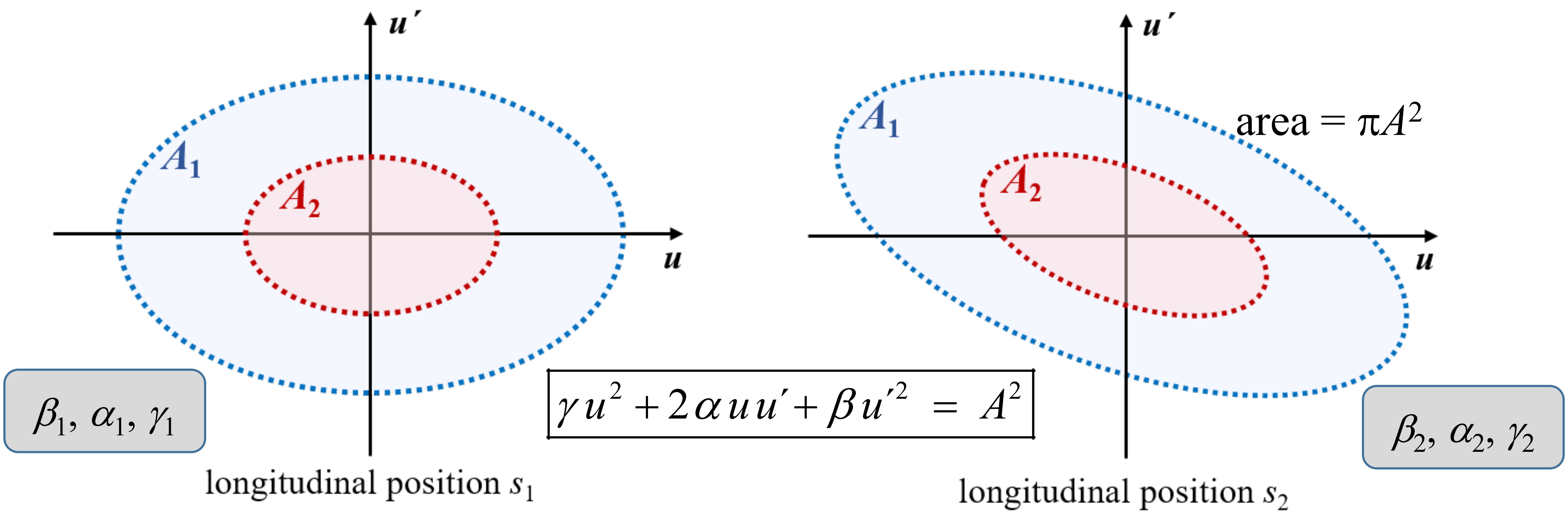
$$\overline{uu'} = -\varepsilon_u \alpha_u$$

Twiss parameters $\alpha, \beta, \gamma, \mu$ with $\frac{1}{\beta} = \mu'$, $\alpha = -\frac{\beta'}{2}$, $\gamma = \frac{1+\alpha^2}{\beta}$



Single Particle Dynamics

Each particle will stay on its own ellipse, which will enclose a constant area in trace space. The amplitude factor ***A* represents the Courant Snyder invariant**! The shape of the ellipse is determined by the Twiss parameters α, β, γ and will change along the magneto-optics system, its area will stay always constant (Rem.: in case of conservative forces and no acceleration). The shape (not the size) of all single particle ellipses are determined by the same Twiss parameters!



$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = A^2$$

Transformation in Trace Space

The transformation of the displacements of a single particle along a beam line may be derived from the transport matrixes.

Important finding: the area A^2 of the corresponding ellipse will remain constant!

So, comparing the displacements at s=0 with those at s, we have for a particle on its ellipse with area A^2

$$\gamma_0 u_0^{\,2} + 2\alpha_0 u_0 u_0' + \beta_0 u_0'^{2} = A^2 = \gamma u^2 + 2\alpha u u' + \beta u'^2$$

Any particle trajectory starting at $s=0$ transforms to $s\neq 0$ by

$$\vec{u} = \mathbf{M}\cdot\vec{u}_0 \quad \rightarrow \quad \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$

which gives via matrix inversion for the inverse transformation

$$\begin{pmatrix} u_0 \\ u_0' \end{pmatrix} = \underbrace{\frac{1}{CS' - C'S} \cdot \begin{pmatrix} S'(s) & -S(s) \\ -C'(s) & C(s) \end{pmatrix}}_{=\mathbf{M}^{-1}} \cdot \begin{pmatrix} u \\ u' \end{pmatrix} \overset{|\mathbf{M}|=1}{=} \begin{pmatrix} S'u - Su' \\ -C'u + Cu' \end{pmatrix}$$

