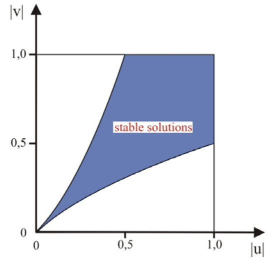
Recap 4th Lecture

Circular Accelerators:

Weak Focusing: Steenbeck criterion: $0 < n = -\dfrac{\rho}{B_0}\dfrac{\partial B_y}{\partial x} < 1 \quad \Rightarrow \quad \beta > \rho$

Strong Focusing: $|n| >> 1 \;\rightarrow$ application of AG focusing

Stability Criterion: $|\mathrm{Tr}\{\mathbf{M}\}| \le 2$



Periodic Lattice: periodic boundary conditions $\alpha = \alpha_0,\; \beta = \beta_0,\; \gamma = \gamma_0$

Periodic FODO: stable necktie, if $f_+ = -f_- = f \quad \rightarrow \quad 4f > L_{\mathrm{FODO}}$

→ optical functions are determined by the lattice only (periodicity)!

FODO lattice: **$\beta(s)$ is always maximal in focusing quads and minimal in defocusing quads of plane considered!** $L_{FODO} = \langle \beta \rangle = \min \;\leftrightarrow\; \mu = 90°$

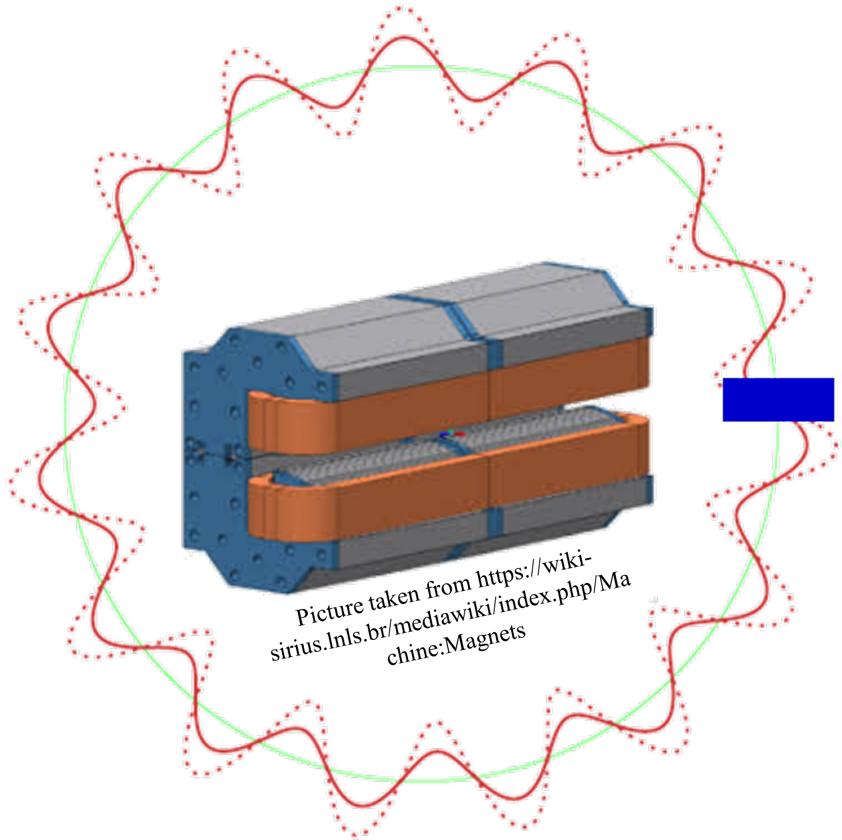
Betatron Tune: indicates the number of transverse oscillations per turn $Q = \dfrac{1}{2\pi}\oint \dfrac{\mathrm{d}s}{\beta(s)}$

Filamentation: beam ellipse will rotate (and enlarge) in case of a non-matched beam

Closed Orbit: equilibrium path of charge center, influenced by field errors, is closed!

Dipole Errors and Integer Tune

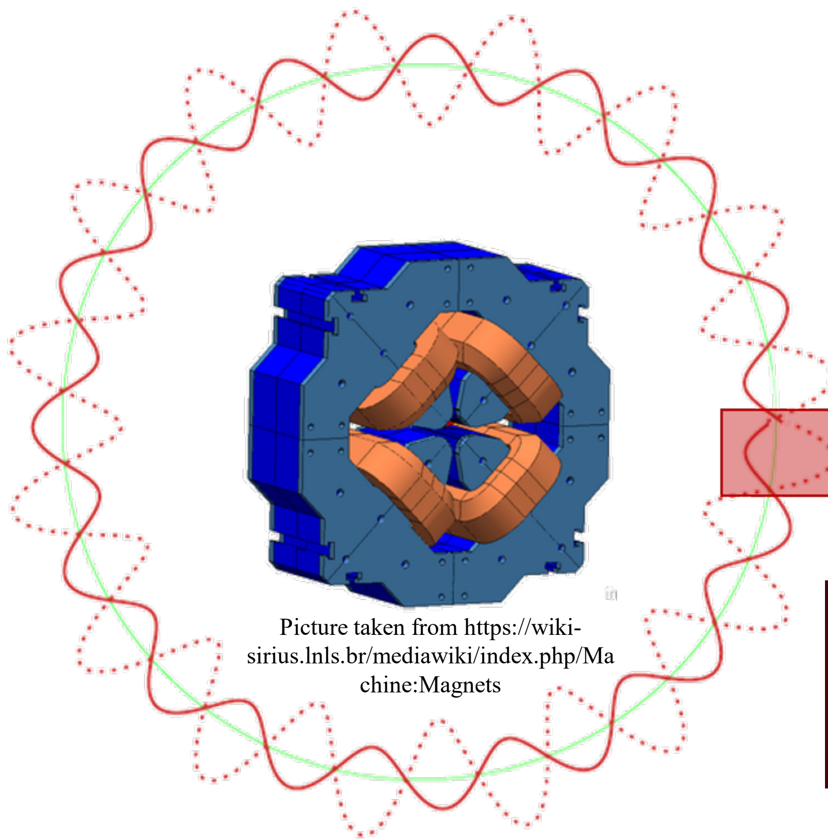
Imagine a single faulty element in a ring and an integer tune …



… and the individual kicks will add up coherently!

