

A first taste of Non-Linear Beam Dynamics

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CERN Accelerator School

Introduction to Accelerator Physics 2024

Spain

22 September – 05 October 2024

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- Introducing aspects of non-linear dynamics
 - Effects of **nonlinear perturbations**
 - Resonances, tune shifts, dynamic aperture
 - **Mathematical tools** for modelling nonlinear dynamics
 - Power series (Taylor) maps and symplectic maps
 - **Analysis** methods
 - Normal forms, frequency map analysis

- Illustrate methods and tools for a simple example of an accelerator
 - **Storage ring**

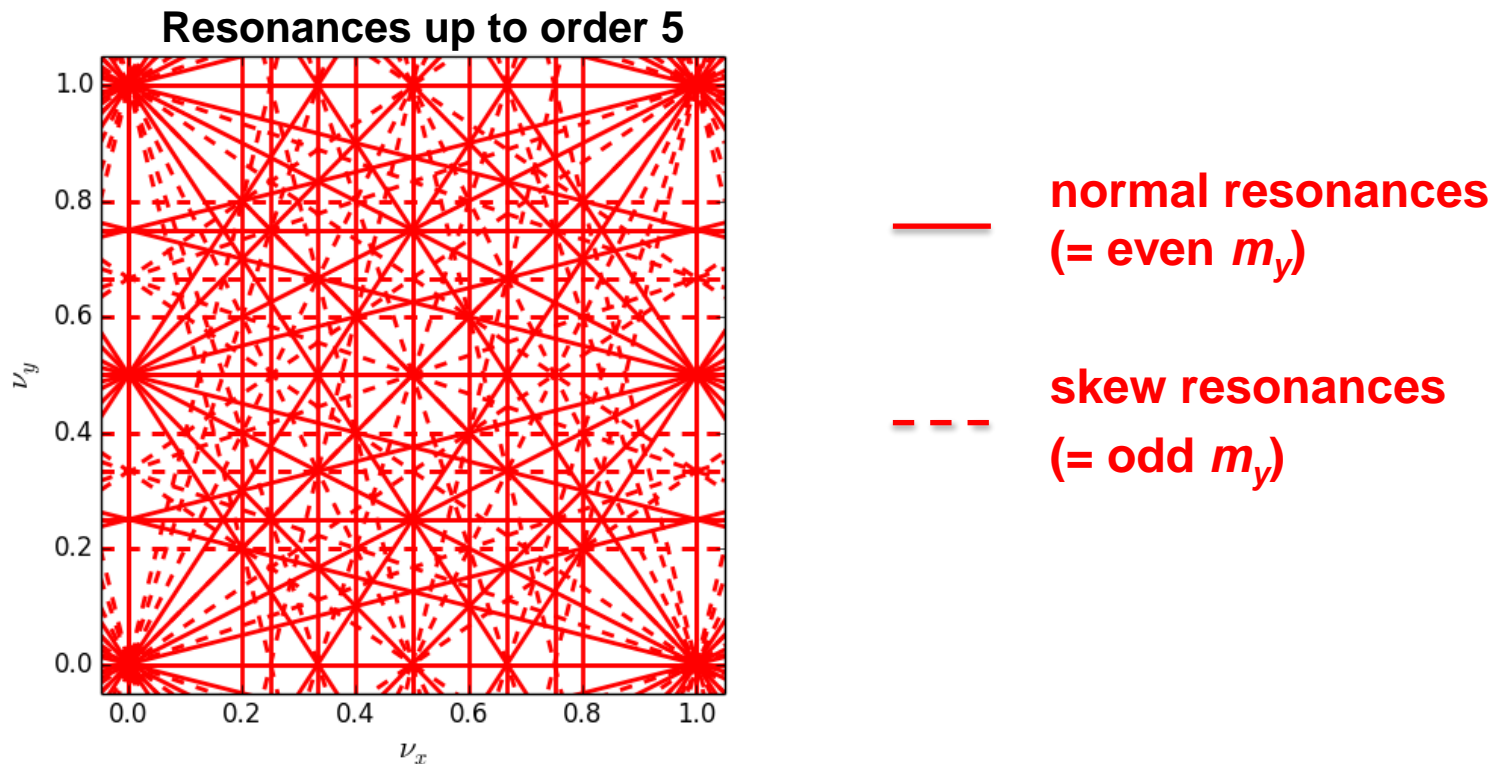
- Define the **resonance conditions** for periodic lattices
- Explain significance of **symplectic maps**, and describe some of the challenges in calculating and applying symplectic maps
- Outline some of the **analysis methods** that can be used to characterise nonlinear beam dynamics in periodic beamlines.

