A first taste of Non-Linear Beam Dynamics

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Purpose of the lecture

- Introducing aspects of non-linear dynamics
  - Effects of **nonlinear perturbations**
    - Resonances, tune shifts, dynamic aperture
  - **Mathematical tools** for modelling nonlinear dynamics
    - Power series (Taylor) maps and symplectic maps
  - **Analysis** methods
    - Normal forms, frequency map analysis

- Illustrate methods and tools for a simple example of an accelerator
  - **Storage ring**

Aim of the 1st Lecture

- Provide an introduction to some of the **key concepts** of nonlinear dynamics in particle accelerators
- Describe some of the **sources** of nonlinearities
- Outline some of the **tools** used for modelling
- Explain the significance and potential **impact** of nonlinear dynamics in some accelerator systems

From Linear to Non-linear

- Particle motion through linear components such as drifts, dipoles and quadrupoles can be represented by **linear transfer maps**
- For example, in a **drift space** of length L, the **horizontal coordinate** and the (scaled) **horizontal momentum** from initial position 0 to a final position 1 are

$$
\begin{aligned}
x_1 &= x_0 + L p_{x0} \\
p_{x1} &= p_{x0}
\end{aligned}
$$

- Note that the horizontal momentum is

$$p_x = \frac{\gamma m v_x}{P_0} \approx \frac{dx}{ds}$$

where γ is the relativistic factor, m is the rest mass of the particle, v_x is the horizontal velocity, and P_0 is the reference momentum

From Linear to Non-linear

- **Linear transfer maps** can be written in terms of **matrices** and for example for a **drift space** of length L

$$\begin{pmatrix} x_1 \\ p_{x1} \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ p_{x0} \end{pmatrix}$$

- In general, a **linear transformation** can be written as

$$\vec{x}_1 = R\vec{x}_0 + \vec{A}$$

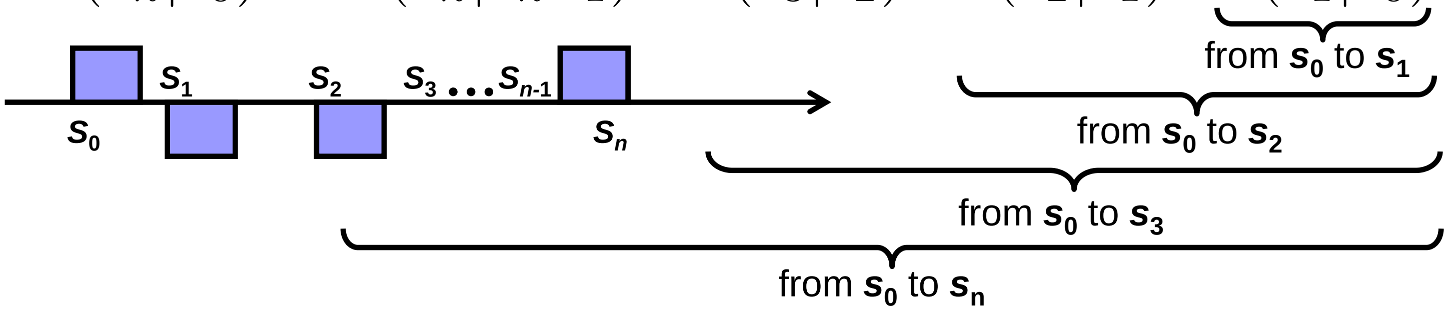
where the phase space vectors are $\vec{x} = (x, p_x)$

- The transfer matrix R and the vector $\vec{A}$ are constants, i.e. they do not depend on $\vec{x}_0$

From Linear to Non-linear

- The transfer matrix for a section of beamline can be found by **multiplying** the **transfer matrices** for the accelerator components within that section

$$R(s_n|s_0) = R(s_n|s_{n-1})...R(s_3|s_2) \cdot R(s_2|s_1) \cdot R(s_1|s_0)$$

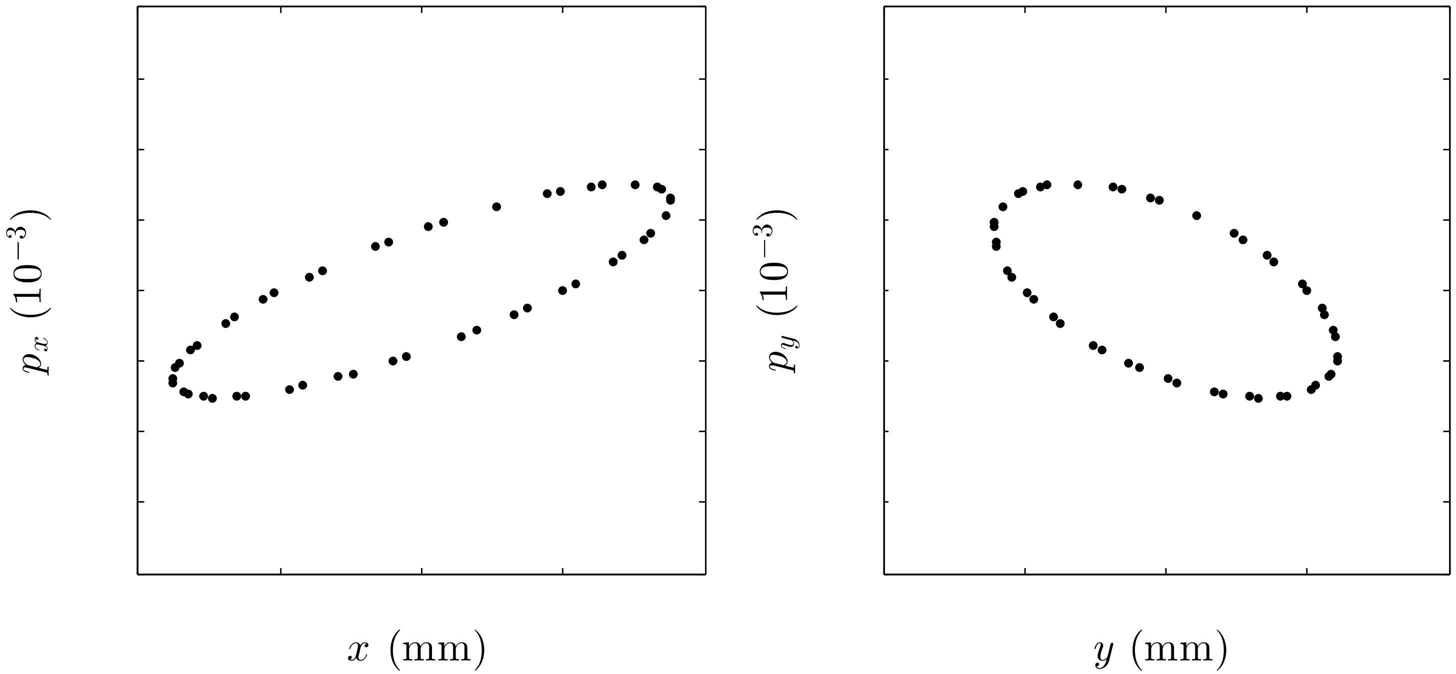


- For **periodic beamlines** (i.e. a beamline constructed from a repeated unit), the transfer matrix for a single period can be parameterised in terms of the **Courant–Snyder parameters** (α, β, γ) and the **phase advance** μ:

$$R = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

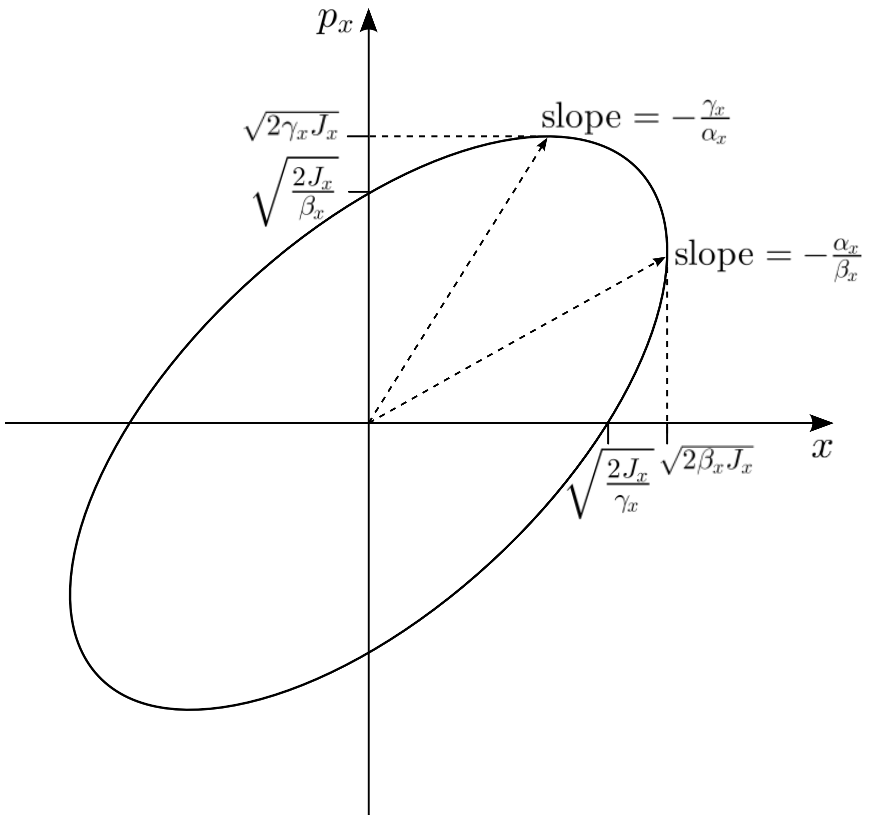
From Linear to Non-linear

- The characteristics of the particle motion can be represented by a **phase space portrait** showing the co-ordinates and momenta of a particle after an increasing number of passes through full periods of the beamline



