Recap 3rd Lecture

very important!

Transverse "Spaces": **configuration space** (x, y); **trace space** (x, x'), (y, y'); **phase space** (x, p_x), (y, p_y)

Beam Size & Divergence: based on 2nd statistical moments: $\sigma_u^2 = \overline{u^2}, \quad \sigma_{u'}^2 = \overline{u'^2}, \quad u = x, y \quad u' = x', y'$

(geom.) Emittance: area/π occupied by the beam in trace space, statistically defined over 2nd moments

$$\varepsilon_u = \sqrt{\sigma_u^2 \sigma_{u'}^2 - \sigma_{uu'}^2} = \sqrt{\overline{u^2}\,\overline{u'^2} - \left(\overline{u\,u'}\right)^2}, \qquad u = x, y \quad u' = x', y'$$

Optical Functions / Twiss Parameters: $\sigma_u = \sqrt{\varepsilon_u \beta_u}, \quad \sigma_{u'} = \sqrt{\varepsilon_u \gamma_u}, \quad \overline{uu'} = -\varepsilon_u \alpha_u, \quad \mu = \text{phase adv.}$

Ellipse Equation: $\varepsilon_u = \gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2$

Courant-Snyder Invariant A: πA^2 = area of single particle ellipse

Twiss parameters are linked via $\dfrac{d}{ds}\mu(s) = \dfrac{1}{\beta(s)}, \quad \dfrac{d}{ds}\beta(s) = -2\alpha(s), \quad \gamma(s) = \dfrac{1+\alpha^2(s)}{\beta(s)}$

Liouville's theorem: ε = const. → separation of beam's internal properties and impact of mag. optics!

Transformation using the Beta Matrix: $\mathbf{B} = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}, \qquad \mathbf{B}(s) = \mathbf{M}(s_0, s) \cdot \mathbf{B}(s_0) \cdot {}^T\mathbf{M}(s_0, s)$

Twiss Matrix: **M** only dependent on $\alpha, \alpha_0, \beta, \beta_0, \gamma, \gamma_0$ and μ

4. Circular Accelerators

- Weak focusing
- Strong (AG) focusing
- Orbit stability
- Periodic FODO
- Betatron tune
- Closed orbit
- Matching & filamentation

Explore the Proton Synchrotron with Google Street View (Image: Google Street View), visit:
https://home.cern/science/accelerators/proton-synchrotron

Weak Focusing

Approach: transverse focusing in both planes at the same time (place):

$$x''(s) + \underbrace{\left(\frac{1}{\rho^2(s)} - k(s)\right)}_{>0} \cdot x(s) = 0$$

$$y''(s) + \overbrace{k(s)} \cdot y(s) = 0$$

→ horizontally defocusing k needs to be compensated by geometrical focusing

$$0 < k = -\frac{q}{p}\frac{\partial B_y}{\partial x} < \frac{1}{\rho^2}$$

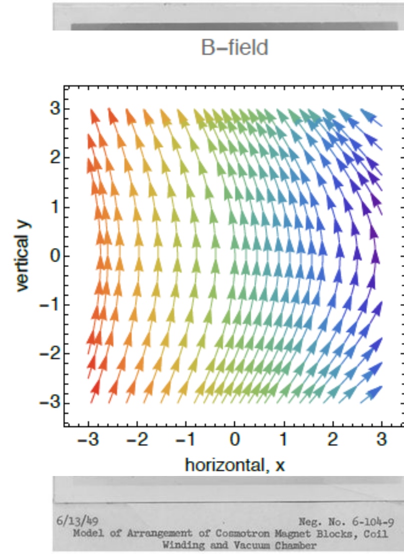
With $p = q\rho B_0$, where B_0 defines the bending field at the design orbit, we obtain the well-known criterion of weak focusing:

$$0 < n = -\frac{\rho}{B_0}\frac{\partial B_y}{\partial x} < 1 \qquad \text{(Steenbeck 1924)}$$

Weak Focusing and Tune

where we have defined the field index n

$$\boxed{n = -\frac{\rho}{B_0}\frac{\partial B_y}{\partial r} = k\rho^2} \quad \leftrightarrow \quad B_y(r) = B_0 \cdot \left(\frac{r}{\rho}\right)^{-n}$$



Thus, a circular accelerator like a synchrotron has to be made of dipole magnets with radially decreasing bending field strength fulfilling the above derived weak focusing condition.

Particles will oscillate around the reference trajectory with the spatial frequency

$$\omega_x = \sqrt{\frac{1}{\rho^2} - k} = \frac{\sqrt{1-n}}{\rho}, \qquad \omega_y = \sqrt{k} = \frac{\sqrt{n}}{\rho}$$

The number Q of oscillations per turn of length $L = 2\pi\rho$ will then be

$$Q_x = \frac{1}{2\pi}\oint\frac{ds}{\beta_x} = \rho\omega_x = \sqrt{1-n} < 1, \qquad Q_y = \frac{1}{2\pi}\oint\frac{ds}{\beta_y} = \rho\omega_y = \sqrt{n} < 1$$

