

LONGITUDINAL beam DYNAMICS in circular accelerators



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Introduction to Accelerator Physics
Santa Susanna, Spain, 22/9 - 5/10/2024

Scope and Summary of the 2 lectures:

The goal of an accelerator is to provide a **stable particle beam**.

The particles nevertheless perform **transverse betatron oscillations**.

We will see that they also perform (so-called **synchrotron**) **oscillations** in the **longitudinal** plane and in **energy**.

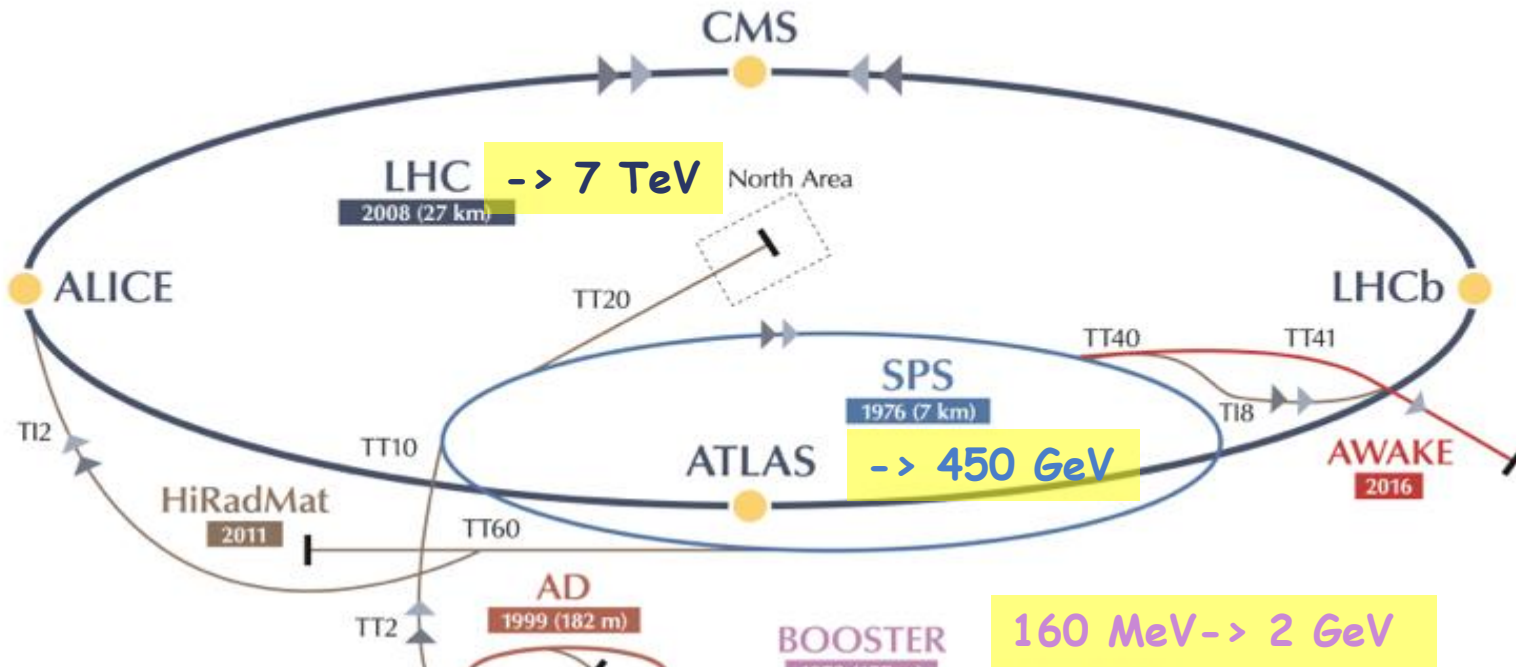
We will look at the stability of these oscillations, their dynamics and derive some basic equations.

- Introduction
- Circular accelerators: Cyclotron / Synchrotron
- Dispersion Effects in Synchrotron
- Stability and Longitudinal Phase Space Motion
- Hamiltonian
- Stationary Bucket
- Injection Matching

More related lectures:

- Linacs
 - RF Systems
 - Electron Beam Dynamics
 - Non-Linear longitudinal Beam Dynamics
 - **Hands-on calculations** - longitudinal in the second week !!!
- David Alesini
 - Christine Völlinger
 - Lenny Rivkin
 - Heiko Damerau

Motivation for circular accelerators



- Linear accelerators scale in size and cost(!) ~linearly with the energy.
 - Circular accelerators can each turn reuse
 - the accelerating system
 - the vacuum chamber 4
 - the bending/focusing magnets
 - beam instrumentation, ...
- > economic solution to reach higher particle energies
 But each accelerator has a limited energy range.

Particle types and acceleration

The accelerating system will depend upon the **evolution** of the **particle velocity**:

- **electrons** reach a **constant velocity** (~speed of light) at relatively low energy
- **heavy particles** reach a constant velocity only at very high energy
 - > need different types of resonators, optimized for different velocities
 - > the **revolution frequency will vary**, so the **RF frequency** will be **changing**
 - > magnetic field needs to follow the momentum increase

Particle rest mass m_0 :

electron 0.511 MeV

proton 938 MeV

^{239}U ~220000 MeV

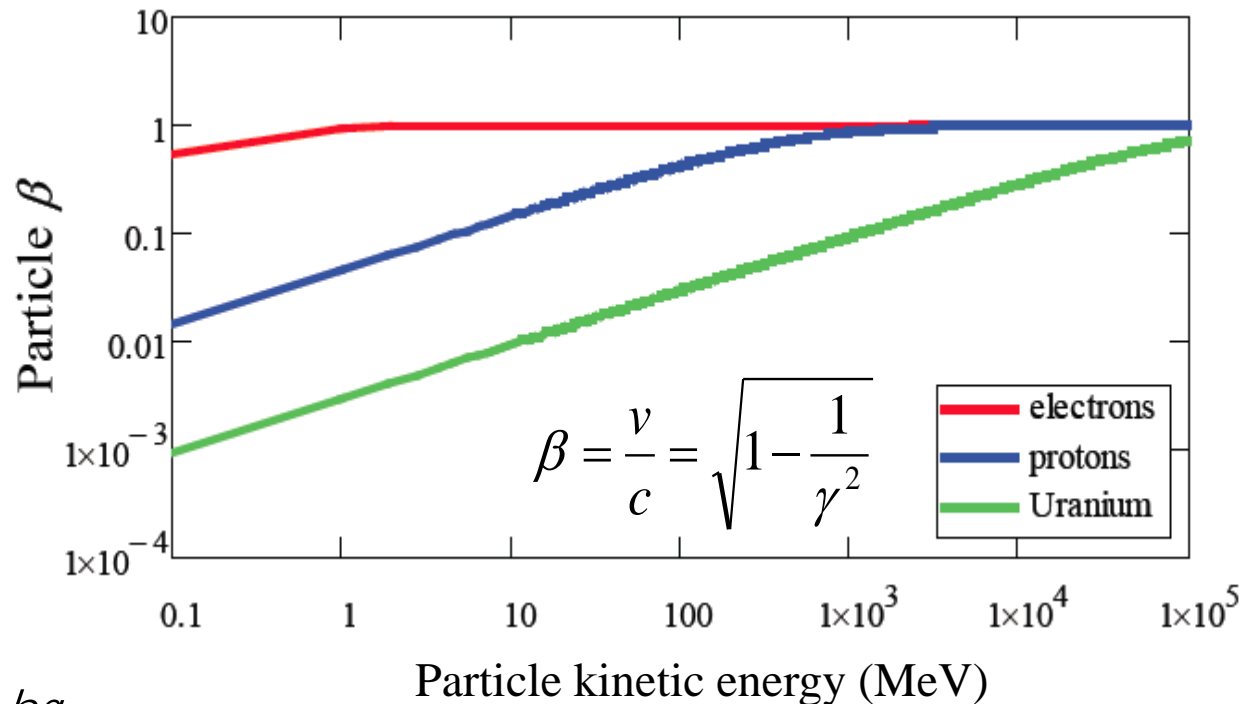
Total Energy: $E = gm_0c^2$

Relativistic
gamma factor:

$$g = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - b^2}}$$

Momentum:

$$p = mv = \frac{E}{c^2} bc = b \frac{E}{c} = bgm_0c$$



Revolution frequency variation

The **revolution and RF frequency** will be **changing** during acceleration
Much **more important for lower energies** (values are kinetic energy - protons).

PS Booster: 50 MeV ($\beta=0.314$) \rightarrow 1.4 GeV ($\beta=0.915$)
(pre LS2) 602 kHz \rightarrow 1746 kHz \Rightarrow **190% frequency increase**
(post LS2): 160 MeV ($\beta=0.520$) \rightarrow 2 GeV ($\beta=0.948$) \Rightarrow **95% increase**

PS: 1.4 GeV ($\beta=0.915$) \rightarrow 25.4 GeV ($\beta=0.9994$)
(pre LS2) 437 kHz \rightarrow 477 kHz \Rightarrow **9% increase**
(post LS2): 2 GeV ($\beta=0.948$) \rightarrow 25.4 GeV ($\beta=0.9994$) \Rightarrow **5% increase**

SPS: 25.4 GeV \rightarrow 450 GeV ($\beta=0.999998$)
 \Rightarrow **0.06% frequency increase**

LHC: 450 GeV \rightarrow 7 TeV ($\beta=0.999999991$)
 \Rightarrow only **2 10^{-6} increase**

RF system needs more flexibility in **lower energy** accelerators.

Question: What about **electrons** and **positrons**?

Acceleration + Energy Gain

May the force be with you!



To accelerate, we need a **force in the direction of motion!**

Newton-Lorentz Force
on a charged particle

Hence, it is necessary
(preferably) **along the**
which changes the mo

In relativistic dynami

$$E^2 = E_0^2 + p^2 c^2$$

The rate of **energy gain**

$$\frac{dE}{dz} = v$$

and the kinetic **energy**

$$dW = dE = qE_z$$



n always perpendicular
on => **no acceleration**

$$\frac{dp}{dt} = qE_z$$

linked by

$$dE = c^2 mv / E dp = v dp$$

z) is then:

is:

- V is a potential
- q the charge

Unit of Energy

Today's accelerators and future projects work/aim at the **TeV energy** range.

LHC: 7 TeV \rightarrow 14 TeV

CLIC: 380 GeV \rightarrow 3 TeV

FCC: 100 TeV

In fact, this energy unit comes from acceleration:

1 eV (electron Volt) is the energy that 1 elementary charge e (like one electron or proton) gains when it is accelerated in a potential (voltage) difference of 1 Volt.

Basic Unit: eV (electron Volt)

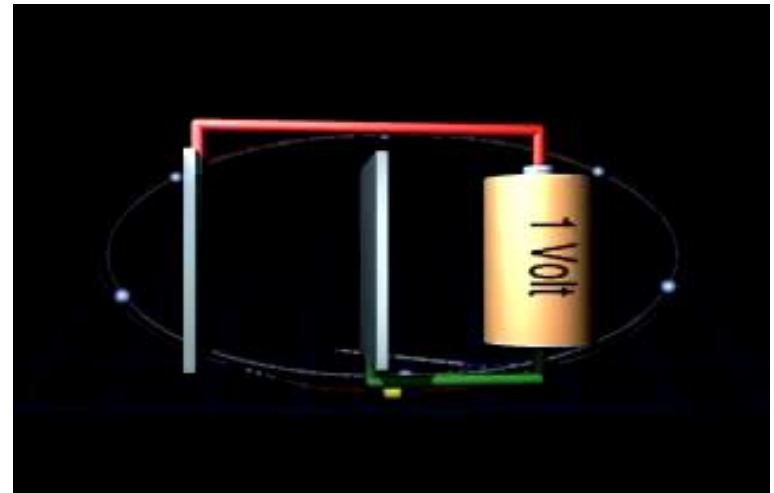
keV = 1000 eV = 10^3 eV

MeV = 10^6 eV

GeV = 10^9 eV

TeV = 10^{12} eV

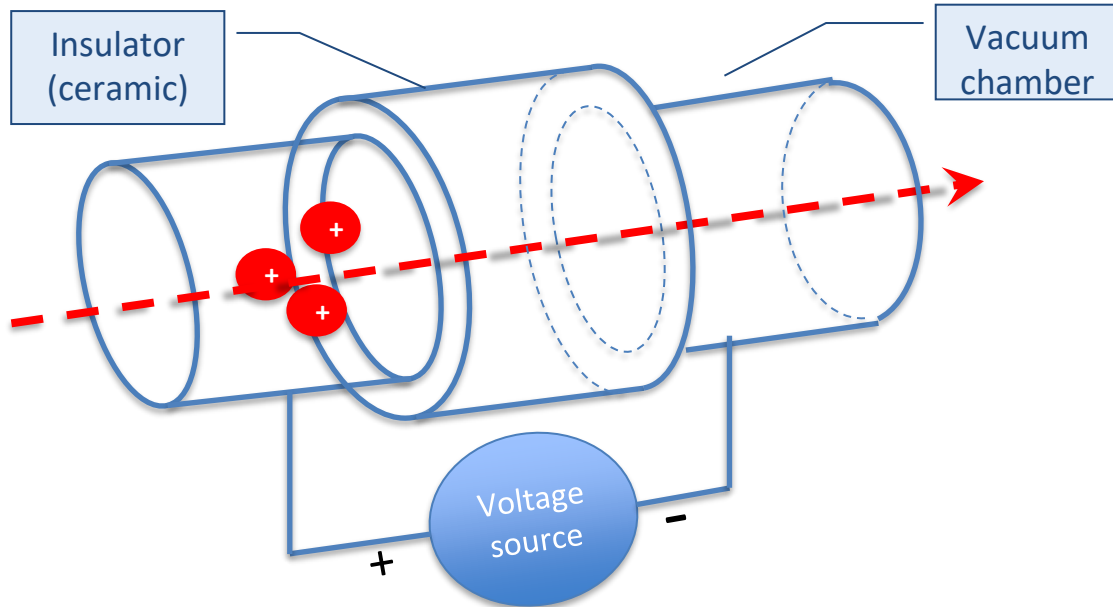
LHC = ~450 Million km of batteries!!!
3x distance Earth-Sun



Methods of Acceleration in circular accelerators

Electrostatic field limited by insulation, magnetic field doesn't accelerate at all.

Circular machine: DC acceleration impossible since $\oint \vec{E} \cdot d\vec{s} = 0$



~~First attracted
Acceleration
Then again attracted
Deceleration~~

no Acceleration

The electric field is derived from a scalar potential ϕ and a vector potential A
 The **time variation of the magnetic field H generates an electric field E**

The solution: \Rightarrow time varying electric fields

- Induction
- RF frequency fields

$$\oint \vec{E} \cdot d\vec{s} = - \iint \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Acceleration by Induction: The Betatron

It is based on the principle of a **transformer**:

- **primary side**: large electromagnet - **secondary side**: electron beam.

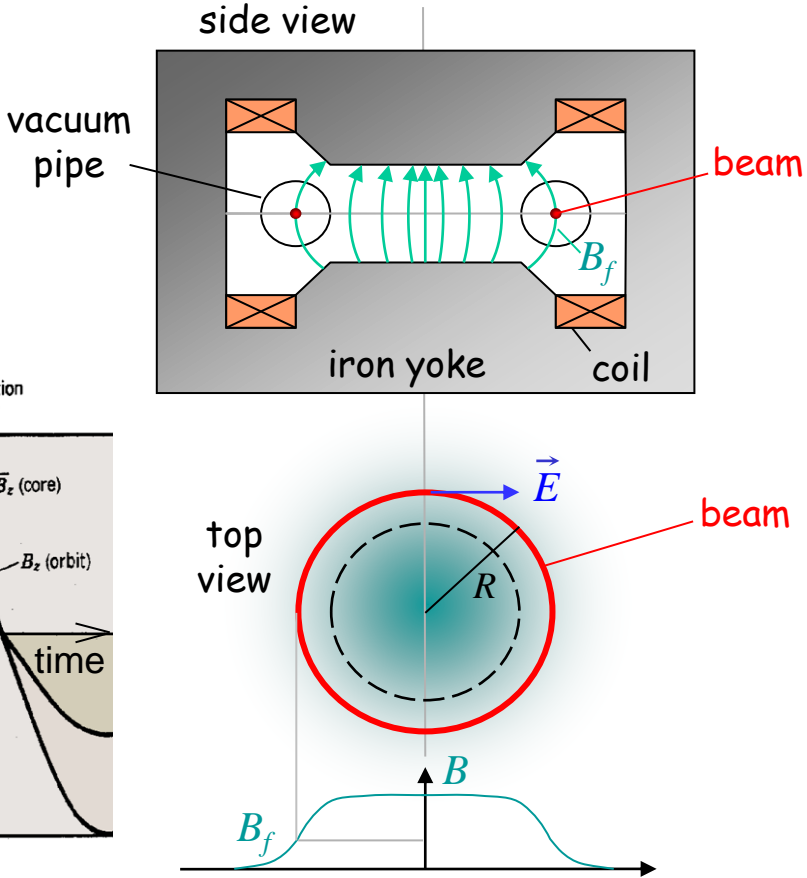
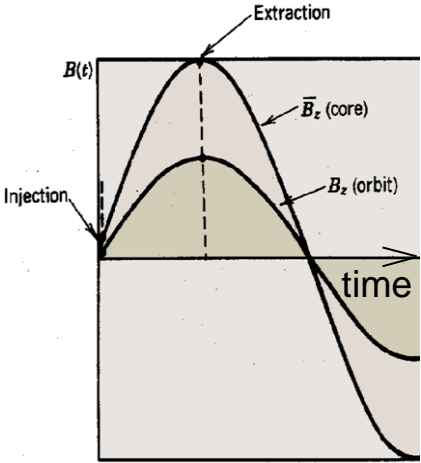
The ramping magnetic field is used to guide particles on a circular trajectory as well as for acceleration.

Limited by saturation in iron (~300 MeV e-)

Used in industry and medicine, as they are compact accelerators for electrons



Donald Kerst with the first betatron, invented at the University of Illinois in 1940



Circular accelerators

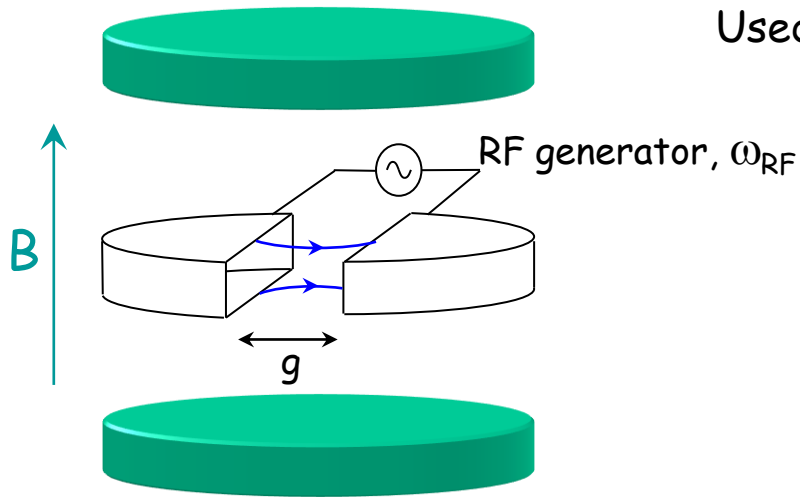
Cyclotron
Synchrotron

Circular accelerators: Cyclotron



Courtesy: EdukiteLearning, <https://youtu.be/cNnNM2ZqIsc>

Circular accelerators: Cyclotron

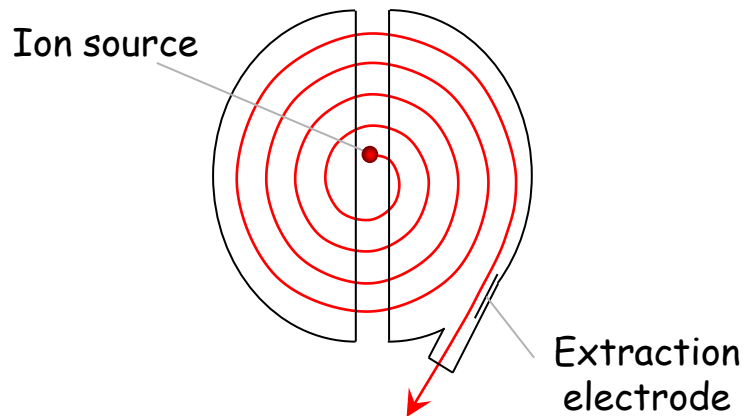


Synchronism condition



$$\omega_s = \omega_{RF}$$

$$2\pi \rho = v_s T_{RF}$$



Ions trajectory

Cyclotron frequency $\omega = \frac{q B}{m_0 \gamma}$

1. γ increases with the energy
 \Rightarrow no exact synchronism
2. if $v \ll c \Rightarrow \gamma \cong 1$

Animation: <https://phyanim.sciences.univ-nantes.fr/Meca/Charges/cyclotron.php>

Circular accelerators: Cyclotron



Courtesy Berkeley Lab,
<https://www.youtube.com/watch?v=cutKuFxeXmQ>

Cyclotron / Synchrocyclotron



TRIUMF 520 MeV cyclotron

Vancouver - Canada



CERN 600 MeV synchrocyclotron

Synchrocyclotron: Same as cyclotron, except a modulation of ω_{RF}

B = constant

$\gamma \omega_{RF}$ = constant

ω_{RF} decreases with time

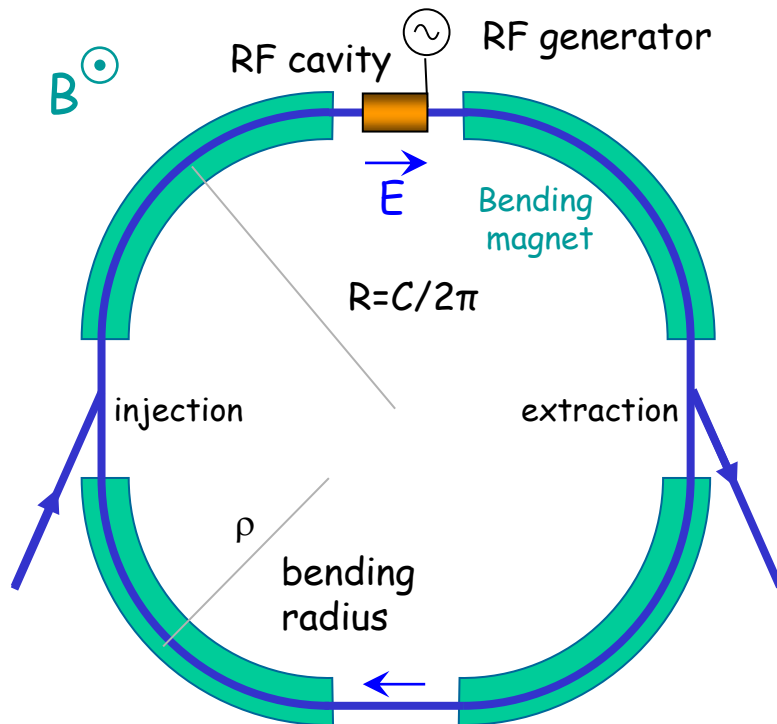
More in
lectures by
Mike Seidel

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q B}{m_0 \gamma(t)}$$

Allows to go beyond the
non-relativistic energies

Circular accelerators: The Synchrotron



1. Constant orbit during acceleration
2. To keep particles on the closed orbit, B should increase with time
3. ω and ω_{RF} increase with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

Synchronism condition \rightarrow

$$\frac{2\pi R}{v_s} = h T_{RF}$$

h integer,
harmonic number:
 number of RF cycles
 per revolution

h is the **maximum number of bunches** in the synchrotron.
 Normally less bunches due to gaps for kickers, collision constraints,...

