# Recap 5th Lecture

## Real Circular Accelerators with Field Errors:

**Dipole Errors:** closed orbit deviations $x_{co}(s) \sim \frac{\sqrt{\beta(s)\beta(s_0)}}{\sin(\pi Q)} \delta(Bl) \quad \leftrightarrow \quad \boxed{Q \neq n}$

**Quadrupole Errors:** effect on beta function (beam size) and tune

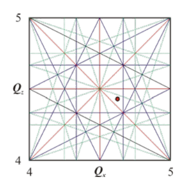
tune shift $\boxed{\Delta Q = \frac{1}{4\pi} \oint \beta(s) \delta K(s) \mathrm{d}s}$

beta beating $\Delta\beta(s) \sim \frac{\beta(s)\,\beta(s_0)}{2\sin(2\pi Q)} \cdot \delta K(s_0) \quad \leftrightarrow \quad \boxed{Q \neq n/2}$

**Optical Resonances:** fractional tunes lead to instabilities → stop bands in tune diagram

general resonance condition: $\boxed{m \cdot Q_x + n \cdot Q_y = k}$ $\quad m + n =$ order of resonance

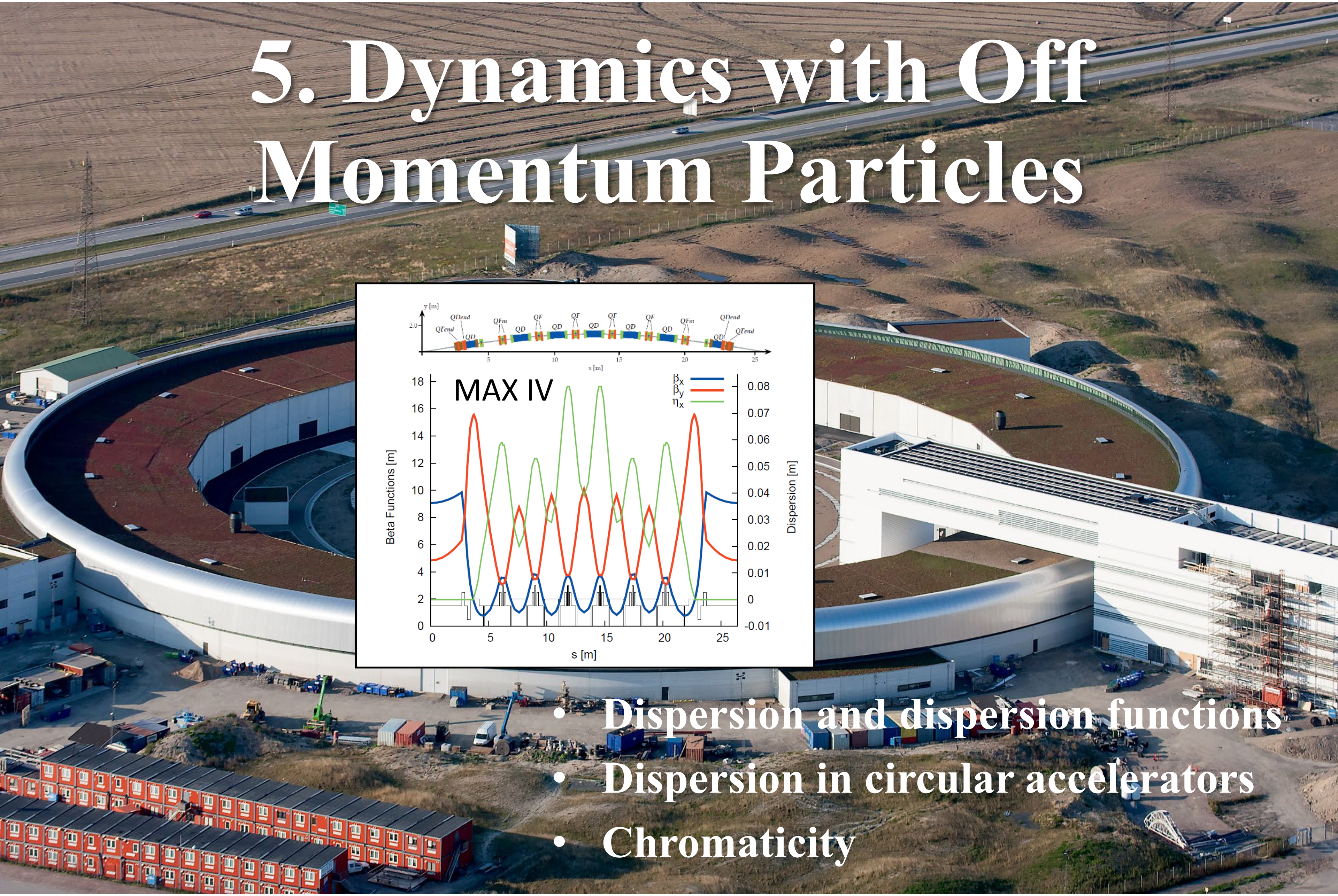
**Tune Diagram:** plot $Q_y$ vs. $Q_x$ with stop bands