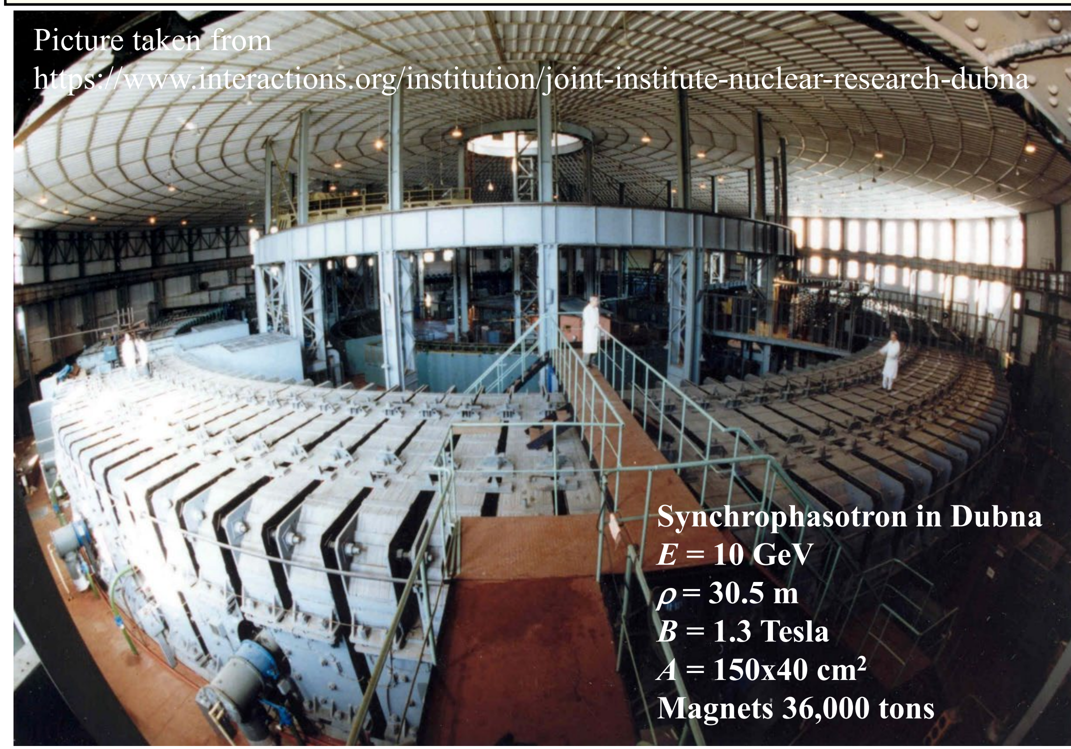
Weak Focusing and Beam Size

Since the beta function is constant, we get

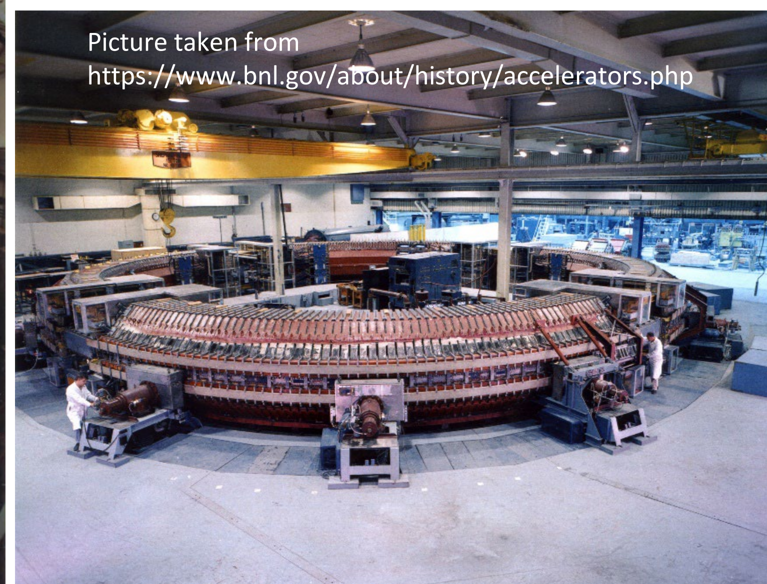
$$\beta_{x,y} > \rho$$

and thus a beam size which will increase with increasing radius according to

$$\boxed{\sigma = \sqrt{\varepsilon\beta} > \sqrt{\varepsilon\rho}}$$

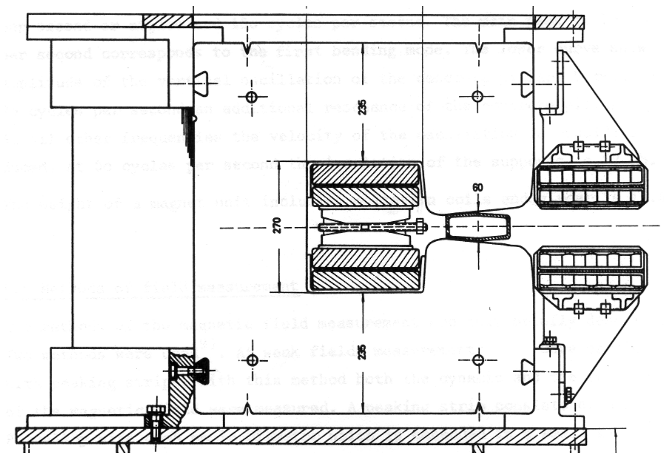


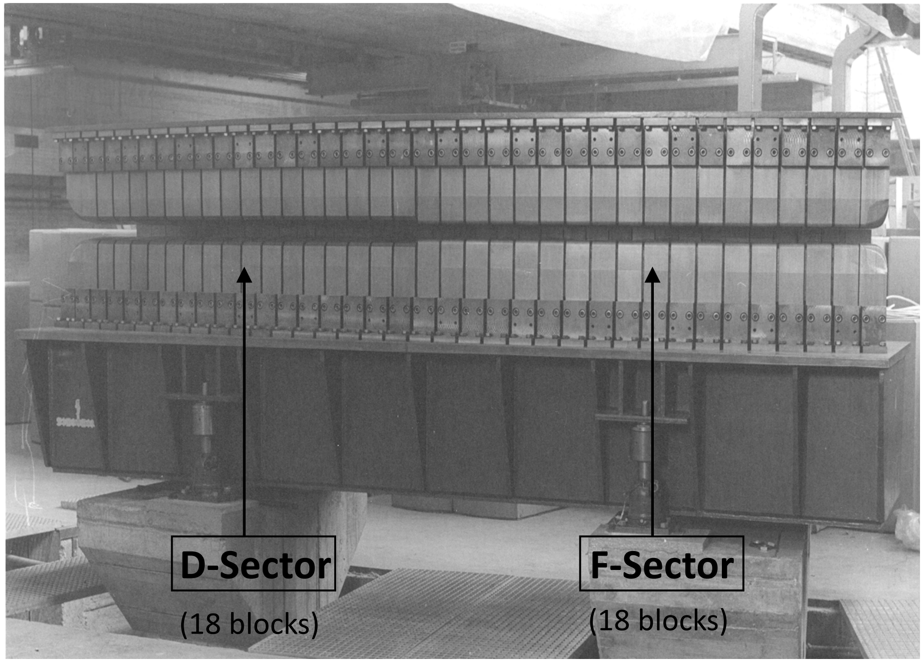
Cosmotron / Brookhaven



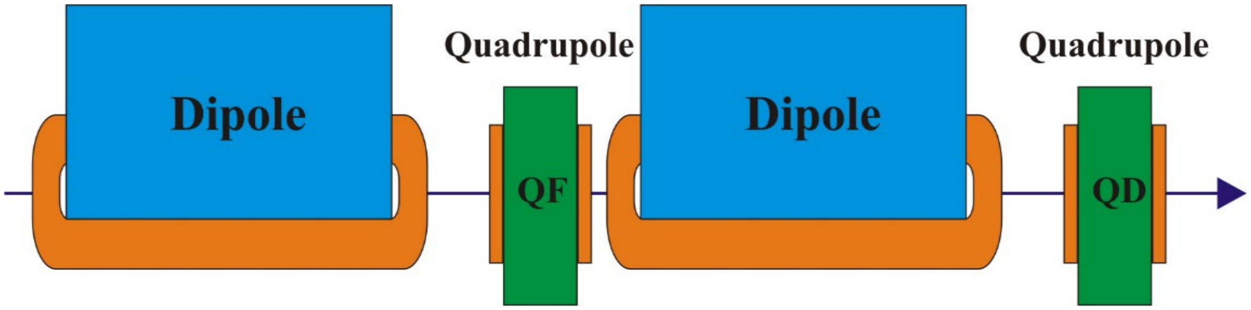
Strong Focusing

AG focusing in the old days:
combined function magnets
with changing gradient



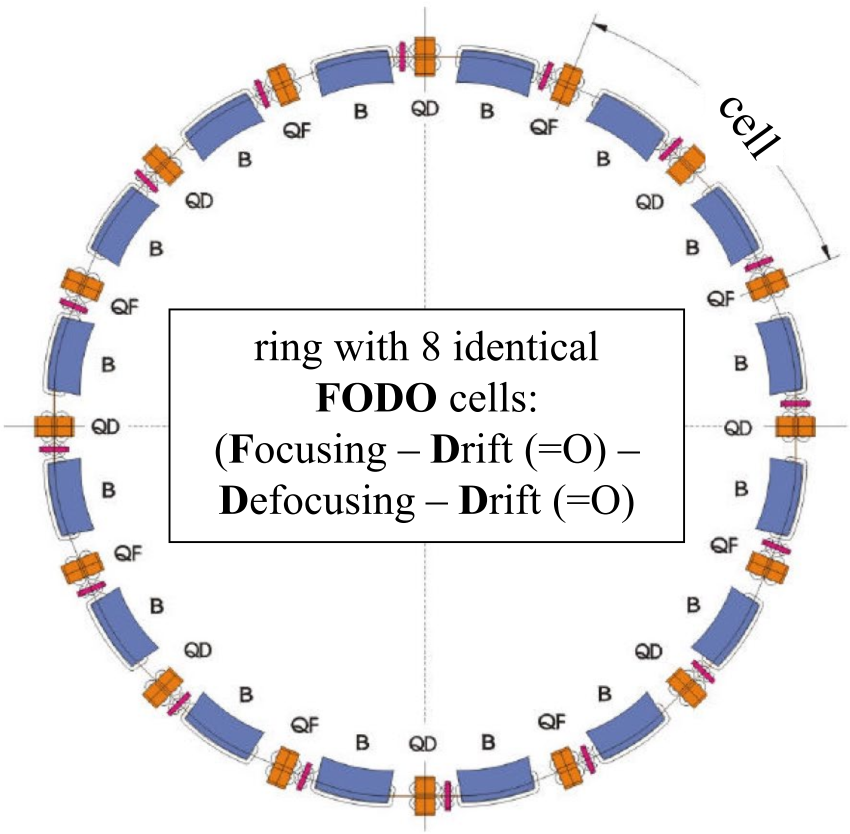


Nowadays: Alternating gradient focusing with strong focusing quadrupoles.
Simplest configuration: FODO lattice, periodic arrangement of identical structures



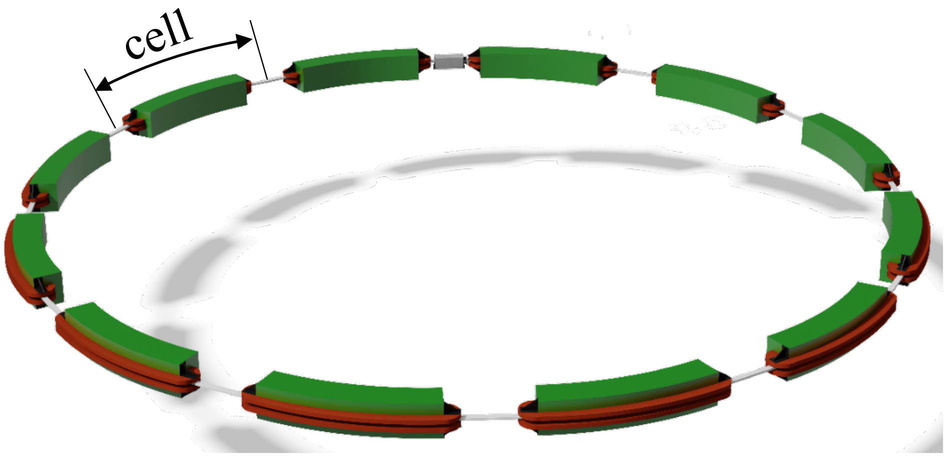
AG Synchrotron

FODO lattice



ring with 8 identical **FODO** cells: (**F**ocusing – **D**rift (=O) – **D**efocusing – **D**rift (=O))

Identical combined function AG magnets



ring with 12 identical combined function FD magnets

Important: due to periodicity, we can choose any position s_0 to define a periodic cell ($s_0 \rightarrow s$) and its transfer matrix $\mathbf{M}(s,s_0) \equiv \mathbf{M}(s-s_0) = \mathbf{M}(L)$

Floquet's Theorem

If $\mathbf{M}(L)$ is the transformation matrix for one periodic cell we will have for N cells:

$$\mathbf{M}(N \cdot L) = \left[\mathbf{M}(L)\right]^N = \text{finite for } N \to \infty \text{ ???}$$

For a full lattice period, we take use of **Floquet's theorem**. Recalling the equations of motions

$$x''(s) + K_x(s) \cdot x(s) = 0 \qquad \text{with} \quad K_x(s) = 1/\rho^2(s) - k(s)$$

$$y''(s) + K_y(s) \cdot y(s) = 0 \qquad \text{with} \quad K_y(s) = k(s)$$

it states (Gaston Floquet, 1847 – 1920) for $u(s) = A\sqrt{\beta_u(s)}\cos\left(\mu_u(s) + \varphi_0\right)$

If K(s) is periodic, the amplitude function (and therefore β(s)) is periodic as well.

In this case we call the EQM **Hill's equation** (George William Hill 1838 – 1914).

Twiss Matrix

Please note and take care:

Floquet's theorem doesn't state that $\mu(s)$ and therewith $x(s)$, $y(s)$ are periodic as well! This would be an exception! (catastrophic, as we will see later)

We will set $s = s_0$ at the beginning of the cell and to s at the end of a cell and set $\beta(s) = \beta$, $\beta(s_0) = \beta_0$ … We recommend periodic boundary conditions (Floquet's theorem)

$$\beta = \beta_0, \quad \alpha = \alpha_0$$

and obtain, using the Twiss parameter representation of the transfer matrix for a cell:

$$\mathbf{M} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix}$$

This matrix was first derived by Twiss from general mathematics principles and is called the **Twiss matrix** (Richard Q. Twiss, 1920 – 2005).