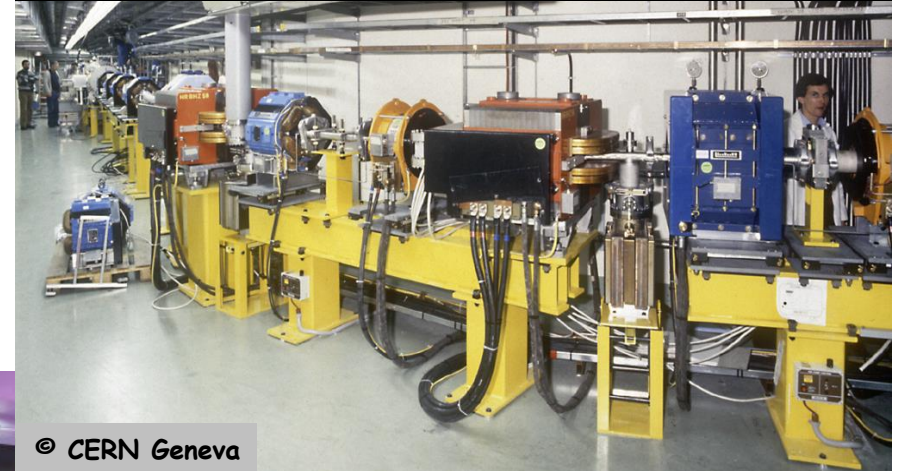
Circular accelerators: The Synchrotron

LEAR (CERN)
Low Energy Antiproton Ring



© CERN Geneva

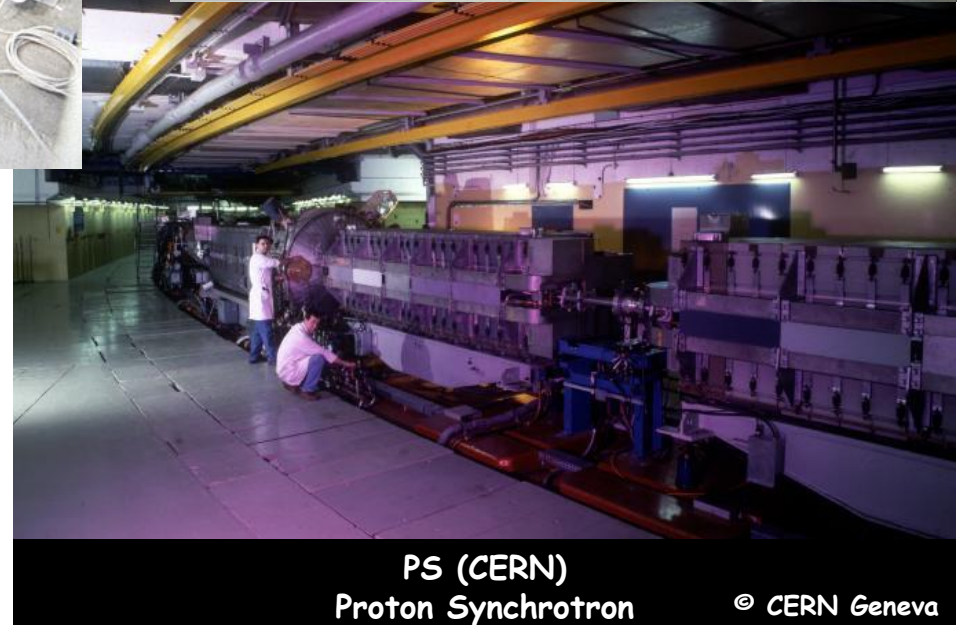
EPA (CERN)
Electron Positron Accumulator



© CERN Geneva

Examples of different
proton and electron
synchrotrons at CERN

+ LHC (of course!)

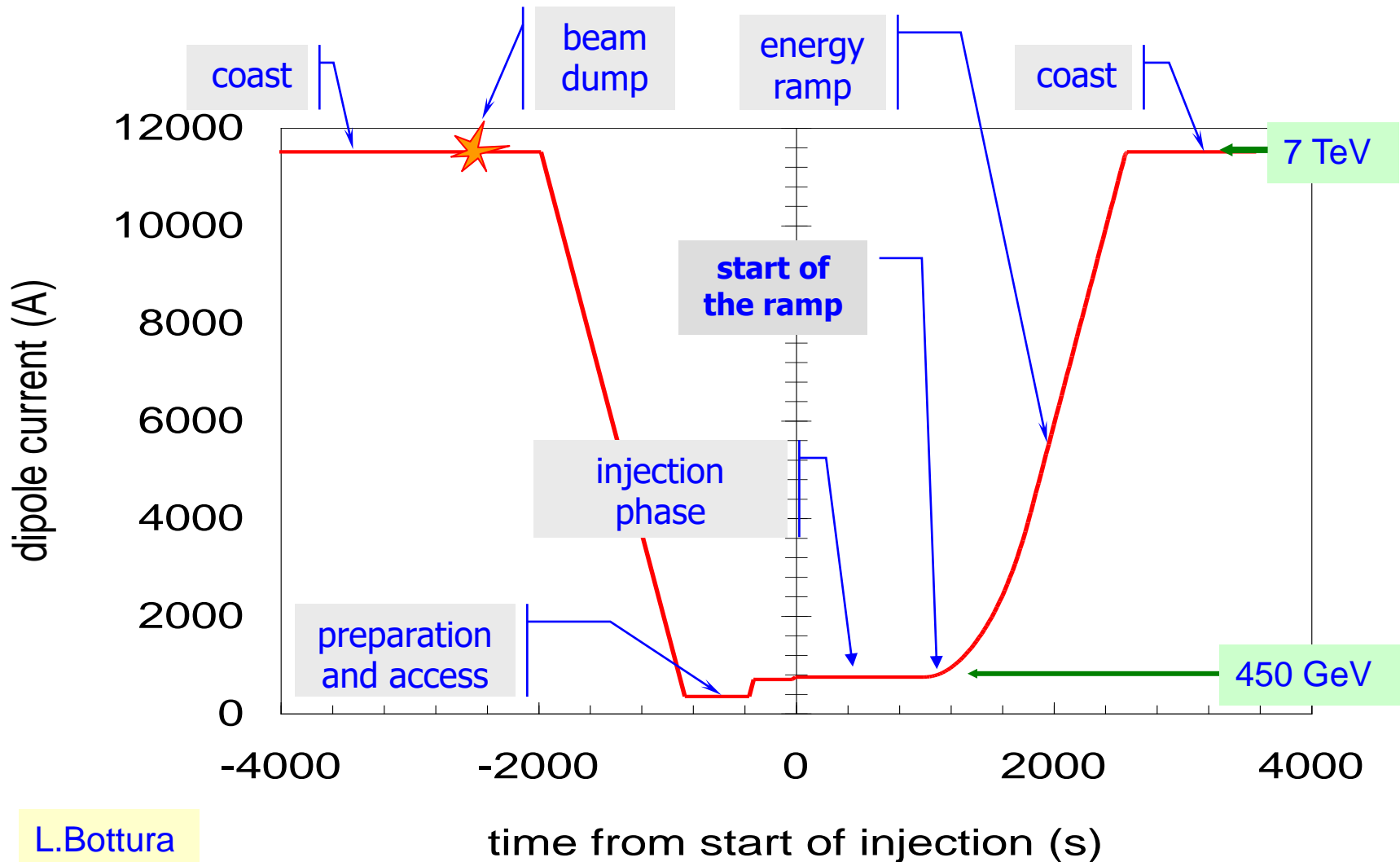


PS (CERN)
Proton Synchrotron

© CERN Geneva

The Synchrotron - LHC Operation Cycle

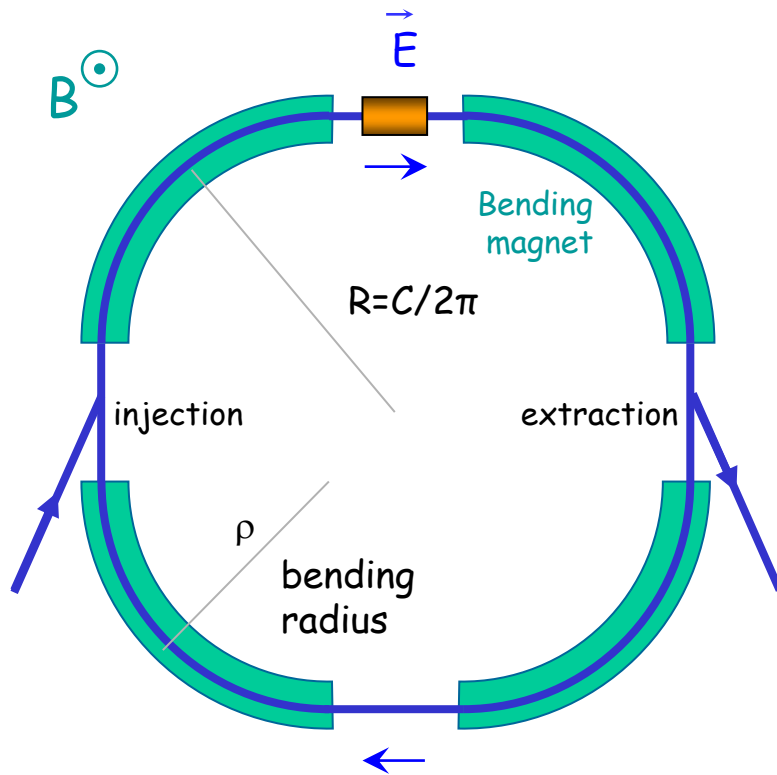
The magnetic **field** (dipole current) is **increased during the acceleration**.



L.Bottura

The Synchrotron

The **synchrotron** is a synchronous accelerator since there is a **synchronous RF phase** for which the energy gain **fits** the **increase of the magnetic field** at each turn. That implies the following operating conditions:



$$q\dot{V} \sin \phi \longrightarrow \text{Energy gain per turn}$$

$$f = f_s \longrightarrow \text{Synchronous particle}$$

$$\omega_{RF} = h\omega \longrightarrow \text{RF synchronism (h - harmonic number)}$$

$$r = cte \quad R = cte \longrightarrow \text{Constant orbit}$$

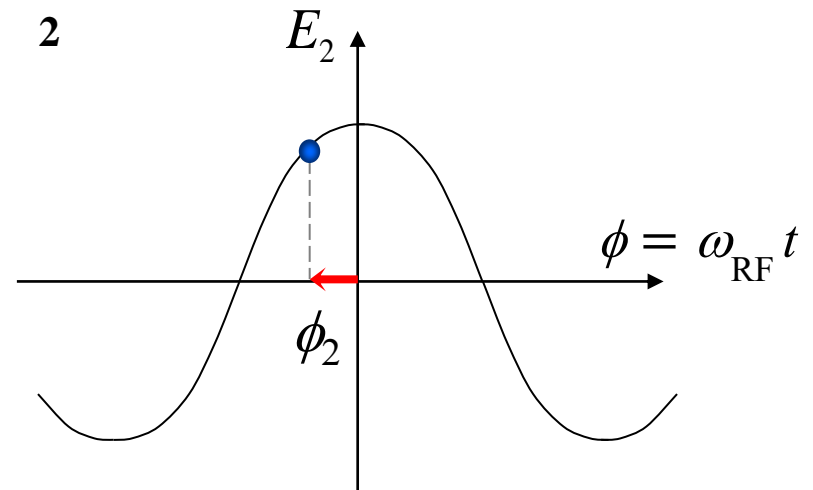
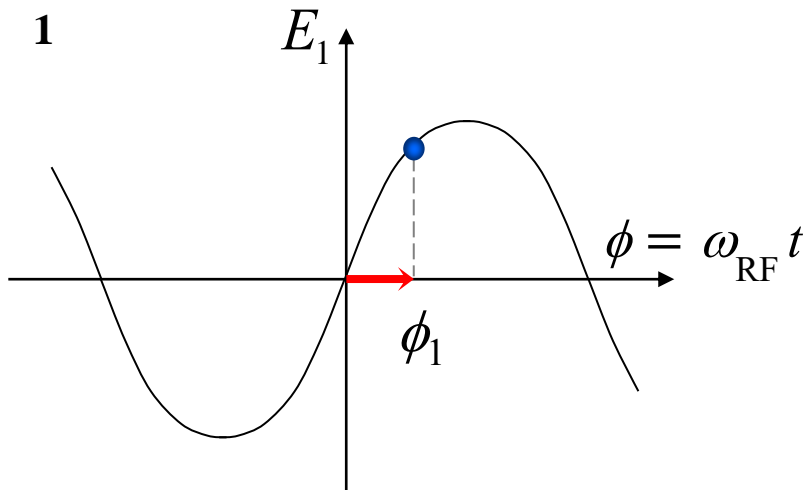
$$B\rho = \frac{p}{q} \supset B \longrightarrow \text{Variable magnetic field}$$

If $v \approx c$, ω hence ω_{RF} remain constant (ultra-relativistic e^-)

Common Phase Conventions

1. For **circular accelerators**, the origin of time is taken at the **zero crossing** of the RF voltage with positive slope
2. For **linear accelerators**, the origin of time is taken at the positive **crest** of the RF voltage

Time $t=0$ chosen such that:



3. I will stick to **convention 1** in the following to avoid confusion

The Synchrotron - Energy ramping

Energy ramping by increasing the B field (frequency has to follow ν):

$$p = qB\rho \quad \rho \text{ const.} \Rightarrow \frac{dp}{dt} = q\dot{B}\rho \quad \Rightarrow \quad (\Delta p)_{\text{turn}} = q\dot{B}\rho T_r = \frac{2\pi q\rho R\dot{B}}{v}$$

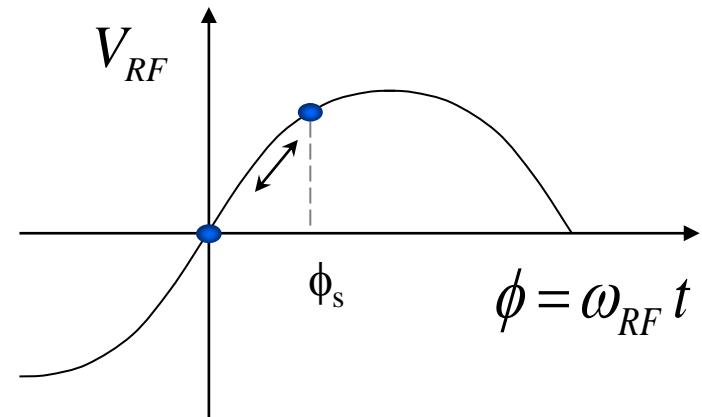
With

$$E^2 = E_0^2 + p^2 c^2 \quad \Rightarrow \quad DE = vDp \quad (\Delta E)_{\text{turn}} = (\Delta W)_s = 2\pi q\rho R\dot{B} = q\hat{V} \sin \phi_s$$

Synchronous phase ϕ_s changes during energy ramping

$$\sin \phi_s = 2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}} \quad \Rightarrow \quad \phi_s = \arcsin \left(2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}} \right)$$

- The **synchronous phase** depends on
 - the **change of the magnetic field**
 - and the **RF voltage**



The Synchrotron - Frequency change

During the energy ramping, **the RF frequency increases** to follow the increase of the revolution frequency :

$$\omega = \frac{\omega_{RF}}{h} = \omega(B, R_S)$$

Hence:
$$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_S} = \frac{1}{2\pi} \frac{qc^2}{E_S(t)} \frac{\rho}{R_S} B(t) \quad (\text{using } p(t) = qB(t)\rho, E = mc^2)$$

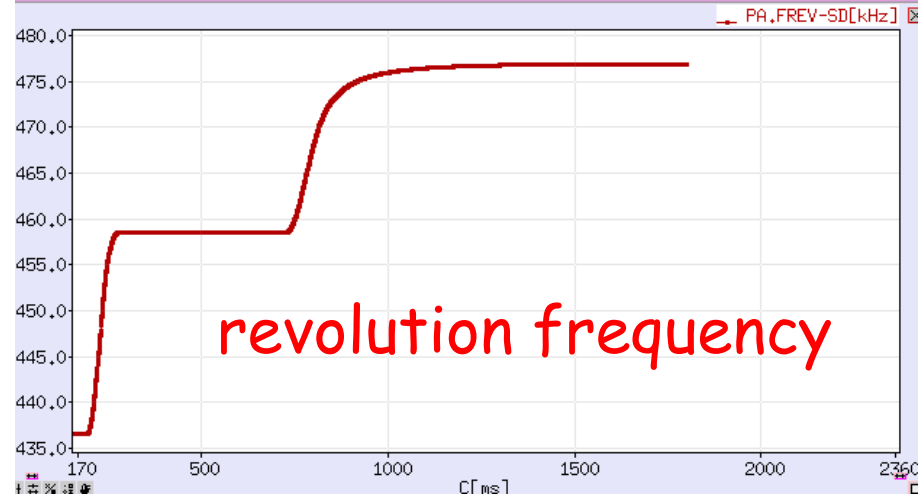
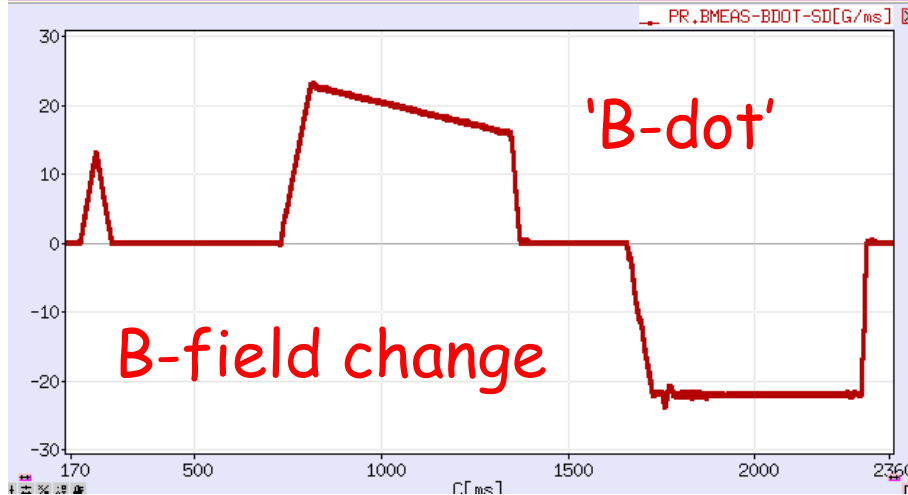
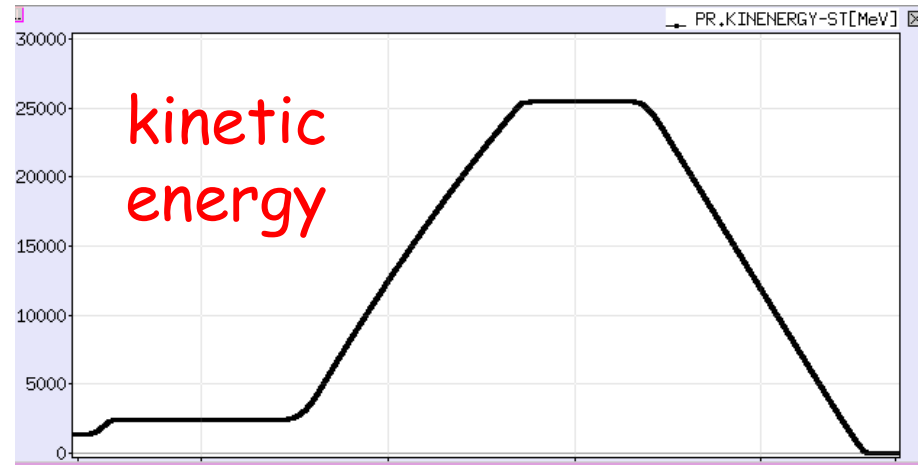
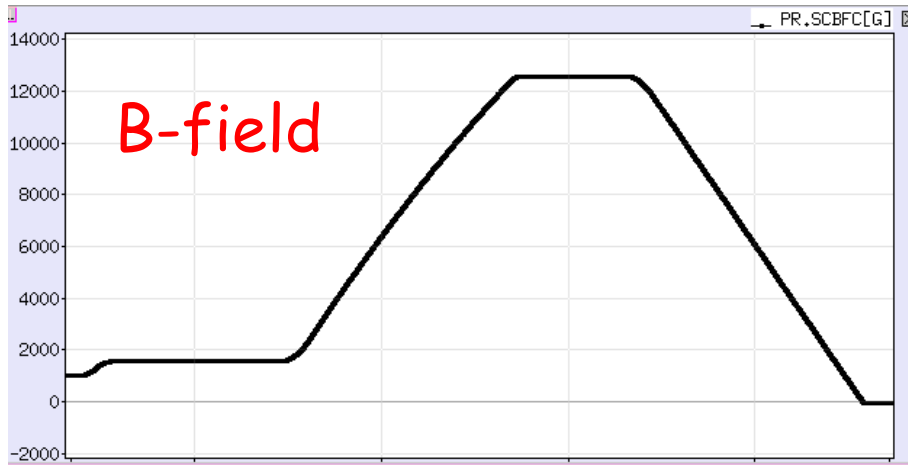
Since $E^2 = (m_0c^2)^2 + p^2c^2$ the RF frequency must follow the variation of the B field with the law

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_S} \left\{ \frac{B(t)^2}{(m_0c^2/qc\rho)^2 + B(t)^2} \right\}^{1/2}$$

RF frequency program during acceleration determined by B-field !

Example: PS - Field / Frequency change

During the energy ramping, the **B-field** and the **revolution frequency** increase



time (ms) →

time (ms) →

Summary of yesterday...