Resonances

- If we include vertical as well as horizontal motion, then we find that **resonances** occur when the tunes satisfy

$$m_x \nu_x + m_y \nu_y = \ell$$

where m_x , m_y and ℓ are integers; resonance is of **order** $|m_x| + |m_y|$

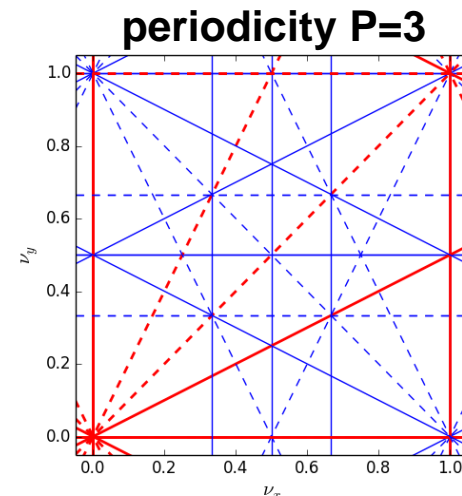
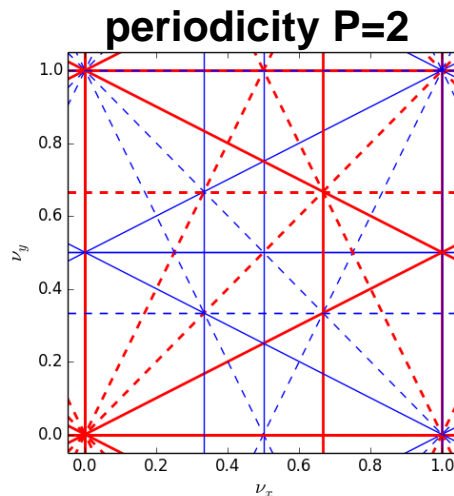
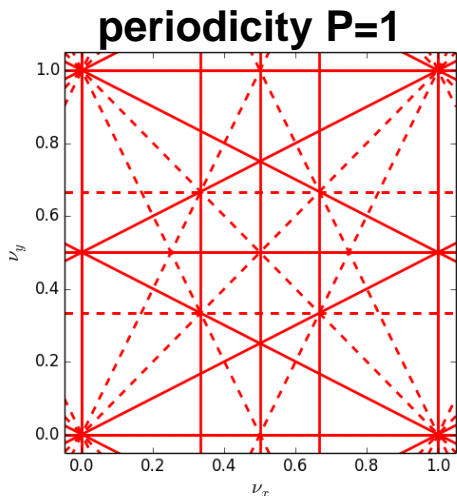


- **Resonances** are associated with **chaotic motion** of particles in storage rings
- The number of **resonance lines** in tune space is **infinite**: any point in tune space will be close to a resonance of some order ...
- This observation raises two questions:
 - How do we know what the real effect of any given resonance line will be?
 - How can we design a storage ring to minimise the adverse effects of resonances?

- By imposing a **periodicity P** in the lattice (i.e. building a machine from P identical cells), the resonance condition becomes

$$m_x \frac{\nu_x}{P} + m_y \frac{\nu_y}{P} = l \quad \rightarrow \quad m_x \nu_x + m_y \nu_y = Pl$$

- i.e. the resonance condition needs to be satisfied by each cell, since for the dynamics there is no difference between passing one cell $P \times N$ turns or passing a lattice consisting of P identical cells for only N turns
- Resonances for which l is integer \rightarrow **systematic**
- If l is NOT integer the resonance cancels \rightarrow **non-systematic**