Eigenvalues

We calculate the eigenvalues of the Twiss matrix from

$$\mathbf{M} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix}$$

using

$$\left|\mathbf{M} - \lambda \cdot \mathbf{I}\right| = \lambda^2 - \mathrm{Tr}\left\{\mathbf{M}\right\} \cdot \lambda + 1 = 0$$

With $\mathrm{Tr}\left\{\mathbf{M}\right\} = 2 \cdot \cos\mu$ we obtain

$$\lambda_{1,2} = \cos\mu \pm i \sin\mu = e^{\pm i\mu}$$

We require that the eigenvalues remain finite thus requiring a real betatron phase μ. This is guaranteed when $\left|\cos\mu\right| \leq 1$ and leads to the general stability condition

$$\left|\mathrm{Tr}\left\{\mathbf{M}\right\}\right| = \left|r_{11} + r_{22}\right| \leq 2$$

Orbit Stability

And now comes the "clou": Rewriting the Twiss matrix using

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \qquad \mathbf{J}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}$$

it can be expressed by

$$\mathbf{M} = \mathbf{I} \cdot \cos\mu + \mathbf{J} \cdot \sin\mu$$

First simple check of $N = 2$:

$$\mathbf{M}^2 = \mathbf{I}\cos^2\mu - \mathbf{I}\sin^2\mu + 2\mathbf{J}\cos\mu\sin\mu$$
$$= \mathbf{I}\cos(2\mu) + \mathbf{J}\sin(2\mu)$$

Similar to Moivre's formula we get for N equal periods

$$\mathbf{M}^N = (\mathbf{I} \cdot \cos\mu + \mathbf{J} \cdot \sin\mu)^N = \mathbf{I} \cdot \cos(N\mu) + \mathbf{J} \cdot \sin(N\mu)$$

and

$$\left|\mathrm{Tr}\left\{\mathbf{M}^N\right\}\right| = \left|2 \cdot \cos(N\mu)\right| \le 2$$

Conclusion:

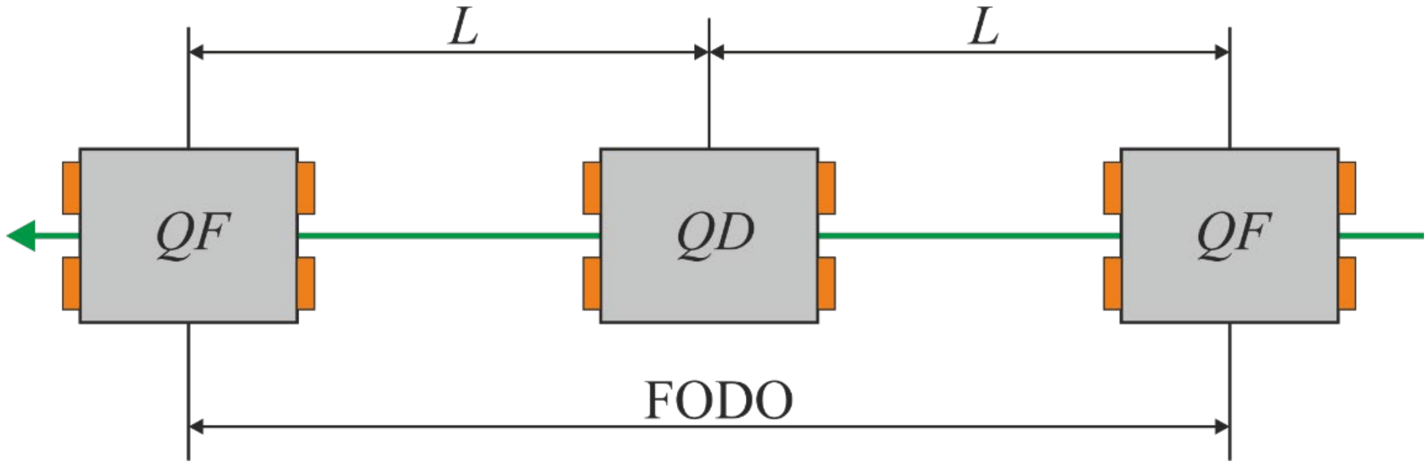
In case of a real betatron phase advance μ, the beam size in a circular accelerator will remain finite (*the 100 Mio $ question in the 50's!*). This can easily be proofed by calculating the trace of the one turn (cell) matrix:

$$\left|\mathrm{Tr}\left\{\mathbf{M}\right\}\right| \le 2$$

Early FODO lattices in HH

General FODO lattice

→ ***Hands-On Lattice Calculations***
recommended: E15, E16



The FODO geometry can be expressed symbolically by the sequence

$$\underbrace{\tfrac{1}{2}\mathrm{QF},\ \mathrm{D},\ \tfrac{1}{2}\mathrm{QD}}_{=\mathbf{M}_{-1/2}},\ \underbrace{\tfrac{1}{2}\mathrm{QD},\ \mathrm{D},\ \tfrac{1}{2}\mathrm{QF}}_{=\mathbf{M}_{1/2}}$$

It is sufficient to use the thin lens approximation $l_Q \ll f$. We will set the focal length to $f_2 = 2\,f_D,\ \ f_1 = 2\,f_F$ and the drift length to L. Defining:

$$1/f^* = 1/f_1 + 1/f_2 - L/\left(f_1 \cdot f_2\right)$$

General FODO lattice

the transformation matrix of half a FODO cell is …  …

$$\mathbf{M}_{1/2} = \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 - L/f_1 & L \\ -1/f^* & 1 - L/f_2 \end{pmatrix}$$

Multiplication with the reverse matrix gives …   …

$$\mathbf{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - 2L/f^* & 2L \cdot (1 - L/f_2) \\ -2/f^* \cdot (1 - L/f_1) & 1 - 2L/f^* \end{pmatrix} \quad \text{and} \quad \left| \mathrm{Tr}\{\mathbf{M}\} \right| = \left| 2 - \frac{4L}{f^*} \right| < 2$$