Quadrupole Errors and Half Integer Tune

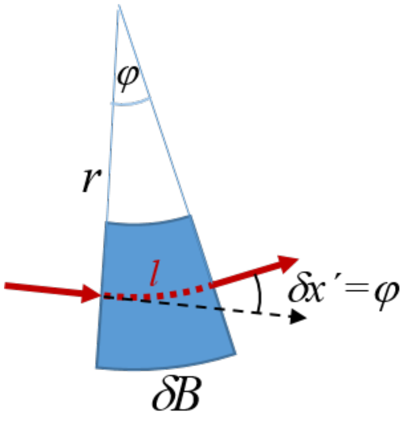
Imagine a single faulty element in a ring and a half integer tune …



Picture taken from https://wiki-sirius.lnls.br/mediawiki/index.php/Machine:Magnets

… and the same will happen caused by the gradient error!

Dipole Kicks



Let us assume a dipole field error produced by a short dipole of length l which makes a small angular kick $\delta x'$ in divergence:

$$l = r \cdot \varphi \approx \frac{p}{q(\delta B)} \cdot \delta x' = \frac{\rho B}{\delta B} \cdot \delta x'$$

We thus get for the angular kick

$$\delta x' = \frac{\delta(B l)}{B \rho}$$

This perturbs the orbit trajectory at the position of the "faulty" element. Elsewhere the orbit obeys the unperturbed Hill's differential equations

$$x''(s) + \left(\frac{1}{\rho^2(s)} - k(s) \right) \cdot x(s) = 0, \qquad y''(s) + k(s) \cdot y(s) = 0$$

Closed Orbit Distortion

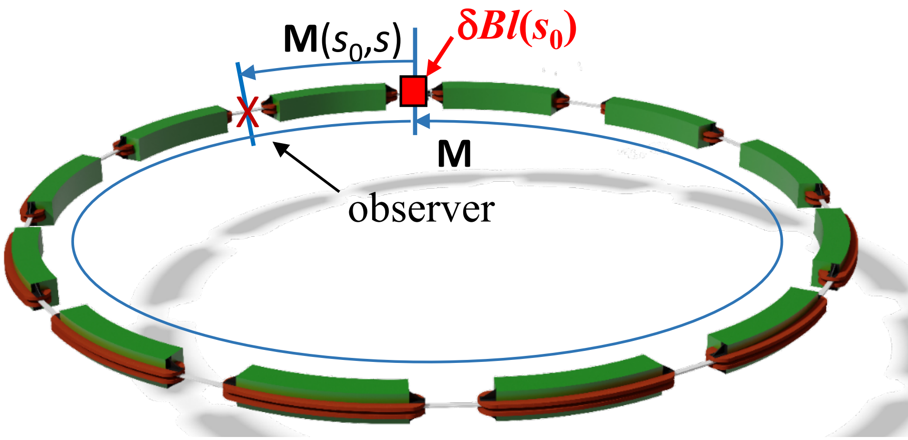
Using matrix algebra, the displacement of the closed orbit x_c at the position s_0 of the field error can be calculated from the displacement just before and after the kick element:

$$\begin{pmatrix} x_{c,0} \\ x_{c,0}' - \delta x' \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} x_{c,0} \\ x_{c,0}' \end{pmatrix} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix} \cdot \begin{pmatrix} x_{c,0} \\ x_{c,0}' \end{pmatrix}$$

and with $\mu = 2\pi Q$

$$x_{c,0} = \frac{\beta_0 \, \delta x'}{2\sin(\pi Q)} \cos(\pi Q)$$

$$x_{c,0}' = \frac{\delta x'}{2\sin(\pi Q)} \left[\sin(\pi Q) - \alpha_0 \cos(\pi Q) \right]$$



The closed orbit displacement $x_c(s)$ is calculated from $\vec{x}_c(s) = \mathbf{M}(s_0, s) \cdot \vec{x}_{c,0}$:

$$\begin{pmatrix} x_c(s) \\ x_c'(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}} (\cos\mu + \alpha_0 \sin\mu) & \sqrt{\beta(s)\beta_0} \sin\mu \\ -\frac{1+\alpha(s)\alpha_0}{\sqrt{\beta(s)\beta_0}} \sin\mu + \frac{1-\alpha(s)\alpha_0}{\sqrt{\beta(s)\beta_0}} \cos\mu & \sqrt{\frac{\beta_0}{\beta(s)}} (\cos\mu - \alpha_0 \sin\mu) \end{pmatrix} \cdot \begin{pmatrix} x_{c,0} \\ x_{c,0}' \end{pmatrix}$$

Integer Resonance

We finally get for the displacement x_c of the closed orbit at s, caused by a dipole kick at s_0:

$$x_c(s) = \sqrt{\beta}\,\eta_0 \cos(Q\psi) = \frac{\sqrt{\beta(s)\,\beta(s_0)}}{2\sin(\pi Q)} \frac{\delta(Bl)}{B\rho}(s_0) \cdot \cos\left(\mu(s) - \mu(s_0) + Q\pi\right)$$