From Linear to Non-linear

- If the transfer map for each period is linear, then the phase space portrait is an **ellipse** with area πJ_x
- The **action** J_x characterises the **amplitude** of the betatron oscillations – it is invariant
- The **shape** of the ellipse is described by the **Courant–Snyder parameters**
- The rate at which particles move around the ellipse (**phase advance** per period) is **independent** of the betatron **action**



From Linear to Non-linear

- Nonlinearities in particle dynamics can come from a number of different **sources**, e.g.
  - **Stray fields** in drift spaces
  - Higher-order **multipole components** in dipoles and quadrupoles
  - Higher-order **multipole magnets** (sextupoles, octupoles...) used to control various properties of the beam;
  - **Electromagnetic fields** generated by a **bunch** of particles, acting on individual particles within the same or another bunch (space-charge forces, beam-beam effects...)
- The effects of nonlinearities can be quite dramatic
- It is paramount to have some understanding of nonlinear dynamics for **optimising** the **design** and **operation** of many accelerator systems

Non-linear transfer maps and effects of non-linearities

Nonlinear transfer map: sextupole

- As example, consider the **vertical field** component in a **sextupole** (i.e. a nonlinear) magnet:

$$\frac{B_y}{B\rho} = \frac{1}{2} k_2 x^2$$

with $B\rho$ the beam rigidity and k_2 the normalized sextupole gradient

- In "thin lens" approximation, the **deflection** of a particle passing through the sextupole of length L is

$$\Delta p_x = -\int \frac{B_y}{B\rho} ds = -\frac{1}{2} k_2 L x^2$$

- The thin lens **transfer map** for the sextupole is

$$\begin{aligned} x_1 &= x_0 \ , \\ p_{x1} &= p_{x0} - \frac{1}{2} k_2 L x^2 \end{aligned}$$

Power series representation

- A **nonlinear transfer map** can be represented as a **power series**

$$x_1 = A_1 + R_{11}x_0 + R_{12}p_{x0} + T_{111}x_0^2 + T_{112}x_0p_{x0} + T_{122}p_{x0}^2 + \ldots$$

$$p_{x1} = A_2 + R_{21}x_0 + R_{22}p_{x0} + T_{211}x_0^2 + T_{212}x_0p_{x0} + T_{222}p_{x0}^2 + \ldots$$

- The **coefficients** R_{ij} are components of the **transfer matrix** R
- The coefficients of the **higher-order** (nonlinear) terms are conventionally represented by T_{ijk} (2nd order), U_{ijkl} (3rd order) and so on…
- The values of the **indices** correspond to **components** of the phase space vector:

index value	1	2	3	4	5	6
component	x	p_x	y	p_y	z	δ

Example of a periodic system: a simple storage ring

A simple storage ring

- As example, consider the transverse dynamics in a **simple storage ring**, assuming:
  - The storage ring is constructed from some number of **identical cells** consisting of dipoles, quadrupoles and sextupoles.
  - The **phase advance** per cell can be tuned from close to zero, up to about $0.5 \times 2\pi$.
  - There is **one sextupole per cell**, which is **located at a point where the horizontal beta function is 1 m, and the alpha function is zero**.
  - Usually, storage rings will contain (at least) two sextupoles per cell, to correct horizontal and vertical chromaticity. To keep things simple, we will use only one sextupole per cell.

Linear dynamics in a storage ring

- The **chromaticity**, and hence the sextupole strength, will normally be a **function** of the **phase advance**
- To investigate the nonlinear effects of sextupoles, we shall keep the **sextupole strength** $k_2 L$ **fixed**, and **change** only the **phase advance**
- We can assume that the **map** from **one sextupole** to the **next** is **linear**, and corresponds to a **rotation** in phase space through an angle equal to the phase advance (as we assumed β_x = 1 m, and α_x = 0):

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto \begin{pmatrix} \cos\mu_x & \sin\mu_x \\ -\sin\mu_x & \cos\mu_x \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

- Again to keep things simple, we shall consider only **horizontal motion**, and assume that the vertical co-ordinate $y = 0$
- Recall that in "thin lens" approximation, the **deflection** of a particle passing through the sextupole of length L is

$$\Delta p_x = -\int \frac{B_y}{B\rho} ds = -\frac{1}{2} k_2 L x^2$$

Nonlinear transfer map: sextupole

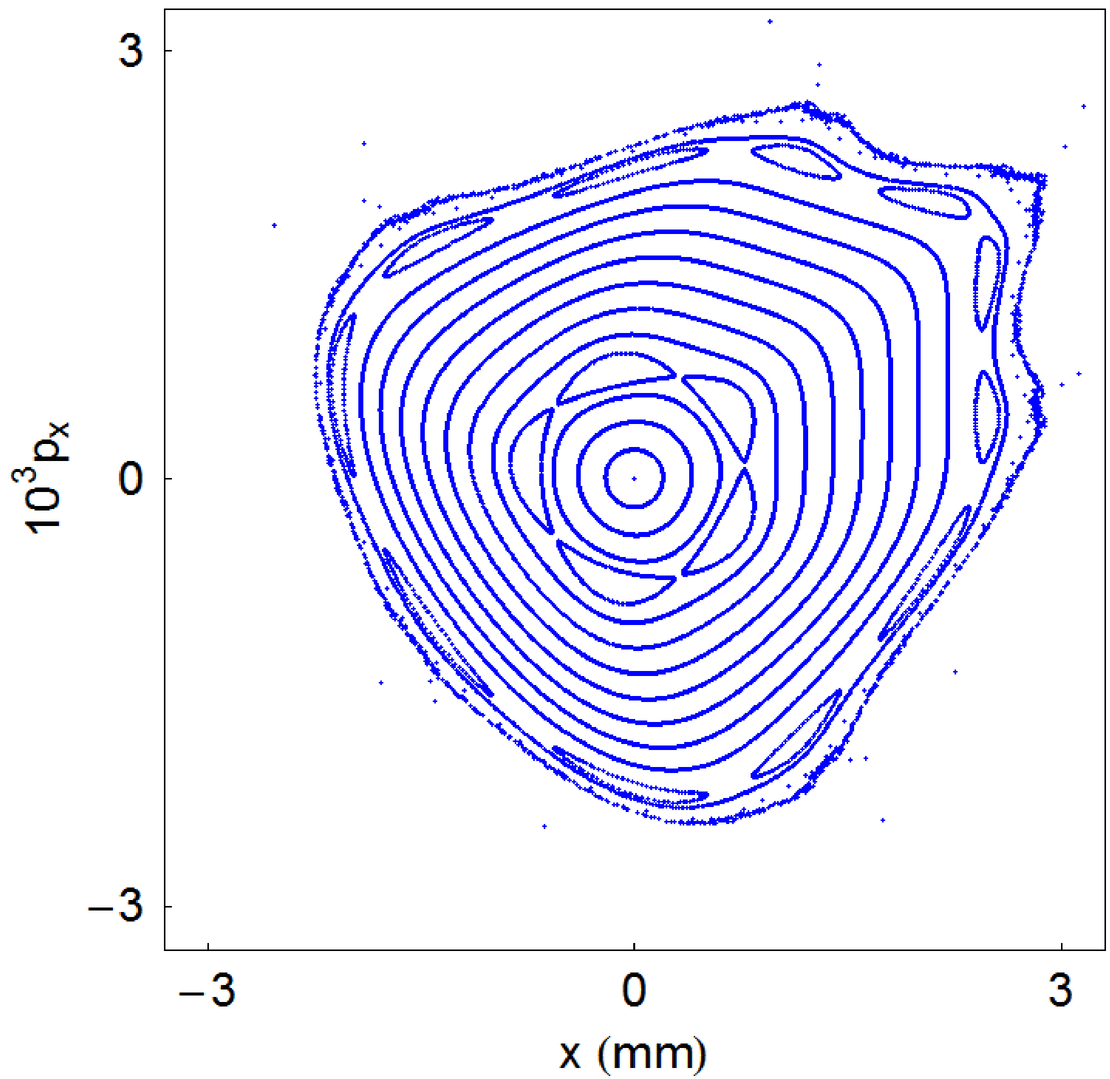
- The map for a particle moving through a short **sextupole** can thus be represented by a **“kick”** in the horizontal momentum:

$$\begin{aligned} x &\mapsto x\ , \\ p_x &\mapsto p_x - \frac{1}{2} k_2 L x^2 \end{aligned}$$