This is equivalent to $0 < \frac{L}{f^*} < 1$, and defining $u = \frac{L}{f_1}$, $v = \frac{L}{f_2}$ we get

$$\boxed{0 < u + v - u \cdot v < 1}$$

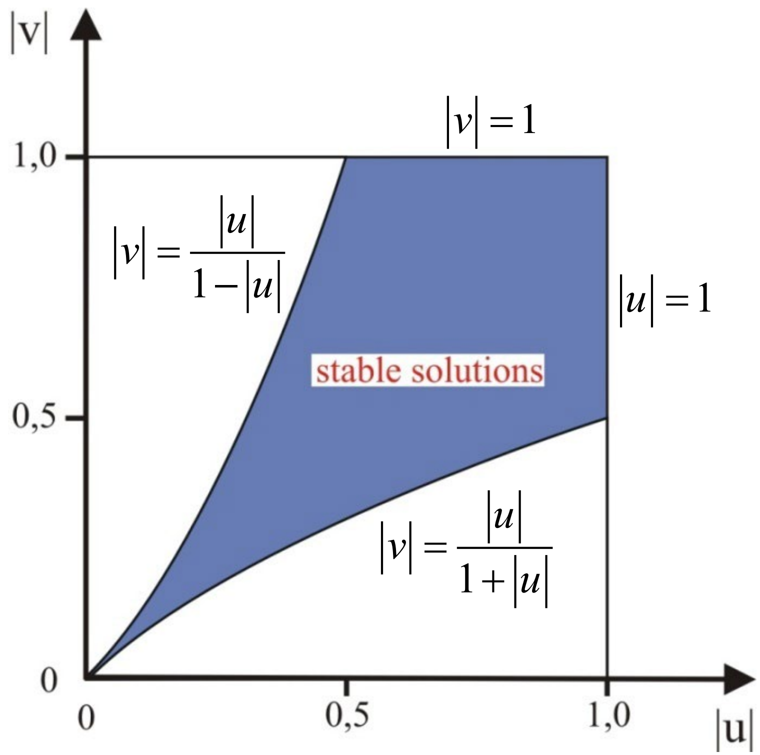
u, v are referred to as **“FODO parameters”**

from which we derive the boundaries of the stability region

$$|u| < 1, \quad |v| < \frac{|u|}{1 - |u|}, \quad |v| < 1, \quad |v| > \frac{|u|}{1 + |u|}$$

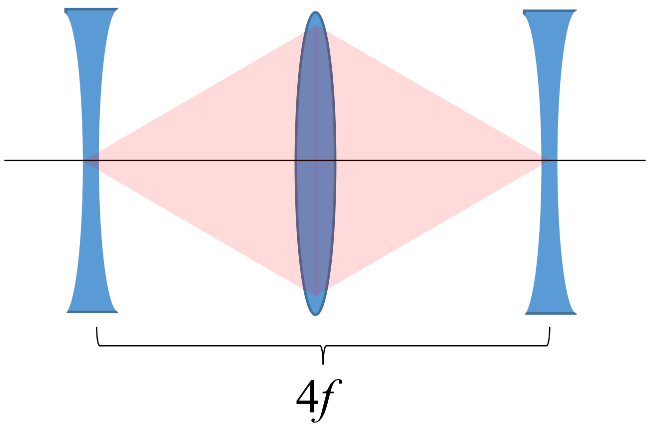
Necktie Diagram

Necktie diagram for thin lens approximation:



→ ***Hands-On Lattice Calculations***
recommended: E16, E23
optional: E3.1Ph

"4*f* imaging"

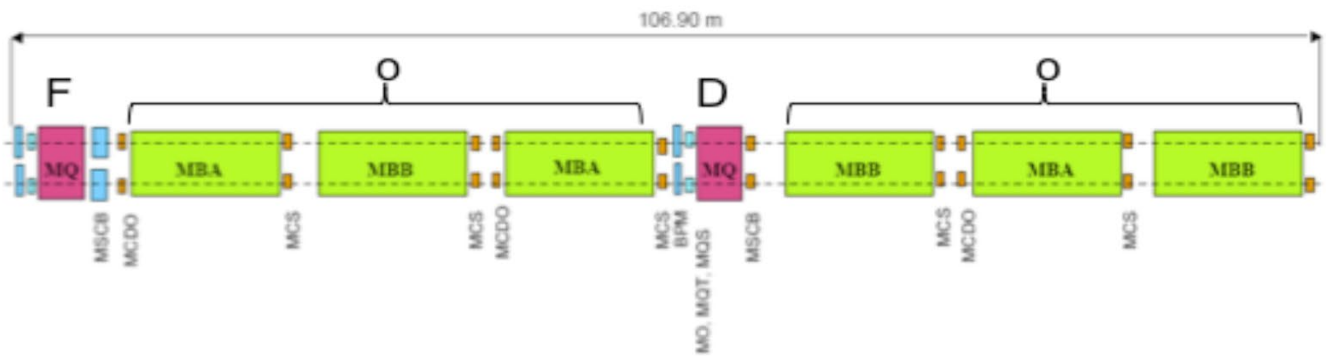


In the simple case of equal focusing strengths, we arrive at

$$|f_1| = |f_2| = \frac{|f_D|}{2} = \frac{|f_F|}{2} = \frac{|f|}{2} \quad \rightarrow \quad \boxed{\left|\frac{L}{2f}\right| = \left|\frac{L_{\text{FODO}}}{4f}\right| < 1}$$

Example: LHC

LHC: Lattice Design
the ARC 90° FoDo in both planes



equipped with additional corrector coils

MB: main dipole
MQ: main quadrupole
MQT: Trim quadrupole
MQS: Skew trim quadrupole
MO: Lattice octupole (Landau damping)
MSCB: Skew sextupole
Orbit corrector dipoles
MCS: Spool piece sextupole
MCDO: Spool piece 8 / 10 pole
BPM: Beam position monitor + diagnostics

Courtesy of Bernhard Holzer, CAS lectures

Periodic Beta Functions

Periodic solutions of a periodic lattice of period-length L will be

$$\beta(s_0+L) = \beta(s_0) = \beta_0$$
$$\alpha(s_0+L) = \alpha(s_0) = \alpha_0$$

Comparing the transfer matrix for one period with its Twiss parameter representation

$$\mathbf{M} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix}$$

we can determine the Twiss parameters at the symmetry points (where $\alpha = 0$!)

$$\alpha_0 = 0, \qquad \beta_0 = \frac{r_{12}}{\sqrt{1-r_{11}^2}}, \qquad \gamma_0 = \frac{-r_{21}}{\sqrt{1-r_{11}^2}}, \qquad \cos\mu = r_{11}$$