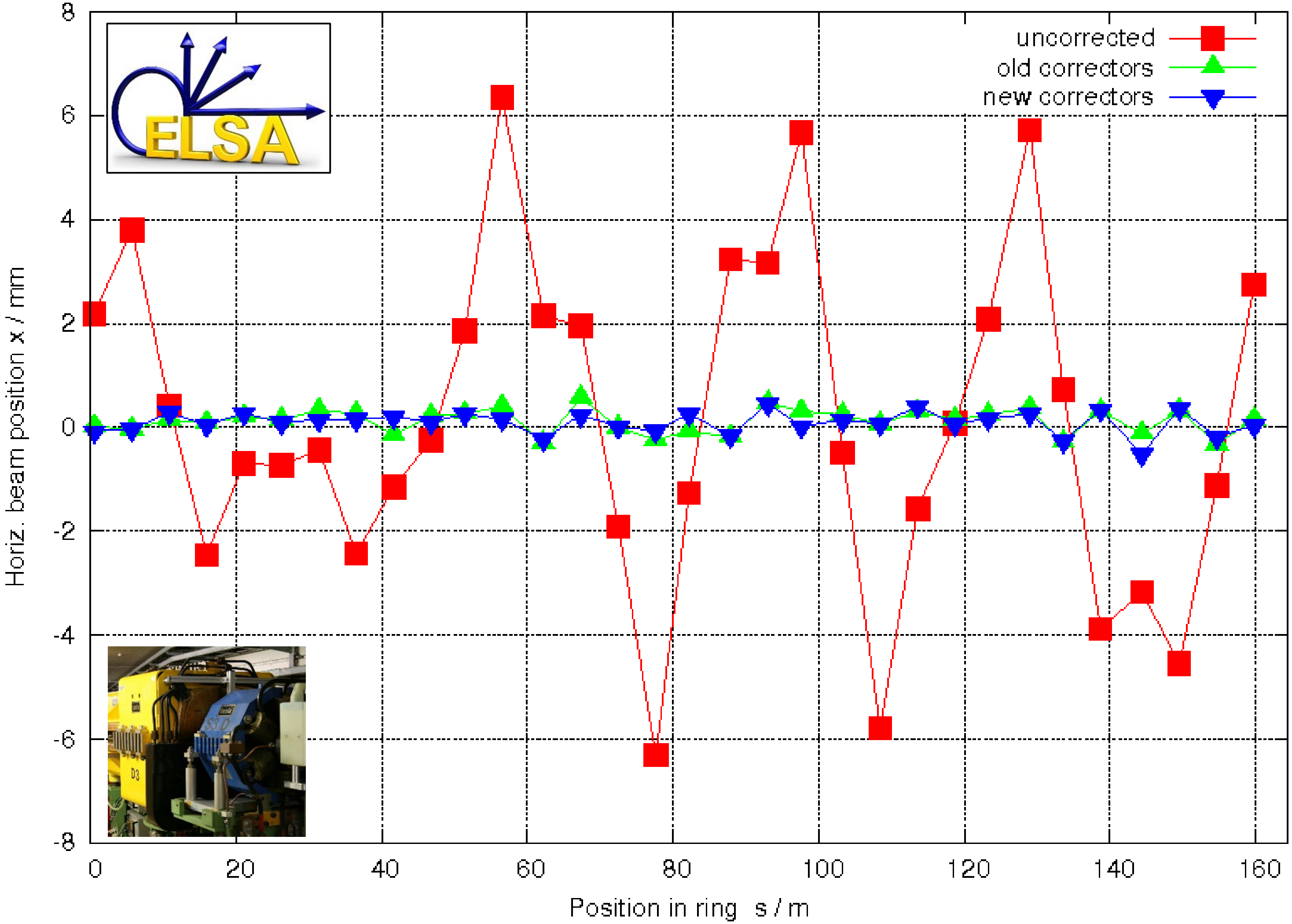
The effect of a random distribution of dipole errors can be estimated from the r.m.s. average, weighted according to the $\beta(s_0)$ values of the kicks δx_i at positions s_i :

$$x_c(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} \cdot \oint_C \sqrt{\beta(s_0)} \cdot \frac{\delta(Bl)}{B\rho}(s_0) \cdot \cos\left(\mu(s) - \mu(s_0) + Q\pi\right) \cdot \mathrm{d}s_0$$

We obtain the very important finding, that for integer tunes the denominator will be zero and the closed orbit displacements will grow to infinite values!

Example: Small Electron Ring

Closed orbit distortions: uncorrected and corrected



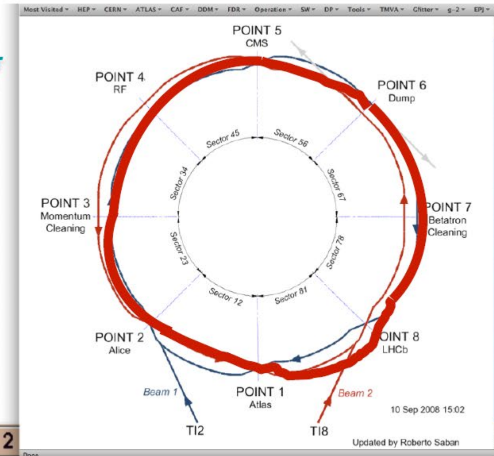
Example: Large Proton Ring

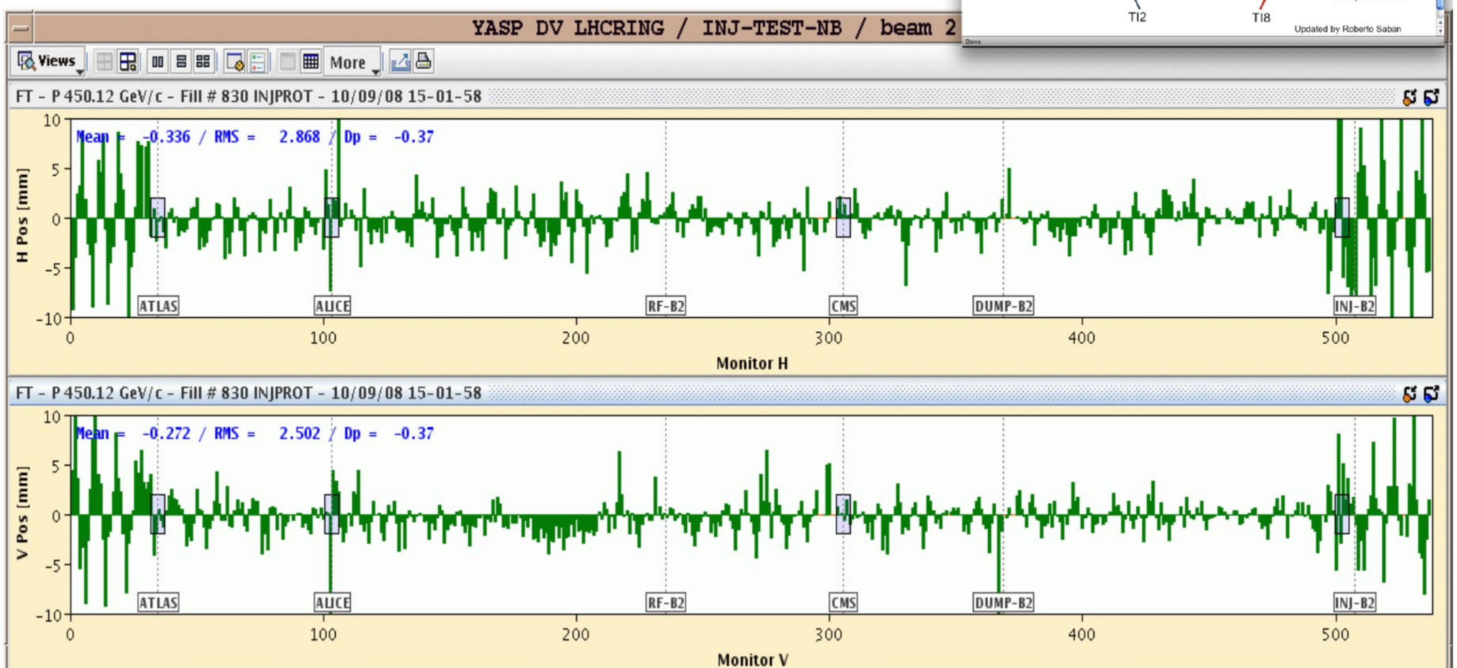
LHC Operation: Beam Commissioning

First turn steering "by sector:"

- One beam at the time
- Beam through 1 sector (1/8 ring),

correct trajectory, open collimator and move on.