**Adiabatic Damping:** Liouville → phase space, $u'$ "shrinks" through acceleration: $u' = \frac{p_u}{m_0 c} \cdot \frac{1}{\beta_r \gamma_r}$

**Normalized Emittance:** remains constant during acceleration $\varepsilon_n = \beta_r \gamma_r \cdot \varepsilon_{geo}$

# 5. Dynamics with Off Momentum Particles




- Dispersion and dispersion functions
- Dispersion in circular accelerators
- Chromaticity

Picture taken from https://en.wikipedia.org/wiki/MAX_IV_Laboratory#/media/File:Max_IV%E2%80%93flygbild_06_september_2014-2.jpg

# Equation of Motion

We will come back to the equation of motion, now explicitly treating the momentum dependent right hand side, depending on the relative momentum deviation $\delta = \Delta p / p_0$

$$x''(s) + \left( \frac{1}{\rho^2(s)} - k(s) \right) \cdot x(s) = \frac{1}{\rho(s)} \frac{\Delta p}{p}$$

$$y''(s) + k(s) \cdot y(s) = 0$$

Since the dynamics of off momentum particles is only affected in the horizontal plane, we will restrict the treatment to 1D including the momentum dependence and concentrate on the horizontal trace space $(x, x')$.

**Remaining task: find a particular solution $x_{ih}$ of the inhomogeneous equation!**

# Dipole Magnet

A particular solution for a non-vanishing $\delta = \Delta p / p$ is $x_{ih} = \rho \cdot \delta$.

Recalling the solution of the homogeneous equation gives

$$x(s) = x_h(s) + x_{ih}(s) = a \cdot \cos\left( s/\rho \right) + b \cdot \sin\left( s/\rho \right) + \rho \cdot \delta$$

The integration constants $a$, $b$ are again derived from the boundary conditions at $s = 0$, but now the inhomogeneous solution has to be included:

$$x(s=0) = a + \rho \cdot \delta = x_0, \quad x'(s=0) = \frac{b}{\rho} = x_0'$$

and by defining the bending angle $\varphi = L/\rho$ of the dipole magnet, we obtain

$$x(L) = x_0 \cdot \cos\varphi + \rho \cdot x_0' \cdot \sin\varphi + \rho(1 - \cos\varphi) \cdot \delta$$
$$x'(L) = -x_0/\rho \cdot \sin\varphi + x_0' \cdot \cos\varphi + \sin\varphi \cdot \delta$$

# Dispersion

This can be easily implemented in the matrix formalism by adding a 3rd component to the particle's position vector dealing with the actual relative momentum deviation compared to the reference particle:

$$\vec{x} = \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} \qquad M_{\text{dipole}} = \begin{pmatrix} \cos\varphi & \rho\sin\varphi & \rho(1-\cos\varphi) \\ -1/\rho\sin\varphi & \cos\varphi & \sin\varphi \\ 0 & 0 & 1 \end{pmatrix}$$

First neglecting the dependence of the quadrupole strength $k$ on the actual particle's momentum, the quadrupole transfer matrices remain "unchanged":

$$M_{\text{QF}} = \begin{pmatrix} \cos\Omega & 1/\sqrt{|k|}\sin\Omega & 0 \\ -\sqrt{|k|}\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\text{QD}} = \begin{pmatrix} \cosh\Omega & 1/\sqrt{|k|}\sinh\Omega & 0 \\ \sqrt{|k|}\sinh\Omega & \cosh\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

→ ***Hands-On Lattice Calculations*** *recommended: E25, E26*

# Dispersion Function

**Important:**

Whereas a quadrupole magnet will not directly cause an impact on the particle's trajectory, **a dipole magnet creates a (horizontal) dispersion**:

$$D = r_{13} = \rho\left(1-\cos\varphi\right), \qquad D' = r_{23} = \sin\varphi$$

The dispersion represents the offset due to a relative momentum deviation $\Delta p/p = 1$.