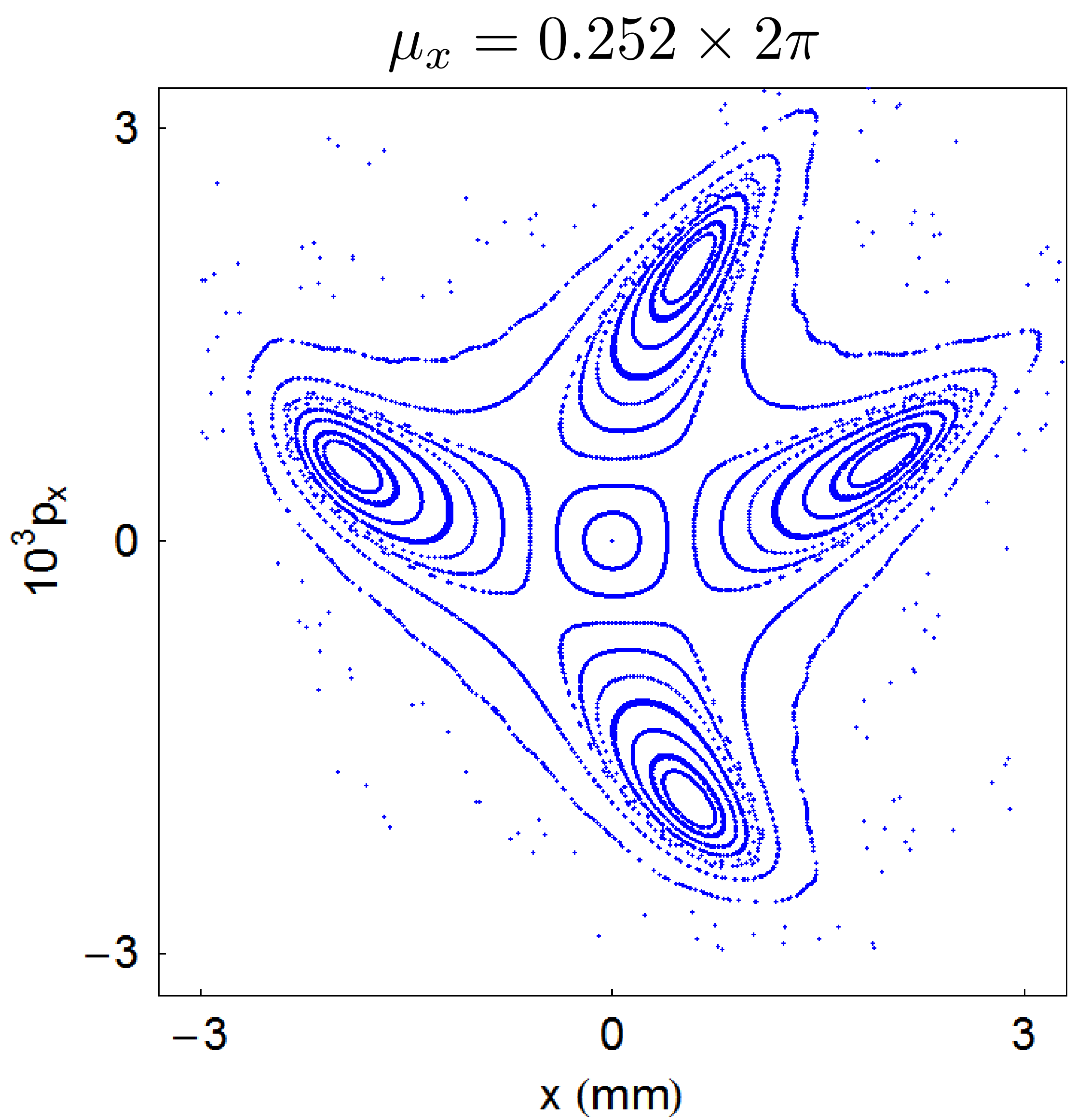
- Let us choose a fixed value $k_2 L = -600\ \mathrm{m}^{-2}$ and look at the effects of the maps for different phase advances.
- For each case, we construct a **phase space portrait** by plotting the values of the dynamical variables after repeated application of the map (rotation + sextupole) for a range of initial conditions.
- First, let us look at the phase space portraits for a range of phase advances from **0.2 × 2π** to **0.5 × 2π**