- **Acceleration by** time-varying **electric fields**, radiofrequency
- Electrons are quickly relativistic, speed does not change
- Protons and ions have large velocity change and need certain RF frequency range

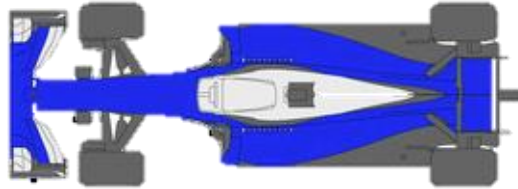
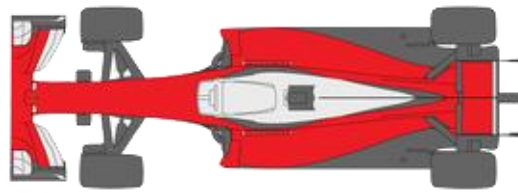
- **Cyclotrons** for proton/ion acceleration below relativistic energy
- **Synchrotrons** for high energies, constant orbit, rising field and frequency synchronously
- **Synchronous phase** depending on acceleration

- **Today:** Particle **oscillation** around synchronous phase

Overtaking in a Formula 1 Race

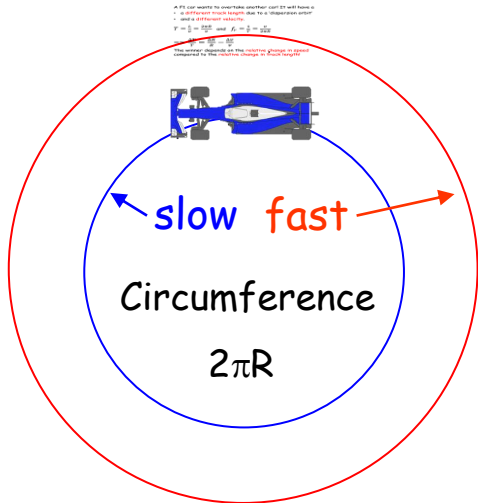


www.formula1.com



Overtaking in a Formula 1 Race

Overtaking in a Formula 1 Race



A F1 car wants to overtake another car! It will have a

- a **different track length** due to a 'dispersion orbit'
- and a **different velocity**.

$$T = \frac{L}{v} = \frac{2\pi R}{v} \quad \text{and} \quad f_r = \frac{1}{T} = \frac{v}{2\pi R}$$

$$\Rightarrow \frac{\Delta T}{T} = \frac{\Delta R}{R} - \frac{\Delta v}{v}$$

The winner depends on the **relative change in speed** compared to the **relative change in track length!**

v =speed of the car

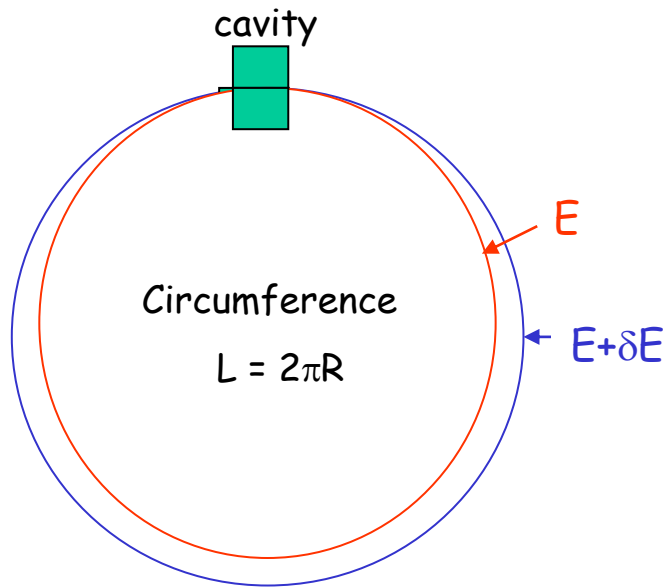
R =track physical radius

T =revolution period

f_r =revolution frequency

If the **relative change in speed** is larger than the **relative change in track length** \Rightarrow the **red** car will win!

Overtaking in a Synchrotron



- A particle slightly shifted in momentum will have a
- dispersion orbit and a **different orbit length**
 - a **different velocity**.

As a result of both effects the revolution period T changes with a "slip factor" η :

$$\eta = \frac{dT/T}{dp/p}$$

Note: you also find η defined with a minus sign!

The "**momentum compaction factor**" is defined as relative orbit length change with momentum:

$$\alpha_c = \frac{dL/L}{dp/p}$$

$$\alpha_c = \frac{p}{L} \frac{dL}{dp}$$

p =particle momentum

R =synchrotron physical radius

T =revolution period

Momentum Compaction Factor

$$\alpha_c = \frac{p \, dL}{L \, dp}$$

$$ds_0 = r \, dq$$

$$ds = (r + x) \, dq$$

The elementary path difference from the two orbits is:

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{r} \stackrel{\text{definition of dispersion } D_x}{=} \frac{D_x}{r} \frac{dp}{p}$$

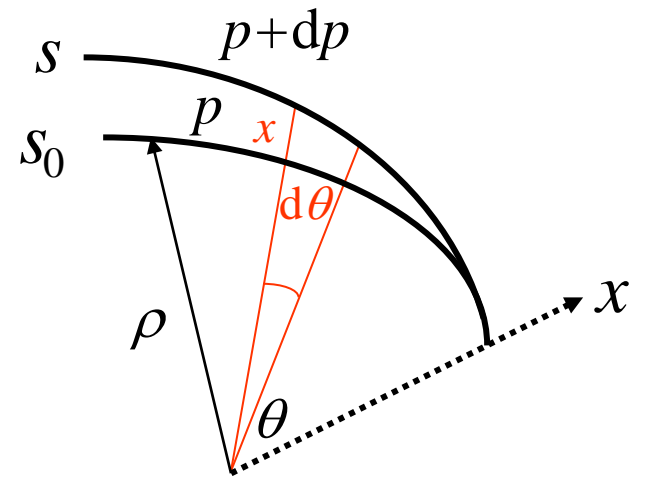
leading to the total change in the circumference:

$$dL = \oint_C dl = \oint_C \frac{x}{r} ds_0 = \oint_C \frac{D_x}{r} \frac{dp}{p} ds_0$$

$$\alpha_c = \frac{1}{L} \int_C \frac{D_x(s)}{\rho(s)} ds_0$$

With $\rho = \infty$ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$



$$x = x_0 + D_x \frac{\Delta p}{p}$$

$\langle \rangle_m$ means that the average is considered over the bending magnet only

Property of the **transverse beam optics!**

Dispersion Effects - Revolution Period

The **two effects** of the **orbit length** and the particle **velocity** change the revolution period as:

$$T = \frac{L}{\beta c} \quad \Rightarrow \quad \frac{dT}{T} = \frac{dL}{L} - \frac{d\beta}{\beta} \stackrel{\uparrow}{=} \alpha_c \frac{dp}{p} - \frac{d\beta}{\beta}$$

definition of momentum
compaction factor

$$\frac{dT}{T} = \left(\alpha_c - \frac{1}{\gamma^2} \right) \frac{dp}{p}$$

$$p = mv = bg \frac{E_0}{c} \quad \triangleright \quad \frac{dp}{p} = \frac{db}{b} + \frac{d(1-b^2)^{-1/2}}{(1-b^2)^{-1/2}} = \underbrace{(1-b^2)^{-1}}_{g^2} \frac{db}{b}$$

**Slip
factor:**

$$\eta = \alpha_c - \frac{1}{\gamma^2}$$

or

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

with

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$

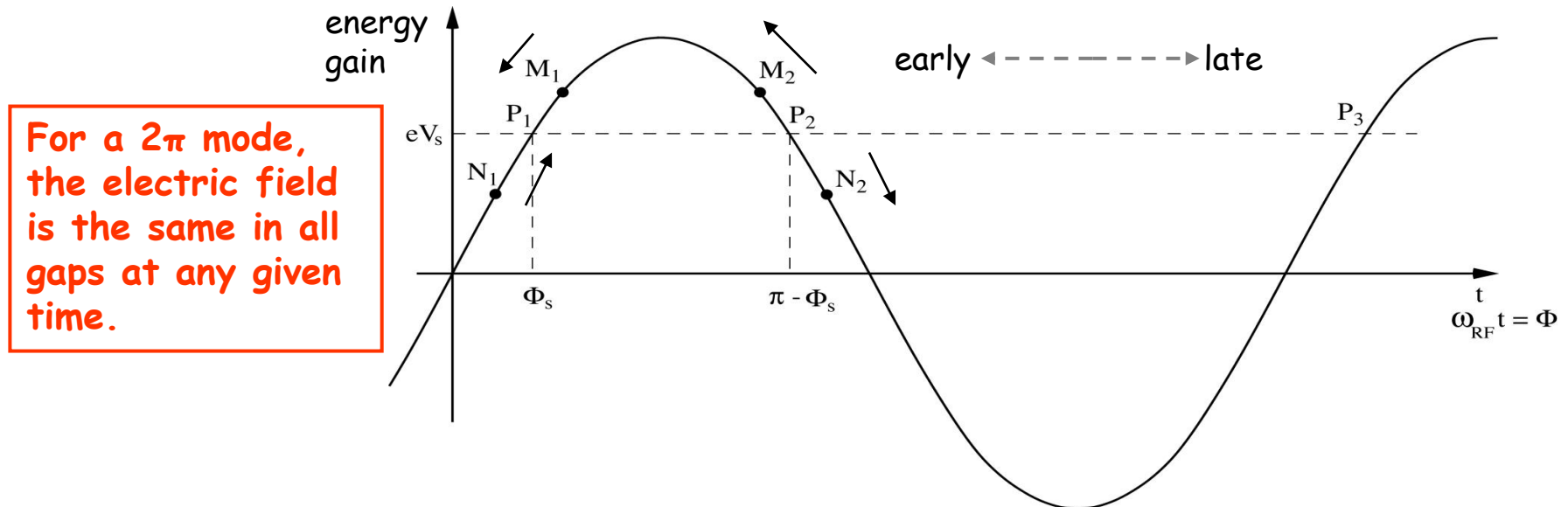
Note: you also find η defined with a minus sign!

At **transition energy**, $\eta = 0$, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

RECAP: Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

$eV_s = e\hat{V} \sin \Phi_s$ is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1, P_2, \dots are fixed points.



For a 2π mode, the electric field is the same in all gaps at any given time.

If an **energy increase** is transferred into a **velocity increase** \Rightarrow

M_1 & N_1 will move towards P_1 \Rightarrow **stable**

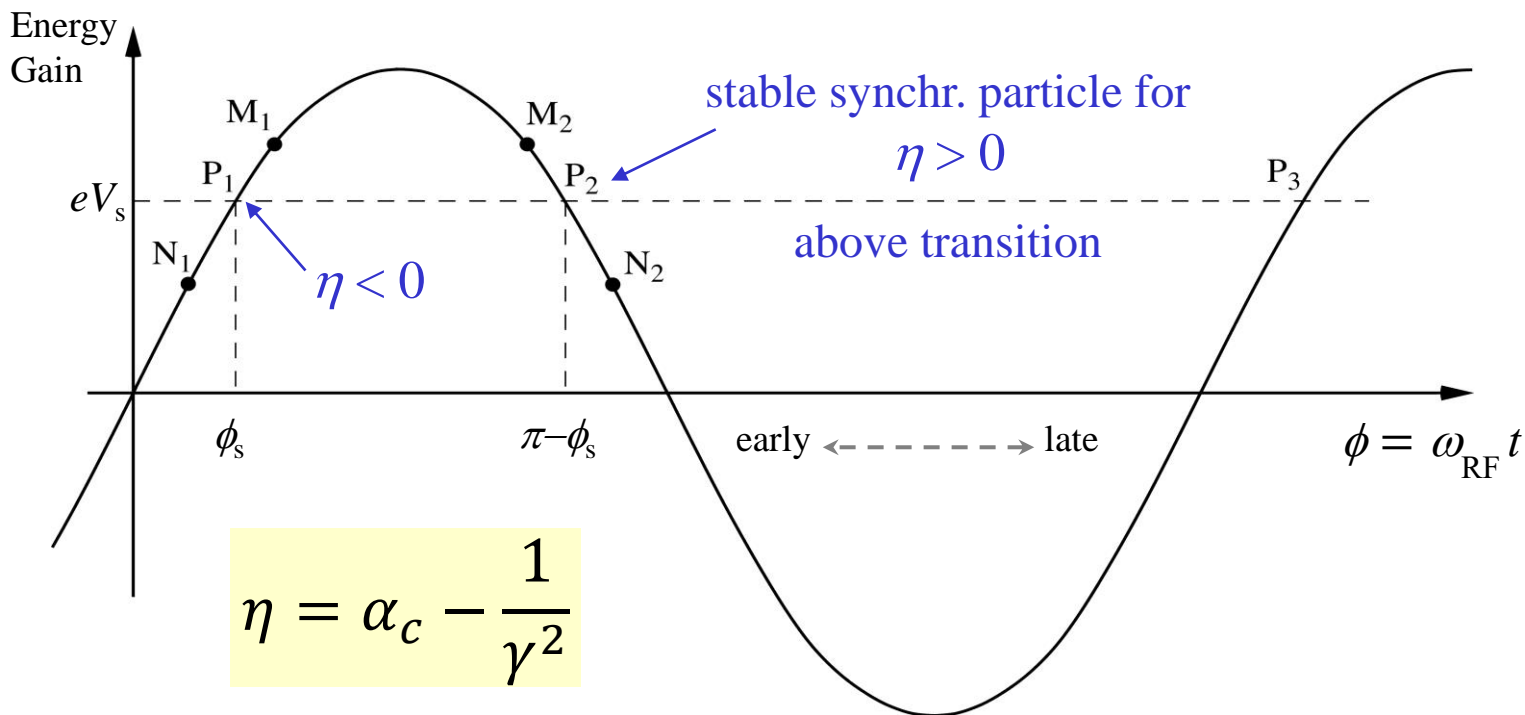
M_2 & N_2 will go away from P_2 \Rightarrow **unstable**

(Highly relativistic particles have no significant velocity change)

Phase Stability in a Synchrotron

From the definition of η it is clear that an **increase in momentum** gives

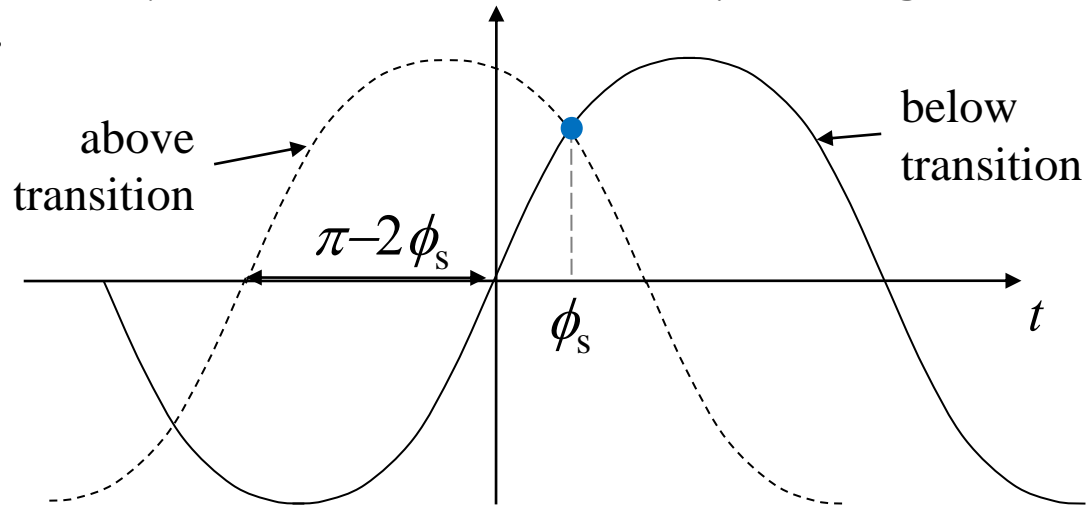
- **below transition** ($\eta < 0$) a **higher revolution frequency** (increase in velocity dominates) while
- **above transition** ($\eta > 0$) a **lower revolution frequency** ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.



Crossing Transition

At **transition**, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a '**phase jump**'.



$$\alpha_c \sim \frac{1}{Q_x^2}$$

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x$$

In the PS: γ_t is at ~ 6 GeV

In the SPS: $\gamma_t = 22.8$, injection at $\gamma = 27.7$

=> no transition crossing!

In the LHC: γ_t is at ~ 55 GeV, also far below injection energy

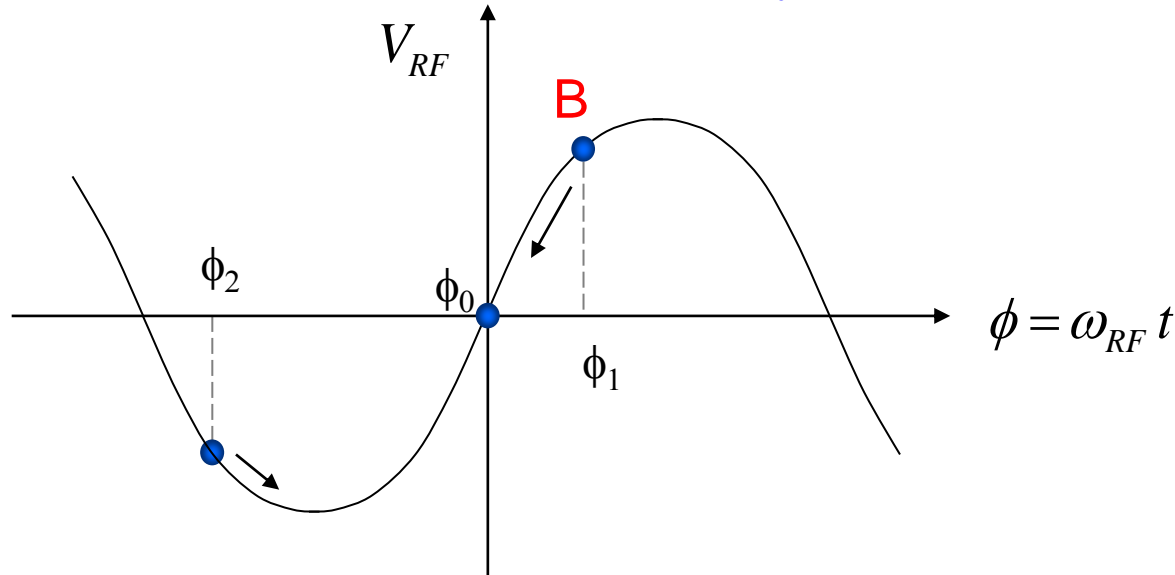
Transition crossing not needed in leptons machines, why?

Dynamics: Synchrotron oscillations

Simple case (no accel.): $B = \text{const.}$, below transition $\gamma < \gamma_t$

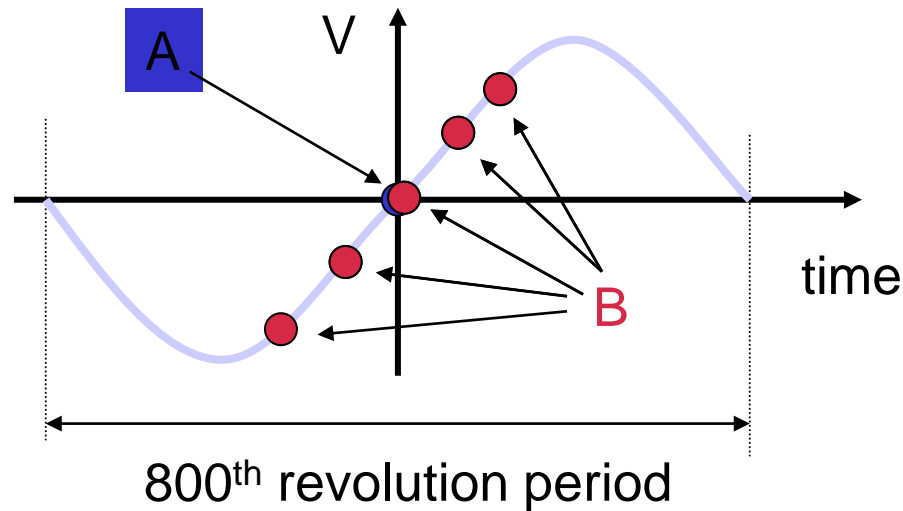
The phase of the synchronous particle must therefore be $\phi_0 = 0$.

- Φ_1
- The particle **B** is accelerated
 - Below transition, an energy increase means an increase in revolution frequency
 - The particle arrives earlier - tends toward ϕ_0



- ϕ_2
- The particle is decelerated
 - decrease in energy - decrease in revolution frequency
 - The particle arrives later - tends toward ϕ_0

Synchrotron oscillations



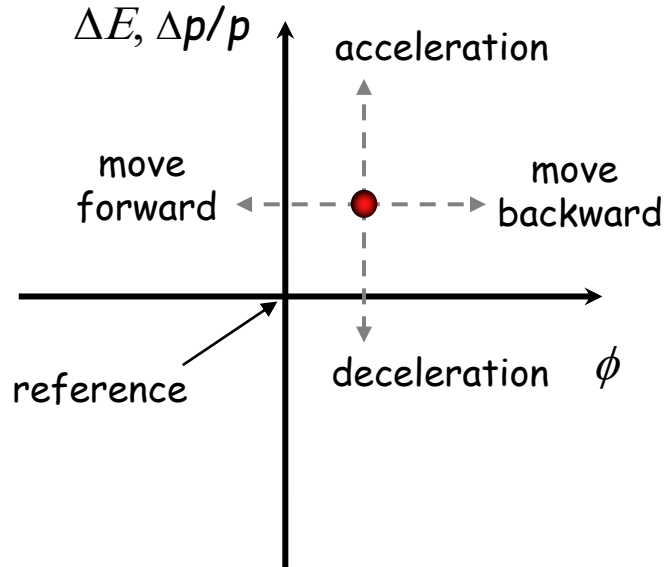
Particle **B** is performing **Synchrotron Oscillations** around synchronous particle **A**.

The amplitude depends on the initial phase and energy.

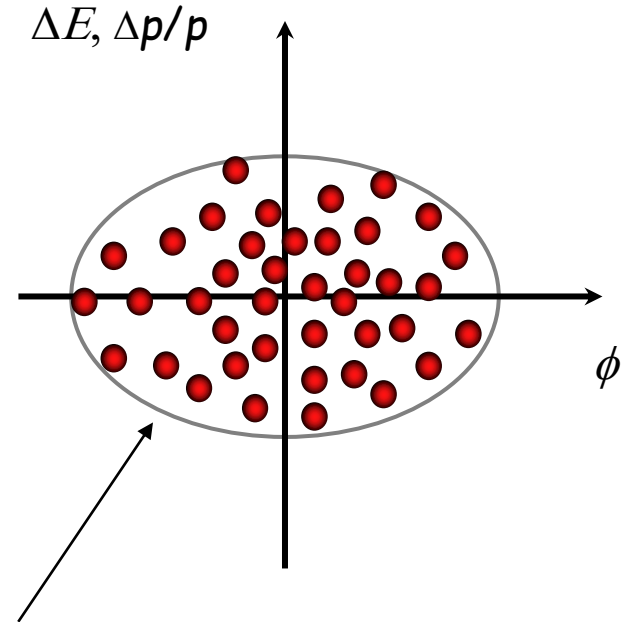
The **oscillation frequency** is much **slower than** in the **transverse** plane. It takes a large number of revolutions for one complete oscillation. Restoring electric force smaller than magnetic force.

Longitudinal phase space

The **energy - phase oscillations** can be drawn in **phase space**:



The particle trajectory in the phase space ($\Delta p/p, \phi$) describes its longitudinal motion.