→ ***Hands-On Lattice Calculations*** *recommended: E17 – E20*

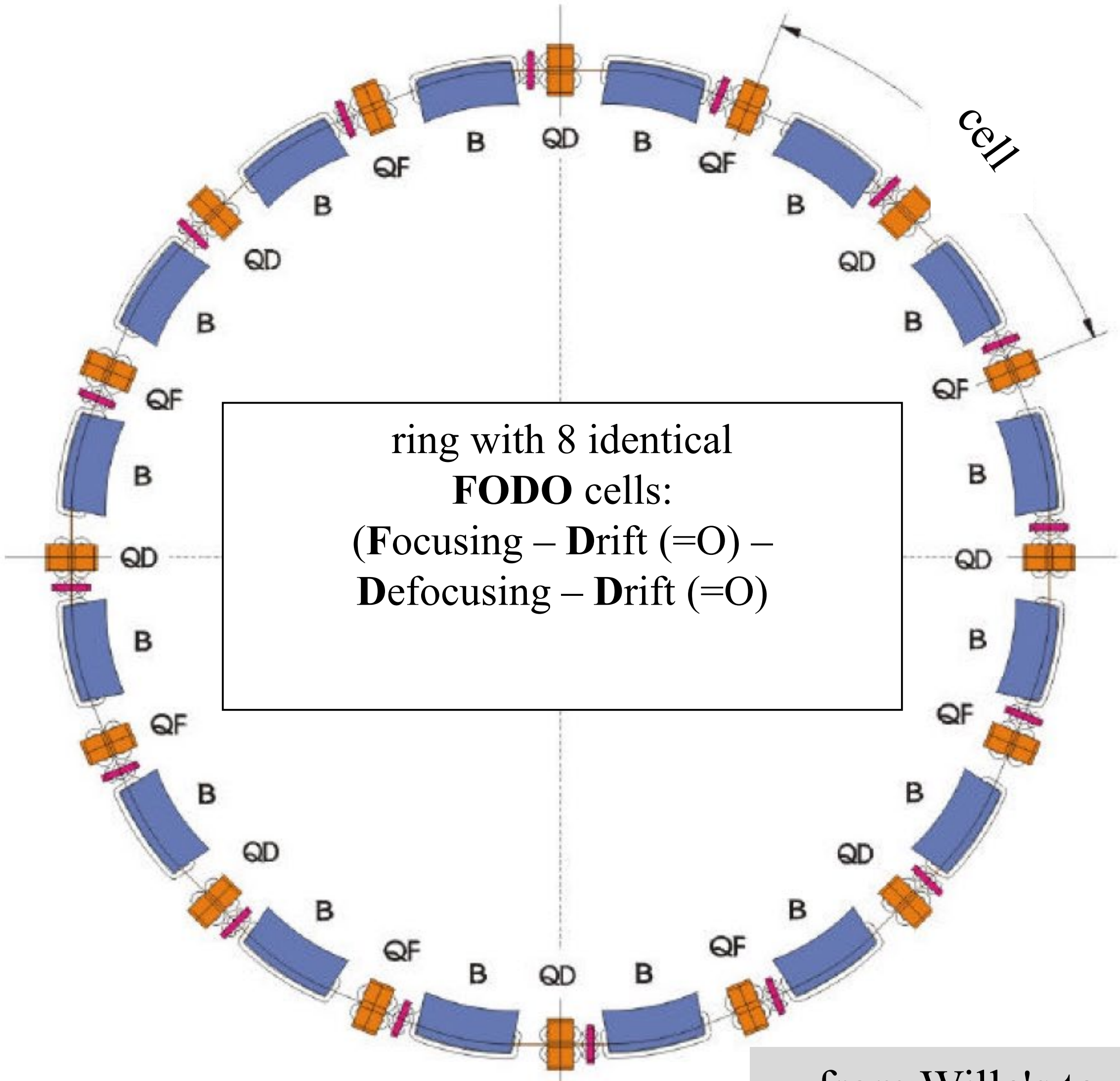
and transform them to any position s using e.g. the beta matrix formalism

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \mathbf{M}(s,s_0) \cdot \begin{pmatrix} \beta_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} \cdot {}^T\mathbf{M}(s,s_0)$$

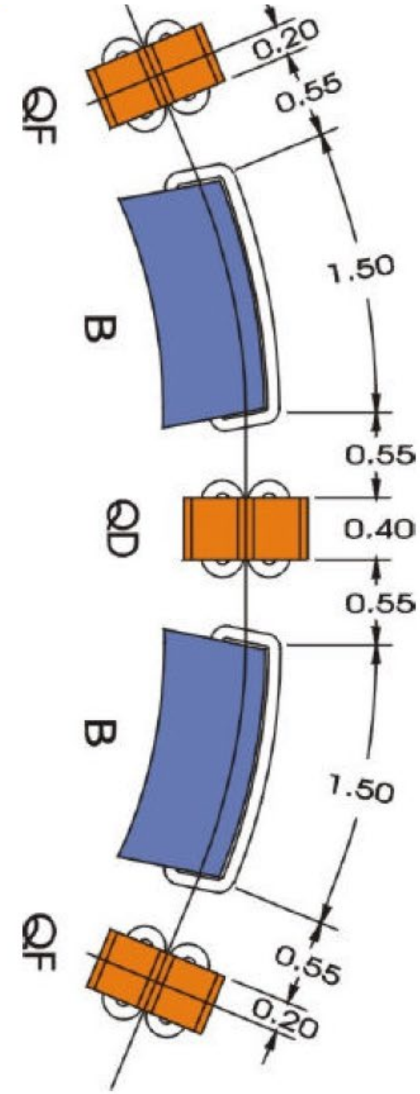
thus revealing the development of β(s), α(s), γ(s).

$$\min = \langle \beta_{FODO} \rangle = L_{FODO}$$
for $\mu \approx 90°$

Example: Toy Ring



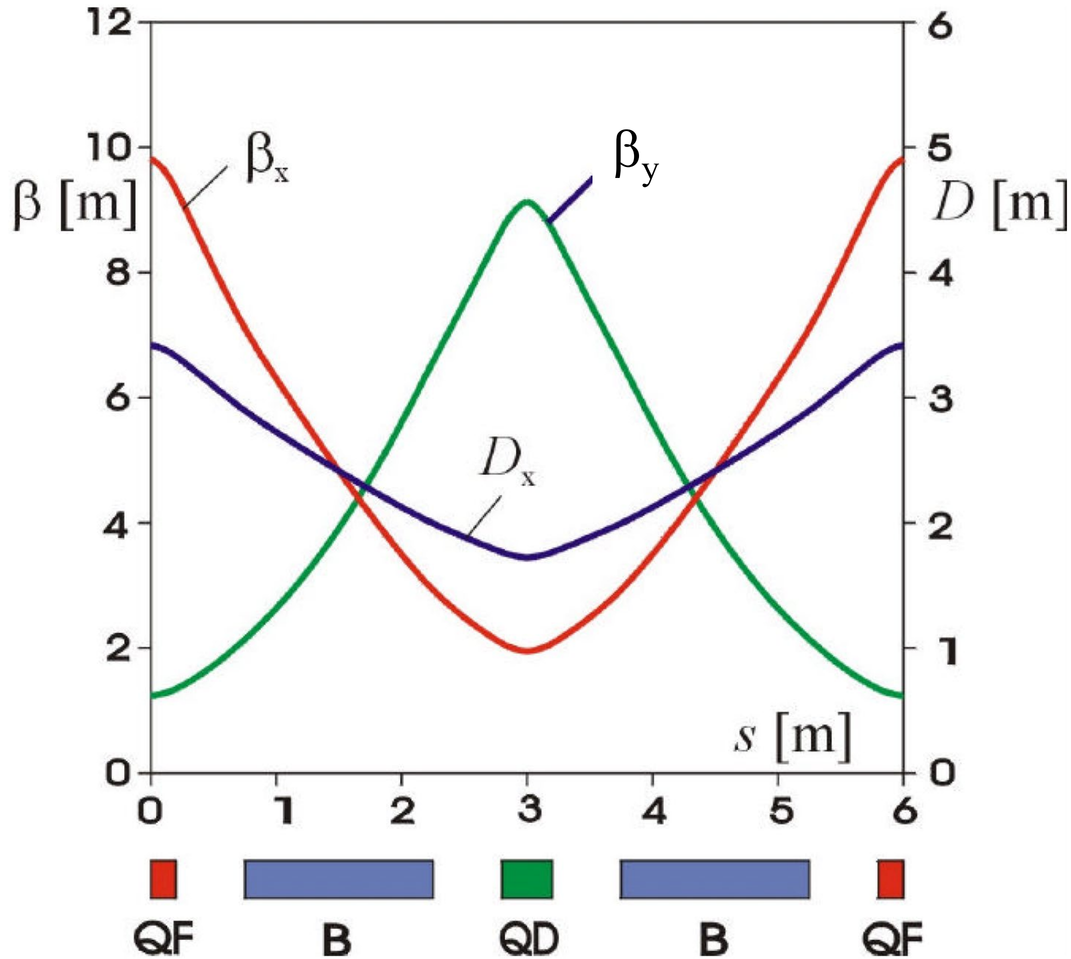
ring with 8 identical
FODO cells:
(**F**ocusing – **D**rift (=O) –
Defocusing – **D**rift (=O)