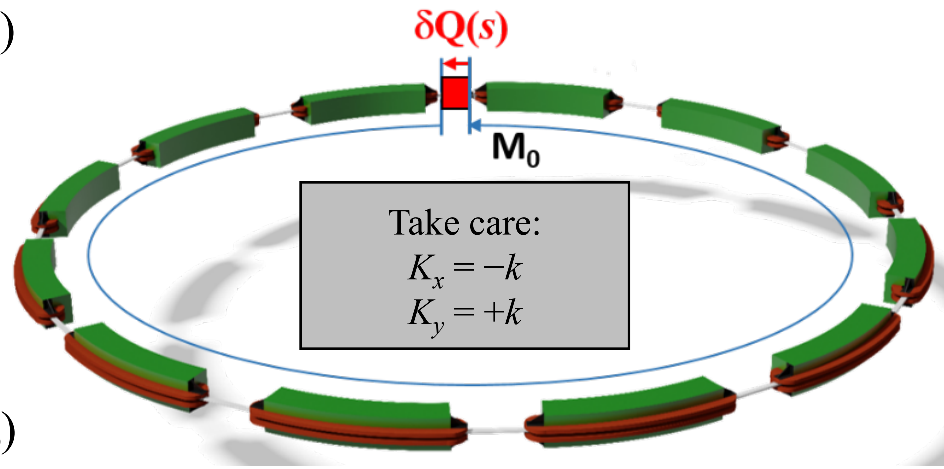




Courtesy of Bernhard Holzer, CAS lectures

Gradient Errors

Consider a small gradient error which affects a quadrupole at position s in the lattice of a circular accelerator. Translated to matrix algebra, we have to multiply a **perturbation matrix** ($K > 0$ means focusing!)



$$\boldsymbol{\delta}\mathbf{Q}(s) = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\delta K(s)\cdot \mathrm{d}s & 1 \end{pmatrix}$$

with the **unperturbed matrix** for one circle staring at s (where $\alpha(s)=\alpha_0$, $\beta(s)=\beta_0$, $\gamma(s)=\gamma_0$)

$$\mathbf{M}_0 = \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ -\gamma_0 \sin\mu_0 & \cos\mu_0 - \alpha_0 \sin\mu_0 \end{pmatrix}$$

giving the one-turn matrix (incl. perturbation) directly after the field error via

$$\tilde{\mathbf{M}}(s) = \boldsymbol{\delta}\mathbf{Q}(s)\cdot \mathbf{M}_0$$

Gradient Errors

$$\tilde{\mathbf{M}}(s) = \begin{pmatrix} \cos\mu_0 + \alpha_0 \sin\mu_0 & \beta_0 \sin\mu_0 \\ -\delta k\, ds\left(\cos\mu_0 + \alpha_0 \sin\mu_0\right) - \gamma_0 \sin\mu_0 & -\delta K\, \mathrm{d}s\, \beta_0 \sin\mu_0 + \cos\mu_0 - \alpha_0 \sin\mu_0 \end{pmatrix}$$

From $\frac{1}{2}\mathrm{Tr}\left\{\tilde{\mathbf{M}}\right\} = \cos\mu$ we can calculate the change in $\cos\mu$

$$\Delta\left(\cos\mu\right) = -\Delta\mu \cdot \sin\mu_0 = -\frac{1}{2}\sin\mu_0\, \beta_0\, \delta K\, \mathrm{d}s$$

$$2\pi\Delta Q = \Delta\mu = \frac{1}{2}\beta(s)\,\delta K(s)\,\mathrm{d}s$$

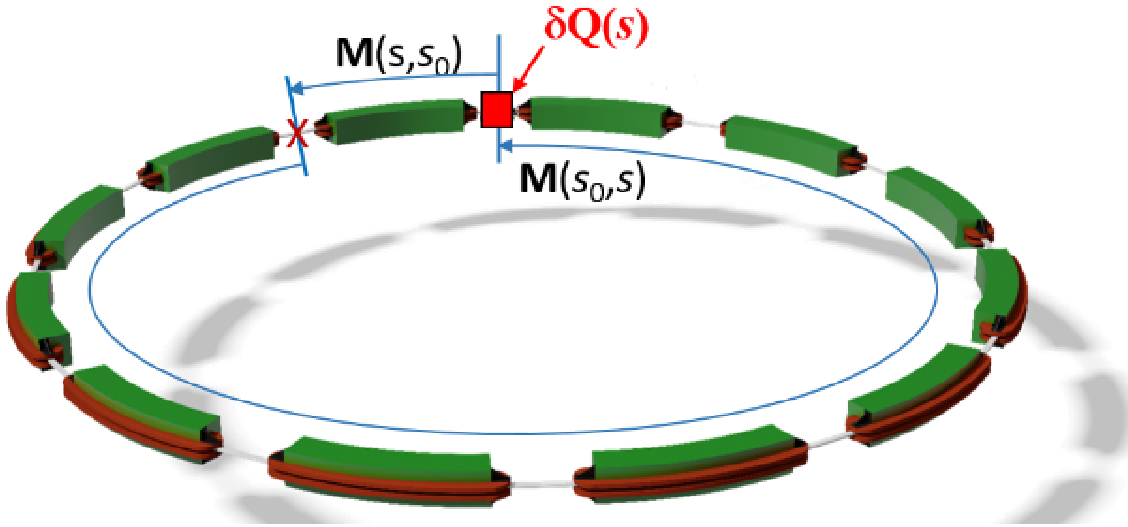
Integrating over the length of the quadrupole perturbation(s), one obtains

$$\Delta Q = \frac{1}{4\pi}\oint \beta(s)\,\delta K(s)\,\mathrm{d}s$$

With this very important formula, we generally can calculate the impact of quadrupole errors on the betatron tune!