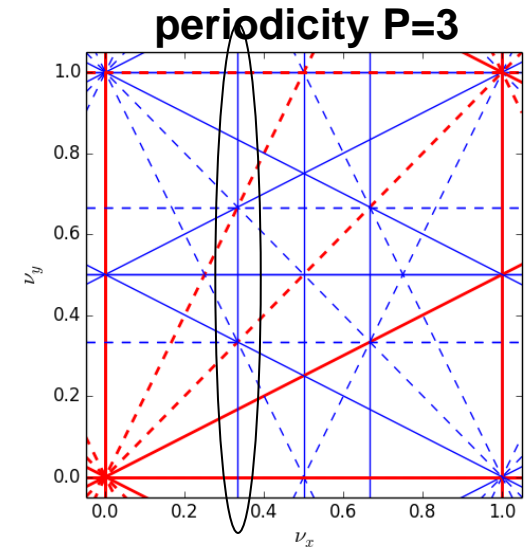
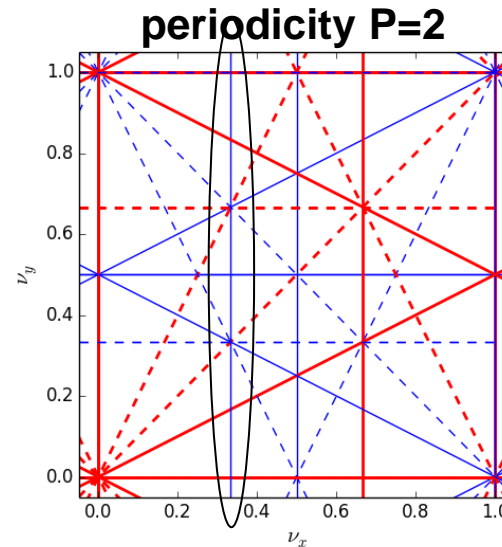
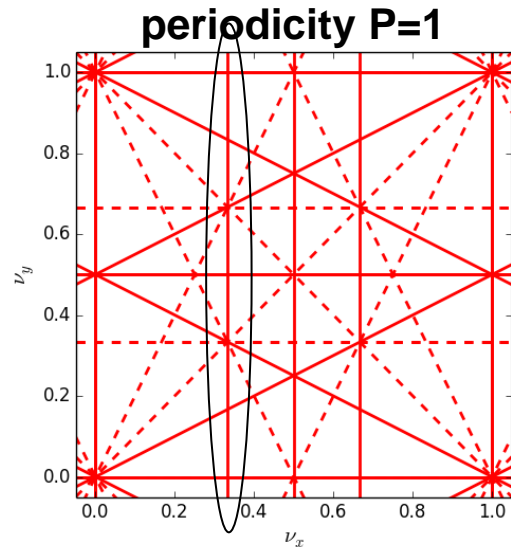
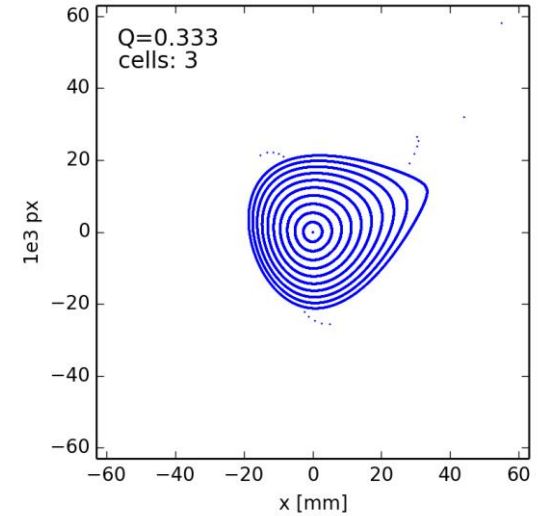
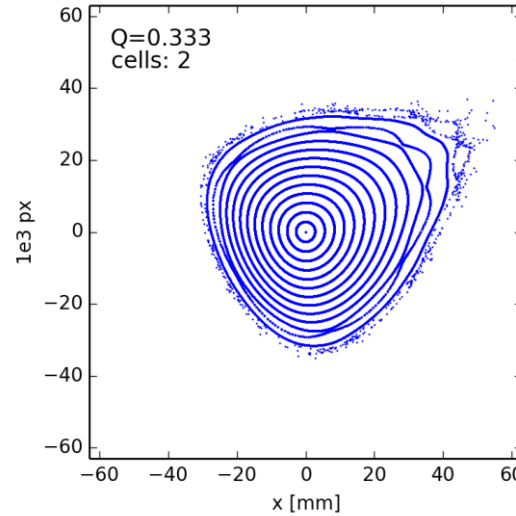
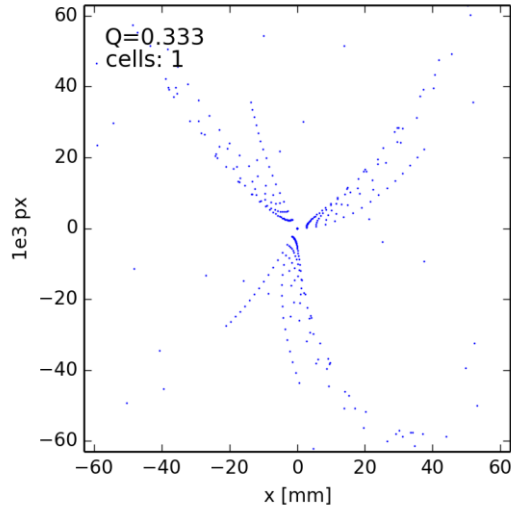