Trafo in Trace Space

We therewith get for the quadratic and mixed terms

$$u_0^{\,2} = S'^2 u^2 - 2SS'uu' + S^2 u'^2$$

$$u_0'^{\,2} = C'^2 u^2 - 2CC'uu' + C^2 u'^2$$

$$u_0 u_0' = -C'S'u^2 + (C'S + CS')uu' - CS\,u'^2$$

which we insert in the ellipse equation $\gamma_0 u_0^{\,2} + 2\alpha_0 u_0 u_0' + \beta_0 u_0'^{\,2} = A^2$ getting

$$\underbrace{\left(S'^2 \cdot \gamma_0 - 2\,S'C' \cdot \alpha_0 + C'^2 \cdot \beta_0\right)}_{=\gamma} \cdot u^2 + 2\,\underbrace{\left(-S\,S' \cdot \gamma_0 + \left(S'C + S\,C'\right) \cdot \alpha_0 - C\,C' \cdot \beta_0\right)}_{=\alpha} \cdot uu'$$

$$+\underbrace{\left(S^2 \cdot \gamma_0 - 2\,S\,C \cdot \alpha_0 + C^2 \cdot \beta_0\right)}_{=\beta} \cdot u'^2 \;=\; A^2$$

This gives the wanted transformation of the Twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2\,S\,C & S^2 \\ -C\,C' & S'C + S\,C' & -S\,S' \\ C'^2 & -2\,S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

Beta Matrix Formalism

Another useful relation may be obtained by defining the Beta matrix **B**

$$\mathbf{B} \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}, \qquad \det(\mathbf{B}) = \beta\gamma - \alpha^2 = 1\,, \qquad \varepsilon \cdot \mathbf{B} = \begin{pmatrix} \sigma_u^2 & \sigma_{uu'} \\ \sigma_{uu'} & \sigma_{u'}^2 \end{pmatrix} \equiv \mathbf{\Sigma}_{\text{beam}}$$

which yields with $\varepsilon = \gamma u^2 + 2\alpha u u' + \beta u'^2$

$$\varepsilon \;=\; {}^T\vec{u} \cdot \mathbf{B}^{-1} \cdot \vec{u} \;=\; {}^T\vec{u}_0 \cdot \mathbf{B}_0^{-1} \cdot \vec{u}_0 \quad \text{with} \quad \mathbf{B}^{-1} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \quad \text{since} \quad \det(\mathbf{B}) = 1$$

Displacement vector $\vec{u}$ transforms according to

$$\vec{u}_1 \;=\; \mathbf{M} \cdot \vec{u}_0\,, \qquad {}^T\vec{u}_1 \;=\; {}^T(\mathbf{M} \cdot \vec{u}_0) \;=\; {}^T\vec{u}_0 \cdot {}^T\mathbf{M}$$

By inserting $\mathbf{1} = \mathbf{M}^{-1} \cdot \mathbf{M}$ we obtain with ${}^T(\mathbf{M}^{-1} \cdot \mathbf{M}) = {}^T\mathbf{M} \cdot {}^T\mathbf{M}^{-1}$

$$\begin{aligned} \varepsilon \;&=\; {}^T\vec{u}_0 \cdot {}^T\mathbf{M} \cdot {}^T\mathbf{M}^{-1} \cdot \mathbf{B}_0^{-1} \cdot \mathbf{M}^{-1} \cdot \mathbf{M} \cdot \vec{u}_0 \\ &=\; {}^T(\mathbf{M} \cdot \vec{u}_0) \cdot \left({}^T\mathbf{M}^{-1} \cdot \mathbf{B}_0^{-1} \cdot \mathbf{M}^{-1} \right) \cdot (\mathbf{M} \cdot \vec{u}_0) \\ &=\; {}^T\vec{u}_1 \cdot \left(\mathbf{M} \cdot \mathbf{B}_0 \cdot {}^T\mathbf{M} \right)^{-1} \cdot \vec{u}_1 \end{aligned}$$

Beta Matrix Formalism

We therewith get the transformation of the beta matrix

$$\mathbf{B}_1 = \mathbf{M} \cdot \mathbf{B}_0 \cdot {}^T\mathbf{M}$$

We thus can transform the Twiss parameters by only taking use of the transfer matrix!