Emittance: phase space area including all the particles

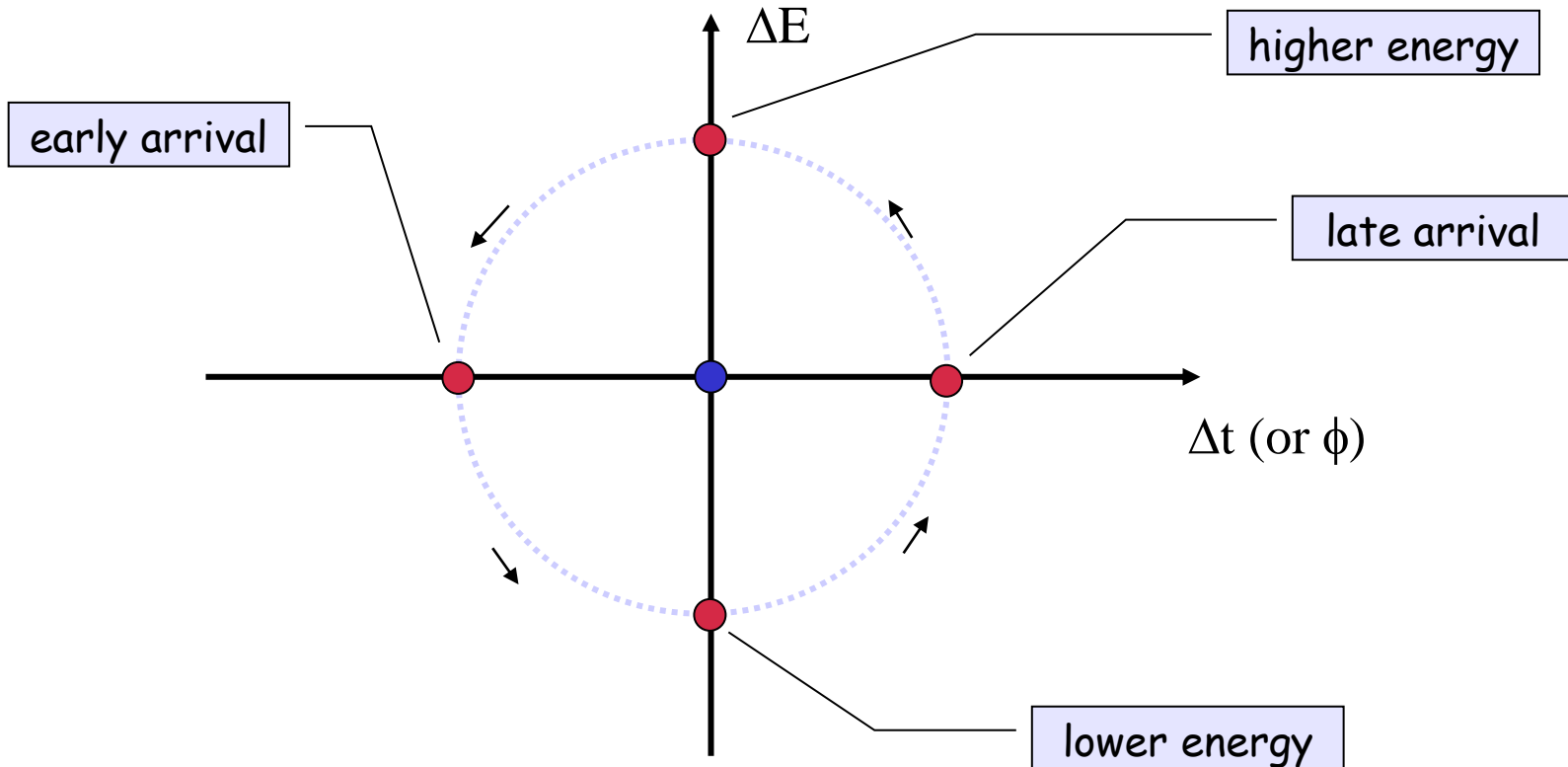
NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

Longitudinal Phase Space Motion

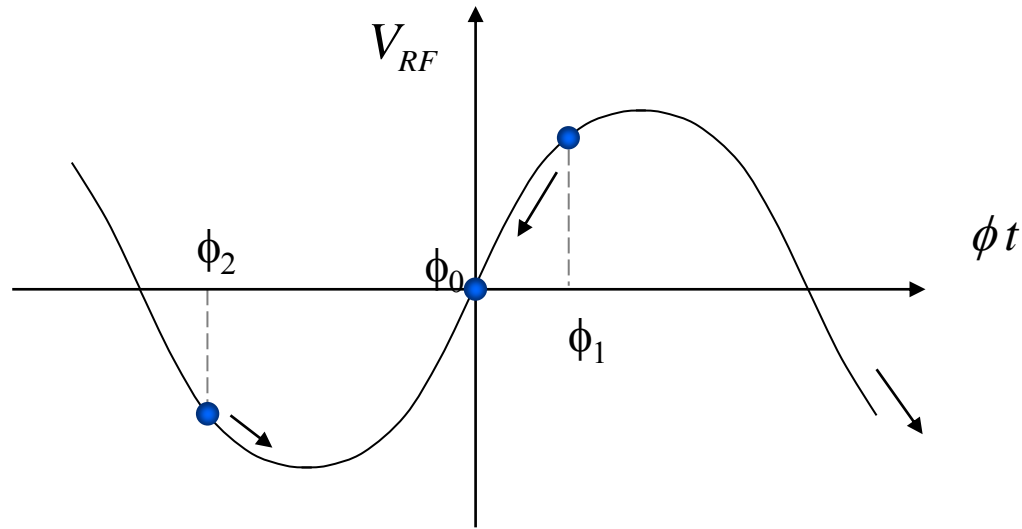
Particle **B** oscillates around particle **A**

This is a synchrotron oscillation

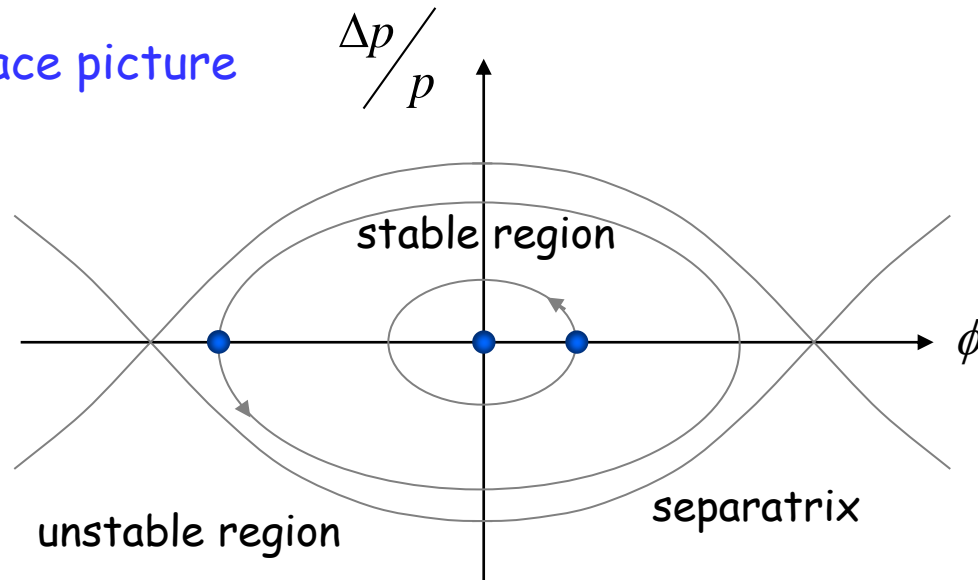
Plotting this motion in longitudinal phase space gives:



Synchrotron oscillations - No acceleration



Phase space picture



Synchrotron motion in phase space

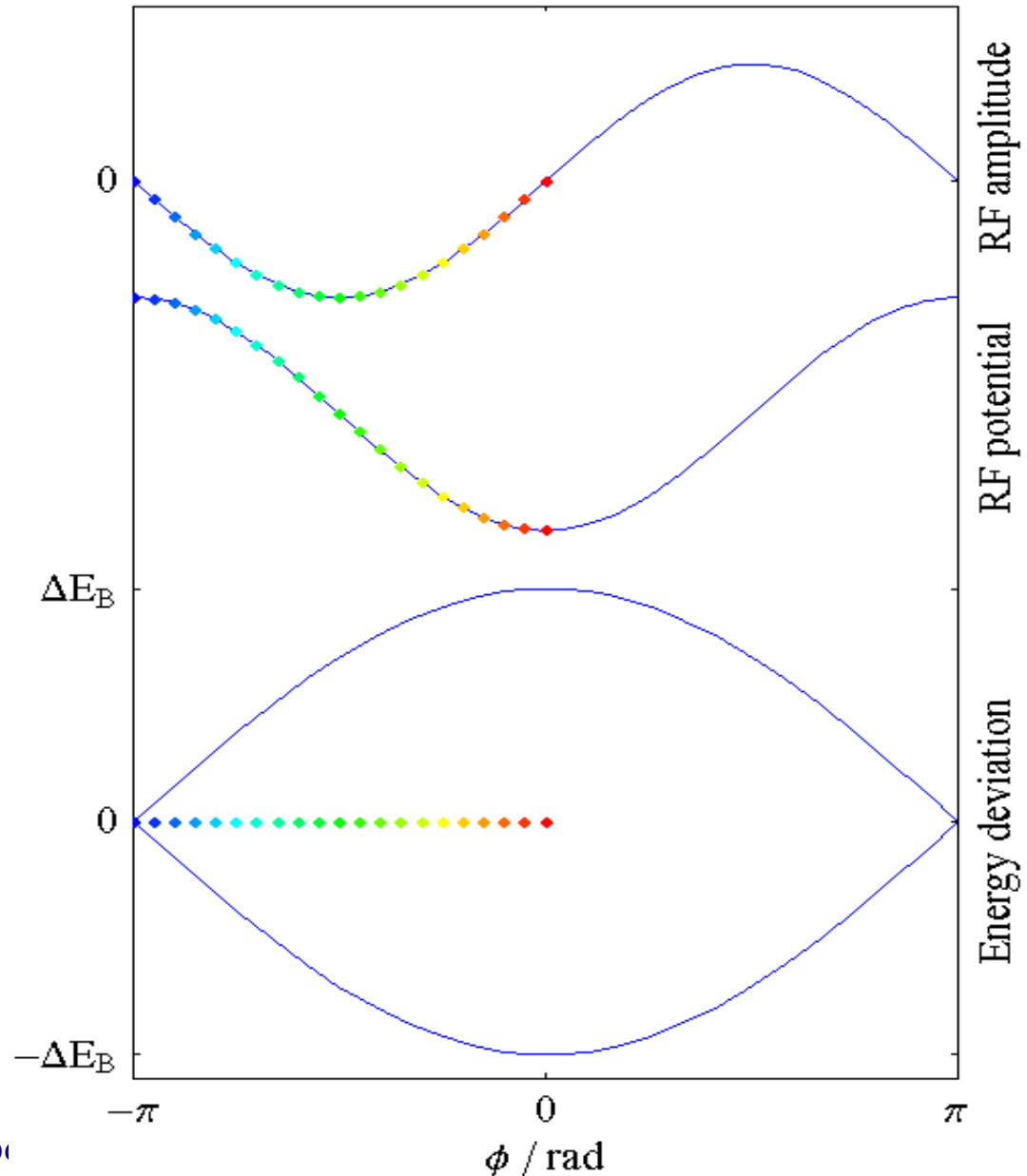
The restoring force is **non-linear**.

⇒ speed of motion depends on position in phase-space

(here shown for a stationary bucket)

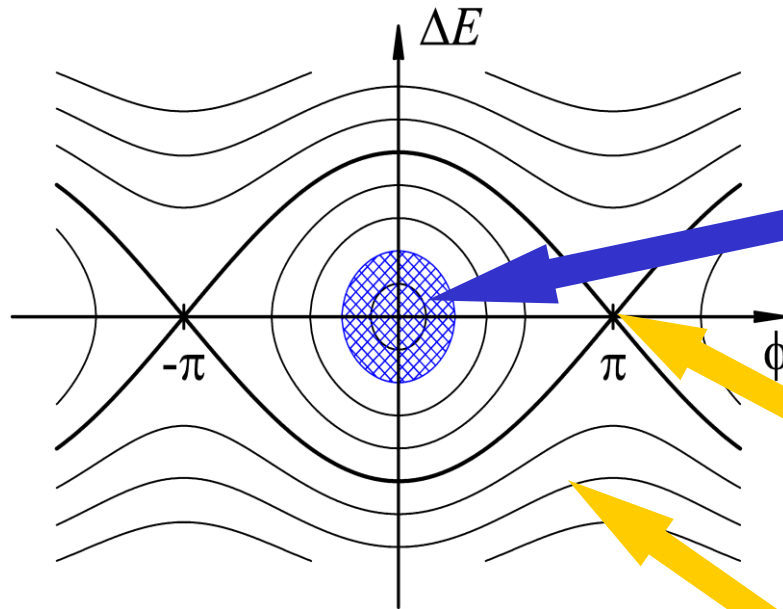
Remark:

Synchrotron frequency **much smaller** than betatron frequency.



Synchrotron motion in phase space

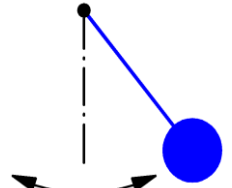
ΔE - ϕ phase space of a **stationary bucket**
(when there is **no acceleration**)



Bucket area: area enclosed by the separatrix
The area covered by particles is the longitudinal emittance

Dynamics of a particle
Non-linear, conservative oscillator \rightarrow e.g. pendulum

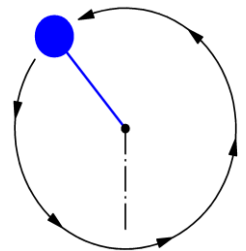
Particle inside the separatrix:



Particle at the unstable fix-point

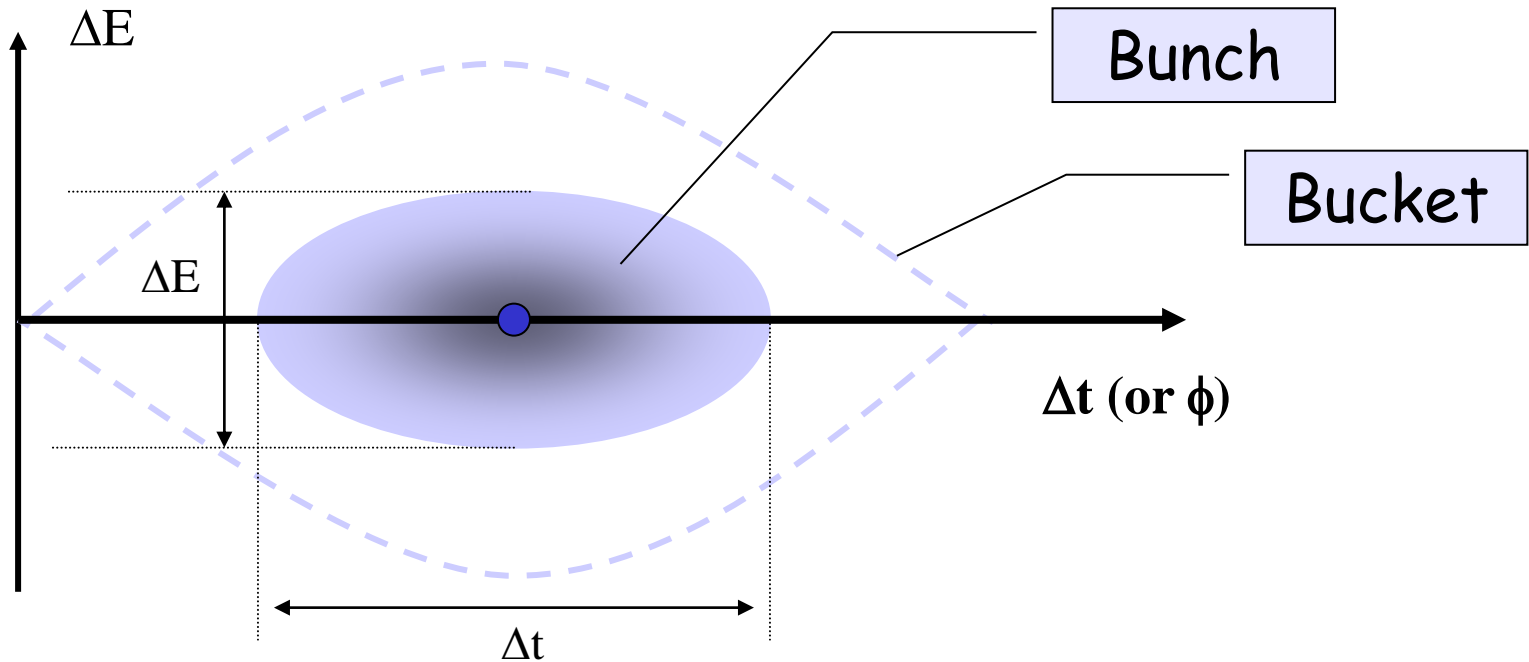


Particle outside the separatrix:



(Stationary) Bunch & Bucket

The **bunches** of the beam **fill** usually **a part** of the **bucket** area.



Bucket area = longitudinal Acceptance [eVs]

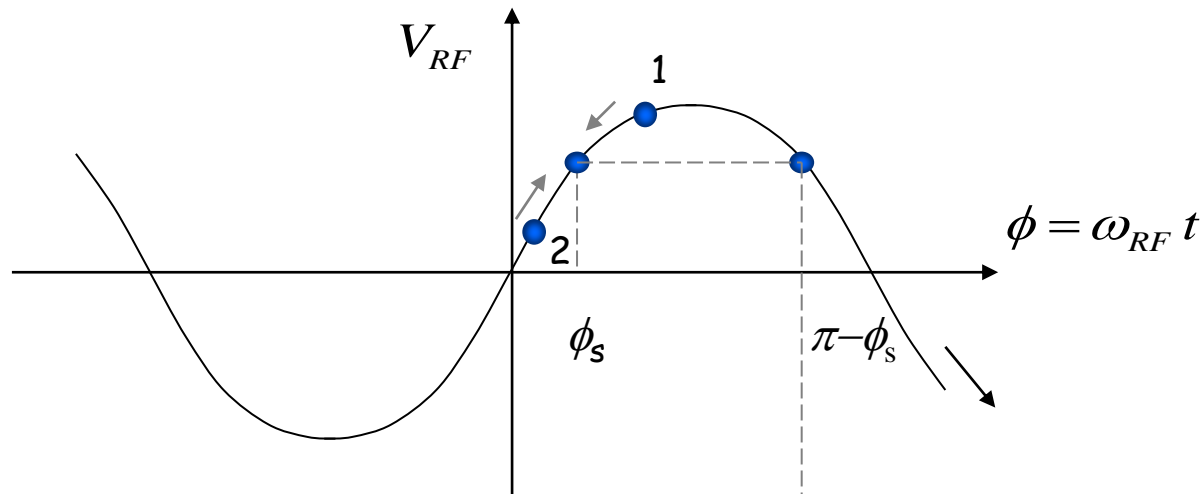
Bunch area = longitudinal beam emittance (rms) = $\pi \sigma_E \sigma_t$ [eVs]

Attention: Different definitions are used!

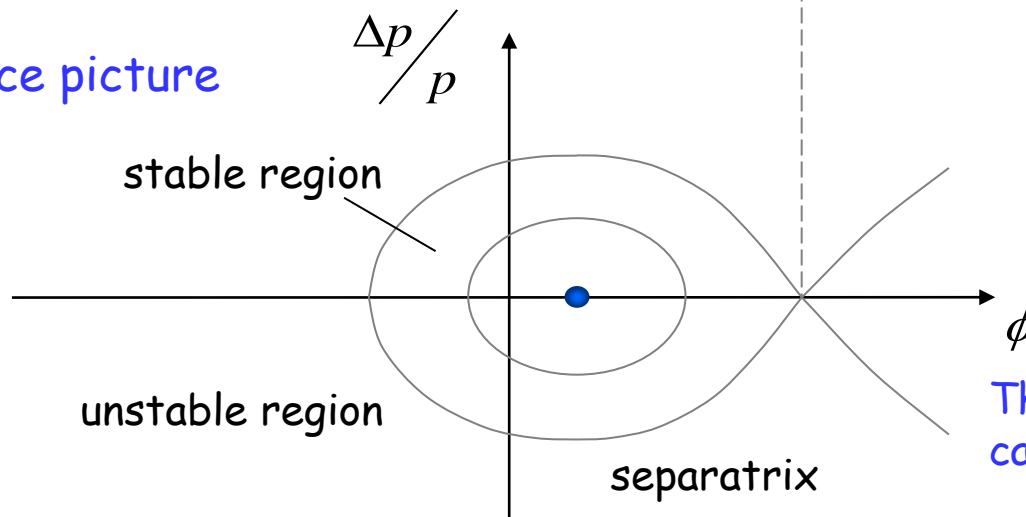
Synchrotron oscillations (with acceleration)

Case with acceleration B increasing

$$\gamma < \gamma_t$$



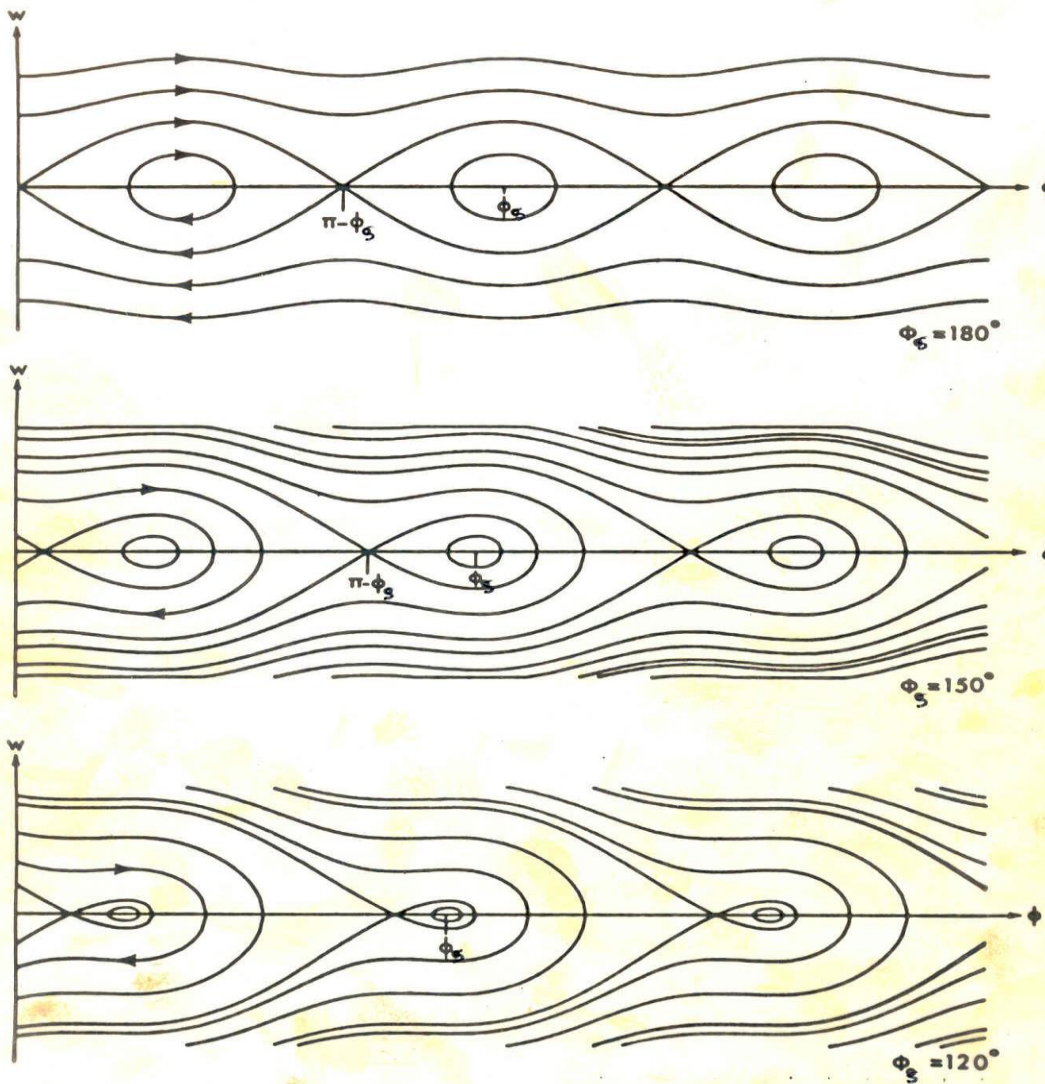
Phase space picture



$$\phi_s < \phi < \pi - \phi_s$$

The symmetry of the case $B = \text{const.}$ is lost

RF Acceptance versus Synchronous Phase



The **areas of stable motion** (closed trajectories) are called "**BUCKET**". The number of circulating buckets is equal to " h ".