In general, we have:

$$x(s) = x_h(s) + x_D(s) = x_h(s) + D(s)\cdot\frac{\Delta p}{p}$$

Here, ***D*(s) is the dispersion function**, a solution of the equation of motion for $\delta = 1$.

This reads in vector notation:

$$\vec{x} = \begin{pmatrix} x \\ x' \\ \delta \end{pmatrix} = \begin{pmatrix} x_h + D\delta \\ x_h{}' + D'\delta \\ \delta \end{pmatrix} = \begin{pmatrix} x_h \\ x_h{}' \\ 0 \end{pmatrix} + \delta\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}$$

# Dispersion Function

**3x3 Formalism**:

Since the transformation of the homogeneous solution is well know from the 2x2 formalism and can be separated, we have

$$\boxed{\vec{x} = \mathbf{M}_{3x3} \cdot \vec{x}_0}$$

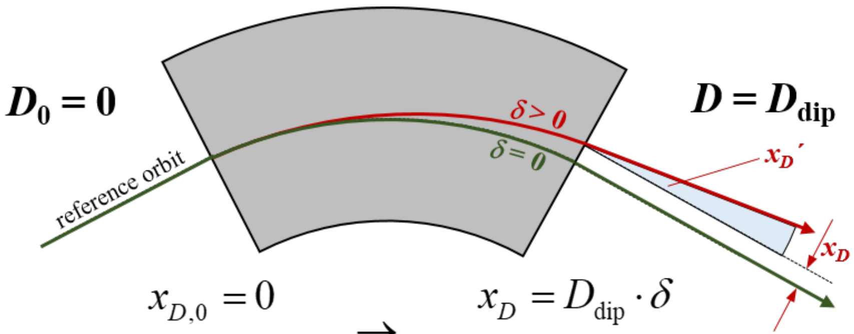
$$\begin{pmatrix} x(s) \\ x'(s) \\ \delta \end{pmatrix} = \begin{pmatrix} x_h \\ x_h' \\ 0 \end{pmatrix} + \delta \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \mathbf{M}_{3x3} \cdot \left[ \begin{pmatrix} x_{h,0} \\ x_{h,0}' \\ 0 \end{pmatrix} + \delta \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} \right] = \begin{pmatrix} x_h \\ x_h' \\ 0 \end{pmatrix} + \delta \cdot \mathbf{M}_{3x3} \cdot \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

and can thus calculate the **transformation of the dispersion function:**

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \mathbf{M}_{3x3} \cdot \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

# Generation of Dispersion

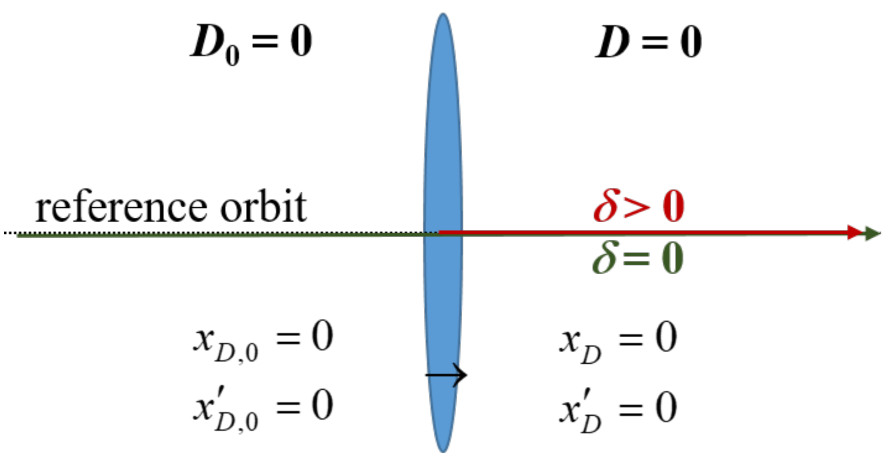
A dipole magnet will create dispersion → dispersion orbit




$$x_{D,0} = 0 \qquad x_D = D_{\text{dip}} \cdot \delta$$
$$x'_{D,0} = 0 \quad \rightarrow \quad x'_D = D'_{\text{dip}} \cdot \delta$$