Explicitly:

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \cdot {}^T\mathbf{M}$$

→ ***Hands-On Lattice Calculations***
recommended: E12, E14

Example: Free Drift

Application of the beta matrix formalism to a drift around a symmetry point of a transfer line where $\alpha_{sym} = 0 \;\rightarrow\; \gamma_{sym} = 1/\beta_{sym}$:

$$\mathbf{B}(s) = \underbrace{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}}_{\mathbf{M}_{\text{drift}}} \cdot \underbrace{\begin{pmatrix} \beta_{sym} & 0 \\ 0 & 1/\beta_{sym} \end{pmatrix}}_{\mathbf{B}_{sym}} \cdot \underbrace{\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}}_{{}^{T}\mathbf{M}_{\text{drift}}} = \begin{pmatrix} \beta_{sym} + \dfrac{s^2}{\beta_{sym}} & \dfrac{s}{\beta_{sym}} \\ \dfrac{s}{\beta_{sym}} & \dfrac{1}{\beta_{sym}} \end{pmatrix}$$

This gives the relations for the beam parameters around a symmetry-point:

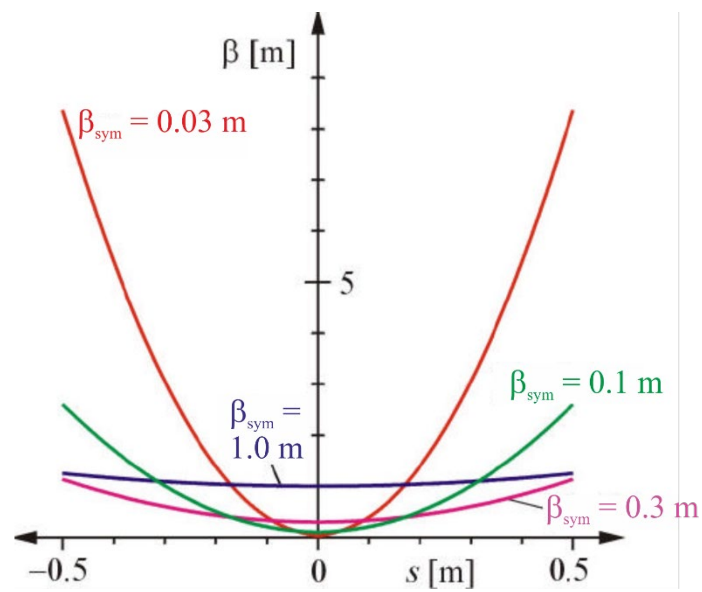
$$\beta(s) = \beta_{sym} + \frac{s^2}{\beta_{sym}}$$

$$\alpha(s) = -\frac{s}{\beta_{sym}}$$

$$\gamma(s) = \frac{1}{\beta_{sym}}$$

The corresponding beam size scales with

$$\sigma(s) = \sqrt{\varepsilon \cdot \beta(s)}$$



Emittance ↔ Wavelength

Evolution of rms beam size $\sigma = \sqrt{\varepsilon\beta}$ and beam divergence $\sigma' = \sqrt{\varepsilon\gamma}$:

$$\sigma(s) = \sigma_0 \cdot \sqrt{1 + \left(\frac{s}{\beta_{sym}}\right)^2} = \sigma_0 \cdot \sqrt{1 + \left(\frac{s}{\sigma_0^2/\varepsilon}\right)^2}, \qquad \sigma'(s) = \frac{\varepsilon}{\sigma_0} = \text{const.}$$

To obtain further insights, let us compare the particle's beam with a Gaussian light beam (TEM_{00}) with wavelength λ. There, we get for the beam radius w

$$\mathrm{w}(s) = \mathrm{w}_0 \cdot \sqrt{1 + \left(\frac{s}{z_R}\right)^2} \quad \text{with the Rayleigh length} \quad z_R = \frac{\pi \mathrm{w}_0^2}{\lambda} = \frac{4\pi\sigma_0^2}{\lambda}$$