Example of a simple storage ring

$$\mu_x = 0.202 \times 2\pi$$

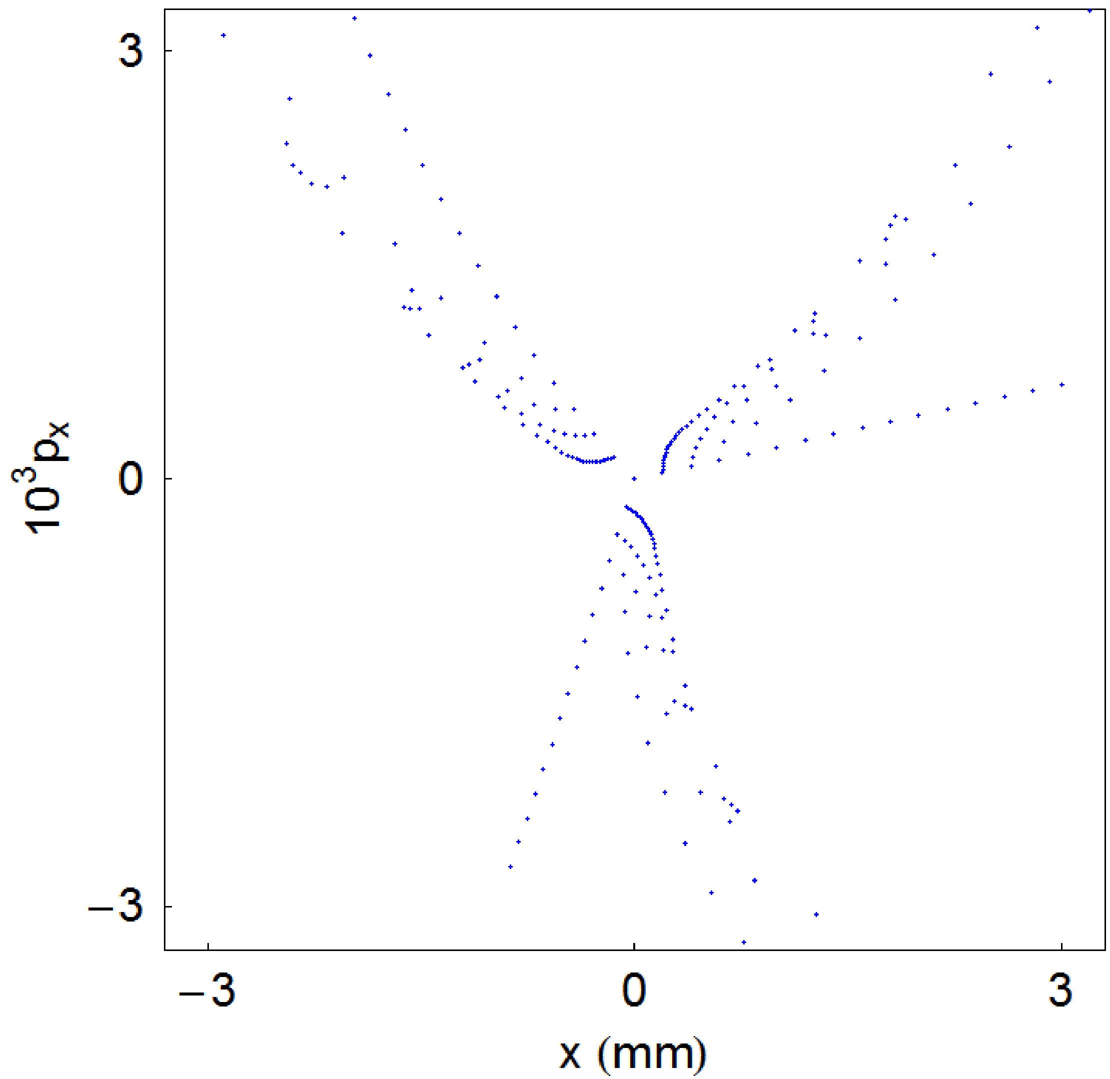


Example of a simple storage ring

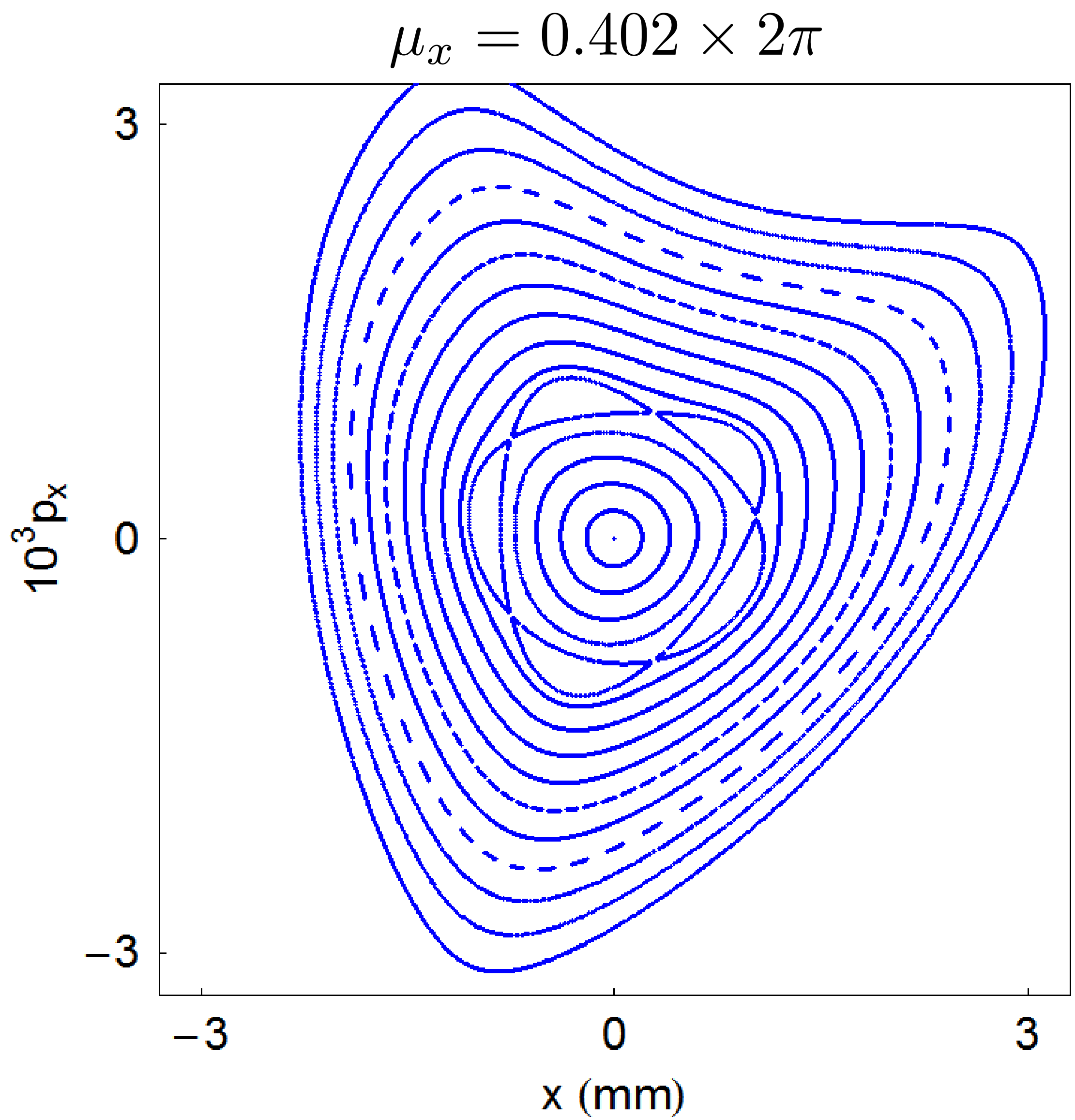


Example of a simple storage ring

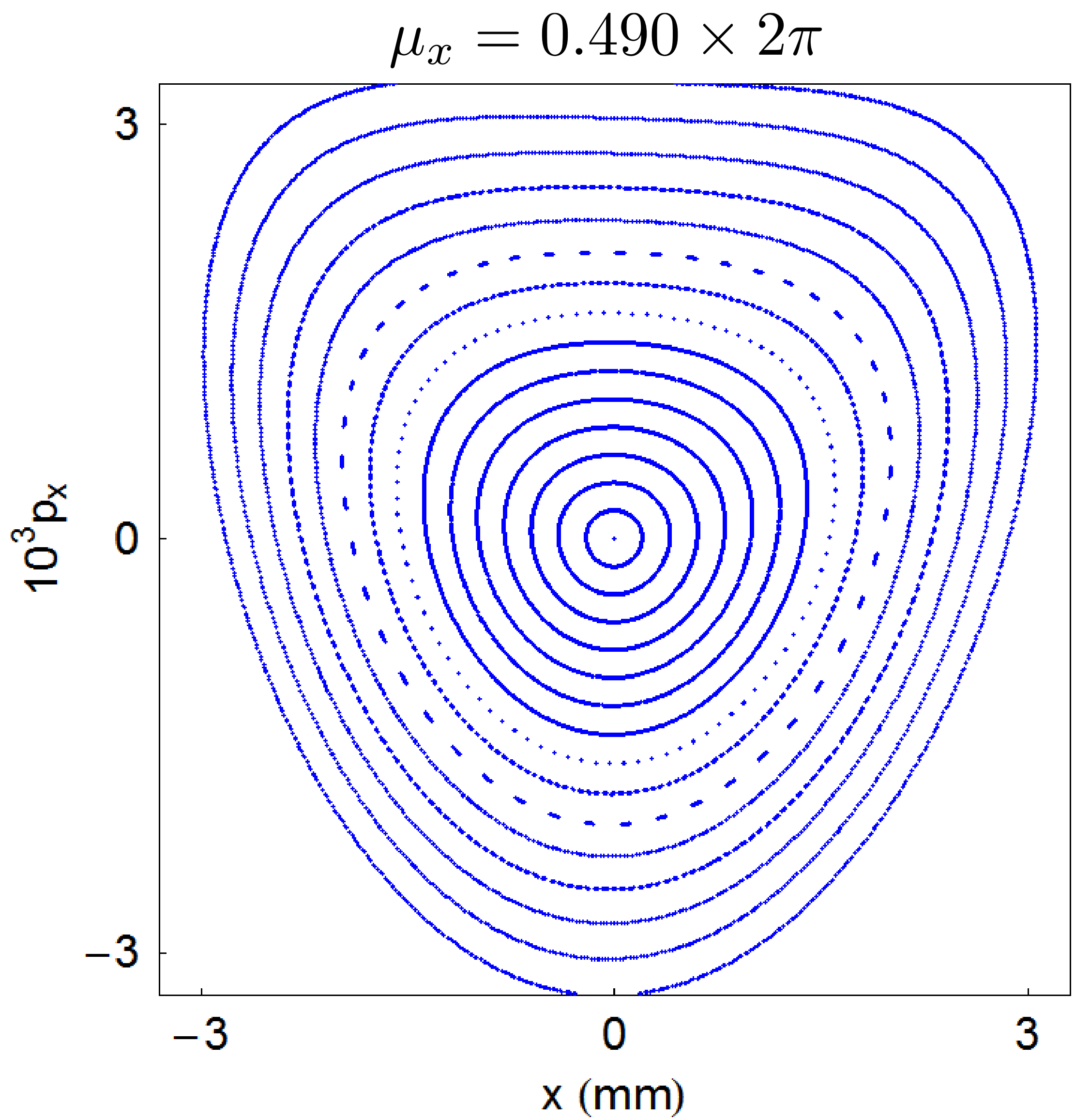
$$\mu_x = 0.330 \times 2\pi$$



Example of a simple storage ring



Example of a simple storage ring



Some observations

- **There are interesting features in these phase space portraits to which it is worth drawing attention:**
  - For small amplitudes (small x and p_x), **particles trace out closed loops around the origin**: this is what we expect for a linear map
  - As the **amplitude is increased**, "**islands**" appear in phase space: the phase advance (for the linear map) is often close to m/p where m is an integer and p is the number of islands
  - Sometimes, a larger number of islands appears at larger amplitude
  - Usually, there is a **closed curve that divides a region of stable motion from a region of unstable motion**. Outside that curve, the amplitude of particles increases without limit as the map is repeatedly applied
  - **The area of the stable region depends strongly on the phase advance**: for a phase advance close to 2π/3, it appears that the stable region almost vanishes altogether
  - As **the phase advance is increased towards π, the stable area becomes large**, and distortions from the linear ellipse become small

Effect of phase advance on nonlinear dynamics

Effect of phase advance

- An important observation is that the **effect** of the sextupole in the periodic cell **depends strongly** on the **phase advance** across the cell
- We can start to understand the significance of the phase advance by considering **two special cases**:
  - Phase advance equal to an **integer** times 2π
  - Phase advance equal to a **half integer** times 2π

Integer tune

- Let us consider first a **phase advance** equal to an **integer** times 2π. In that case, the linear part of the map is just the identity

$$
\begin{aligned}
x &\mapsto x\,, \\
p_x &\mapsto p_x
\end{aligned}
$$

- The **combined effect** of the **linear map** and the **sextupole kick** is:

$$
\begin{aligned}
x &\mapsto x\,, \\
p_x &\mapsto p_x - \frac{1}{2}k_2 L x^2
\end{aligned}
$$

- Clearly, the **horizontal momentum** will **increase** without limit
- There are **no stable regions** of phase space, apart from $x = 0$

Half-Integer tune

- Now consider what happens if the phase advance of a cell is a **half integer** times 2π, so the linear part of the map is just a rotation through π
- If a **particle** starts at the entrance of a sextupole with $x = x_0$ and $p_x = p_{x0}$, then at the **exit** of that sextupole:

$$
\begin{aligned}
x_1 &= x_0 \ , \\
p_{x1} &= p_{x0} - \frac{1}{2} k_2 L x_0^2
\end{aligned}
$$

- Then, after passing to the **entrance** of the **next sextupole**, the co-ordinates will be:

$$
\begin{aligned}
x_2 &= \cos(\pi) x_1 = -x_1 = -x_0 \ , \\
p_{x2} &= \cos(\pi) p_{x1} = -p_{x1} = -p_{x0} + \frac{1}{2} k_2 L x_0^2
\end{aligned}
$$

Half-Integer tune

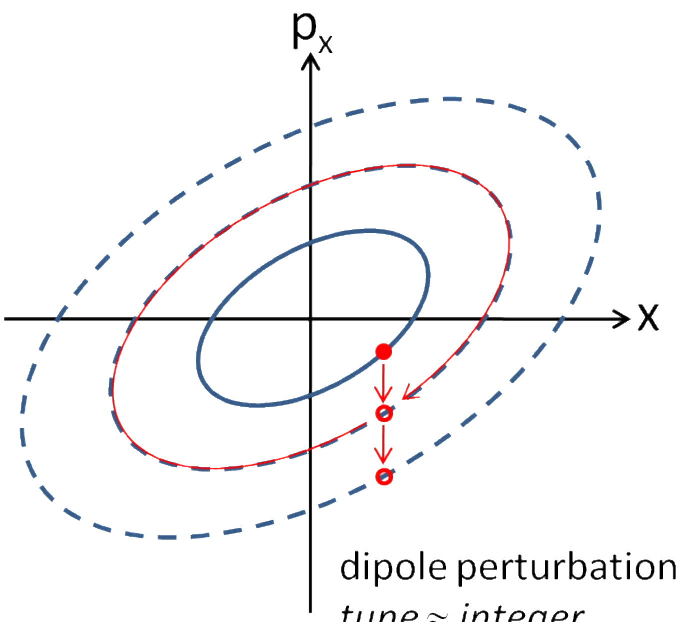
- Finally, after **passing** through the **second sextupole**:

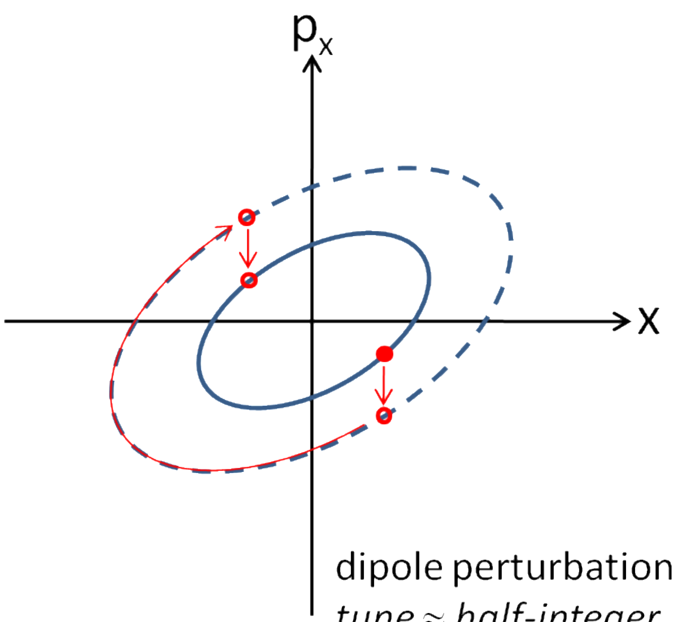
$$x_3 = x_2 = -x_0 \ ,$$
$$p_{x3} = p_{x2} - \frac{1}{2} k_2 L x_2^2 = -p_{x0}$$

- In other words, the **momentum kicks** from the two sextupoles exactly **cancel** each other
- The resulting map is a purely **linear phase space rotation** by π.
- In this situation, we expect the **motion** to be **stable** (and periodic), independent of the amplitude

Impact of phase advance

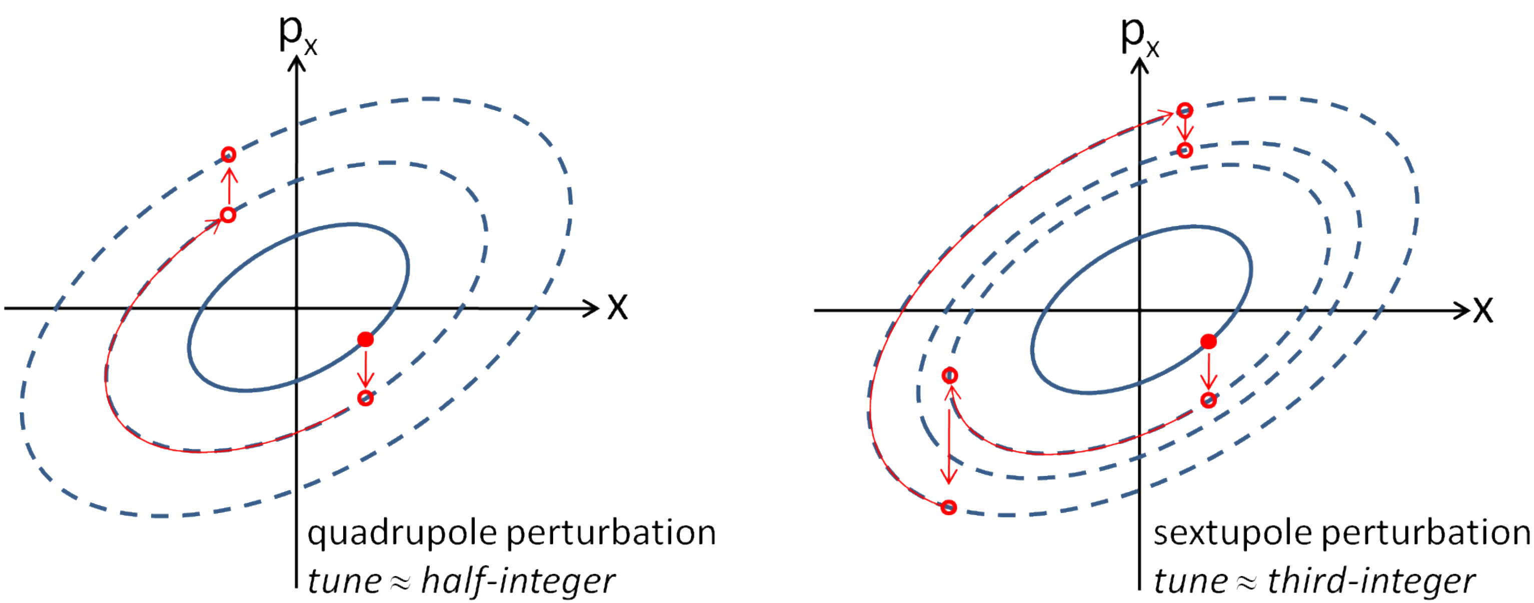
- The effect of the phase advance on the sextupole “kicks” is **similar** to the effect on **perturbations** arising from **dipole** and **quadrupole errors** in a storage ring
- In the case of **dipole errors**, the **kicks add up** if the phase advance is an **integer**, and **cancel** if the **phase advance** is a **half integer**





Impact of phase advance

- In the case of **quadrupole errors**, the **kicks add up** if the phase advance is a **half integer** times 2π
- **Higher-order multipoles** drive **higher-order resonances** but the effects are less easily illustrated on a phase space diagram



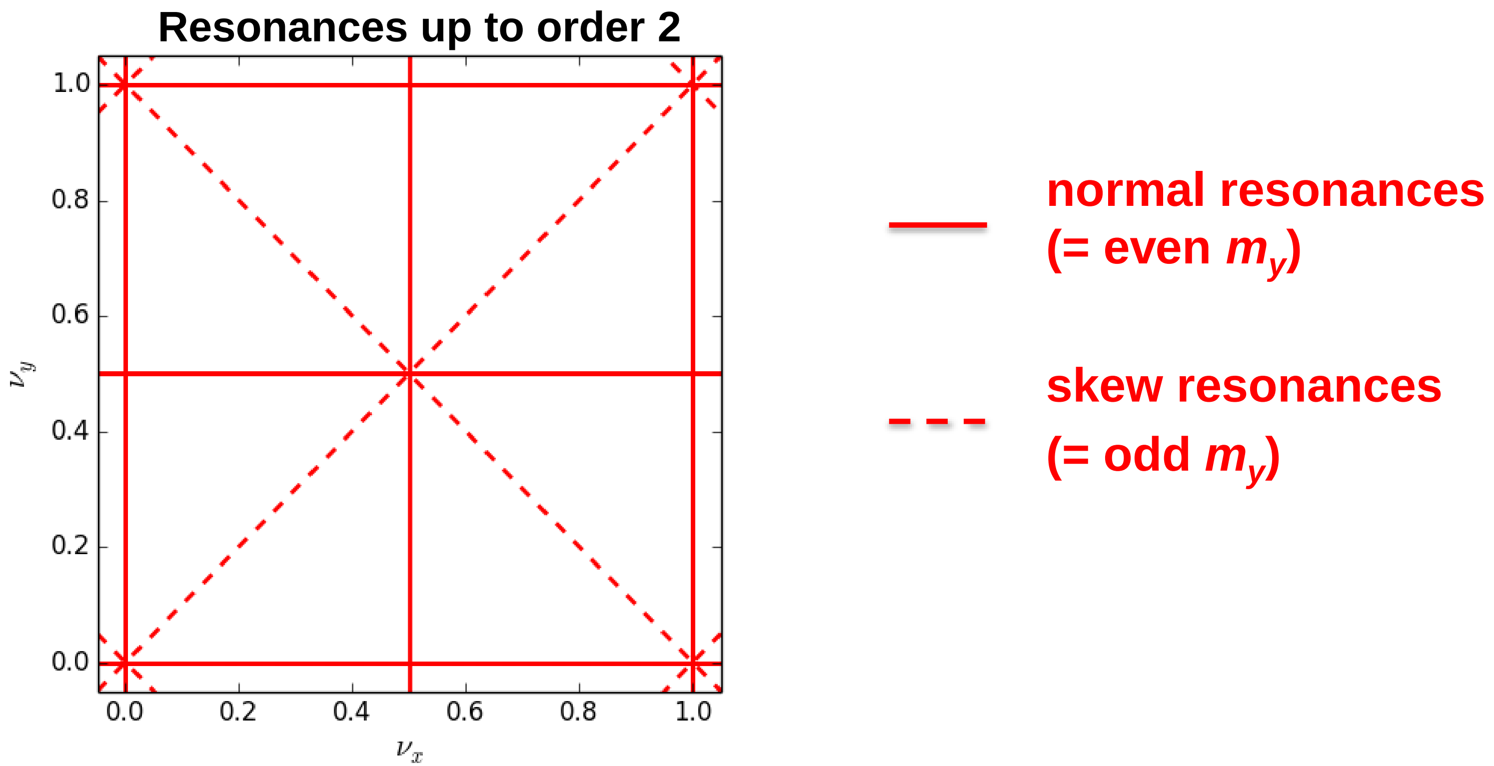
Resonances

Resonances

- If we include vertical as well as horizontal motion, then we find that **resonances** occur when the tunes satisfy

$$m_x \nu_x + m_y \nu_y = \ell$$

where m_x, m_y and l are integers; resonance is of **order** $|m_x| + |m_y|$



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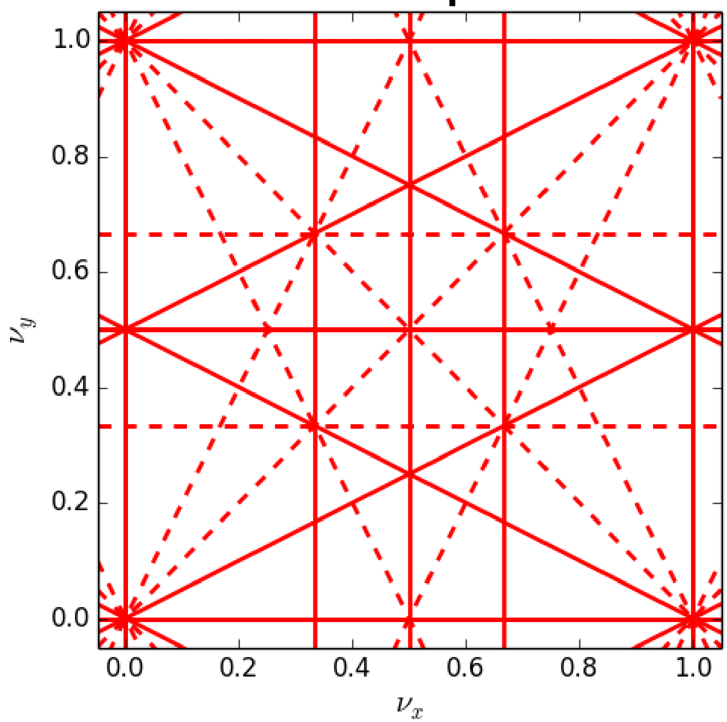
Resonances up to order 3



normal resonances (= even m_y)

skew resonances (= odd m_y)