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M}_{\text{dipole}}} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} r_{13} \\ r_{23} \\ 1 \end{pmatrix}$$

A quadrupole magnet will not create any dispersion:




$$x_{D,0} = 0 \qquad x_D = 0$$
$$x'_{D,0} = 0 \quad \rightarrow \quad x'_D = 0$$

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M}_{\text{quadrupole}}} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

# Dispersion Function

**But now take care**:

Due to $x(s) = x_h(s) + x_D(s)$ we will observe a change of the **dispersion orbit** $x_D(s)$ when passing a dipole magnet or a quadrupole magnet!!

Both dipole and quadrupole magnets therefore will modify an existing dispersion according to

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M}_{\text{dipole}}} \cdot \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} \qquad \begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M}_{\text{quadrupole}}} \cdot \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

# Beams with Dispersion

The influence of dispersion on the effective beam size, divergence and emittance can be calculated using the statistical definition and replacing $\delta$ by the r.m.s. relative momentum spread $\sigma_\delta$:

$$\tilde{\sigma}_x^2 = \frac{1}{N}\sum_{i=1}^{N}\left(x_i + D\delta_i\right)^2 = \frac{1}{N}\sum_{i=1}^{N}\left(x_i\right)^2 + \frac{1}{N}D^2\sum_{i=1}^{N}\left(\delta_i\right)^2 + \frac{2}{N}D\sum_{i=1}^{N}\left(x_i\delta_i\right) = \sigma_x^2 + \left(D\cdot\sigma_\delta\right)^2$$

$$\tilde{\sigma}_{x'}^2 = \frac{1}{N}\sum_{i=1}^{N}\left(x_i' + D'\delta_i\right)^2 = \frac{1}{N}\sum_{i=1}^{N}\left(x_i'\right)^2 + \frac{1}{N}D'^2\sum_{i=1}^{N}\left(\delta_i\right)^2 + \frac{2}{N}D'\sum_{i=1}^{N}\left(x_i'\delta_i\right) = \sigma_{x'}^2 + \left(D'\cdot\sigma_\delta\right)^2$$

and thus

$$\tilde{\sigma}_x = \sqrt{\varepsilon_x\beta_x + \left(D\cdot\sigma_\delta\right)^2}, \qquad \tilde{\sigma}_x' = \sqrt{\varepsilon_x\gamma_x + \left(D'\cdot\sigma_\delta\right)^2}$$

→ ***Hands-On Lattice Calculations***
*recommended: E27, E28*

The effective emittance has to be obtained from a lengthy calculation first defining a generalized 6-dimensional intensity distribution $I(\vec{U})$ and then integrating over unwanted coordinates and thus deriving an intensity distribution $I(x,x',\delta)$ in which finally appears:

$$\tilde{\varepsilon}_x = \sqrt{\varepsilon_x^2 + \varepsilon_x\cdot\sigma_\delta^2\cdot\left(\gamma_x D^2 + 2\alpha_x DD' + \beta_x D'^2\right)}$$

# Periodic Dispersion Functions

In a periodic lattice, the dispersion function has – as well as the beta function – to fulfill periodic boundary conditions:

$$D(s_0 + L) = D(s_0)$$

Thus the dispersion function can obtained from applying the 3x3 transport matrix **M**($L$):

$$\begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} = \mathbf{M}(L) \cdot \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} D_0 \\ D_0' \\ 1 \end{pmatrix}$$

yielding

$$D_0 = \frac{r_{13}\left(1 - r_{22}\right) + r_{12}\, r_{23}}{2 - r_{11} - r_{22}} = \frac{r_{13}\left(1 - r_{22}\right) + r_{12}\, r_{23}}{2\left(1 - \cos\mu\right)}$$