solid lines: normal resonances

dashed lines: skew resonances

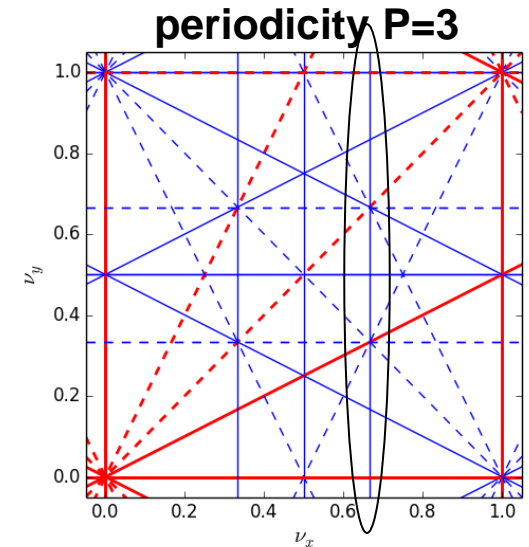
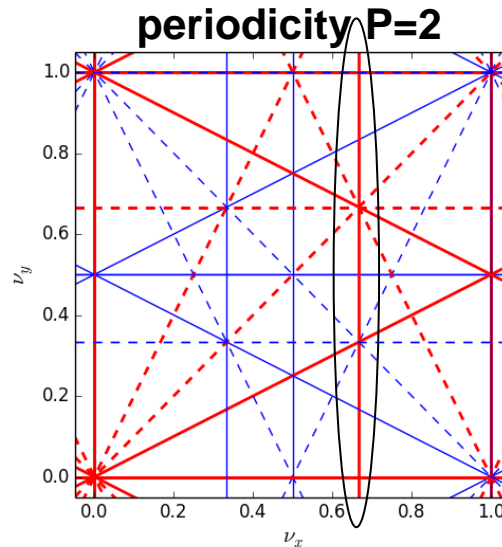
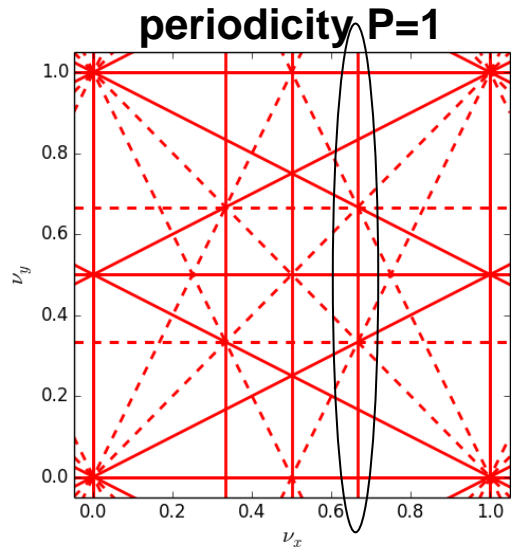