Beta Beating

We now want to look at the impact of gradient errors on the beta function



A gradient error will not influence the closed orbit but the betatron function of the lattice. In order to calculate the betatron amplitude modulation, we have to determine the single turn transport matrix starting at a given observer position s, introducing a small gradient perturbation at position s_0

$$\tilde{r}_{12} = \Delta(\beta \cos \mu)$$

$$\tilde{\mathbf{M}}_s = \begin{pmatrix} \square & \tilde{r}_{12} \\ \square & \square \end{pmatrix} = \mathbf{M}(s,s_0) \cdot \boldsymbol{\delta}\mathbf{Q}(s_0) \cdot \mathbf{M}(s_0,s) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\delta K \, \mathrm{d}s_0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Beta Beating

It is only necessary to evaluate the element $\tilde{r}_{12}$ which is

$$\tilde{r}_{12} = b_{11}a_{12} + b_{12}\left(-\delta K\,\mathrm{d}s \cdot a_{12} + a_{22}\right) = r_{12} - \delta K\,\mathrm{d}s_0 \cdot a_{12}b_{12}$$

where r_{12} from the unperturbed matrix found by putting $\delta K\,\mathrm{d}s_0 = 0$

Thus the variation in the r_{12} term due to the perturbation is

$$\Delta\left[\beta(s)\sin\left(2\pi Q_0\right)\right] = -\delta K\,\mathrm{d}s_0\,\beta(s)\,\beta(s_0)\cdot\sin\left(\mu(s)-\mu(s_0)\right)\cdot\sin\left(\mu(s_0)-\mu(s)\right)$$
$$= -\delta K\,\mathrm{d}s_0\,\beta(s)\,\beta(s_0)\cdot\sin\left(\mu(s)-\mu(s_0)\right)\cdot\sin\left[2\pi Q_0-\left(\mu(s)-\mu(s_0)\right)\right]$$

Using $\sin\alpha\cdot\sin\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right)-\cos\left(\alpha+\beta\right)\right]$ the left-hand and right-hand sides can be expanded to give

$$\Delta\beta(s)\sin\left(2\pi Q_0\right) + \beta(s)\cdot\underbrace{2\pi\Delta Q}\cdot\cos\left(2\pi Q_0\right) =$$

$$\uparrow = \downarrow$$

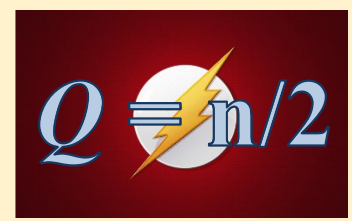
$$\overbrace{\frac{1}{2}\delta K\,\mathrm{d}s_0\,\beta(s_0)}\,\beta(s)\left\{\cos\left(2\pi Q_0\right)-\cos\left[2\left(\mu(s)-\mu(s_0)-\pi Q_0\right)\right]\right\}$$

Beta Beating

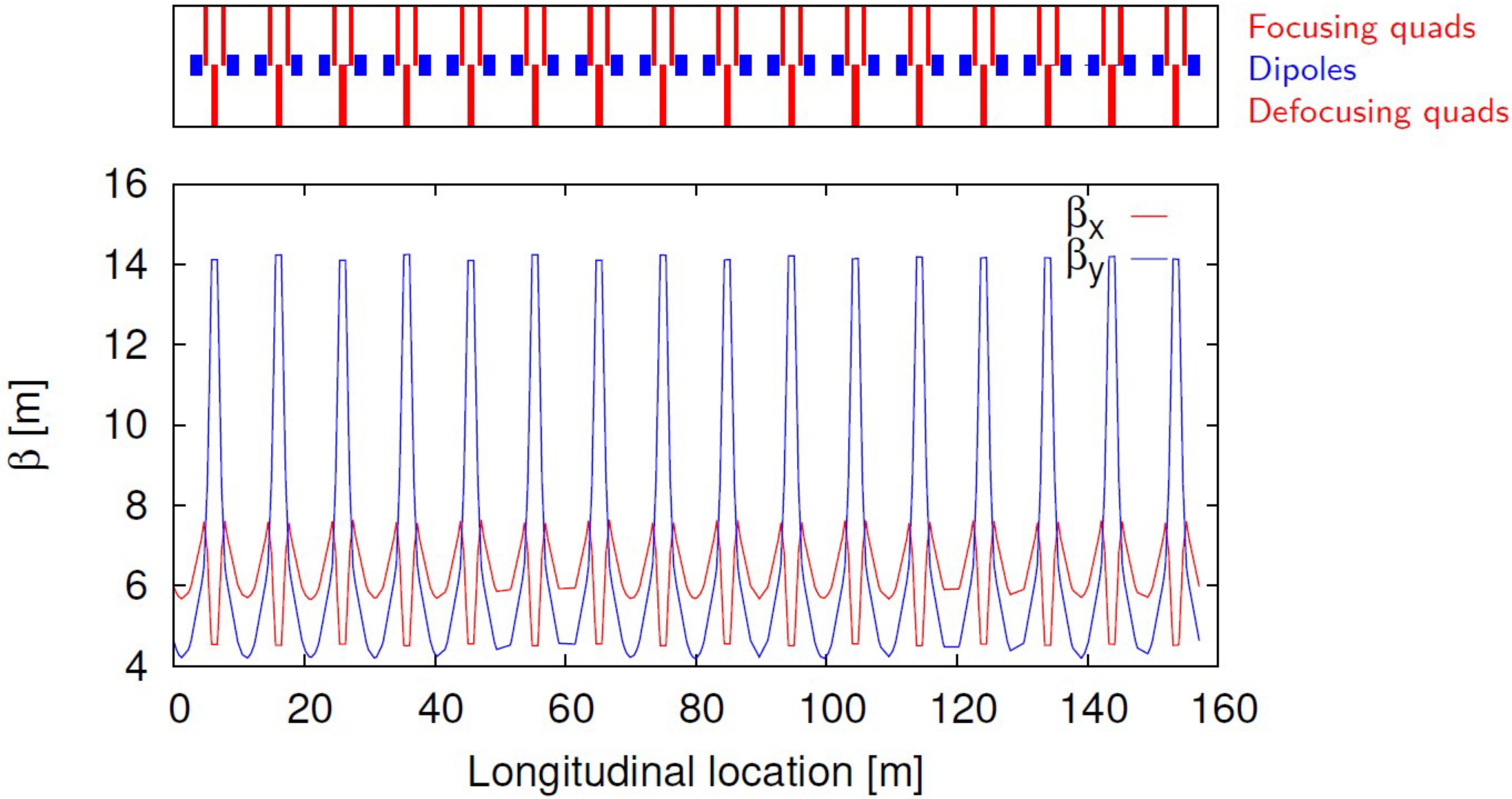
This leaves the final expression for the betatron amplitude modulation (the so-called **beta beating**):

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin\left(2\pi Q\right)}\cdot\oint_C \delta K(s_0)\beta(s_0)\cos\left[2\left(\mu(s)-\mu(s_0)-\pi Q_0\right)\right]\cdot \mathrm{d}s_0$$

We obtain the very important finding, that for half integer tunes the denominator will be zero and the beta function and therewith the beam size will grow to infinite values!