The phase extension of the **bucket is maximum** for $\phi_s = 180^\circ$ (or 0°) which means **no acceleration**.

During **acceleration**, the buckets get **smaller**, both in length and **energy acceptance**.

=> **Injection** preferably **without acceleration**.

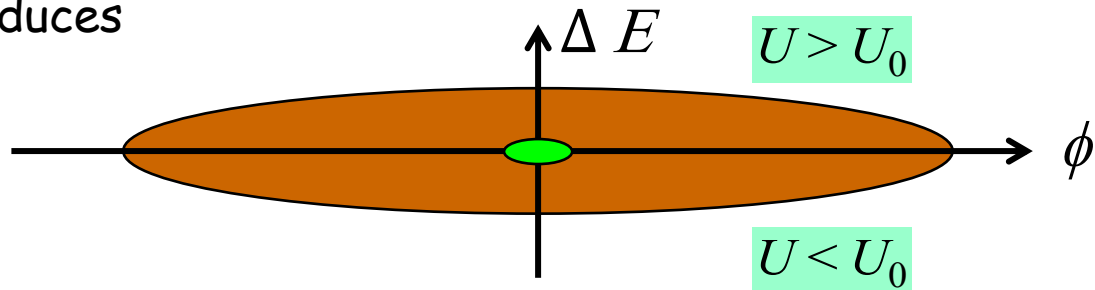
Longitudinal Motion with Synchrotron Radiation

Synchrotron radiation energy-loss energy dependant:

$$U_0 = \frac{4}{3} \frac{r_{ep}}{(m_0 c^2)^3} \frac{E^4}{\rho}$$

During one period of synchrotron oscillation:

- when the particle is in the upper half-plane, it loses more energy per turn, its energy gradually reduces



- when the particle is in the lower half-plane, it loses less energy per turn, but receives U_0 on the average, so its energy deviation gradually reduces

The phase space trajectory spirals towards the origin (limited by quantum excitations)

=> The **synchrotron motion** is **damped** toward an **equilibrium bunch length** and **energy spread**.

$$\sigma_\tau = \frac{\alpha}{\Omega_s} \left(\frac{\sigma_\varepsilon}{E} \right)$$

More details in the lectures on *Electron Beam Dynamics*

Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the "synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the **energy** gained by the particle and the **RF phase** experienced by the same particle.

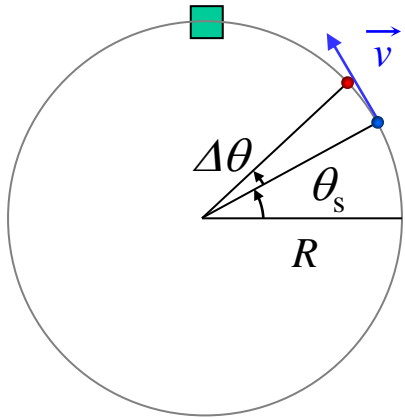
Since there is a **well-defined synchronous particle** which has always the same **phase** ϕ_s , and the nominal **energy** E_0 , it is sufficient to follow other particles with respect to that particle.

So let's introduce the following **reduced variables**:

particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_0$
particle energy	:	$\Delta E = E - E_0$
angular frequency	:	$\Delta\omega = \omega - \omega_0$
azimuth orbital angle:		$\Delta\theta = \theta - \theta_s$

Look at **difference**
from **synchronous**
particle

First Energy-Phase Equation



$$f_{RF} = h f_r \Rightarrow Df = -h Dq \quad \text{with} \quad q = \int w dt$$

particle ahead arrives earlier
 \Rightarrow smaller RF phase ϕ

Revolution frequency change vs RF phase change:

$$\Delta\omega_{..} = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h} \frac{d}{dt}(\Delta\phi) = -\frac{1}{h} \frac{d\phi}{dt}$$

Since: $\eta = -\frac{p_0}{\omega_0} \left(\frac{d\omega}{dp} \right)_s$

and

$$E^2 = E_0^2 + p^2 c^2$$

$$\Delta E = v_s \Delta p = \omega_0 R \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_0} = \frac{p_0 R}{h\eta\omega_0} \frac{d(\Delta\phi)}{dt} = \frac{p_0 R}{h\eta\omega_0} \dot{\phi}$$

Second Energy-Phase Equation

The rate of energy gained by a particle is:

$$\frac{dE}{dt} = q \hat{V} \sin \phi \frac{\omega_r}{2\pi}$$

The difference of energy gain with respect to the reference particle $\Delta E = E - E_0$ leads to the second energy-phase equation:

$$2\pi \frac{d}{dt} \left(\frac{\Delta E}{\omega_0} \right) = q \hat{V} (\sin \phi - \sin \phi_s)$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_0} = \frac{p_0 R}{h\eta\omega_0} \dot{\phi}$$

$$2\pi \frac{d}{dt} \left(\frac{\Delta E}{\omega_0} \right) = q \hat{V} (\sin \phi - \sin \phi_s)$$

deriving and combining

$$\frac{d}{dt} \left[\frac{-p_0 R}{h\eta\omega_0} \frac{d\phi}{dt} \right] + \frac{q \hat{V}}{2\pi} (\sin \phi - \sin \phi_s) = 0$$

This second order equation is non-linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will study some cases in the following...

Small Amplitude Oscillations

Let's assume constant parameters R , p_0 , ω_0 and η :

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0 \quad \text{with} \quad \Omega_s^2 = \frac{-q \hat{V}_{RF} \eta h \omega_0}{2\pi R p_0} \cos \phi_s$$

Consider now **small phase deviations** from the reference particle:

$$\sin \phi - \sin \phi_s = \sin(\phi_s + \Delta\phi) - \sin \phi_s \cong \cos \phi_s \Delta\phi \quad (\text{for small } \Delta\phi)$$

and the corresponding linearized motion reduces to a **harmonic oscillation**:

$$\ddot{f} + W_s^2 D f = 0 \quad \text{where } \Omega_s \text{ is the } \text{synchrotron angular frequency}.$$

The **synchrotron tune** ν_s is the number of synchrotron oscillations per revolution:

$$\nu_s = \Omega_s / \omega_0$$

Typical values are $\ll 1$, as it takes several 10 - 1000 turns per oscillation.

- proton synchrotrons of the order 10^{-3}
- electron storage rings of the order 10^{-1}

Stability condition for ϕ_s

$$\Omega_s^2 = \frac{-q\hat{V}_{RF}\eta h\omega_0}{2\pi R p_0} \cos \phi_s \Leftrightarrow \Omega_s^2 = \omega_0^2 \frac{-q\hat{V}_{RF}\eta h}{2\pi\beta^2 E} \cos \phi_s \quad \text{with } Rp = \frac{\beta^2 E}{\omega}$$

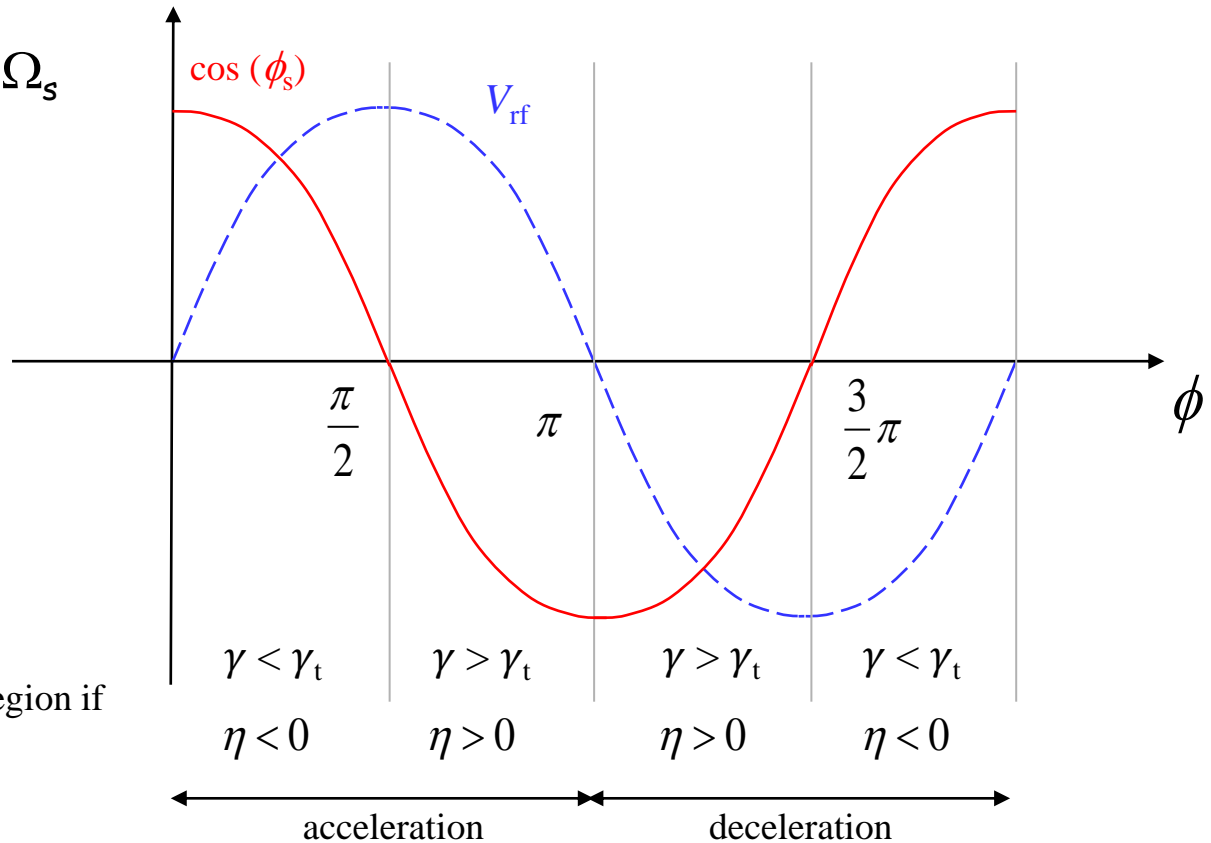
Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 > 0$$

⇕

$$\eta \cos \phi_s < 0$$

Stable in the region if



Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0 \quad (\Omega_s \text{ as previously defined})$$

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = I$$

which for small amplitudes reduces to:

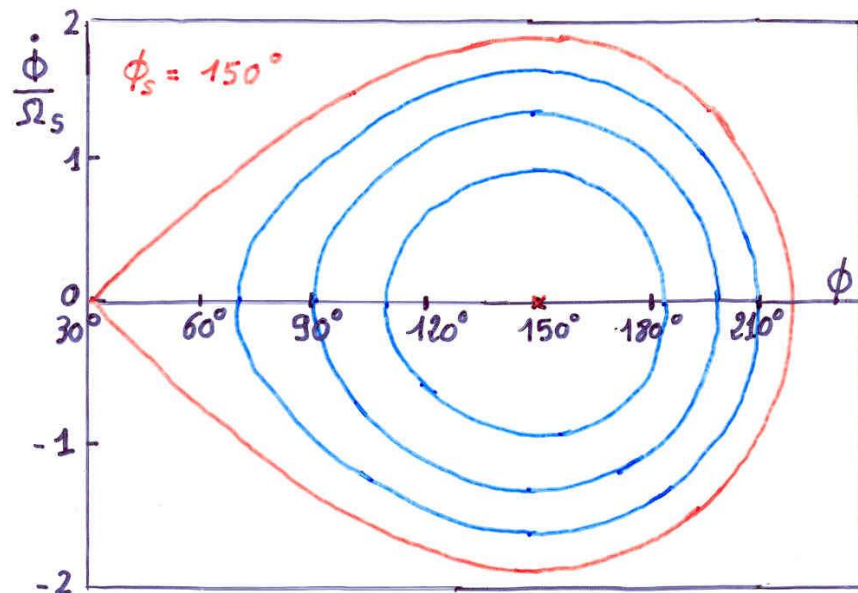
$$\frac{\dot{\phi}^2}{2} + W_s^2 \frac{(D\mathcal{F})^2}{2} = I' \quad (\text{the variable is } \Delta\phi, \text{ and } \phi_s \text{ is constant})$$

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi - \phi_s$ the force goes to zero and beyond it becomes non restoring.

Hence $\pi - \phi_s$ is an extreme amplitude for a stable motion which in the phase space $(\frac{\dot{\phi}}{\Omega_s}, \phi)$ is shown as closed trajectories.



Equation of the **separatrix**:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = -\frac{\Omega_s^2}{\cos \phi_s} (\cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos \phi_m + \phi_m \sin \phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s$$

Energy Acceptance

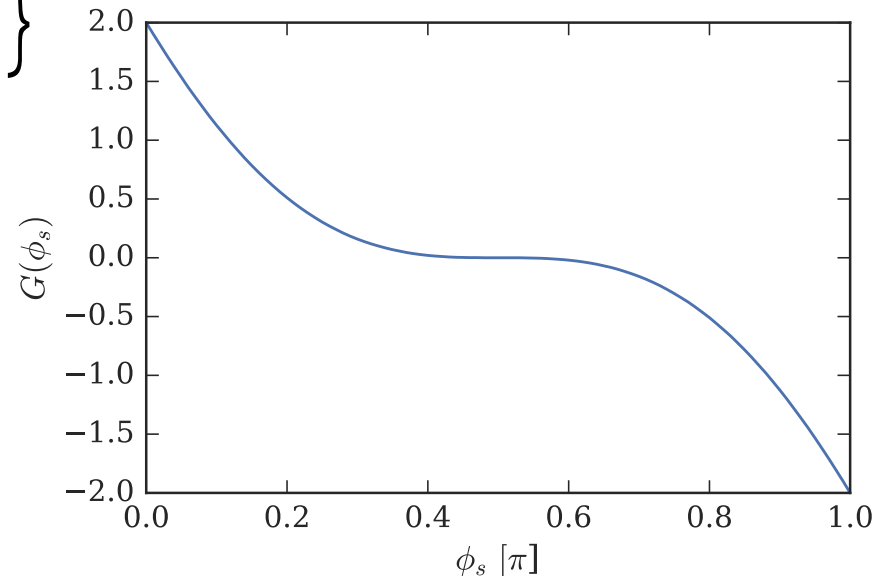
From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme at $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{f}_{\max}^2 = 2W_s^2 \left\{ 2 + (2f_s - \rho) \tan f_s \right\}$$

That translates into an **energy acceptance**:

$$\left(\frac{\Delta E}{E_0} \right)_{\max} = \pm \beta \sqrt{\frac{-q\hat{V}}{\pi h \eta E_0} G(\phi_s)}$$

$$G(\phi_s) = \frac{1}{2} \left[2 \cos \phi_s + (2\phi_s - \rho) \sin \phi_s \right]$$



This “**RF acceptance**” depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

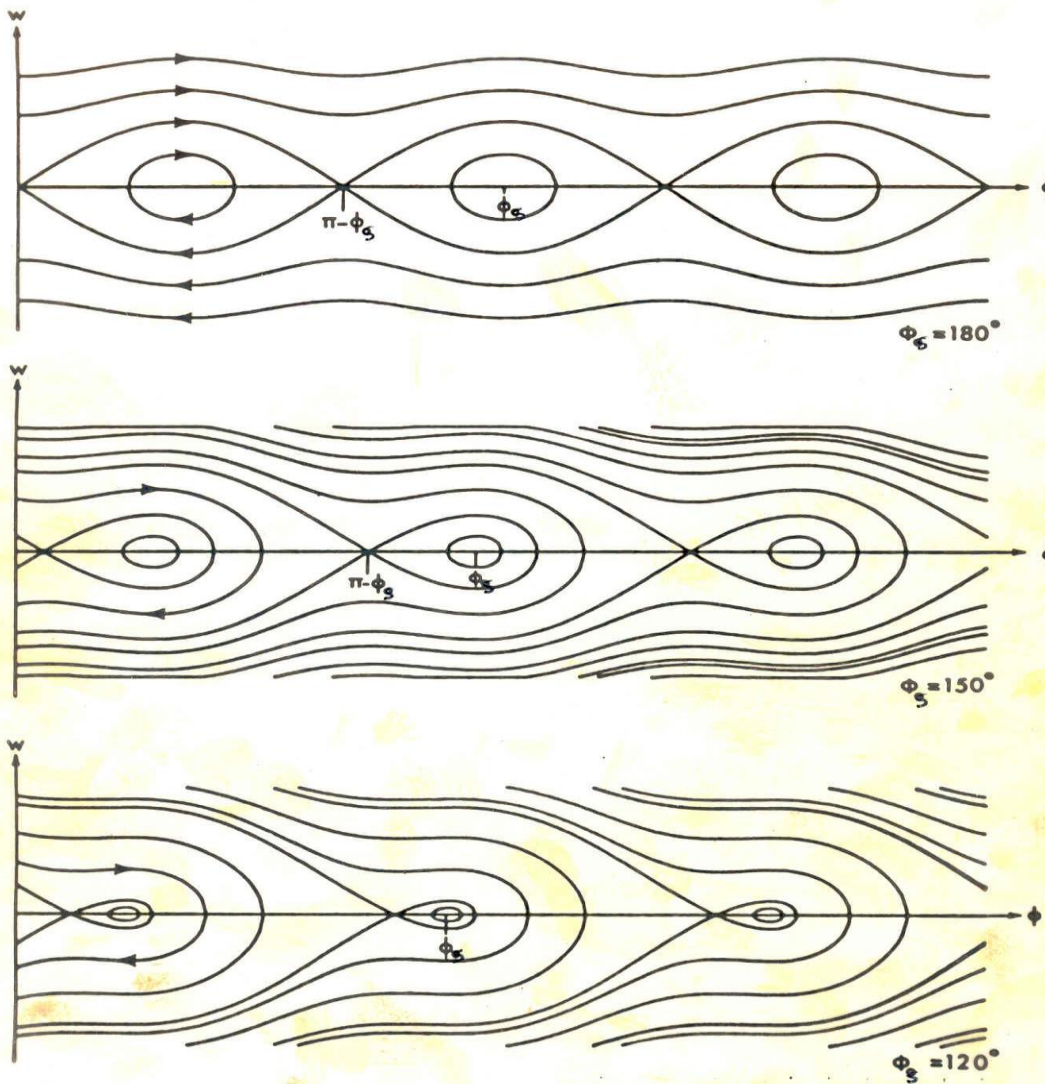
It's **largest** for $\phi_s=0$ and $\phi_s=\pi$ (**no acceleration**, depending on η).

It becomes smaller during acceleration, when ϕ_s is changing

Need a **higher RF voltage** for **higher acceptance**.

For the **same RF voltage** it is **smaller** for **higher harmonics h**.

RF Acceptance versus Synchronous Phase



The **areas of stable motion** (closed trajectories) are called "**BUCKET**". The number of circulating buckets is equal to " h ".

The phase extension of the **bucket is maximum** for $\phi_s = 180^\circ$ (or 0°) which means **no acceleration**.

During **acceleration**, the buckets get **smaller**, both in length and **energy acceptance**.

=> Injection preferably without acceleration.

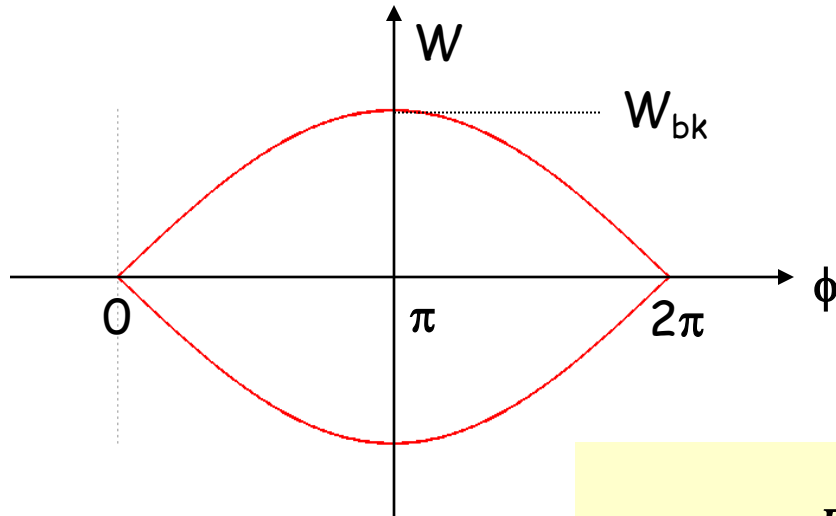
Stationnary Bucket - Separatrix

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s=\pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos \phi = \Omega_s^2$$

$$\frac{\dot{\phi}^2}{2} = 2\Omega_s^2 \sin^2 \frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W :



with $C=2\pi R$

$$W = \frac{\Delta E}{\omega_0} = \frac{p_0 R}{h\eta\omega_0} \dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

$$W = \pm \frac{R}{c} \sqrt{\frac{2q\hat{V}E_0}{\pi h\eta}} \sin \frac{\phi}{2} = \pm W_{bk} \sin \frac{\phi}{2}$$

Bucket height - bucket area

Setting $\phi=\pi$ in the previous equation gives the **height** of the **stationary bucket**:

$$W_{bk} = \frac{R}{c} \sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}}$$

The **bucket area** is: $A_{bk} = 2 \int_0^{2\pi} W d\phi$

Since: $\int_0^{2\pi} \sin \frac{\phi}{2} d\phi = 4$

one gets: $A_{bk} = 8 W_{bk} = \frac{8R}{c} \sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}} \longrightarrow W_{bk} = \frac{A_{bk}}{8}$

For an **accelerating bucket**, this **area** gets **reduced** by a factor depending on Φ_s :

$$\alpha(\phi_s) \approx \frac{1 - \sin \phi_s}{1 + \sin \phi_s}$$

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W , leads to the 1st order equations:

$$W = \frac{\Delta E}{\omega_0}$$



$$\frac{d\phi}{dt} = \frac{h\eta\omega_0}{p_0R} W$$

$$\frac{dW}{dt} = \frac{e\hat{V}}{2\pi} (\sin \phi - \sin \phi_s)$$

The two variables ϕ, W are canonical since these equations of motion can be derived from a Hamiltonian $H(\phi, W, t)$:

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W}$$

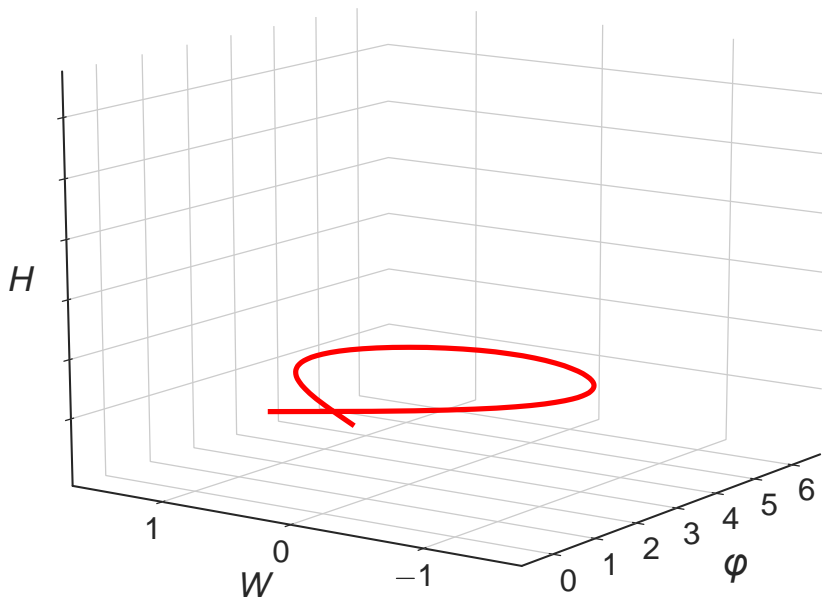
$$\frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W) = \frac{1}{2} \frac{h\eta\omega_0}{p_0R} W^2 + \frac{e\hat{V}}{2\pi} [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s]$$

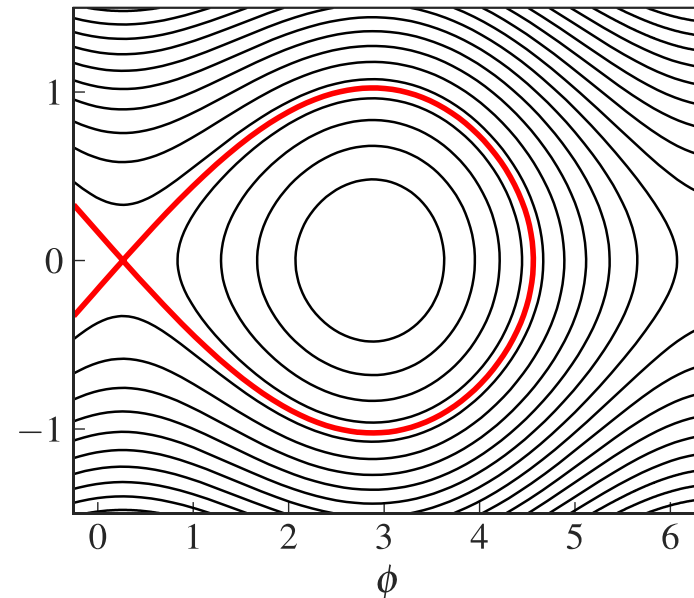
Hamiltonian of Longitudinal Motion

What does it represent? The total energy of the system!