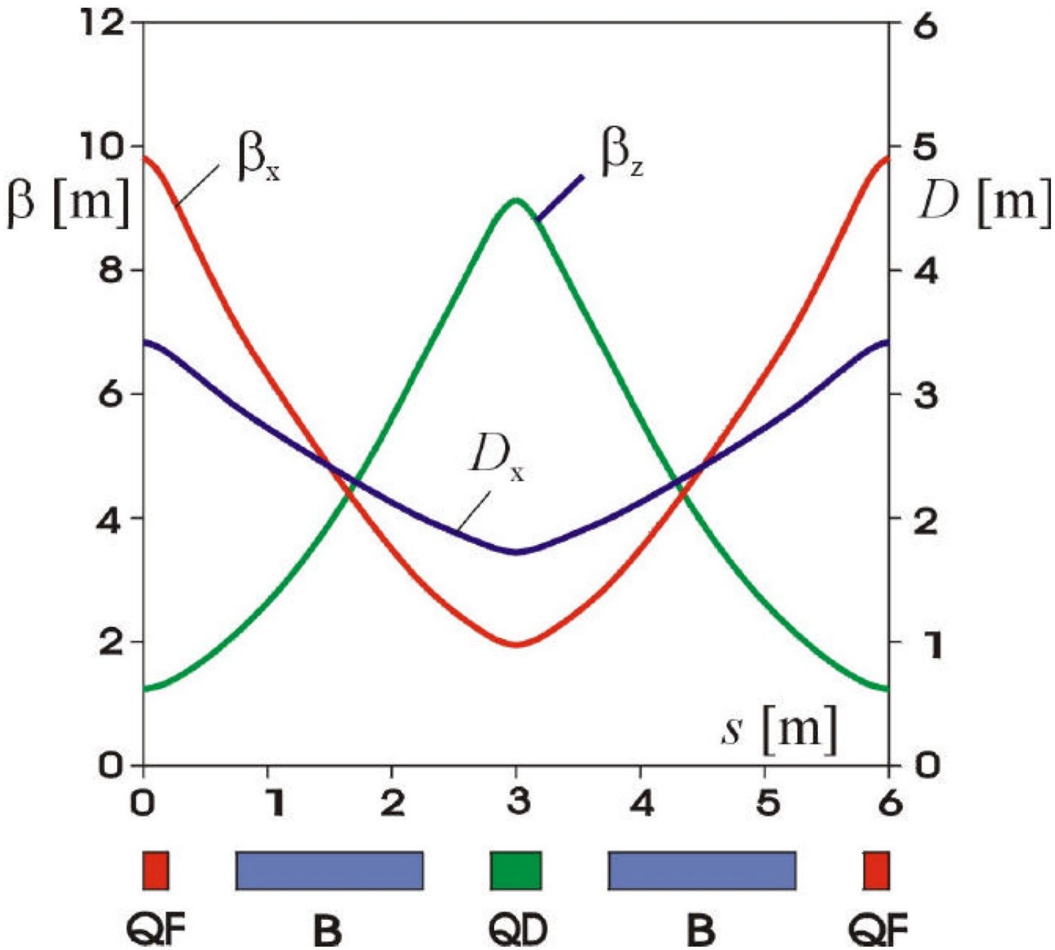
$$D_0' = \frac{r_{13}\left(1 - r_{11}\right) + r_{21}\, r_{13}}{1 - r_{11} - r_{22}} = \frac{r_{13}\left(1 - r_{11}\right) + r_{21}\, r_{13}}{2\left(1 - \cos\mu\right)}$$

which for a symmetry point, where $D'(s) = 0$, simplifies to

$$D_0^{\text{sym}} = \frac{r_{13}}{1 - r_{11}}$$

# Dispersive Lattice

Applying this to our model toy synchrotron, we can derive the dispersion function which is plotted in blue:




The dispersion shows the same general behaviour as the corresponding beta function!

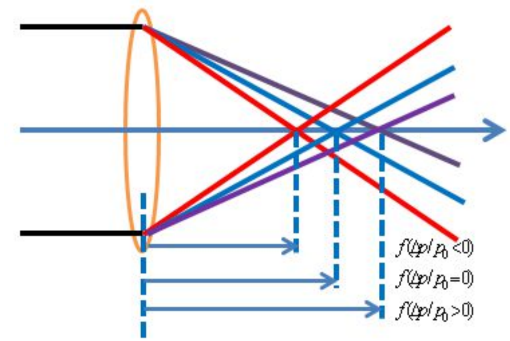
**Please note that the total beam width is given by**

$$\sigma_x = \sqrt{\varepsilon_x \beta_x + \left(D_x \sigma_\delta\right)^2}$$

→ ***Hands-On Lattice Calculations***
*recommended: E29*

# Chromaticity

Since the quadrupole strength k was normalized to the reference momentum $p_0$, the focusing of off-momentum particles will be different. Thus, we expect a different tune for off-momentum particles!