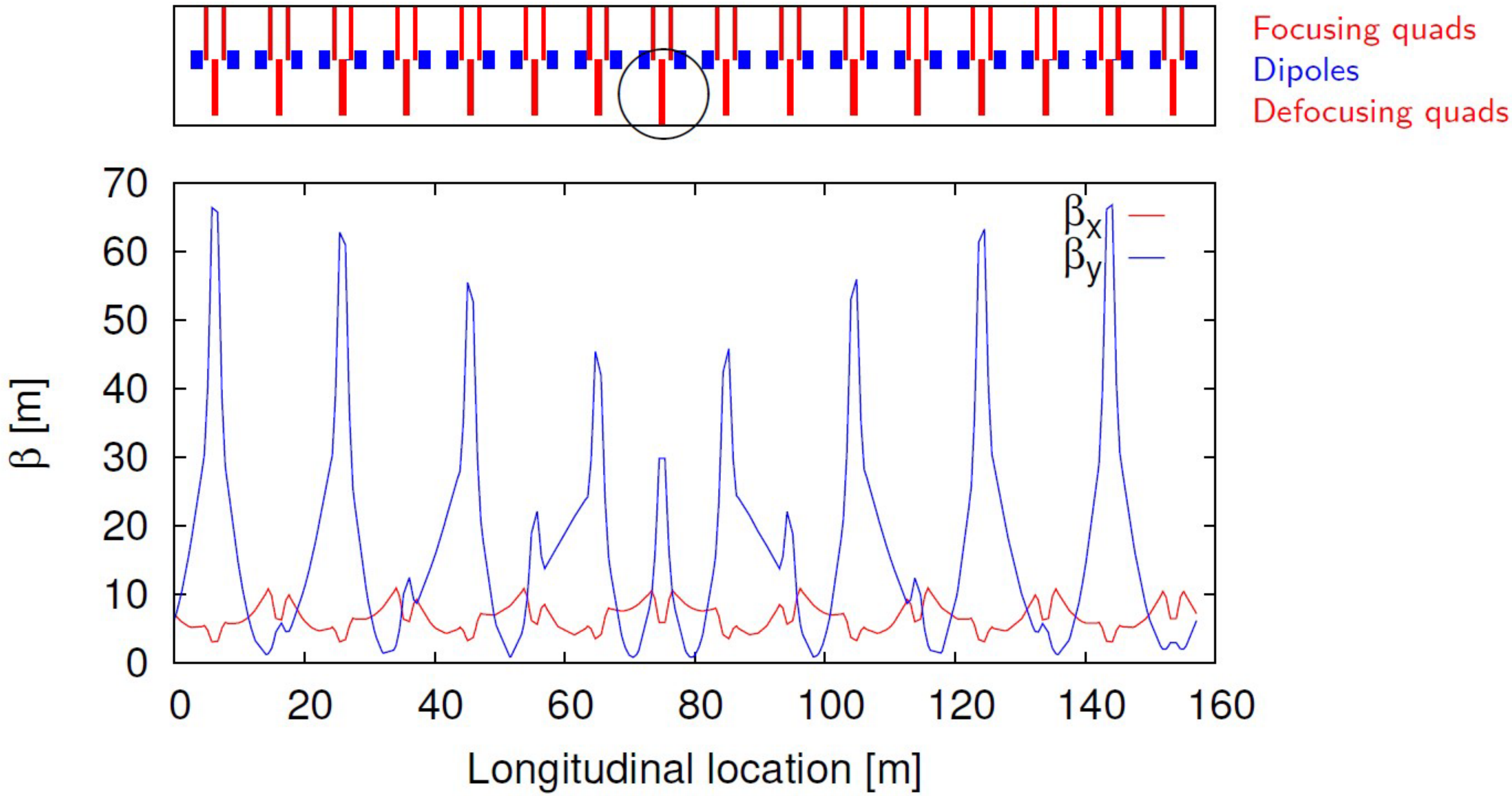


Ideal World



Focusing quads
Dipoles
Defocusing quads
16
14
12
10
8
6
4
β [m]
β_x
β_y
0
20
40
60
80
100
120
140
160
Longitudinal location [m]

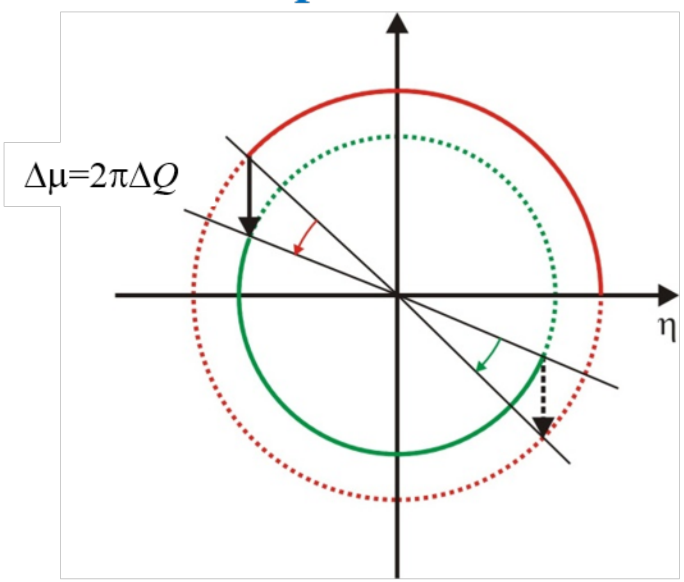
Single Quadrupole Error



Optical Resonances

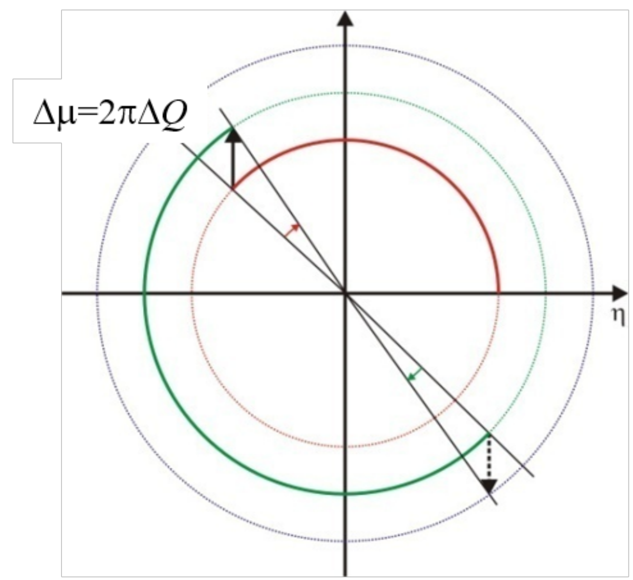
*Dipole errors will give a large closed orbit displacement when the tune is close to an integer value. Gradient errors will produce an average tune shift ΔQ and an amplitude modulation of the beta function which will explode for half integer Q values. These phenomena are called **optical resonances**. Due to the turn by turn modulation of the tune, there exist regions of instability called stop bands around the resonance conditions. The width of these stop bands are given by the tune modulation amplitude. These effects can be studied best when regarding the normalized phase space, where the particles ellipses transform to circles:*