Surface of $H(\varphi, W)$



Contours of $H(\varphi, W)$

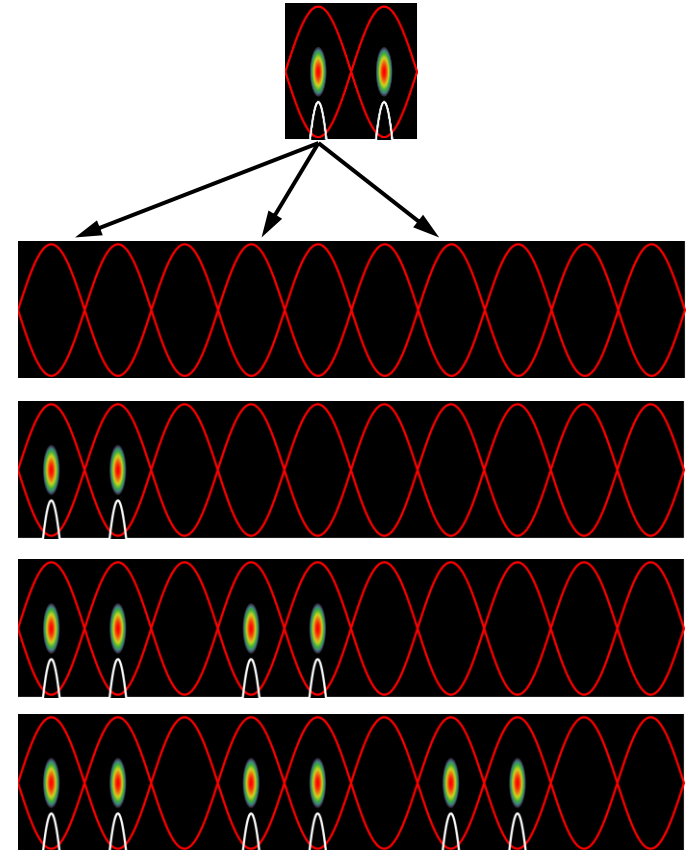
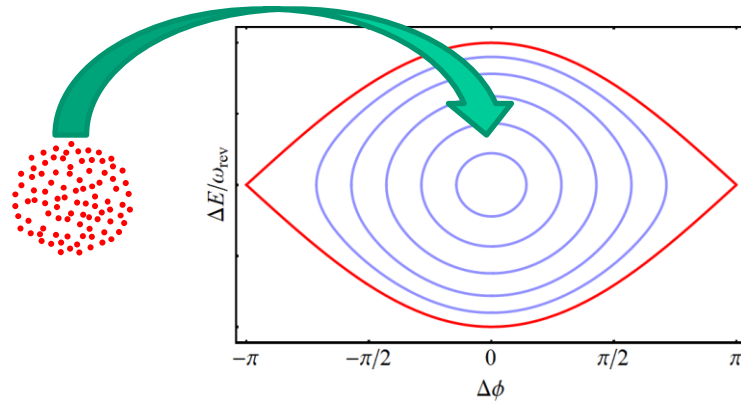


Contours of constant H are particle trajectories in phase space!
(H is conserved)

Hamiltonian Mechanics can help us understand some fairly complicated dynamics (multiple harmonics, bunch splitting, ...)

Injection: Bunch-to-bucket transfer

- Bunch from sending accelerator into the bucket of receiving

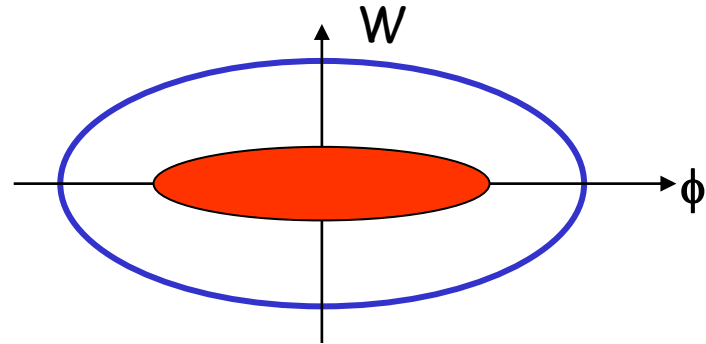
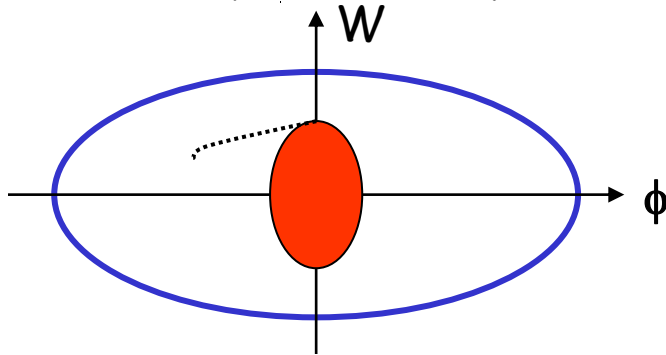


Advantages:

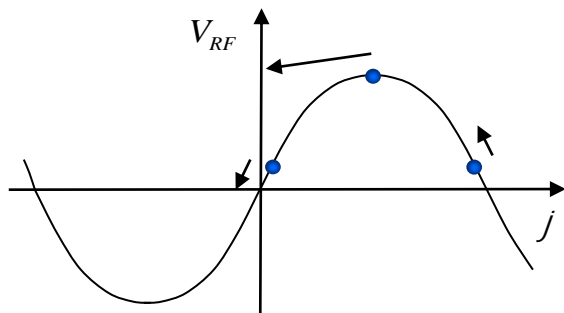
- Particles always subject to longitudinal focusing
- No need for RF capture of de-bunched beam in receiving accelerator
- No particles at unstable fixed point
- Time structure of beam preserved during transfer

Effect of a Mismatch

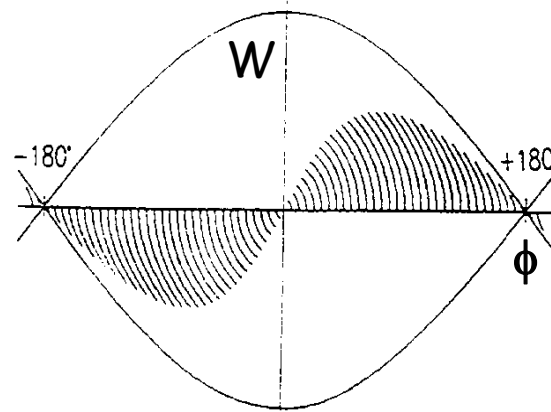
Injected bunch: short length and large energy spread
 after 1/4 synchrotron period: longer bunch with a smaller energy spread.



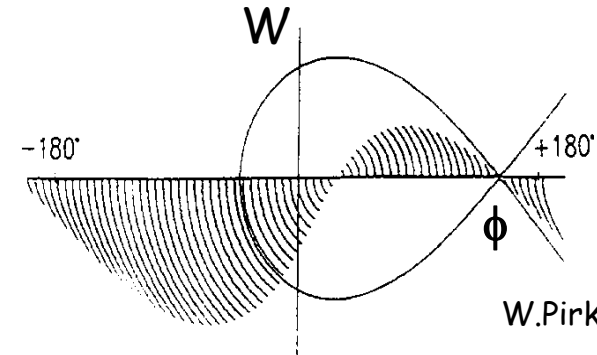
For **larger amplitudes**, the angular phase space motion is slower
 (1/8 period shown below) \Rightarrow can lead to **filamentation** and **emittance growth**



restoring force is non-linear



stationary bucket



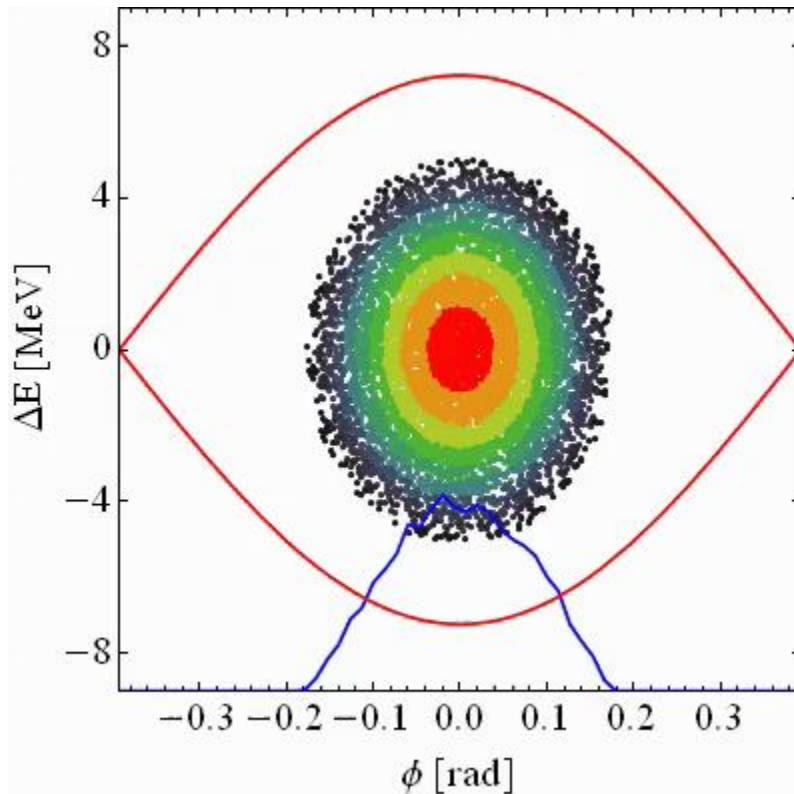
accelerating bucket

W.Pirkel

Effect of a Mismatch (2)

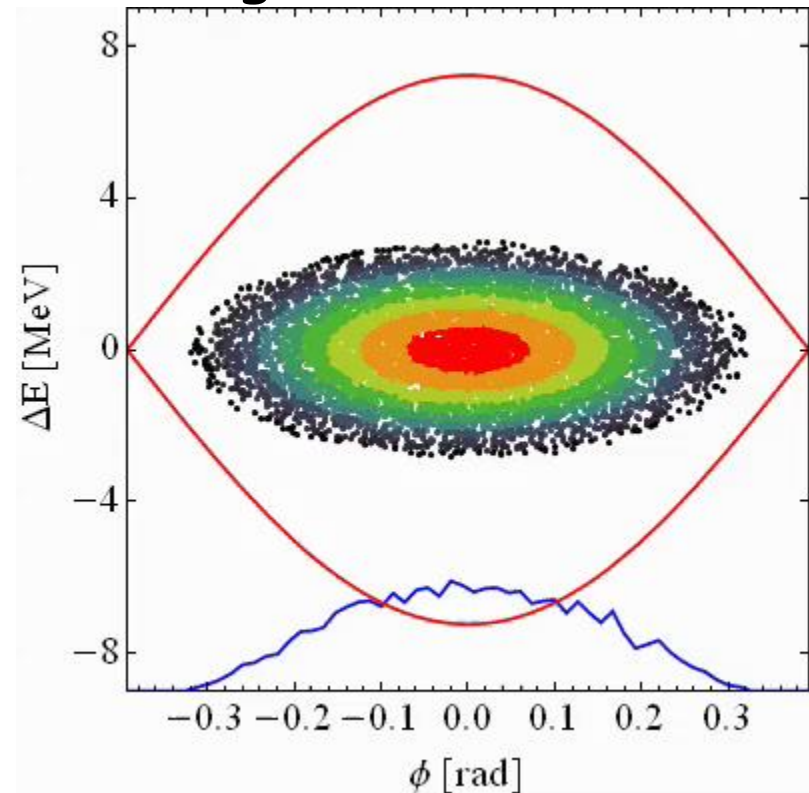
- Long. emittance is only preserved for **correct RF voltage**

Matched case



→ Bunch is fine, longitudinal emittance remains constant

Longitudinal mismatch

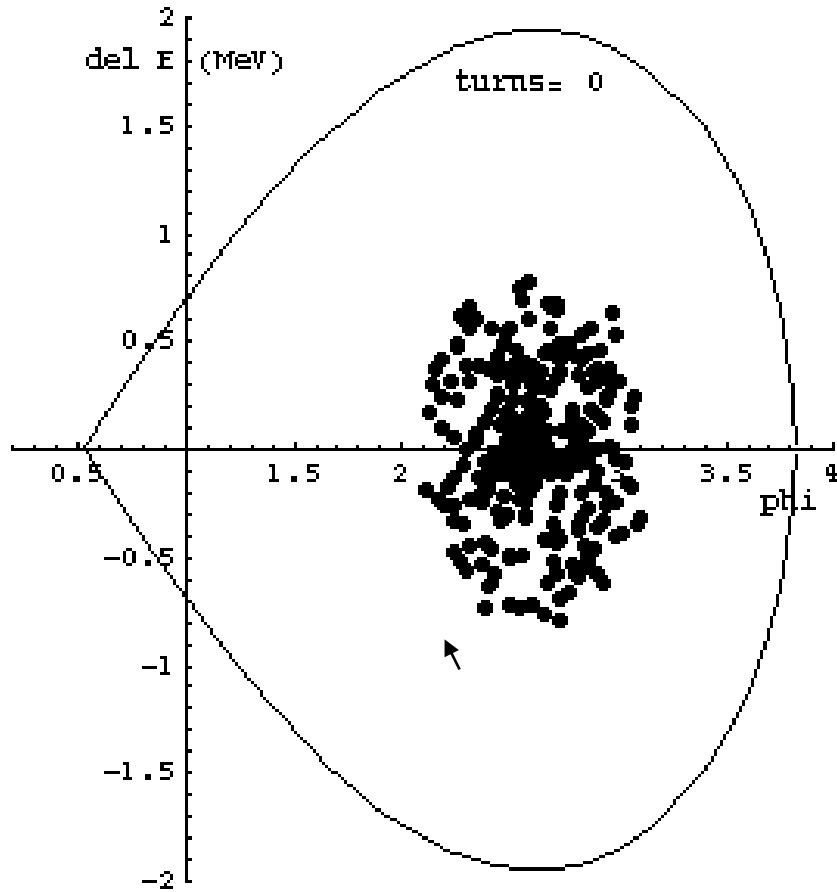


→ Dilution of bunch results in increase of long. emittance

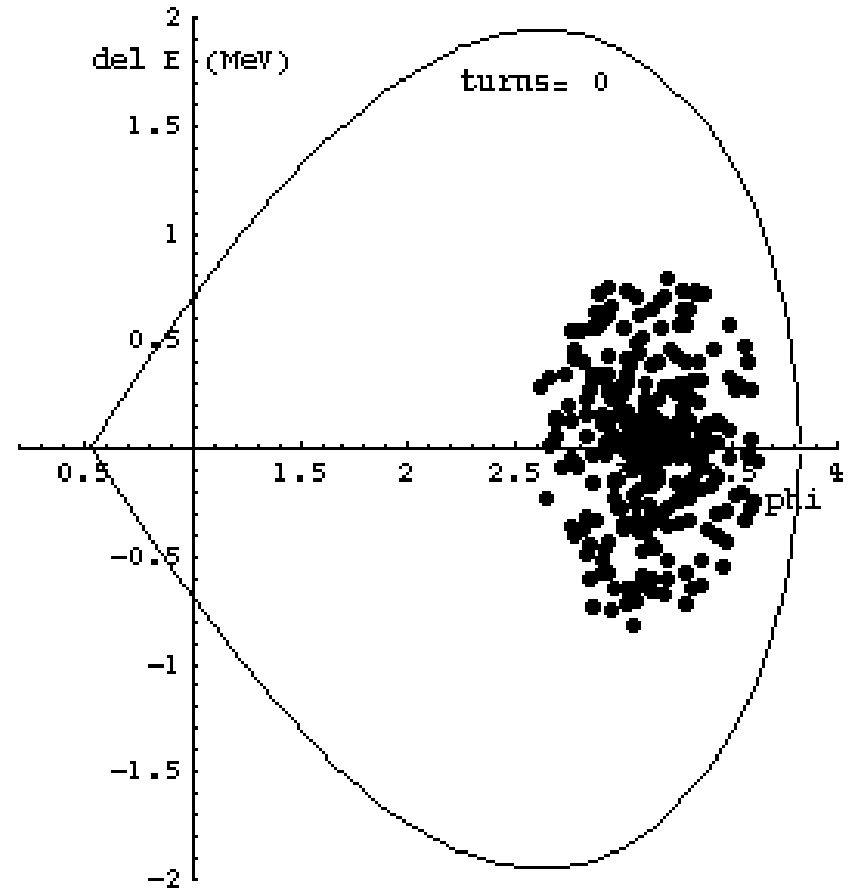
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.

For a mismatched transfer, the emittance increases (right).



matched beam

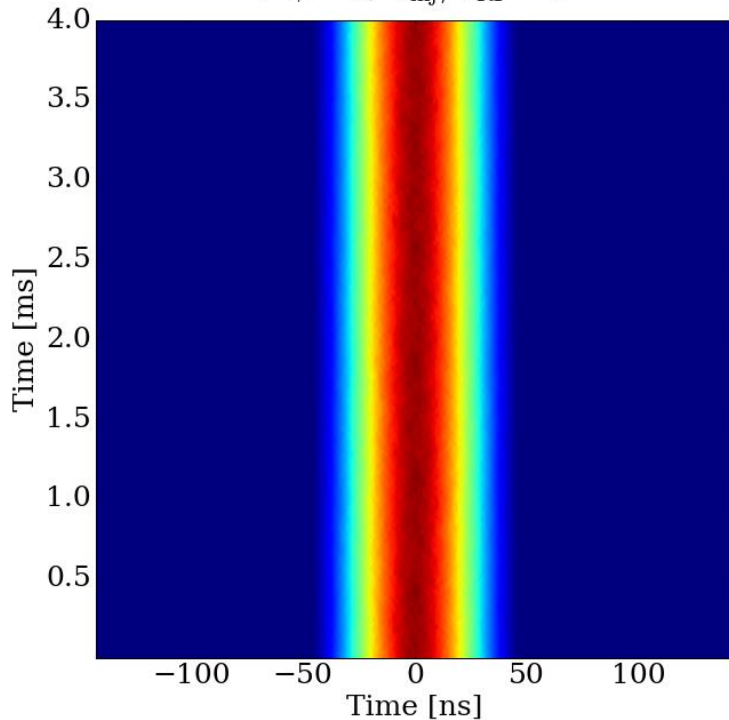


mismatched beam - phase error

Longitudinal matching - Beam profile

Matched case

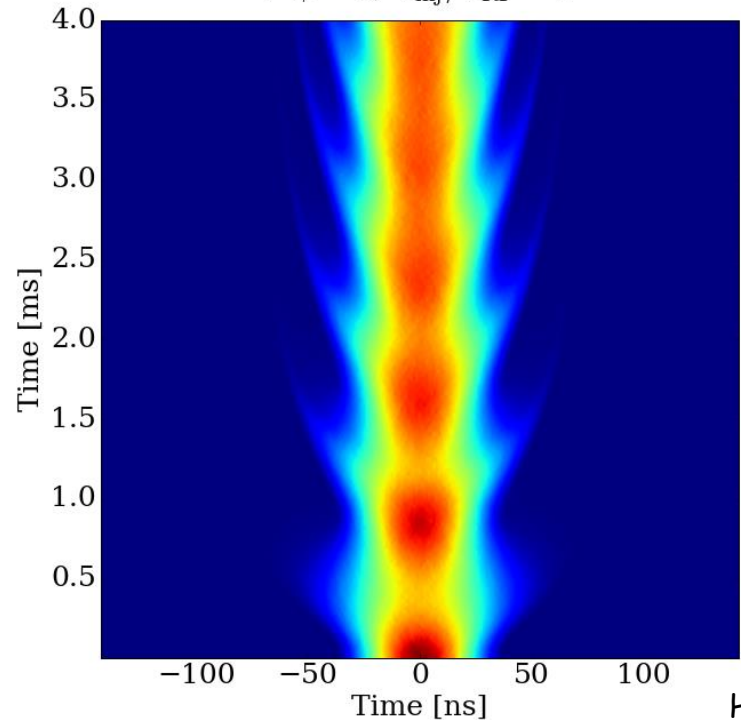
$$\Delta\phi = 0, V_{\text{inj}}/V_{\text{RF}} = 1$$



→ Bunch is fine, longitudinal emittance remains constant

Longitudinal mismatch

$$\Delta\phi = 0, V_{\text{inj}}/V_{\text{RF}} = 2$$

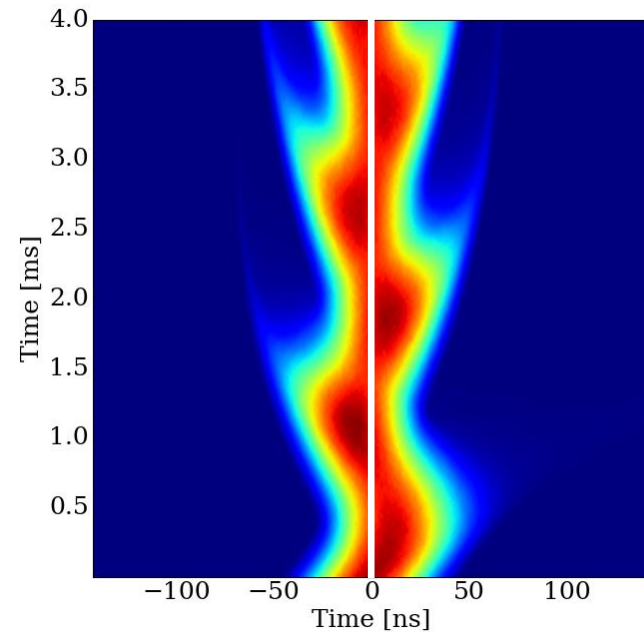
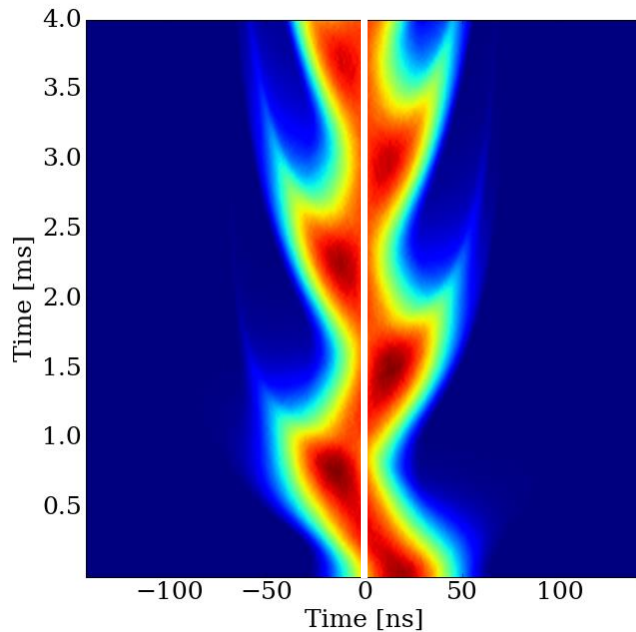


→ Dilution of bunch results in increase of long. emittance

H.Damerau

Matching quiz!