The variation of tunes is called **chromaticity** and is defined by the factor $\xi$ in

$$\Delta Q_{x,y} = \xi_{x,y} \cdot \frac{\Delta p}{p_0}$$

We distinguish between **natural chromaticity** created by the chromatic aberration of quadrupole magnets and perturbations derived from non-linear perturbations in the particles trajectories (e.g. produced by sextupole magnets).

# Natural Chromaticity

The quadrupole strength scales with the particles momentum:

$$k_0 = \frac{q}{p_0} g \quad \xrightarrow{\Delta k = \frac{\partial k}{\partial p}\Delta p} \quad \Delta k = -k_0 \cdot \Delta p / p_0$$

and the tune shift can therefore be calculated from the perturbation formula:

$$\Delta Q_{x,y} = \underbrace{-\frac{1}{4\pi}\oint \beta_{x,y}(\tilde{s}) \cdot K_{x,y}(\tilde{s}) \cdot \mathrm{d}\tilde{s}}_{=\xi_{x,y}} \cdot \frac{\Delta p}{p_0}$$

Take care:
$K_x = -k$
$K_y = +k$

**Natural chromaticity:** $\xi_{x,y} = -\frac{1}{4\pi}\oint \beta_{x,y}(\tilde{s}) \cdot K_{x,y}(\tilde{s}) \cdot \mathrm{d}\tilde{s}$

Since in a linear lattice, the horizontal beta function is always maximal in a focusing quadrupole where $K > 0$ and minimal in a defocusing quadrupole where $K < 0$, the natural chromaticity is always negative!

# Chromaticity produced by Sextupoles

A beam of particles moving on a dispersion orbit through a sextupole magnet is (de-)focused by the nonlinear field due to horizontal displacement $x = D \cdot \Delta p / p_0$. We thus can derive a position dependent focusing strength from

$$\frac{q}{p}\vec{B}_{sext} = m\,x\,y\,\hat{e}_x + \frac{1}{2}m\left(x^2 - y^2\right)\hat{e}_y$$

giving a dispersion dependent $K_{x,sext}$ and $K_{y,sext}$ to:

$$K_{x,sext} = \frac{q}{p}\cdot\frac{\partial B_y}{\partial x} = m \cdot x = m \cdot D \cdot \frac{\Delta p}{p_0}$$

$$K_{y,sext} = -\frac{q}{p}\cdot\frac{\partial B_x}{\partial y} = -m \cdot x = -m \cdot D \cdot \frac{\Delta p}{p_0}$$