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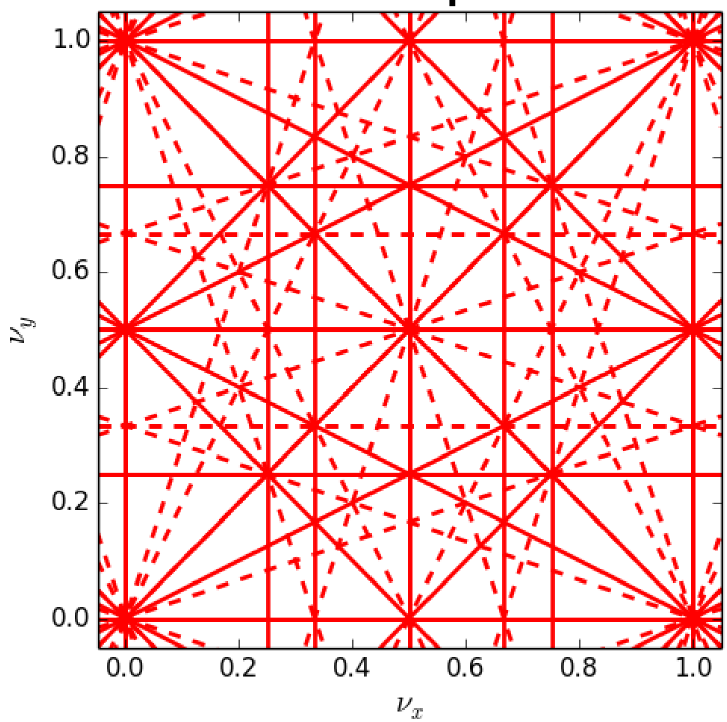
Resonances up to order 4



normal resonances (= even m_y)

skew resonances (= odd m_y)

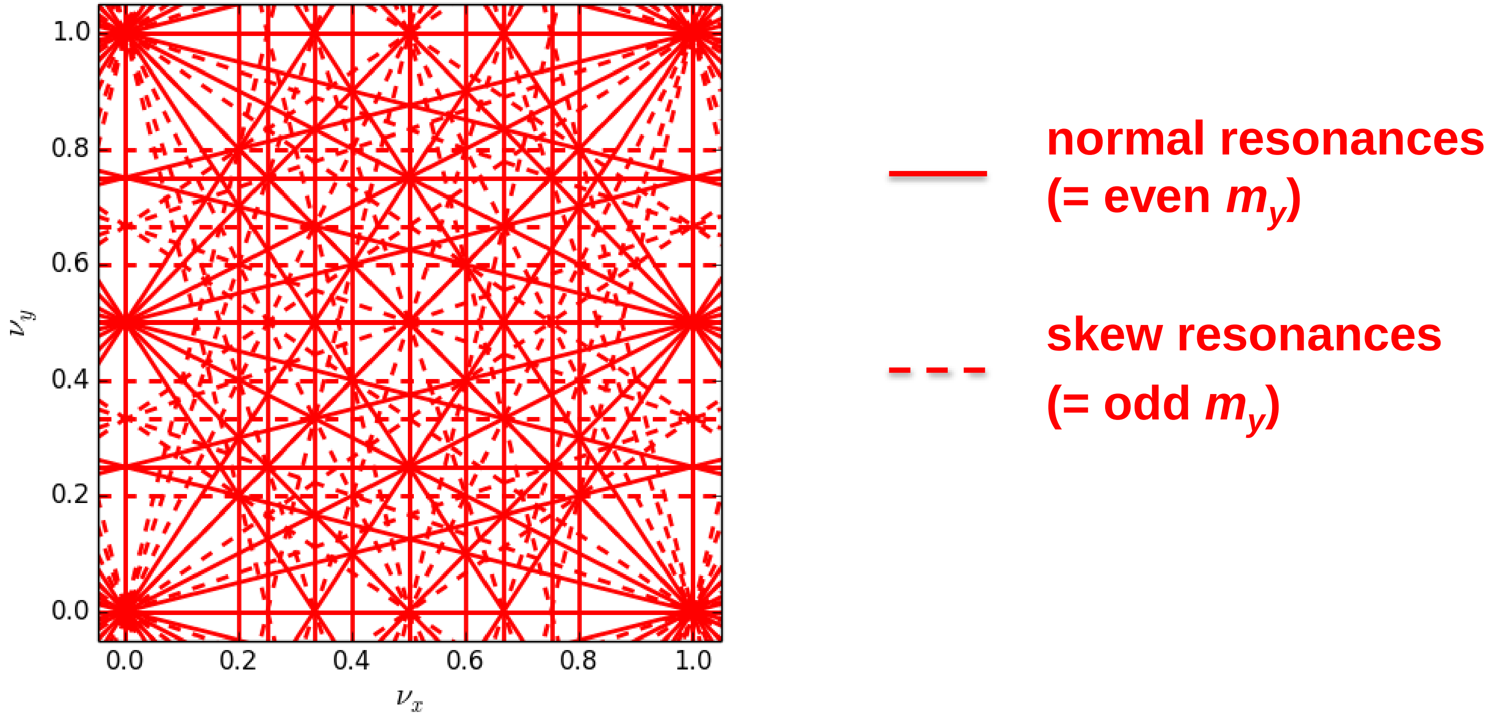
Resonances

- If we include vertical as well as horizontal motion, then we find that **resonances** occur when the tunes satisfy

$$m_x \nu_x + m_y \nu_y = \ell$$

where m_x, m_y and l are integers; resonance is of **order** $|m_x| + |m_y|$

Resonances up to order 5



Conclusions and Summary

Some conclusions

- **Nonlinear effects** can **limit** the **performance** of an accelerator system
- Sometimes the **effects** are **small enough** that they can be **ignored**
- In many cases, a **system designed without** taking account of **nonlinearities** will **not achieve** the specified **performance**
- If we analyse and understand the **nonlinear behaviour** of a system, then, we may be able to devise means of **compensating** any adverse effects

Summary

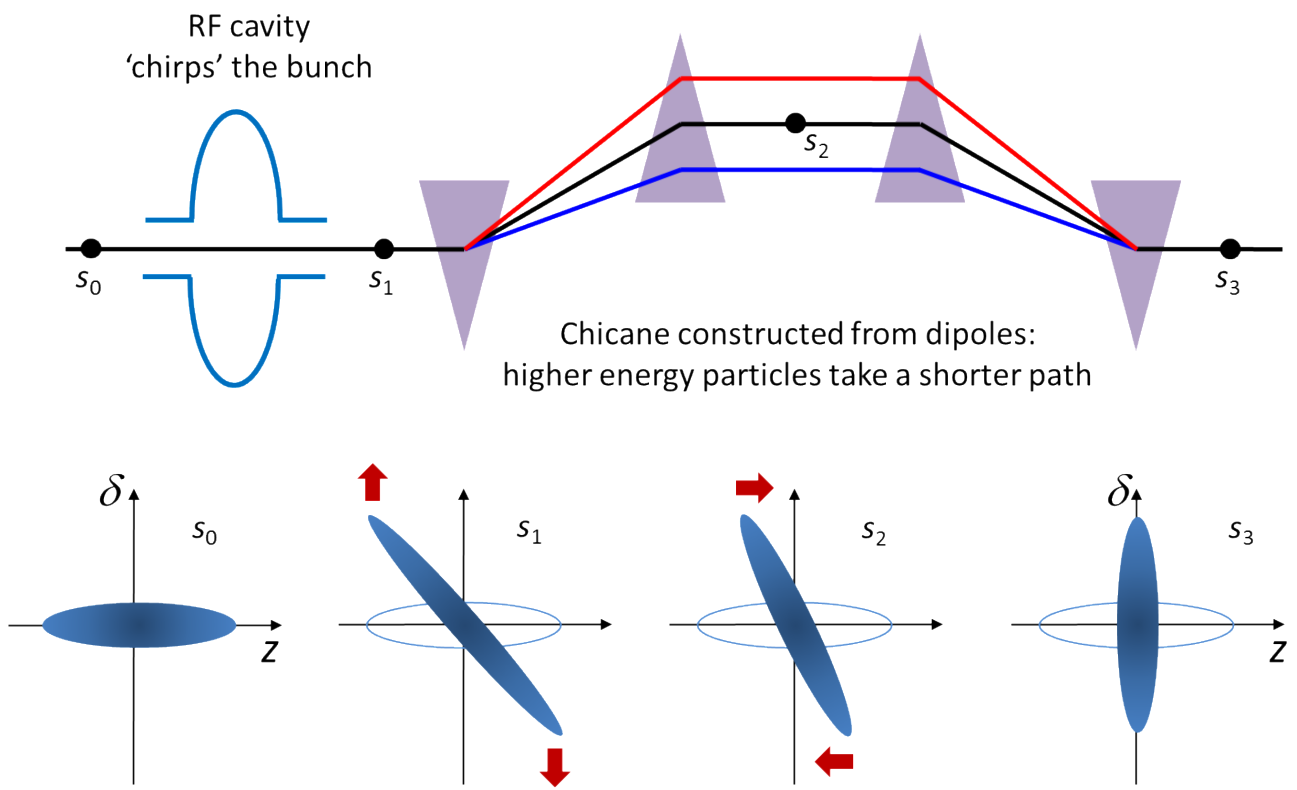
- **Nonlinear effects** can arise from a number of **sources** in accelerators, including stray fields, higher-order multipole components in magnets, space-charge, ...
- The **transfer map** for a nonlinear element (such as a sextupole) may be represented as a **power series** in the initial values of the phase space variables
- The effects of **nonlinearities** in accelerator systems vary widely, depending on the **type of system** in which they occur (e.g. a periodic accelerator)
- In some cases, **nonlinear effects** can limit the **performance** of an accelerator system. In such cases, it is important to take nonlinearities into account in the **design** of the system