Dipole errors:



No average tune shift $\Delta Q = 0$
Tune modulation amplitude $\mathrm{d}Q$

Quadrupole errors:



Average tune shift $\Delta Q = \frac{1}{4\pi}\beta\delta(kl)$
Tune modulation amplitude $\mathrm{d}Q = \Delta Q$

Auto Locking

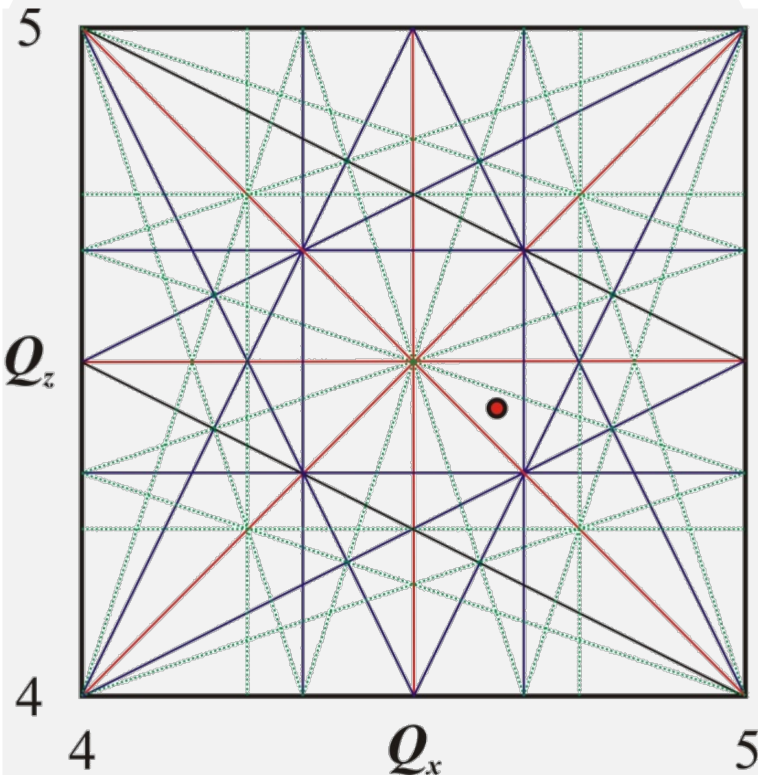
Any particle whose unperturbed Q lies in the stop band width dQ will lock into resonance and is lost! The width of the stopband is determined by the strength of the field error and the order (integer, half integer, third integer, …) of the resonance!

We may generalize and give a list of resonances and their driving multipoles:

resonance type	driving multipole
integer resonance $Q = n$	dipole errors
half-integer resonance $2Q = n$	quadrupole errors
third-integer resonances $3Q = n$	sextupole + dipole errors
…	…

Tune Diagram

Example:



"small" ring:

- 164 m circumference
- 16 dipoles
- 32 quadrupoles

Due to betatron coupling, perturbations may depend on the betatron amplitude in both planes. These coupling terms lead to the generalized resonance condition