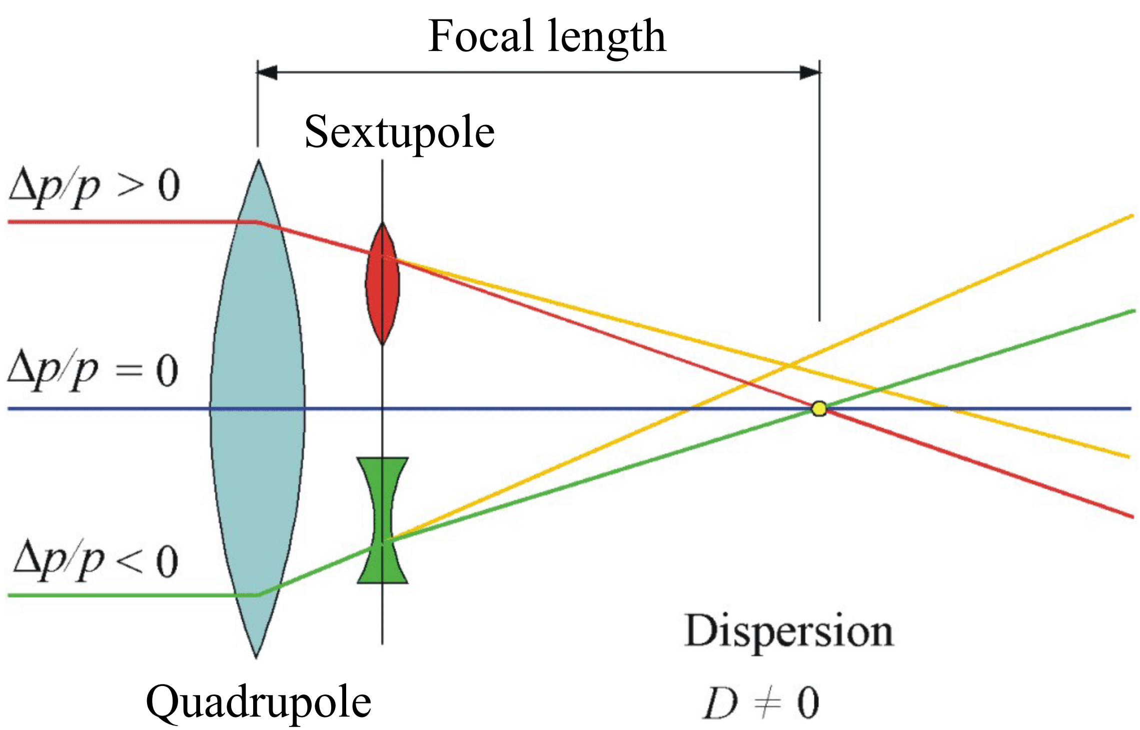
Take care:
$K_x = -k$
$K_y = +k$

This adds to the natural chromaticity and gives in total:

$$\xi_{x,y} = -\frac{1}{4\pi}\oint \beta_{x,y}(\tilde{s})\cdot\left[K_{x,y}(\tilde{s}) \mp m(\tilde{s})\cdot D(\tilde{s})\right]\cdot \mathrm{d}\tilde{s}$$