BACK UP

Nonlinear effects in a bunch compressor

Bunch compressors

- A **bunch compressor** reduces the **length** of a bunch, by performing a **rotation** in longitudinal phase space
- Bunch compressors are used, for example, in **free electron lasers** to increase the peak current



Distribution of particles 'rotates' in longitudinal phase space (area is conserved).

Bunch compressors

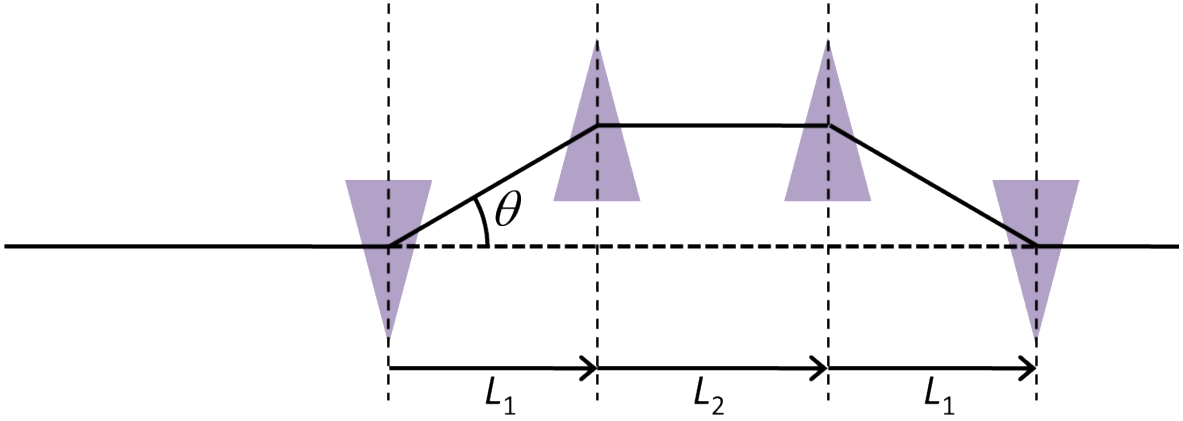
- The **RF cavity** is designed to "chirp" the bunch, i.e. to provide a **change** in **energy deviation** as a function of **longitudinal position** z within the bunch
- The **energy deviation** δ of a particle with energy E from a reference energy E_0 is defined as:

$$\delta = \frac{E - E_0}{E_0}$$

- The **transfer map** for the **RF cavity** in the bunch compressor with voltage V and frequency $\frac{\omega}{2\pi}$ is:

$$\begin{aligned} z_1 &= z_0 \ , \\ \delta_1 &= \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c}\right) \end{aligned}$$

- Neglecting synchrotron radiation, the chicane does not change the energy of the particles. However, the **path length** L **depends** on the **energy** of the particle.



- If we assume that the bending angle in a dipole is small:

$$L = \frac{2L_1}{\cos\theta} + L_2$$

- The **bending angle** is a function of the **energy** of the particle:

$$\theta = \frac{\theta_0}{1+\delta}$$

Bunch compressors linear dynamics

- The **change** in the co-ordinate z is the **difference** between the **nominal** path length, and the length of the path actually taken by the particle
- Hence, the **chicane transfer map** can be written:

$$\begin{aligned} z_2 &= z_1 + 2L_1 \left(\frac{1}{\cos\theta_0} - \frac{1}{\cos(\theta(\delta_1))} \right) , \\ \delta_2 &= \delta_1 \end{aligned}$$

where θ_0 is the nominal bending angle of each dipole in the chicane, and $\theta(\delta)$ is given by

$$\theta(\delta) = \frac{\theta_0}{1+\delta}$$

- Clearly, the **complete transfer map** for the bunch compressor is **nonlinear**, but how important are the nonlinear terms?

Bunch compressors linear dynamics

- To understand the effects of the nonlinear part of the map, we will study a **specific example**
- First, we will “**design**” a bunch compressor using only the **linear part** of the map
- The linear part of a transfer map can be obtained by **expanding** the map as a **Taylor series** in the dynamical variables, and keeping only the **first-order** terms
- After finding appropriate values for the various **parameters** using the **linear transfer map**, we shall see how our **design** has to be **modified** to take account of the **nonlinearities**

Bunch compressors linear dynamics

- To **first order** in the dynamical variables, the **map** for the **RF cavity** can be written:

$$\begin{aligned} z_1 &= z_0 \,, \\ \delta_1 &= \delta_0 + R_{65} z_0 \end{aligned} \quad \text{with} \quad R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}$$

- The **map** for the **chicane** is

$$\begin{aligned} z_2 &= z_1 + R_{56}\delta_1 \,, \\ \delta_2 &= \delta_1 \end{aligned} \quad \text{with} \quad R_{56} = 2L_1 \frac{\theta_0 \sin\theta_0}{\cos^2\theta_0}$$

Bunch compressors linear dynamics

- As a specific example, consider a bunch compressor for the International Linear Collider (ILC)

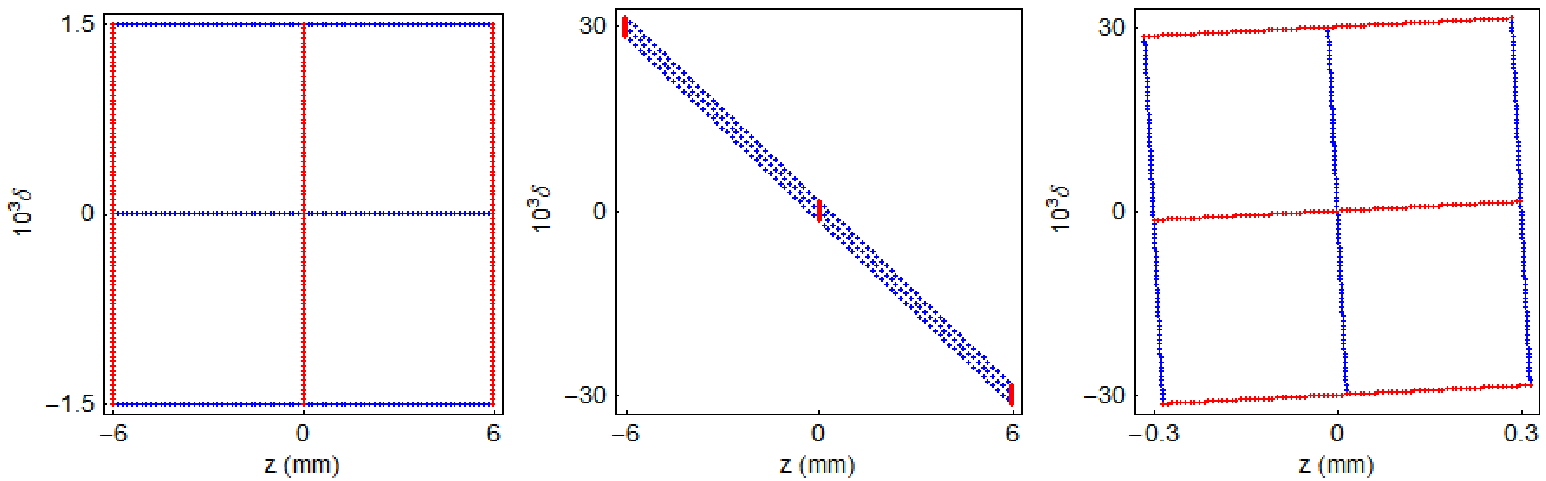
Initial rms bunch length	$\sqrt{\langle z_0^2 \rangle}$	6 mm
Initial rms energy spread	$\sqrt{\langle \delta_0^2 \rangle}$	0.15%
Final rms bunch length	$\sqrt{\langle z_2^2 \rangle}$	0.3 mm

- **Two constraints** determine the values of R_{65} and R_{56}
  - The **bunch length** should be **reduced** by a **factor 20**
  - There should be **no "chirp"** on the bunch at the exit of the bunch compressor
- With these constraints, we find (see Appendix):

$$R_{65} = -4.9937 \text{ m}^{-1} \qquad R_{56} = 0.19975 \text{ m}$$

Bunch compressors linear dynamics

- We can illustrate the effect of the linearised bunch compressor map on **phase space** using an artificial **“window frame” distribution**:



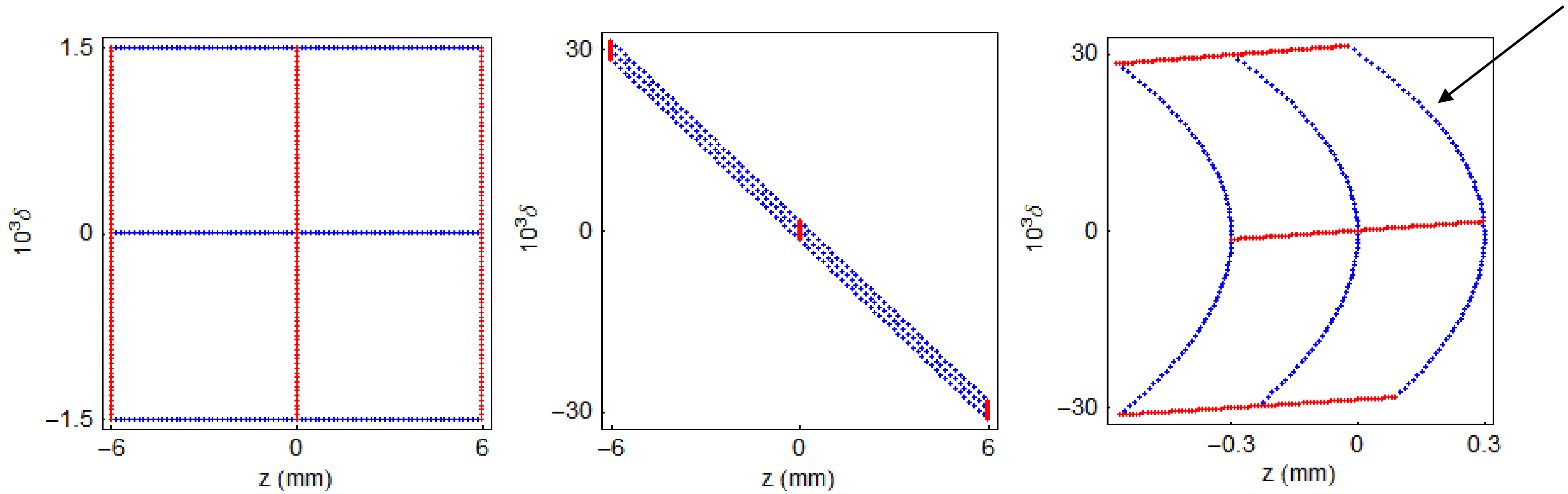
- The rms **bunch length** is **reduced** by a factor of 20 as required, but the **rms energy spread** is **increased** by the same factor, because the transfer map is **symplectic**, so phase space areas are conserved under the transformation

Bunch compressors non-linear dynamics

- Let's apply now the **full nonlinear map** for the bunch compressor.
- We need first to make some **assumptions** for the **RF voltage** and **frequency**, and the dipole **bending angle** and chicane **length** in order for the coefficient R_{65} and R_{56} to have the appropriate values

Beam (reference) energy	E_0	5 GeV
RF frequency	$f_{\rm rf}$	1.3 GHz
RF voltage	$V_{\rm rf}$	916 MV
Dipole bending angle	θ_0	3°
Dipole spacing	L_1	36.3 m

- As before, we illustrate the effect of the bunch compressor map on phase space using a “window frame” distribution:



- Although the **bunch length** has been **reduced**, there is significant **distortion** of the distribution: the **rms bunch length** will be **significantly longer** than what we are aiming for
- To reduce the distortion, we need to understand where it comes from
- In the phase space shown above, we see a **quadratic dependence** of the final particle position z_2 on the initial particle position z_0.

Bunch compressors non-linear dynamics

- Consider a particle entering the bunch compressor with initial phase space co-ordinates z_0 and δ_0. We can write the co-ordinates z_1 and δ_1 of the particle after the **RF cavity** to **2nd order** in z_0 and δ_0:

$$\begin{aligned} z_1 &= z_0 \ , \\ \delta_1 &= \delta_0 + R_{65} z_0 + T_{655} z_0^2 \end{aligned}$$

- The co-ordinates of the particle after the **chicane** are (to **2nd order**):

$$\begin{aligned} z_2 &= z_1 + R_{56}\delta_1 + T_{566}\delta_1^2 \ , \\ \delta_2 &= \delta_1 \end{aligned}$$

- If we **combine** the **maps** for the RF and the chicane, we get:

$$\begin{aligned} z_2 = &(1 + R_{56}R_{65})z_0 + R_{56}\delta_0 \\ &+ (R_{56}T_{655} + R_{65}^2 T_{566})z_0^2 \\ &+ 2R_{65}T_{566} z_0 \delta_0 + T_{566}\delta_0^2, \\ \delta_2 = &\ \delta_0 + R_{65} z_0 + T_{655} z_0^2 \end{aligned}$$

Bunch compressors non-linear dynamics

- In order to eliminate the strong **non-linear distortion**, we have to **eliminate** the **second term**, i.e.

$$R_{56}T_{655} + R_{65}^2 T_{566} = 0$$

- By expanding the original map,

$$z_2 = z_1 + 2L_1\left(\frac{1}{\cos\theta_0} - \frac{1}{\cos(\theta(\delta_1))}\right)$$

as a Taylor series in δ, we find that for small angles:

$$T_{566} \approx -3L_1\theta_0^2$$

- Now, it remains to determine T_{655}, i.e. the **coefficient** for the **second-order** dependence of the **energy deviation** on **longitudinal position**

Bunch compressors non-linear dynamics

- The **map** of the **energy deviation**

$$\delta_1 = \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c}\right)$$

contains only **odd order terms** unless the RF cavity is operated **out of phase**, i.e.

$$\delta_1 = \delta_0 - \frac{eV}{E_0} \sin\left(\frac{\omega z_0}{c} + \phi_0\right)$$

- The **first** and **second order** coefficients in the transfer map for the **energy deviation** are:

$$R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}\cos\phi_0 \quad \text{and} \quad T_{655} = -\frac{1}{2}\frac{eV}{E_0}\left(\frac{\omega}{c}\right)^2 \sin\phi_0$$

Bunch compressors non-linear dynamics

- Recall that $R_{65} = -4.9937 \text{ m}^{-1}$ and $R_{56} = 0.19975 \text{ m}$
- We also obtain

$$T_{566} \approx -3L_1\theta_0^2 = -0.29963 \text{ m}$$

- By imposing $R_{56}T_{655} + R_{65}^2 T_{566} = 0$, we have that

$$T_{655} = 37.406 \text{ m}^{-2}$$

- Using the expressions

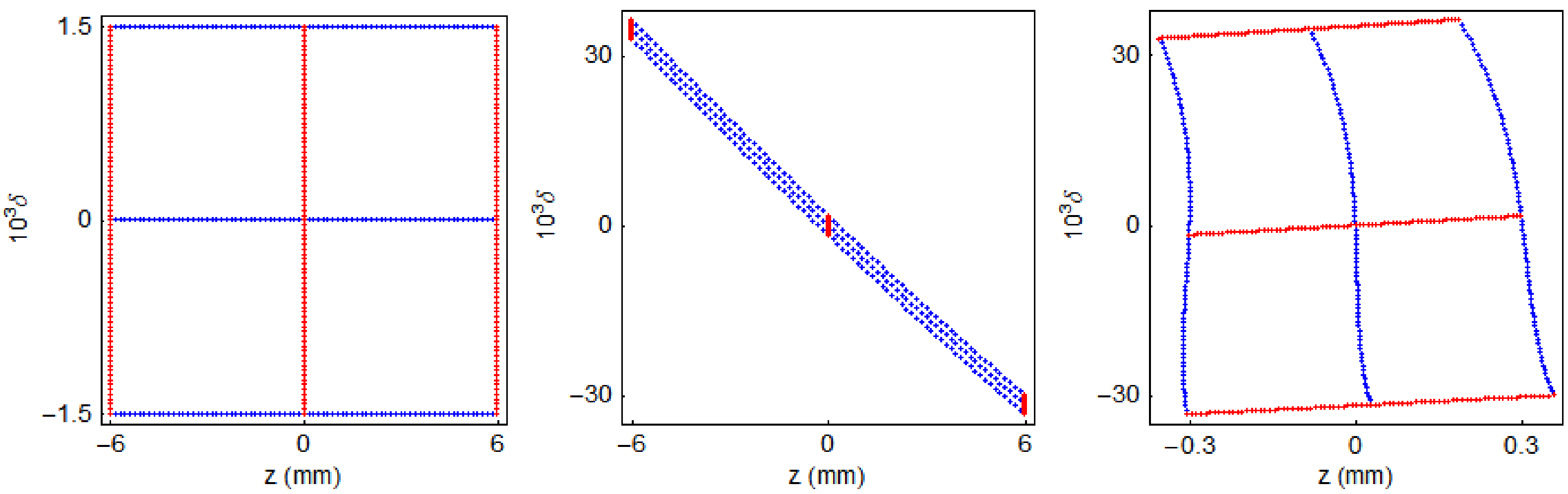
$$R_{65} = -\frac{eV}{E_0}\frac{\omega}{c}\cos\phi_0 \quad \text{and} \quad T_{655} = -\frac{1}{2}\frac{eV}{E_0}\left(\frac{\omega}{c}\right)^2 \sin\phi_0$$

the **voltage** and **phase** are determined as

$$V = 1046 \text{ MV} \quad \text{and} \quad \phi_0 = 28.8^\circ$$

Bunch compressors non-linear dynamics

- As before, we illustrate the effect of the bunch compressor on phase space using a "window frame" distribution, using the parameters determined above, to try to compress by a factor 20, while minimising the second-order distortion:



- The **dominant distortion** now appears to be **3rd order**, and looks **small enough** that it should not significantly affect the performance