- Find the difference!



H.Damerau

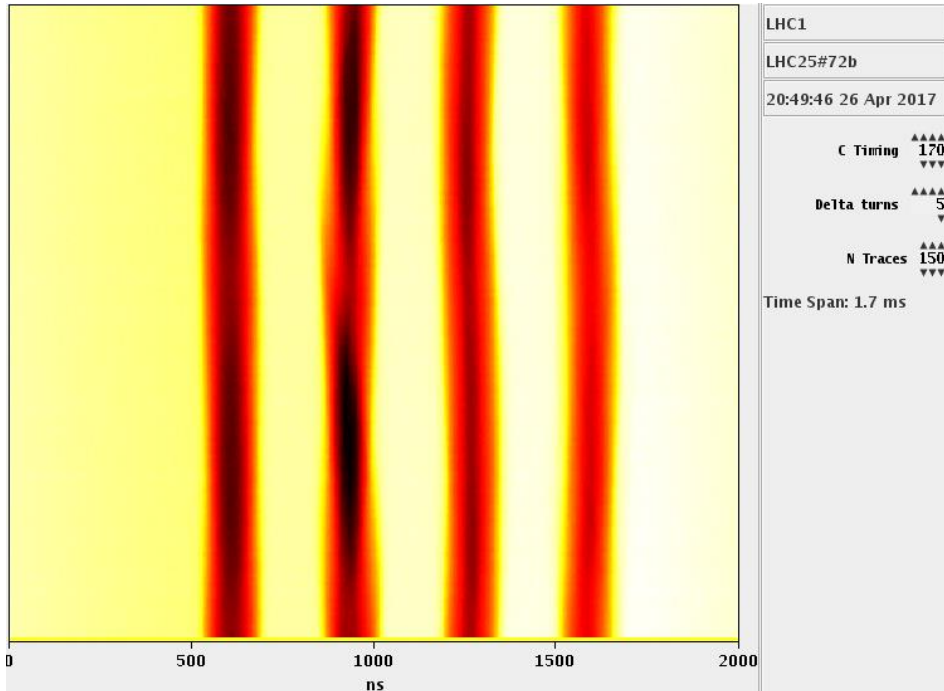
- -45° phase error at injection
- Can be easily corrected by bucket phase

- Equivalent energy error
- Phase does not help: requires beam energy change

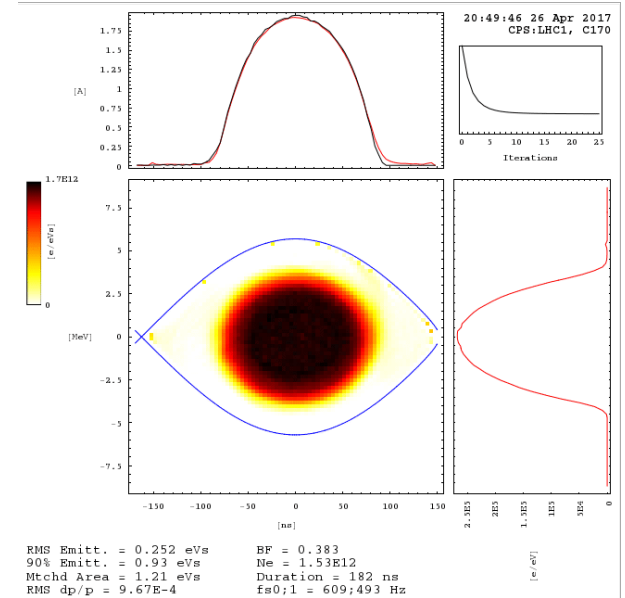
Phase Space Tomography

We can reconstruct the phase space distribution of the beam.

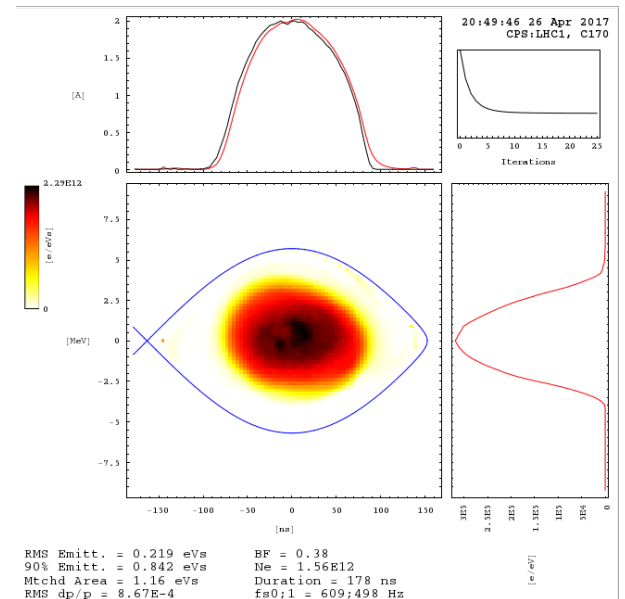
- Longitudinal bunch profiles over a number of turns
- Parameters determining Ω_s



1st
bunch



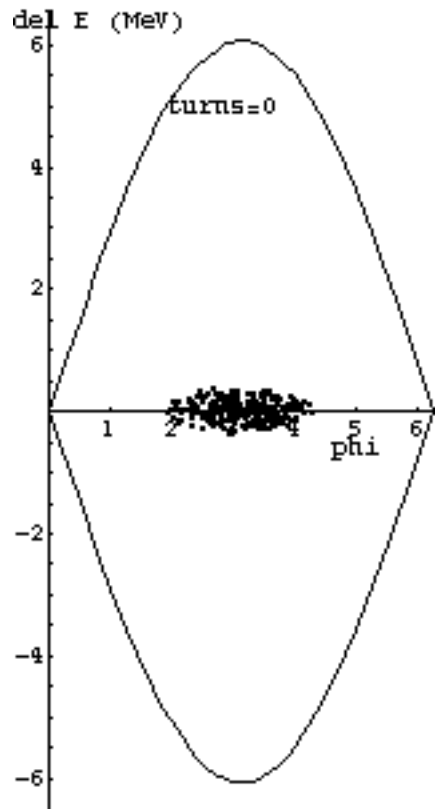
2nd
bunch



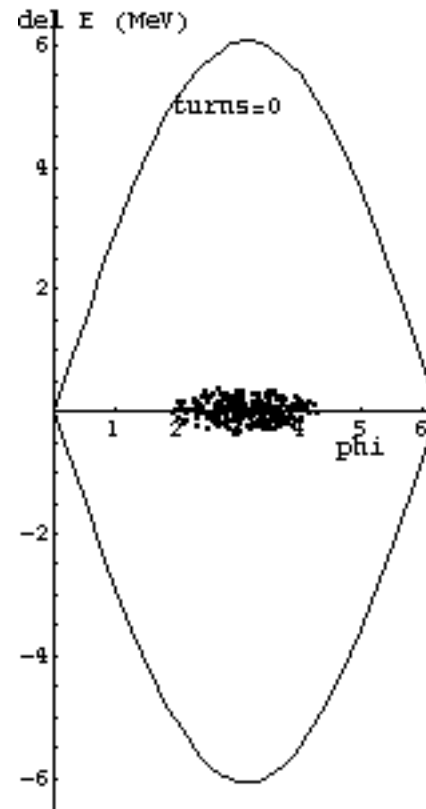
Bunch Rotation

Phase space motion can be used to make short bunches.

Start with a long bunch and extract or recapture when it's short.

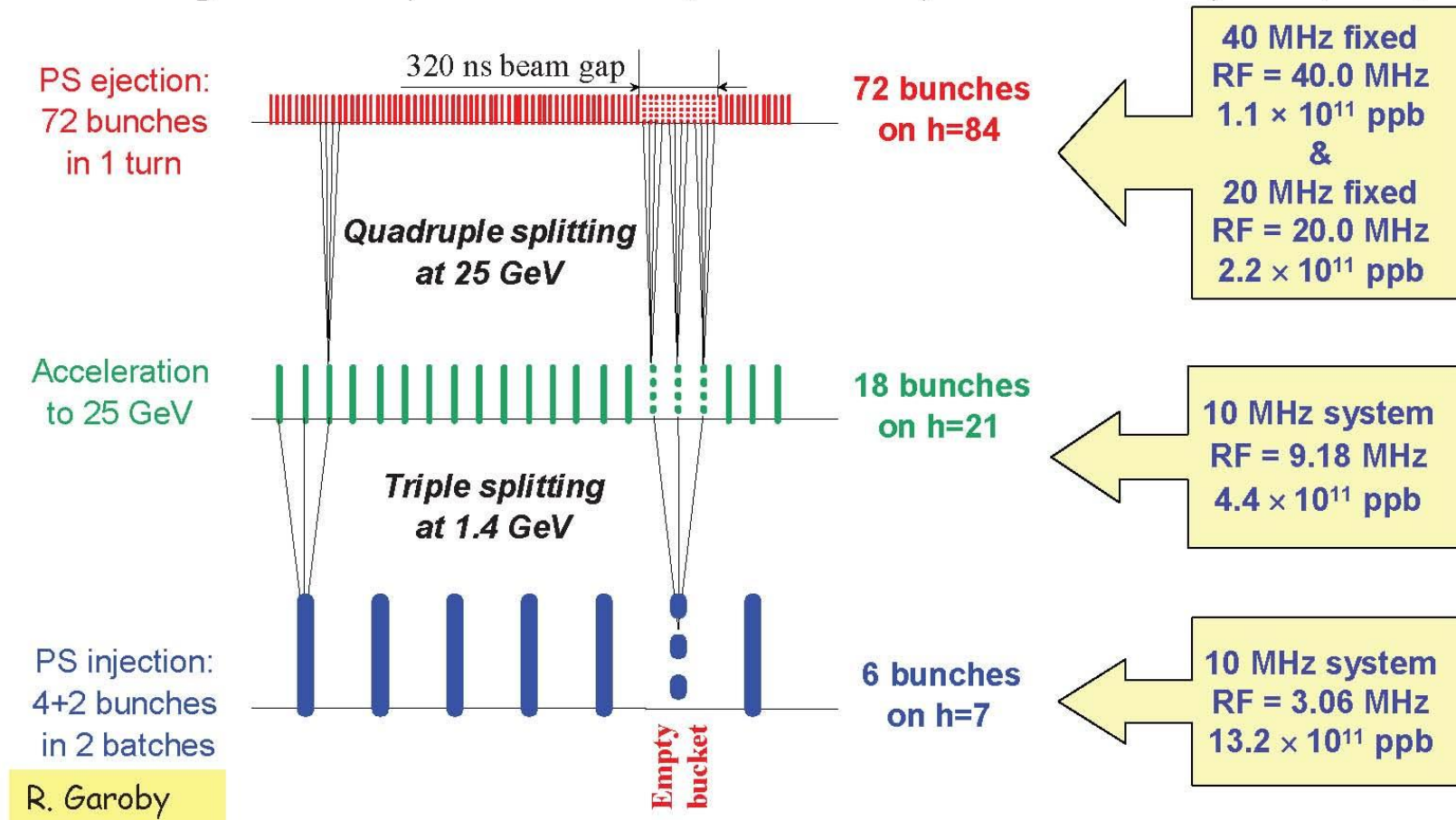


initial beam



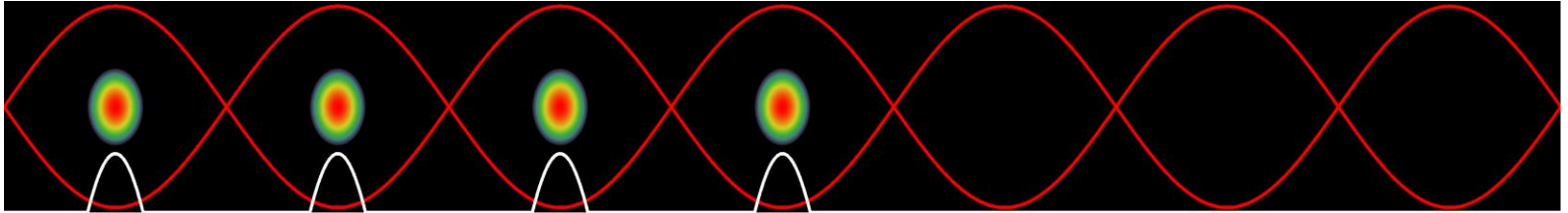
Generating a 25ns LHC Bunch Train in the PS

- **Longitudinal bunch splitting (basic principle)**
 - Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)



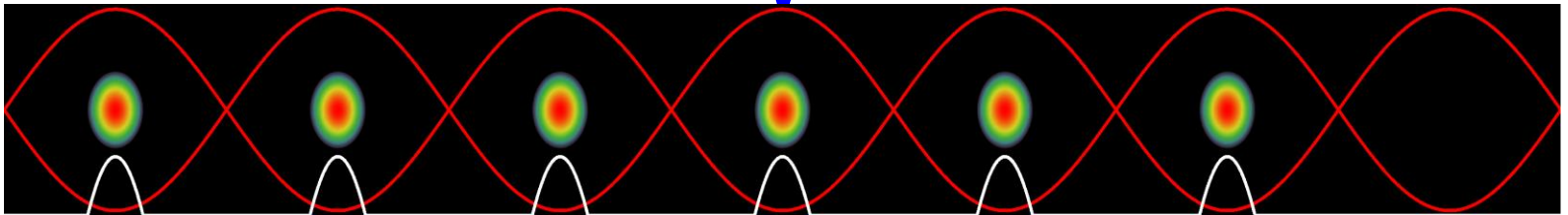
Production of the LHC 25 ns beam

1. Inject four bunches ~ 180 ns, 1.3 eVs

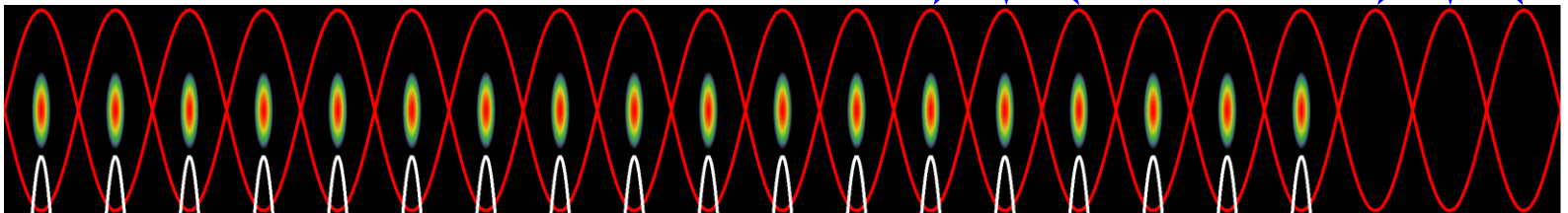


Wait 1.2 s for second injection

2. Inject two bunches



3. Triple split after second injection

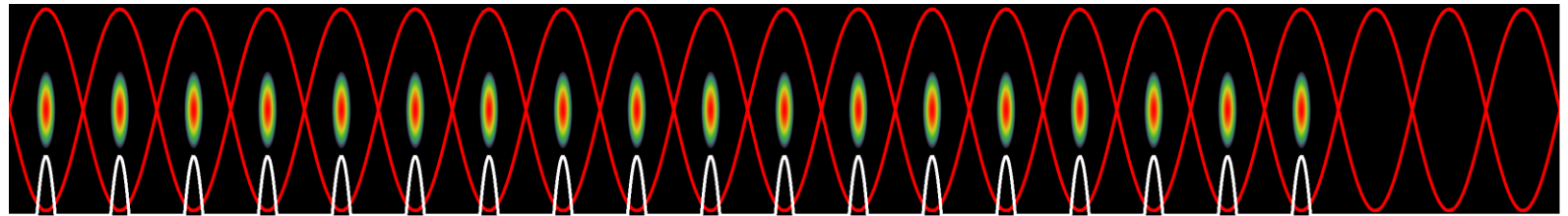


~ 0.7 eVs

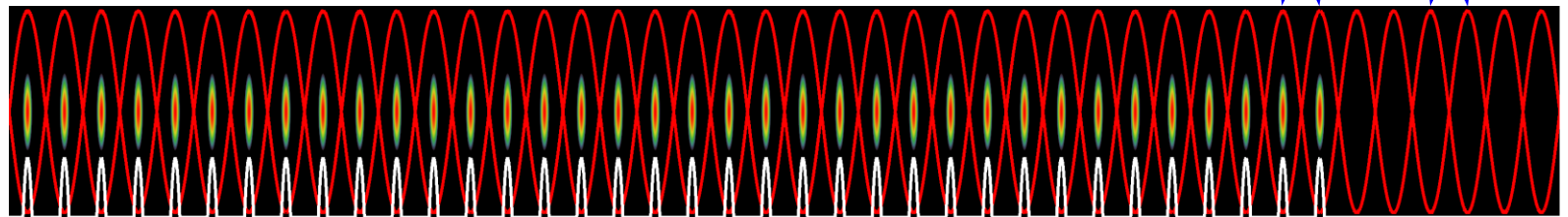
4. Accelerate from 1.4 GeV (E_{kin}) to 26 GeV

Production of the LHC 25 ns beam

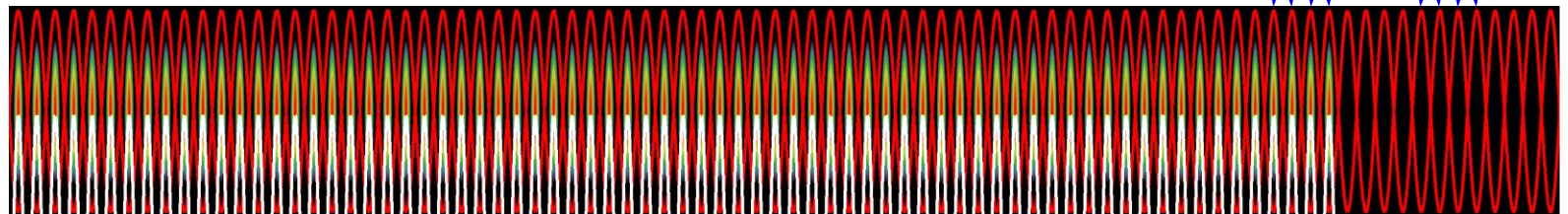
5. During acceleration: longitudinal emittance blow-up: $0.7 - 1.3$ eVs



6. Double split ($h_{21} \rightarrow h_{42}$)

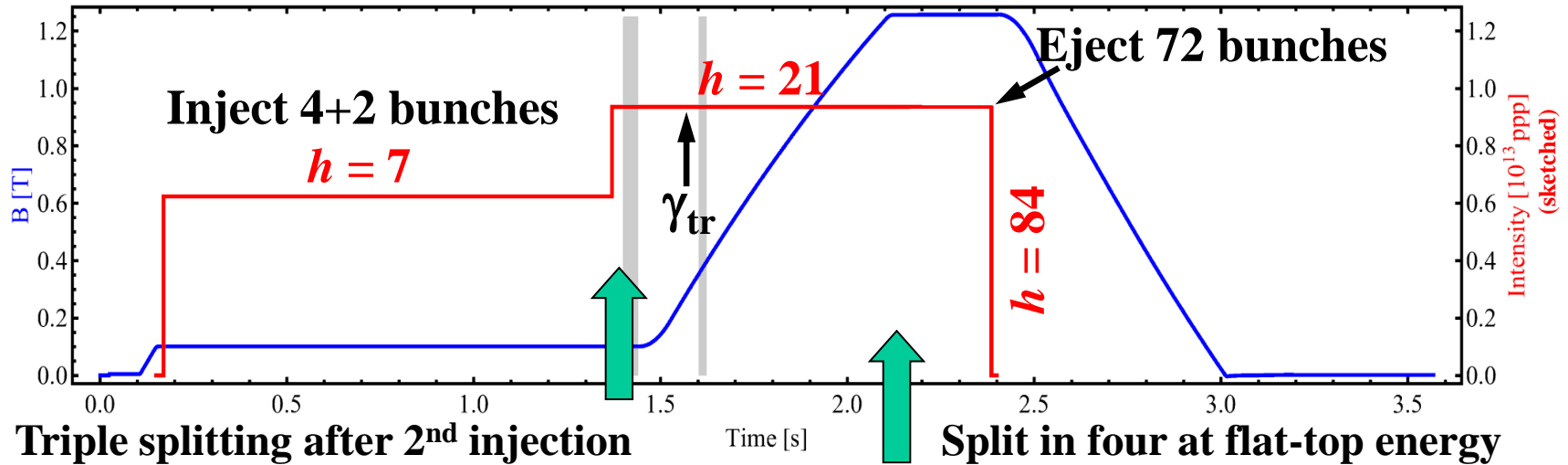


7. Double split ($h_{42} \rightarrow h_{84}$) ~ 0.35 eVs, 4 ns

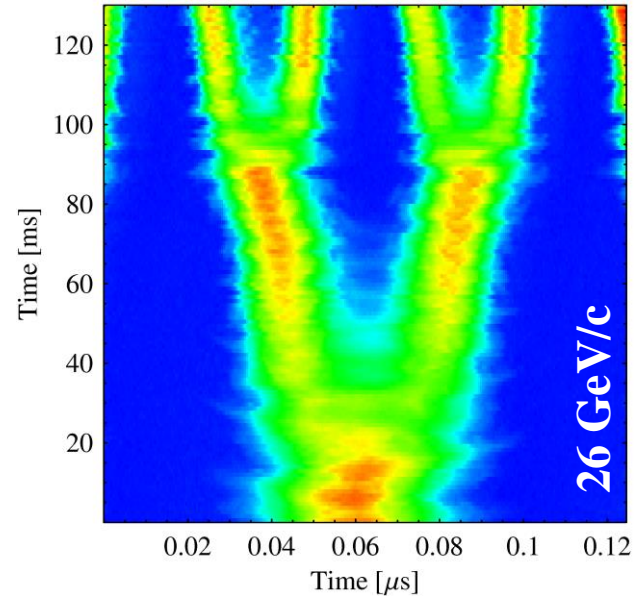
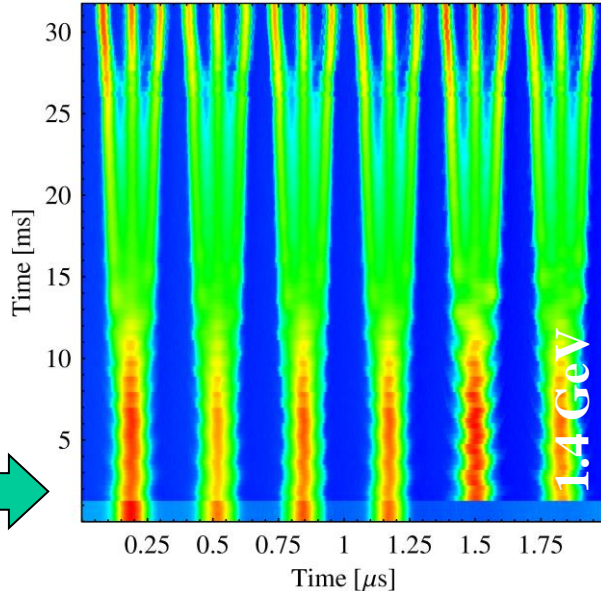
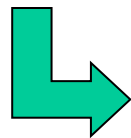