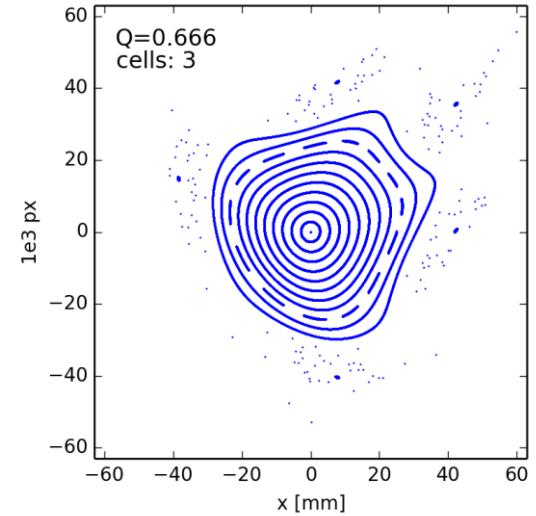
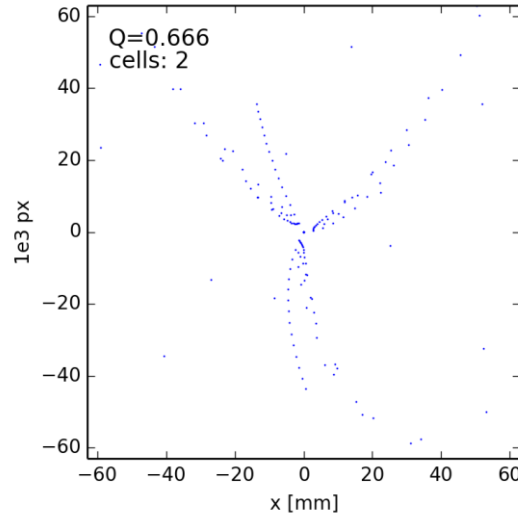
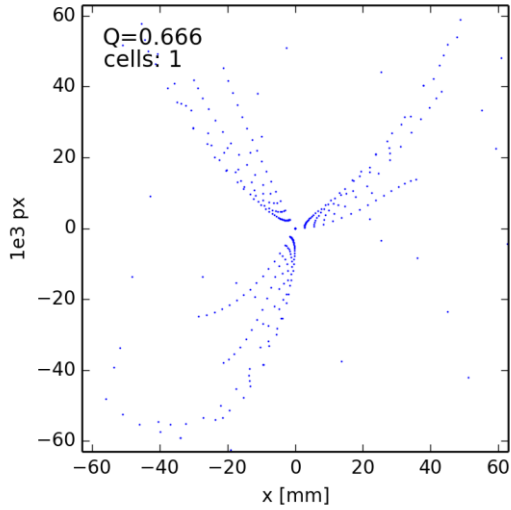
$$m_x \nu_x + m_y \nu_y = P\ell$$