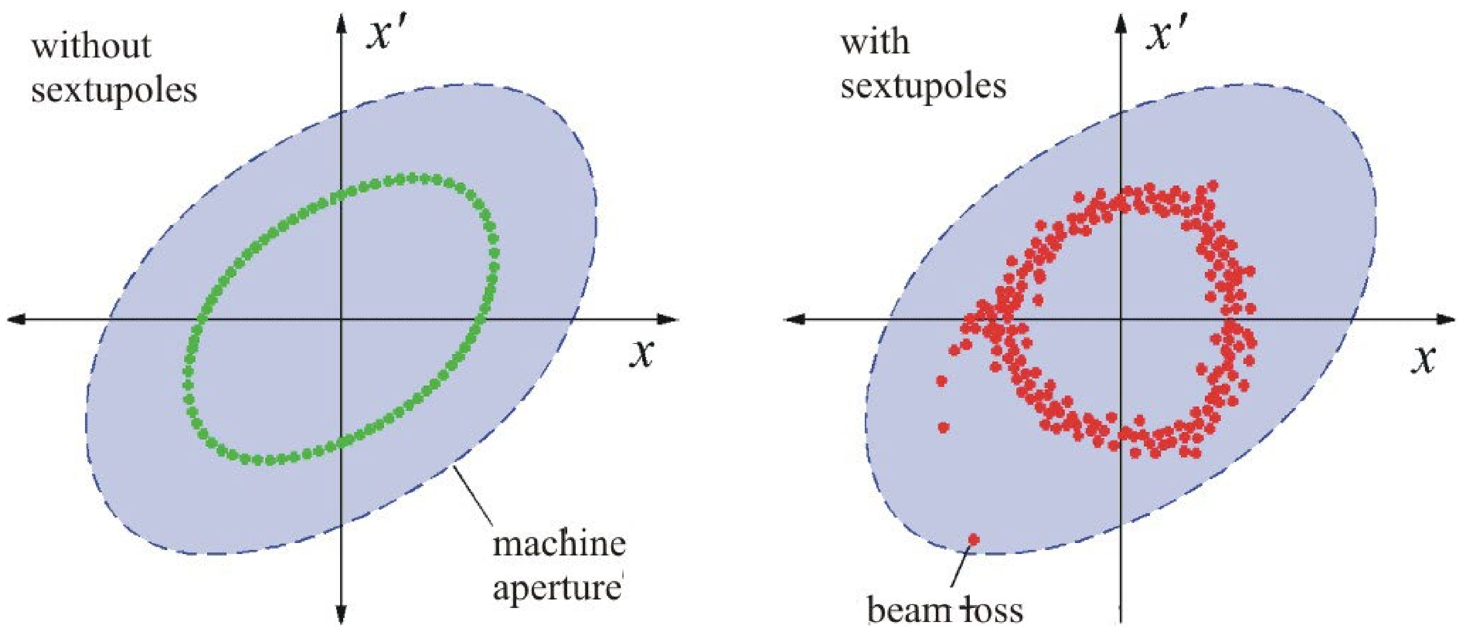
# Chromaticity Correction

In order to avoid a large tune spread, chromaticity has to be corrected by the use of additional sextupole magnets right after focusing and defocusing quadrupoles where the horizontal dispersion does not vanish



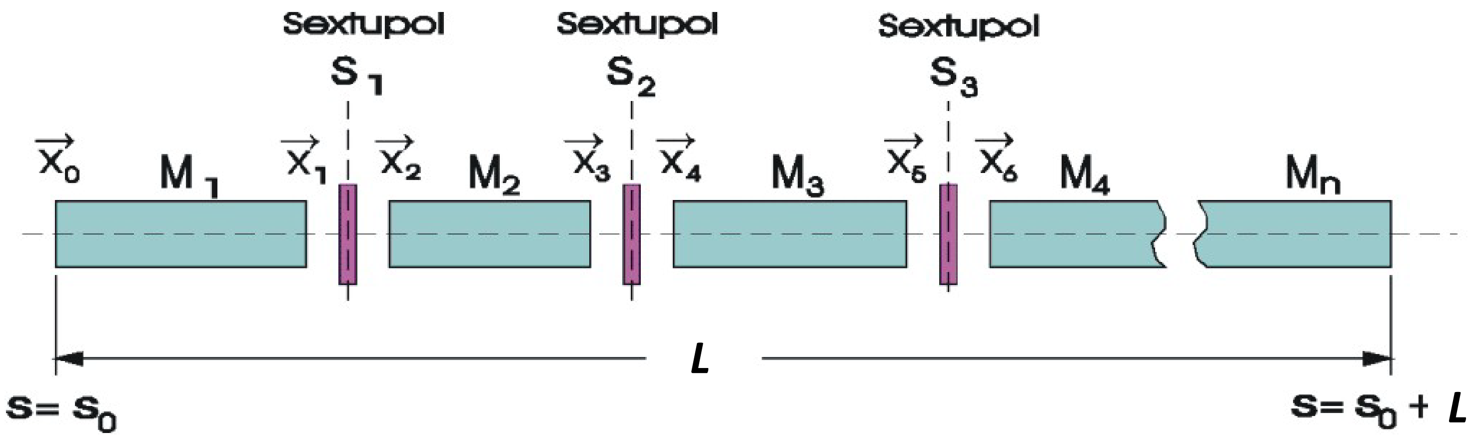
# Dynamic Aperture

This correction will have an influence on the stability of the beam and the maximum aperture given by nonlinear effects (so called dynamic aperture):




The dynamic aperture can be calculated from a tracking of the particles orbit through the accelerator where the nonlinear effect of sextupole magnets has to be treated as step by step correction in linear beam matrix optics:

# Dynamic Aperture




The orbit vector is transformed from $s_0$ to $s_1$ by matrix transformation $\vec{U}_1 = \mathbf{M}_1 \cdot \vec{U}_0$

A sextupole of length $l$ will produce an angular kick in the horizontal and vertical orbit of

$$\Delta x_1' = \frac{1}{2} m\, l \cdot \left( x_1^2 - y_1^2 \right) \qquad \Delta y_1' = m\, l \cdot x_1\, y_1$$

which gives an orbit vector right after the sextupole of

$$\vec{U}_2 = \begin{pmatrix} x_1 \\ x_1' + \Delta x_1' \\ y_1 \\ y_1' + \Delta y_1' \end{pmatrix}$$

By this method a randomly chosen distribution of start vectors $\vec{U}_0$ is tracked through the accelerator for many revolutions and the resulting dynamic aperture is derived from the phase space representation.

# The End!


**Thanks for listening!**

**Enjoy the coming lectures**

**and *Hands-ON Calculations* !**