... from Wille's textbook

Example: Toy Ring

Choosing $|k_{QF}| = |k_{QD}| = 1.20\text{m}$ we can calculate the transfer matrix **M** and extract the Twiss parameters, obtaining (please ignore the blue D_x curve – comes later)



Important:

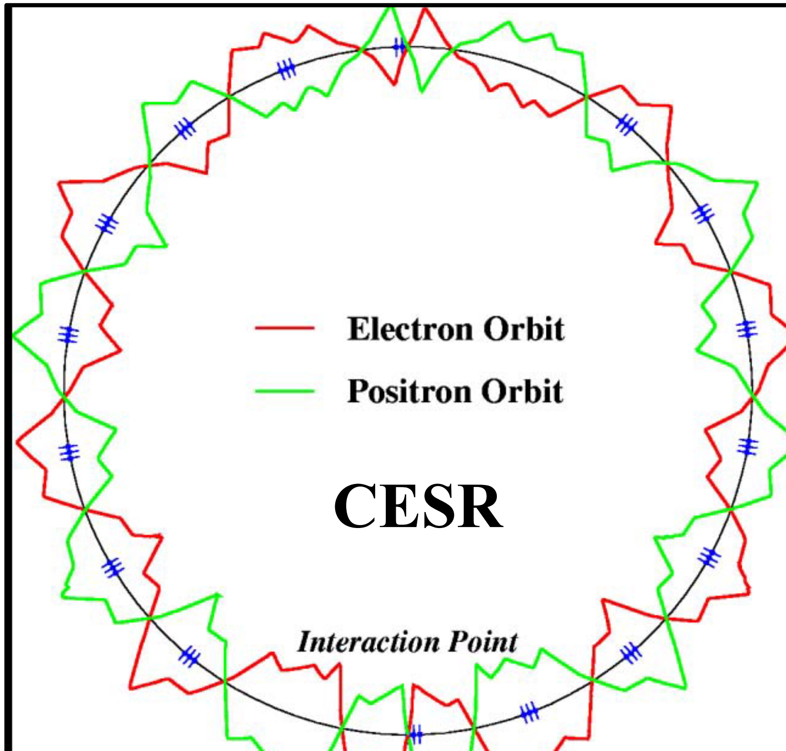
- $\beta_{x,y}$ are completely determined by the magnet lattice!
- β_x is maximal in all QF and minimal in all QD
- β_y is maximal in all QD and minimal in all QF

→ ***Hands-On Lattice Calculations***
recommended: E24
optional: E3.4Ph, E3.5Ph

Closed Orbit

Remember: In circular accelerators the amplitude function is periodic according to Floquet's theorem and reproduces itself after one turn.

This implies, that the charge center of the beam also moves on a closed trajectory, which is called the closed orbit!



Picture taken from Crittenden et al., TUPAS056, Proc. PAC 2007

The shape of the closed orbit is determined by the magnets and can – due to errors and misalignments – significantly deviate from the design orbit!

Dedicated steerer magnets (small dipoles), which have to be installed around the ring, are used to correct closed orbit deviations.

→ ***Hands-On Lattice Calculations***
recommended: E11 … and think about closed trajectories!

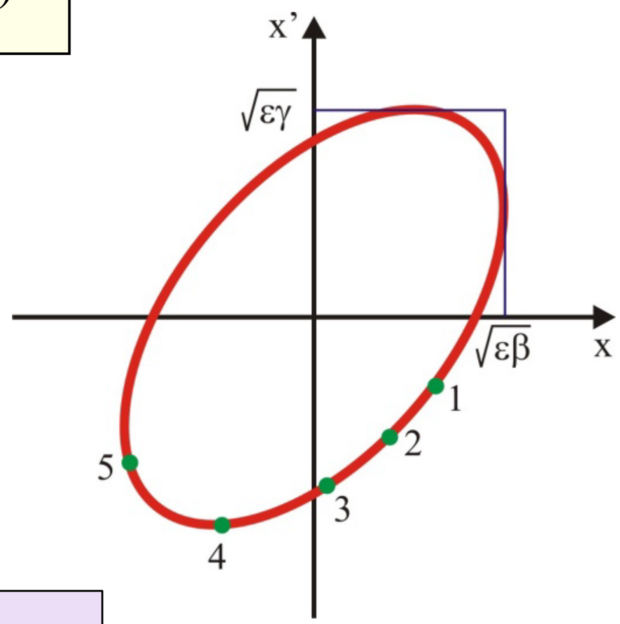
Betatron Tune

The betatron tune Q is defined as the number of oscillations per revolution:

$$Q_{x,y} = \frac{\mu_{x,y}(L)}{2\pi} = \frac{1}{2\pi} \cdot \oint \frac{ds}{\beta_{x,y}(s)}$$

If one regards the phase space at an arbitrarily chosen point, a single particle moves on its phase space ellipse.

The points represents the parameters after 1,2, … 5 revolutions.



→ ***Hands-On Lattice Calculations***
recommended: E21 – E22, optional: E3.2Ph, E3.3Ph

The betatron tune is one of the most important parameter in circular accelerators!