3 Qx = 1 resonance



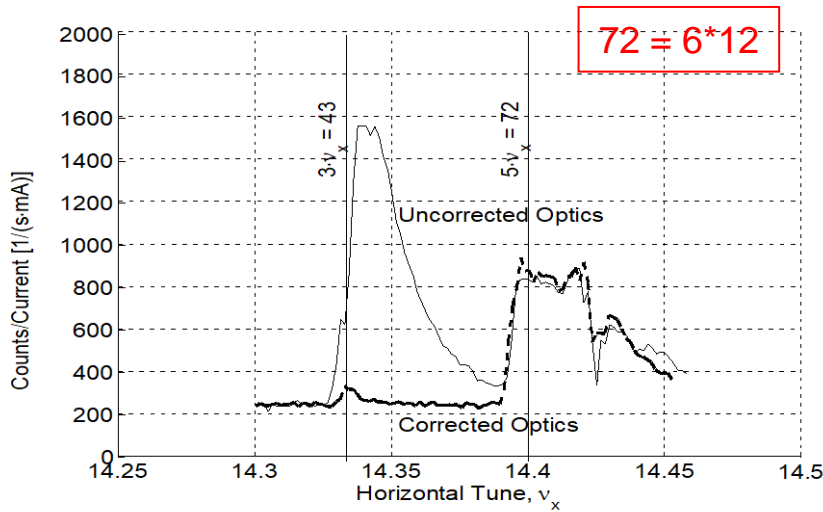
$$m_x \nu_x + m_y \nu_y = P \ell$$

3 Qx = 2 resonance



Advanced Light Source, design lattice periodicity: 12

Measurement of beam loss as function of tune



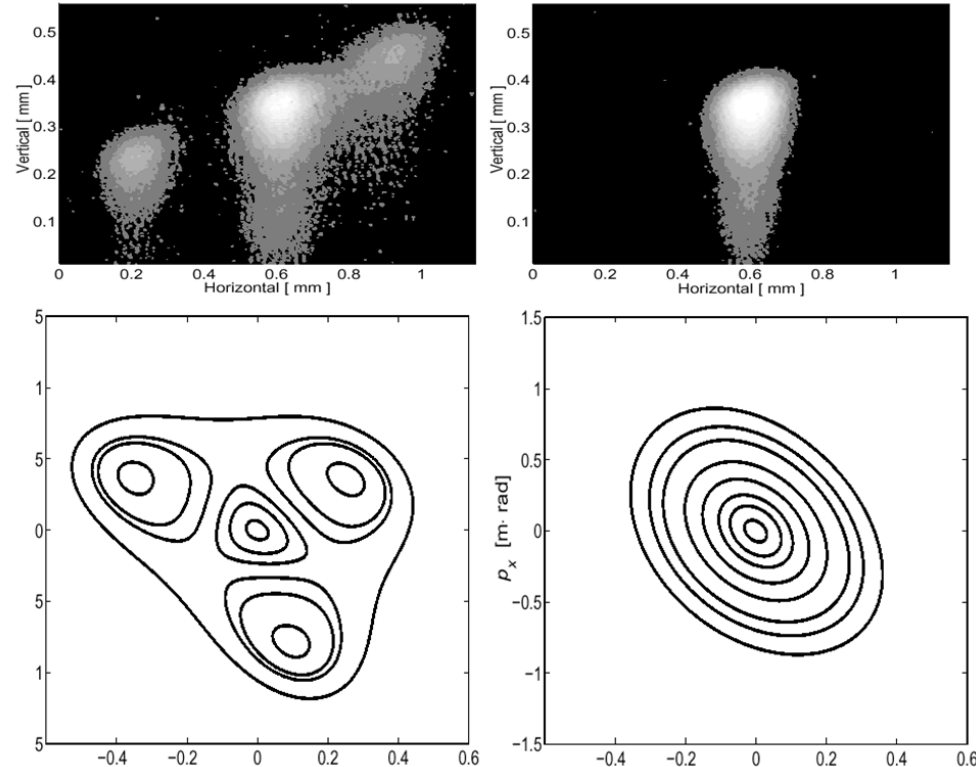
Beta beating

Before optics correction: ~30%
 After optics correction: <1%

Synchrotron light beam spot

Uncorrected optics

Corrected optics

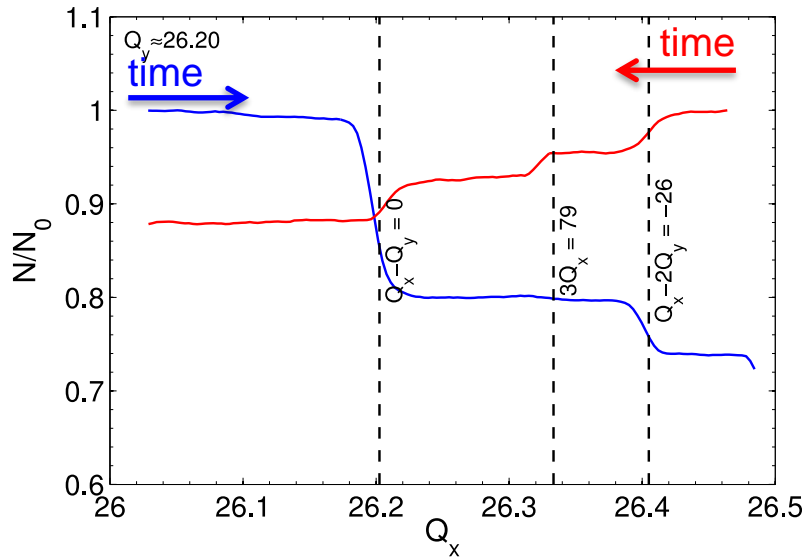


Simulated phase space

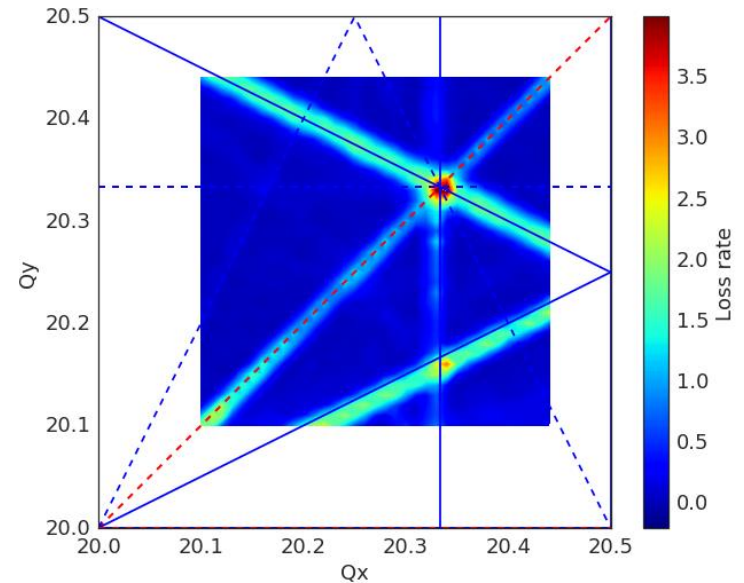
D. Robin, C. Steier, J. Safranek, W. Decking, "Enhanced performance of the ALS through periodicity restoration of the lattice", proc. EPAC 2000.

- CERN SPS (hadron machine) has design lattice periodicity of 6
- **Sextupole resonances can be clearly identified although they should be suppressed by lattice periodicity**
- Unfortunately the SPS has no individual quadrupoles to restore optics functions distortions (and typical beta beating is only around 10-20%)

Measured losses during tune scan



Measured loss rate in 2D scan



Non-linear map representation

- For any dynamical variable x_j the **Taylor map** up to **3rd order** can be written as

$$x_j^{\text{new}} = \sum_{k=1}^6 R_{jk} x_k + \sum_{k=1}^6 \sum_{l=1}^6 T_{jkl} x_k x_l + \sum_{k=1}^6 \sum_{l=1}^6 \sum_{m=1}^6 U_{jklm} x_k x_l x_m$$

- **Taylor series** provide a convenient way of systematically representing **transfer maps** for **beamline components**, or sections of beamline
- The **main drawback** of Taylor series is that in general, transfer maps can only be represented exactly by series with an **infinite number of terms**
- In practice, we have to **truncate** a **Taylor map** at some order, and we then lose certain desirable properties of the map. In particular, a **truncated map** will be usually **non-symplectic**

- Consider two sets of canonical variables \vec{x}_i, \vec{x}_f , which represent the evolution of the system between two points in phase space
- A **map** $\mathcal{M} : \vec{x}_i \mapsto \vec{x}_f$ describes the transformation from one set to the other
- This map is **symplectic**, i.e. **it conserves phase space volumes when it is applied**, if