10. Fine synchronization, bunch rotation \rightarrow Extraction!

The LHC25 (ns) cycle in the PS

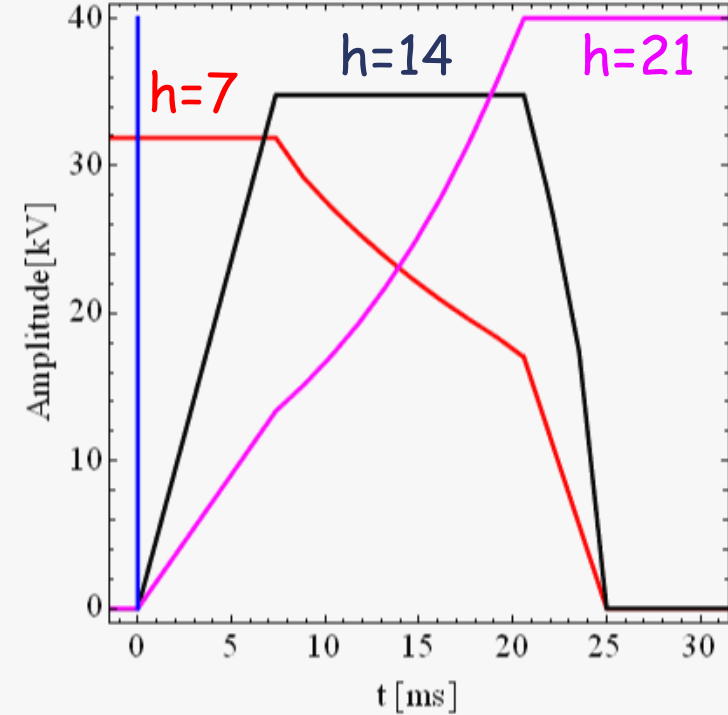
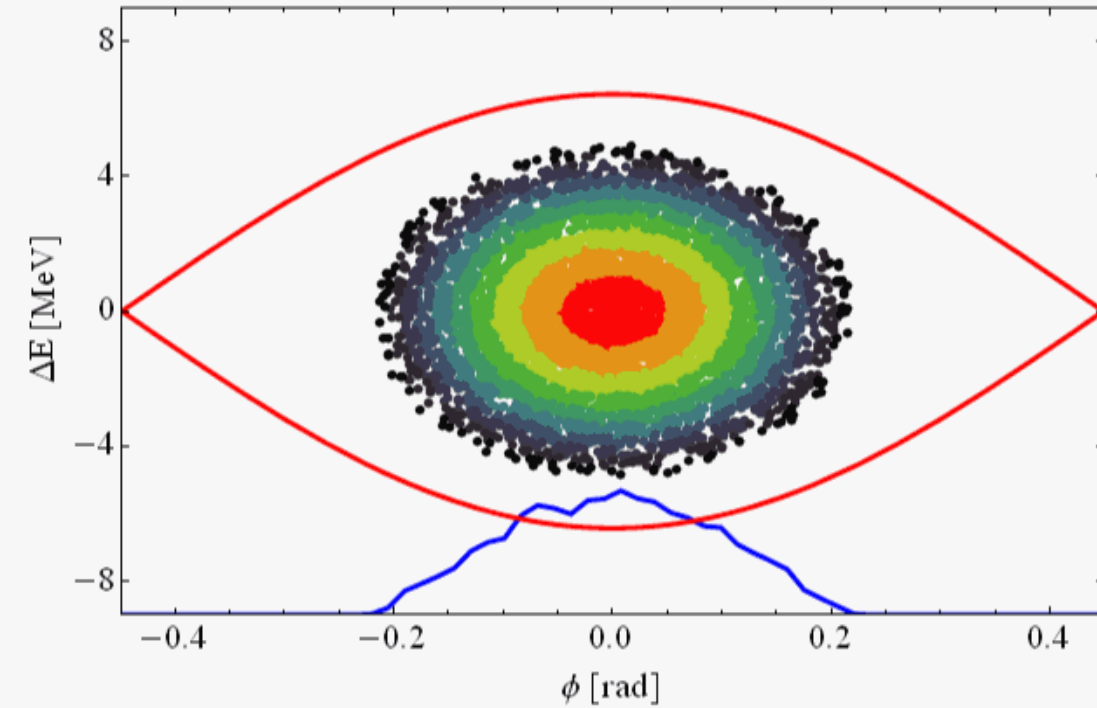


2nd injection



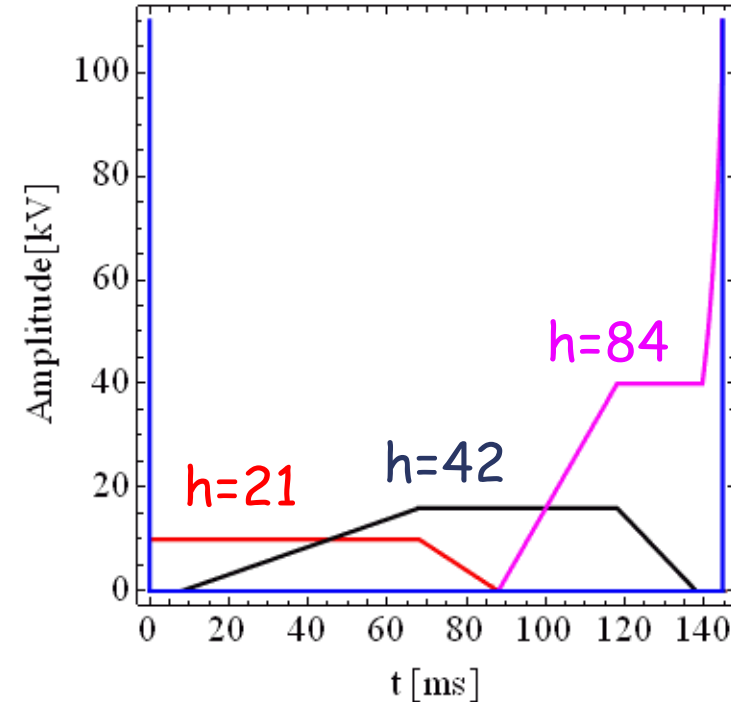
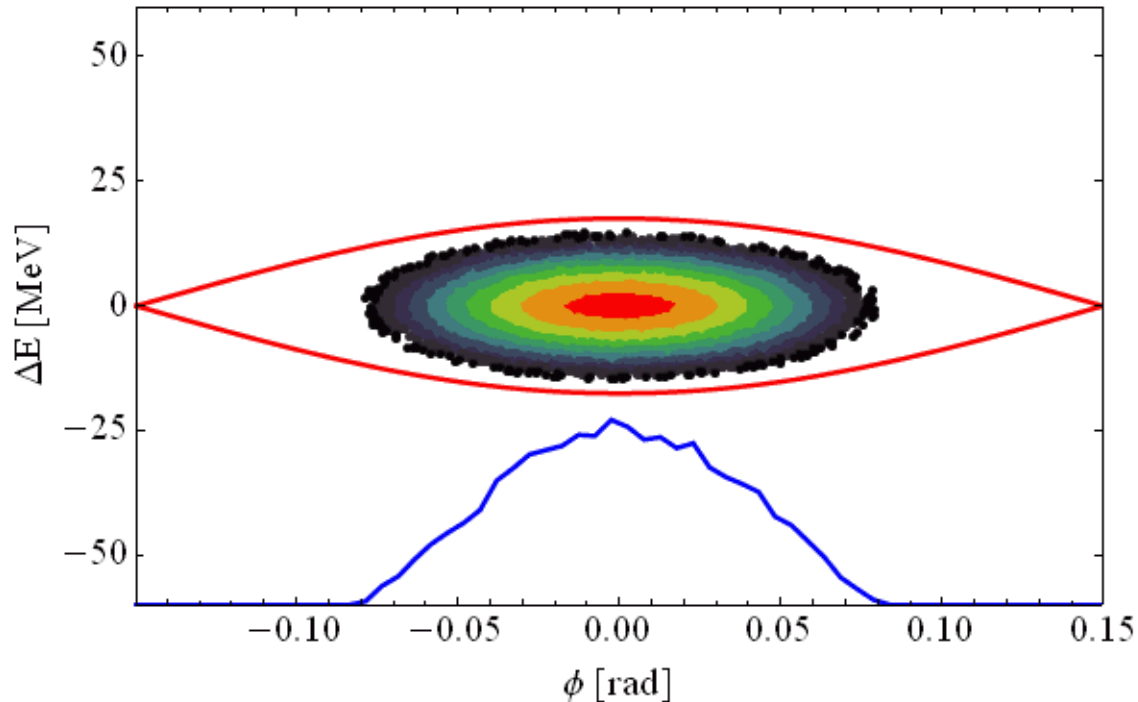
→ Each bunch from the Booster divided by 12 → $6 \times 3 \times 2 \times 2 = 72$

Triple splitting in the PS



Two times double splitting in the PS

Two times double splitting and bunch rotation:



- Bunch is divided twice using RF systems at $h = 21/42$ (10/20 MHz) and $h = 42/84$ (20/40 MHz)
- Bunch rotation: first part $h=84$ only + $h=168$ (80 MHz) for final part

Summary

- Cyclotrons/Synchrocyclotrons for low energy
- **Synchrotrons** for high energies, constant orbit, synchronously rising field and frequency
- Particles with higher energy have a longer orbit (normally) but a higher velocity
 - at low energies (below transition) velocity increase dominates
 - at high energies (above transition) velocity almost constant
- Particles perform **oscillations around synchronous phase**
 - synchronous phase depending on acceleration
 - below or above transition
- **Hamiltonian** approach can deal with fairly complicated dynamics
- **Bucket** is the stable region in phase space inside the **separatrix**
- **Matching** the shape of the bunch to the bucket is essential

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And CERN Accelerator Schools (CAS) Proceedings
In particular: <https://arxiv.org/abs/2011.02932>
Longitudinal Beam Dynamics in Circular Accelerators

Acknowledgements

End of our
crash course in
longitudinal dynamics



www.formula1.com

Appendix: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F} = \frac{d\vec{p}}{dt} = e \left(\vec{E} + \vec{v} \times \vec{B} \right)$

2nd term always perpendicular to motion => no acceleration

Relativistic Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \quad g = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \beta^2}}$$

$$p = mv = \frac{E}{c^2} bc = b \frac{E}{c} = bg m_0 c$$

$$E^2 = E_0^2 + p^2 c^2 \quad \longrightarrow \quad dE = v dp$$

$$\frac{dE}{dz} = v \frac{dp}{dz} = \frac{dp}{dt} = e E_z$$

$$dE = dW = e E_z dz \quad \rightarrow \quad W = e \int E_z dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin W_{RF} t = \hat{E}_z \sin f(t)$$

$$\int \hat{E}_z dz = \hat{V}$$

$$W = e \hat{V} \sin \phi$$

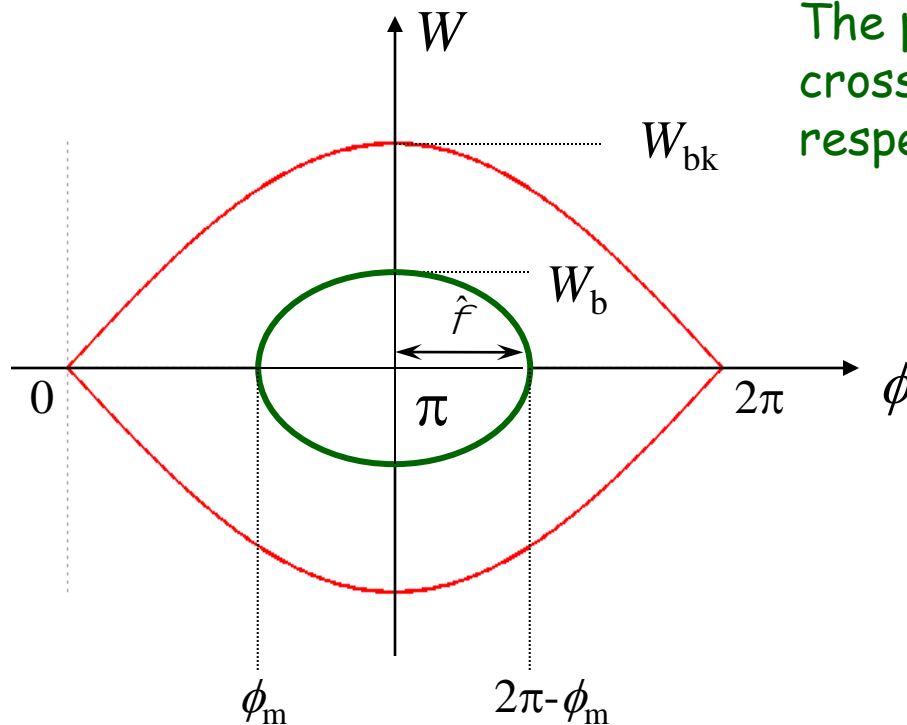
(neglecting transit time factor)

The field will change during the passage of the particle through the cavity
=> effective energy gain is lower

Phase Space Trajectories inside Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = I \quad \xrightarrow{\phi_s = \pi} \quad \frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos \phi = I$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos \phi = \Omega_s^2 \cos \phi_m$$

$$\dot{\phi} = \pm \Omega_s \sqrt{2(\cos \phi_m - \cos \phi)}$$

$$W = \pm W_{bk} \sqrt{\cos^2 \frac{j}{2} \frac{m}{2} - \cos^2 \frac{j}{2}}$$

$$\cos(f) = 2 \cos^2 \frac{f}{2} - 1$$

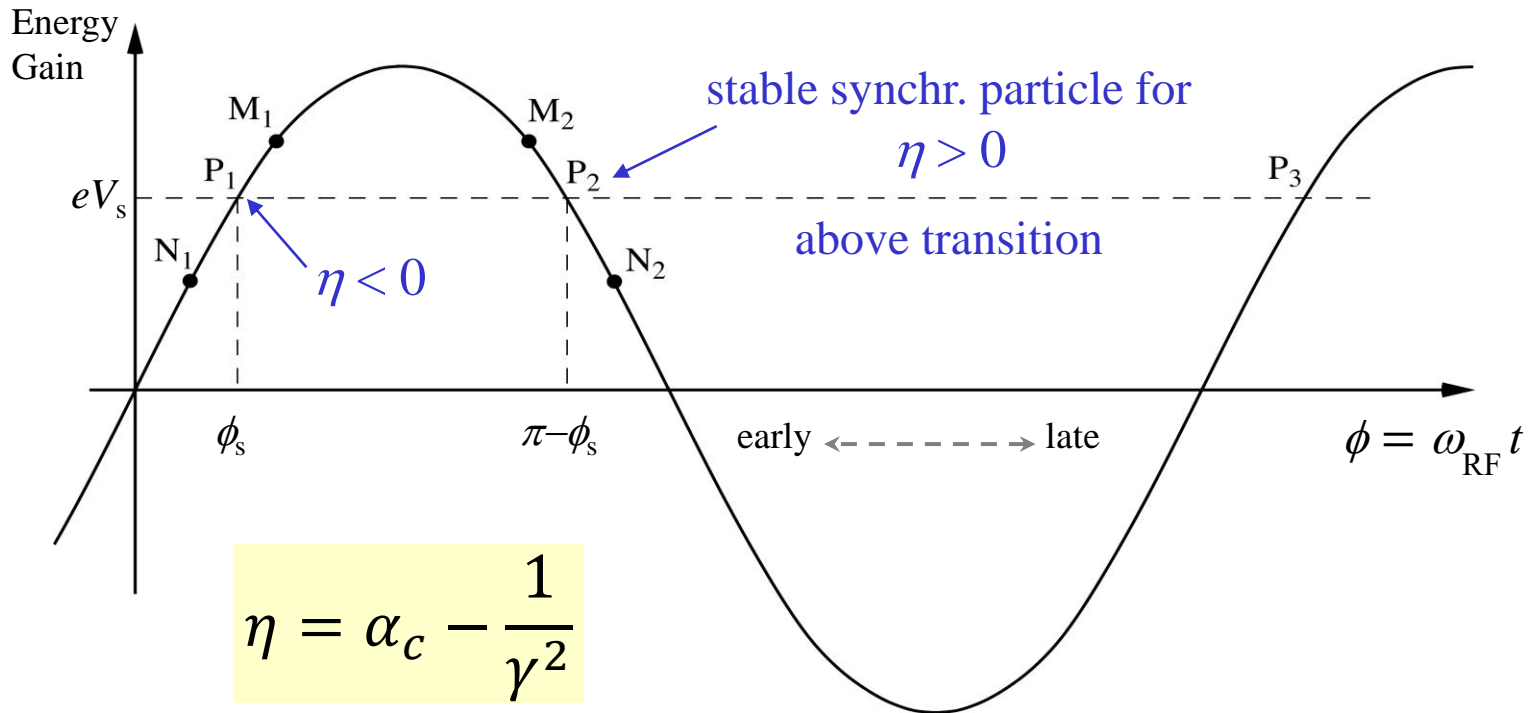
The Synchrotron - Energy ramping

Energy ramping by increasing the B field (frequency has to follow v).
 Stable phase ϕ_s changes during energy ramping

$$\sin \phi_s = 2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}}$$



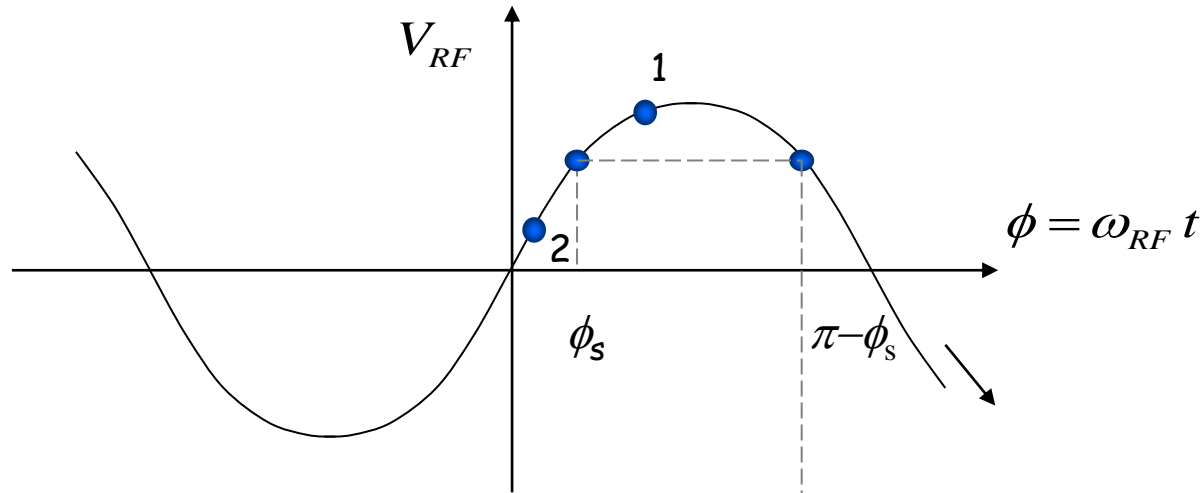
$$\phi_s = \arcsin \left(2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}} \right)$$



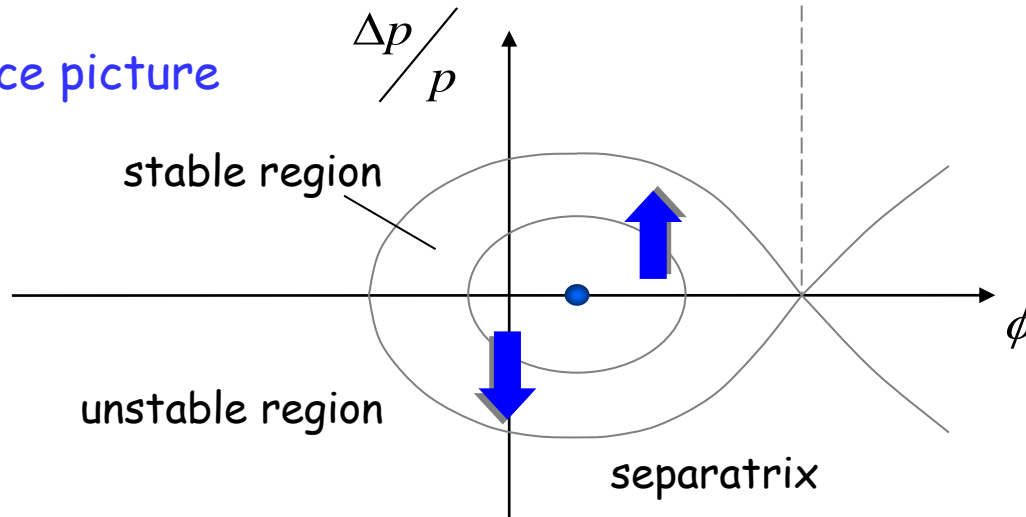
Transition

Case with acceleration B increasing

$$\gamma < \gamma_t$$



Phase space picture



$$\phi_s < \phi < \pi - \phi_s$$

Particles will accumulate energy difference each turn