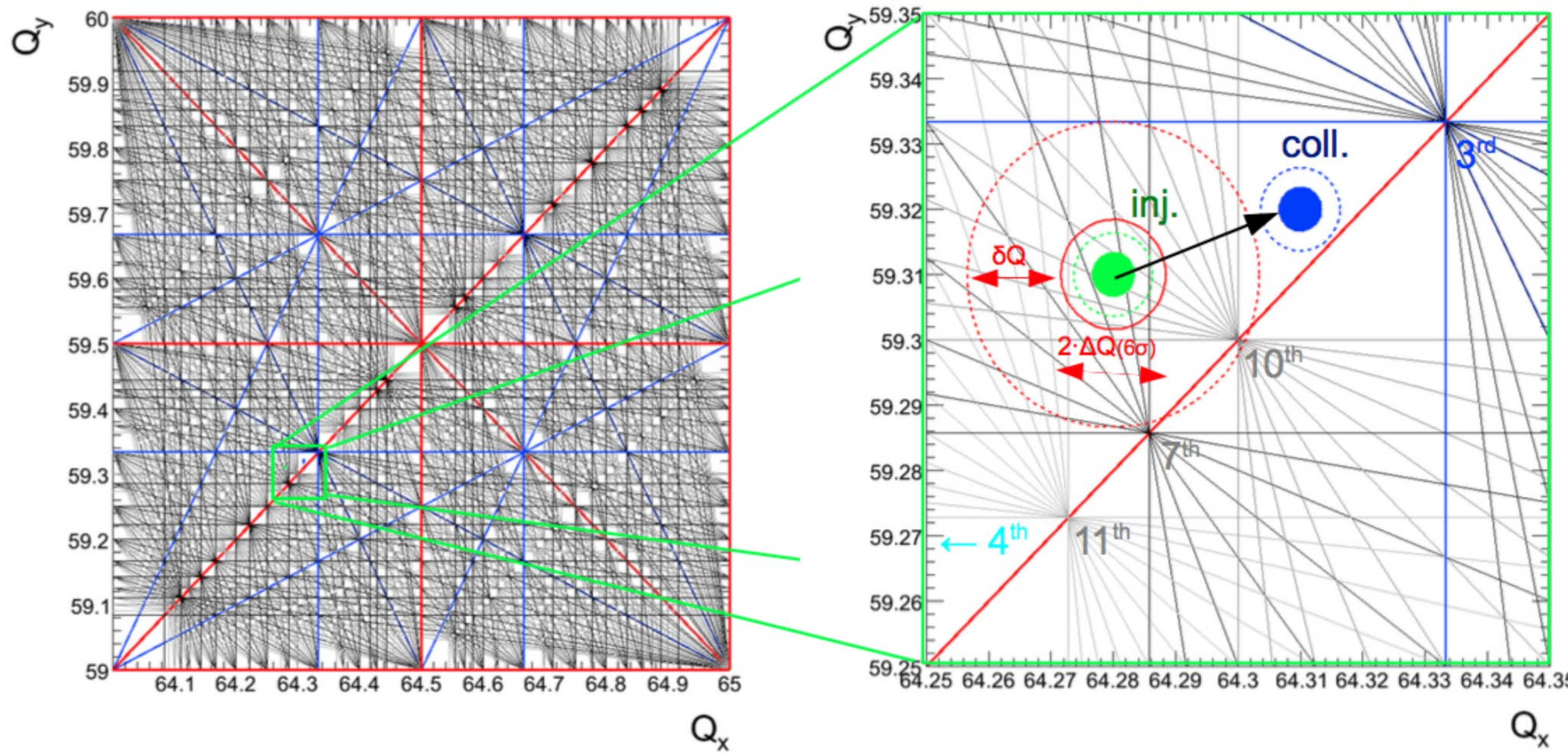
$$j \cdot Q_x + l \cdot Q_z = N$$

where $j+l$ indicates the **order** of the resonance. The circle represents the tune on the energy ramp of ELSA.

Tune Diagram

Example:

LHC



Tune stability requirements: $\Delta Q < 0.001$ vs exp. Drifts ~ 0.06

Note: need to stay much further of resonances due to finite tune width (chromaticity, momentum spread), space charge, beam-beam, etc., and finite width of stop bands.

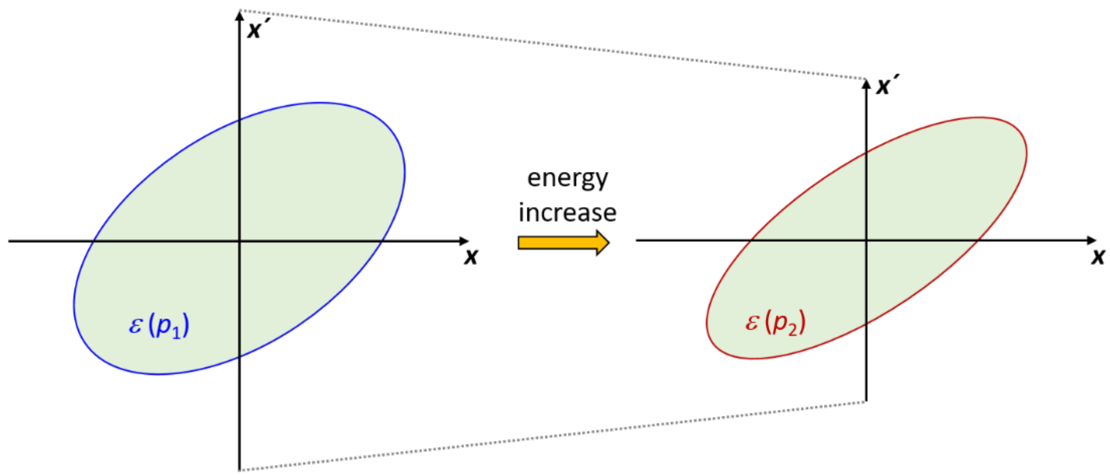
Adiabatic Damping

Trace space in accelerator physics $\neq$ **Phase space** in classical mechanics:

coordinates x, x' $\leftrightarrow$ canonical coordinates x, p_x

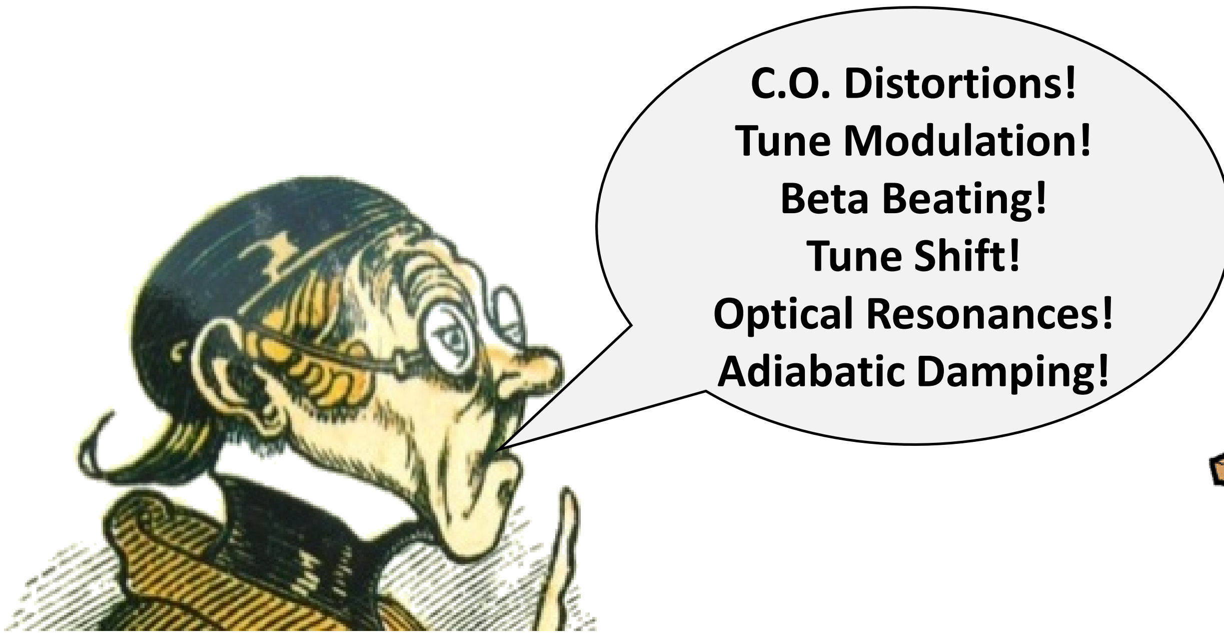
$$p_x = m \cdot \dot{x} = m \cdot \dot{s} \cdot x' \approx p_0 \cdot x' = \beta_r \gamma_r \cdot m_0 c \cdot x' \quad \rightarrow \quad \boxed{p_x \sim \beta_r \gamma_r \cdot x'}$$

Beam acceleration (momentum increase) causes compression of x' axis and therewith decrease of the beam emittance, which is called **adiabatic damping**:



→ Define normalized emittance, which is conserved: $\varepsilon_n = \beta_r \gamma_r \varepsilon$

End of 5th Lecture!



Questions?