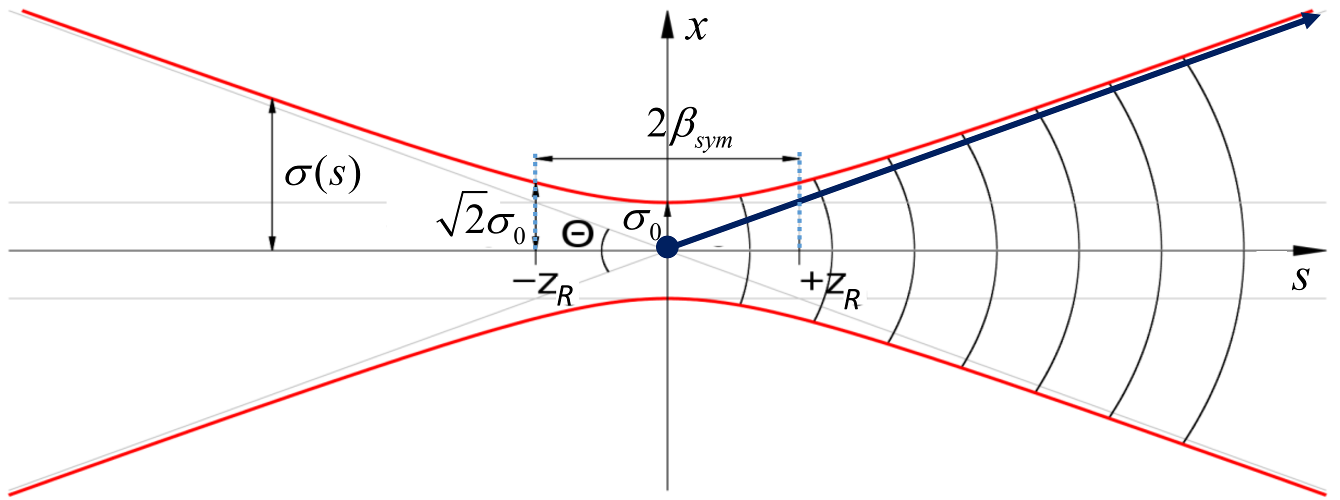
Direct comparison reveals the important relation

$$4\pi \cdot \varepsilon \mathrel{\hat{=}} \lambda$$

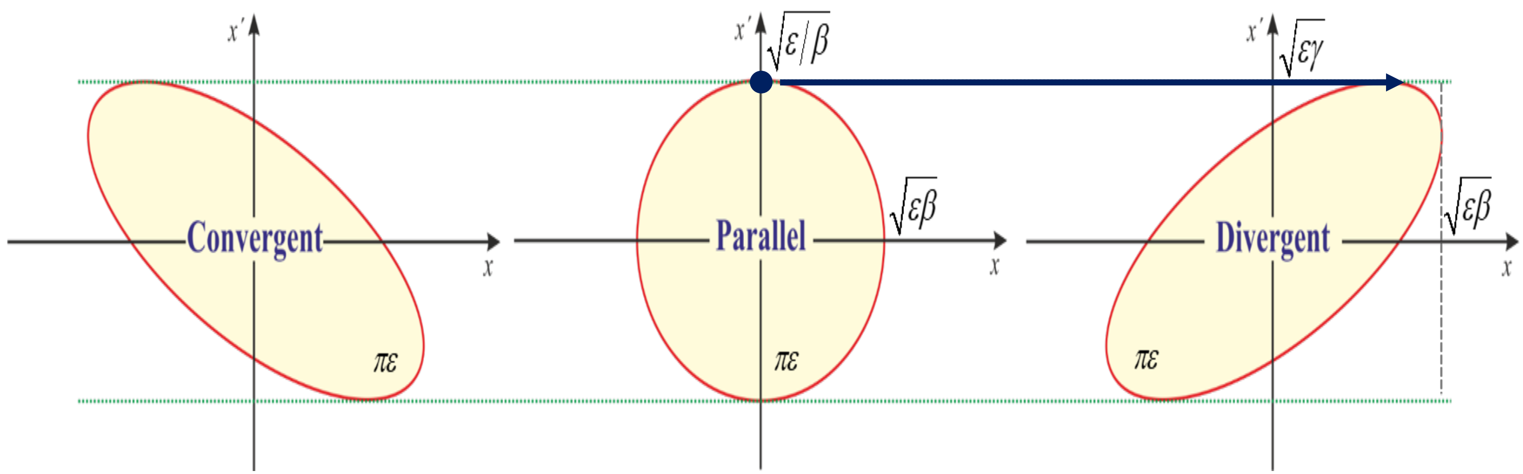
A charged particle's beam with emittance ε "behaves" like a Gaussian TEM_{00} light beam with wavelength $\lambda\,/\,(4\pi)$!