Filamentation

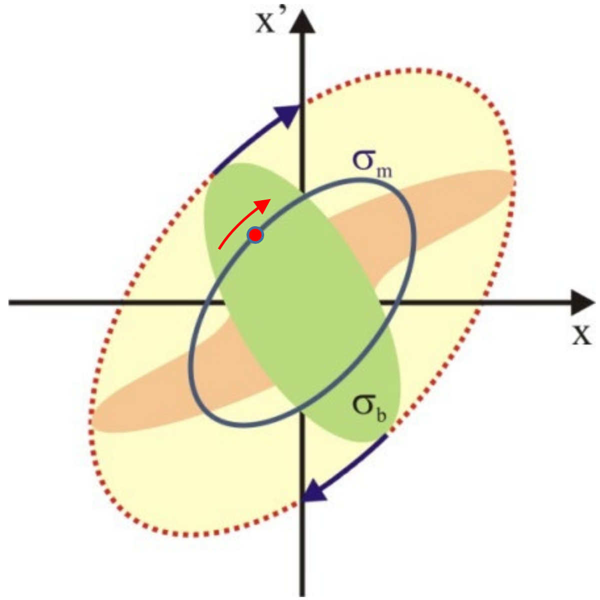
If the envelope ellipse β_b of the beam is not matched to the ellipse β_m of the periodic lattice, it will start to rotate with a phase advance per revolution of $2\pi Q$.

Beam matrix:

$$\Sigma_{\text{beam}} = \begin{pmatrix} \sigma_x^2 & \overline{xx'} \\ \overline{xx'} & \sigma_{x'}^2 \end{pmatrix}$$

$$\Sigma_{\text{beam}} = \varepsilon \cdot \mathbf{B}_{\text{beam}}$$

$$\mathbf{B}_{\text{beam}} = \begin{pmatrix} \beta_{\text{b}} & -\alpha_{\text{b}} \\ -\alpha_{\text{b}} & \gamma_{\text{b}} \end{pmatrix}$$



Matching:

$\beta_{\text{b}} = \beta_{\text{m}}$

$\alpha_{\text{b}} = \alpha_{\text{m}}$

$\gamma_{\text{b}} = \gamma_{\text{m}}$

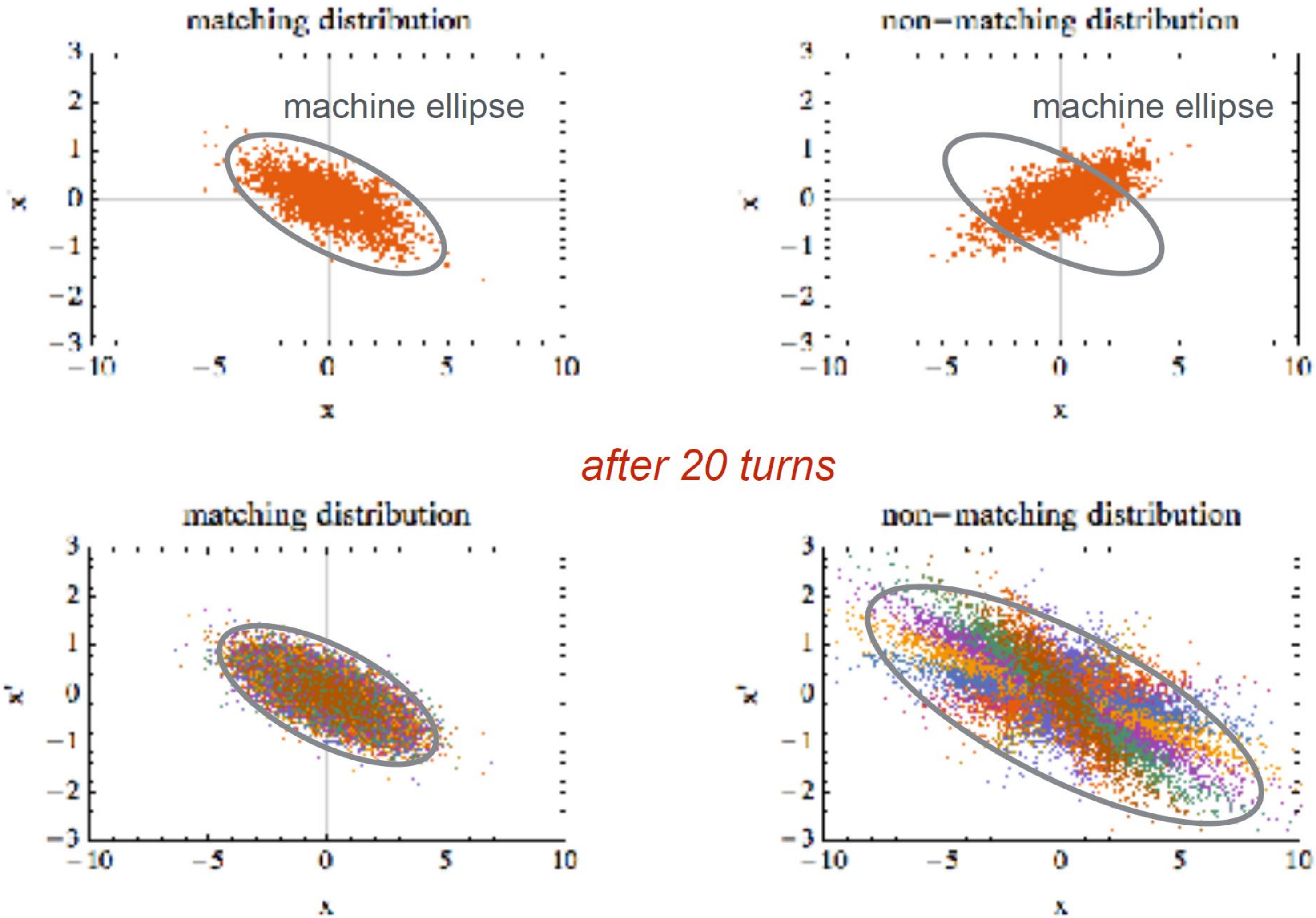
b = beam
m = machine

Due to effects of higher order the quadrupole strengths and therefore the phase advance depends on the amplitude (horizontal and vertical displacements). In case of mismatch, the beam phase space distribution starts to filament. After a large number of revolutions, the distribution may be surrounded by a large ellipse of the form of the lattice ellipse.

→ ***Hands-On Lattice Calculations***
optional: E3.6Ph

Filamentation

Example for an unmatched and matched beam (courtesy of B. Schmidt):



End of 4th Lecture!

FODO Lattice!
Stable Necktie!
Closed Orbit!
Betatron Tune!
Beam Matching!
Filamentation!

Questions?