Example: Free Drift

→ *Hands-On Lattice Calculations*
optional: E2.3Ph



$\alpha > 0$ | $\alpha = 0$ | $\alpha < 0$



Transfer Matrix from Twiss

The transformation matrix **M** can be derived also from the Twiss parameters. With

$$u(s) = \sqrt{\varepsilon\beta}\cos(\mu+\varphi_0) = \sqrt{\varepsilon}\cdot\sqrt{\beta}\cdot\{\cos\mu\cdot\cos\varphi_0 - \sin\mu\cdot\sin\varphi_0\}$$

$$u'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta}}\cdot\{\alpha\cdot[\cos\mu\cdot\cos\varphi_0 - \sin\mu\cdot\sin\varphi_0] - \sin\mu\cdot\cos\varphi_0 + \cos\mu\cdot\sin\varphi_0\}$$

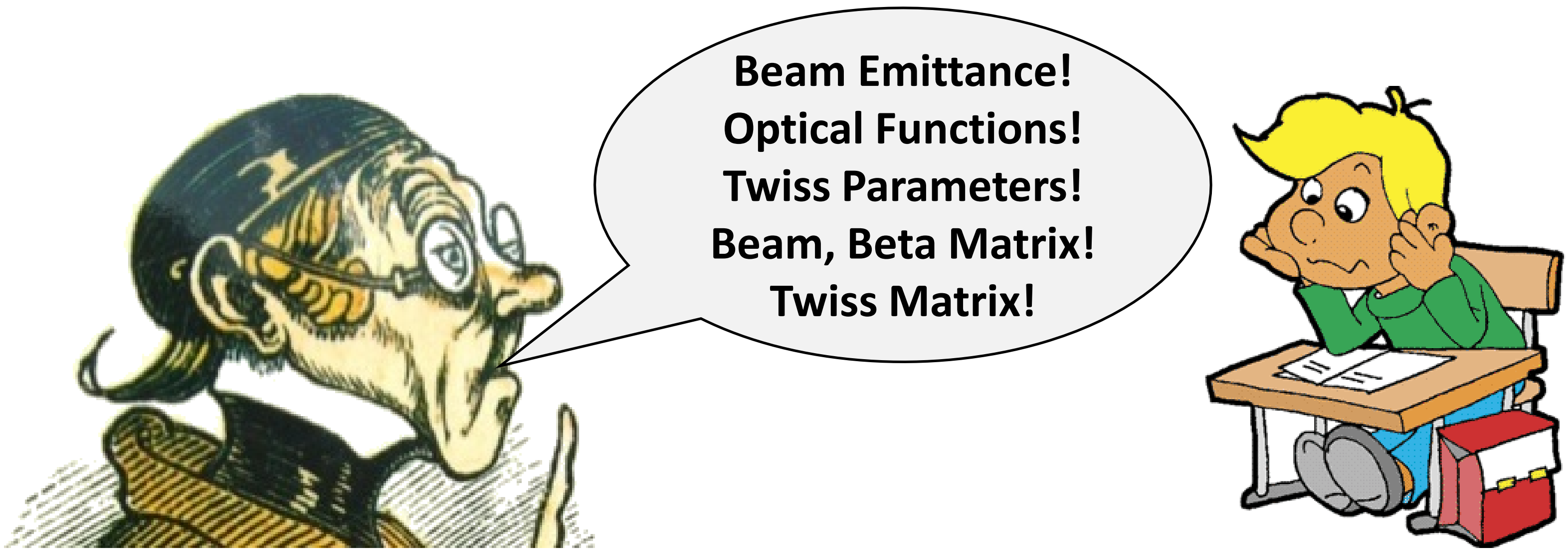
and the starting conditions $u(0) = u_0$, $u'(0) = u_0'$, $\mu(0) = 0$, which transforms to

$$\cos\varphi_0 = \frac{u_0}{\sqrt{\varepsilon\,\beta_0}}, \qquad \sin\varphi_0 = -\frac{1}{\sqrt{\varepsilon}}\left(u_0'\sqrt{\beta_0} + \frac{\alpha_0 u_0}{\sqrt{\beta_0}}\right)$$

we obtain **M**, only dependent on the initial and final Twiss parameters (and μ!)

$$\mathbf{M}(s) = \begin{pmatrix} \frac{\sqrt{\beta}}{\sqrt{\beta_0}}(\cos\mu + \alpha_0\sin\mu) & \sqrt{\beta\,\beta_0}\sin\mu \\ \frac{\alpha_0-\alpha}{\sqrt{\beta\,\beta_0}}\cos\mu - \frac{1+\alpha\,\alpha_0}{\sqrt{\beta\,\beta_0}}\sin\mu & \frac{\sqrt{\beta_0}}{\sqrt{\beta}}(\cos\mu - \alpha\sin\mu) \end{pmatrix}$$

End of 3rd Lecture!



Questions?