$$J^T S J = S$$

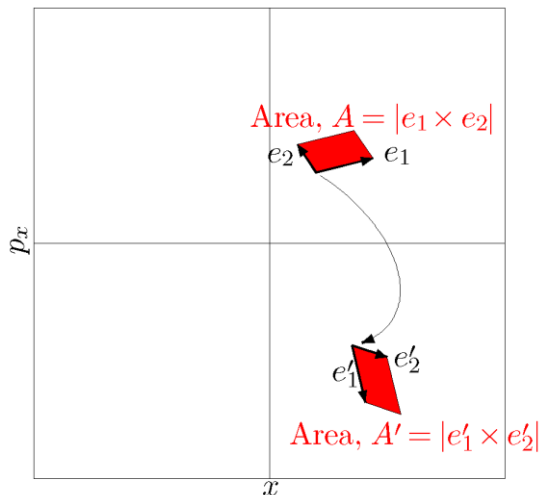
symplecticity condition

$$J_{mn} \equiv \frac{\partial x_{m,f}}{\partial x_{n,i}}$$

Jacobian matrix
of the map

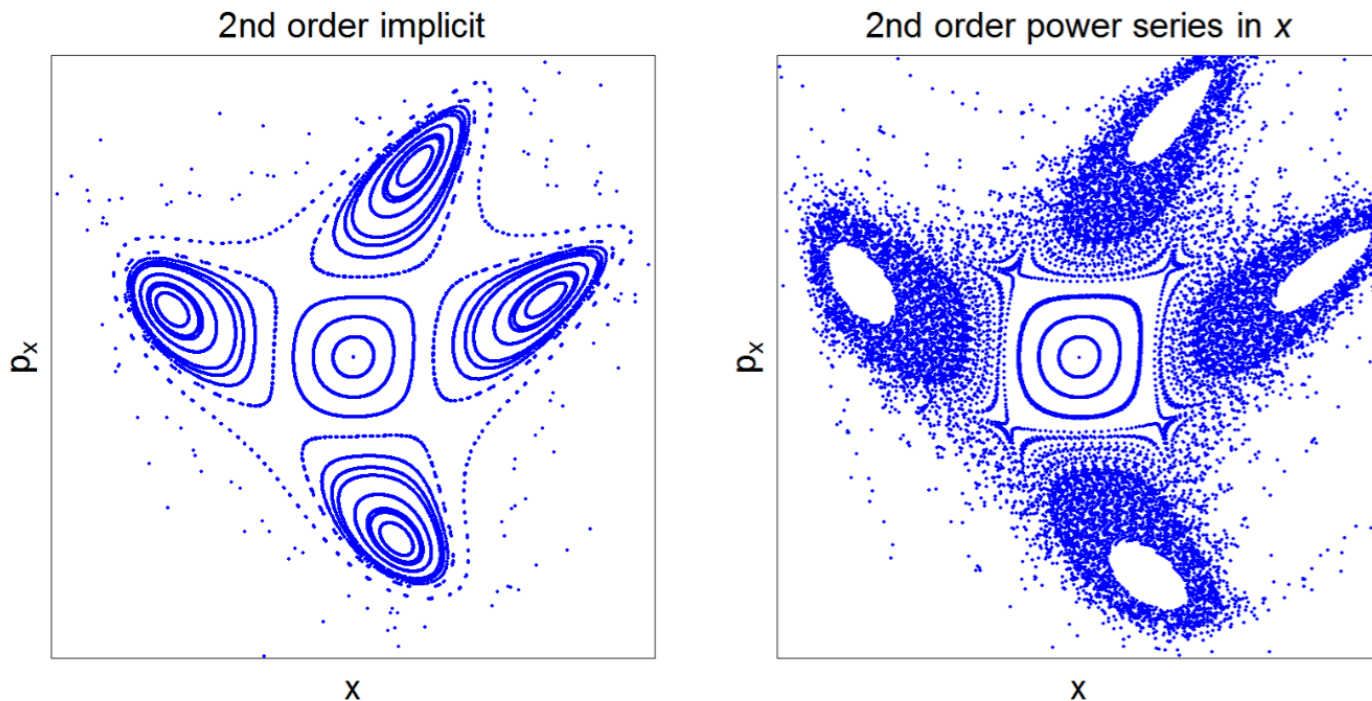
$$S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

antisymmetric matrix
with block diagonals



... this is Liouville's theorem, and is a property of charged particles moving in electromagnetic fields, in the absence of radiation

- The effect of **losing symplecticity** becomes apparent if we compare phase space portraits constructed using symplectic (below, left) and non-symplectic (below, right) transfer maps.



- Modelling a storage ring using non-symplectic maps can lead to an inaccurate estimate of the dynamic aperture and the beam lifetime

- There are a number of ways of representing transfer maps to ensure symplecticity. These include:
 - Taylor maps can be specially constructed to retain symplecticity with a certain (finite) number of terms. Taylor maps are explicit: once the coefficients have been calculated, the map can be applied simply by substitution of values for the dynamical variables.
 - Lie transformations provide a symplectic finite representation for infinite Taylor series, and are useful for analytical studies

- **Symplectic Taylor maps** with a finite number of terms can be constructed for multipole magnets of any order using the “**kick**” **approximation**.

- Consider a sextupole with equations of motion:

$$\frac{dx}{ds} = p_x, \quad \frac{dp_x}{ds} = -\frac{1}{2}k_2x^2$$

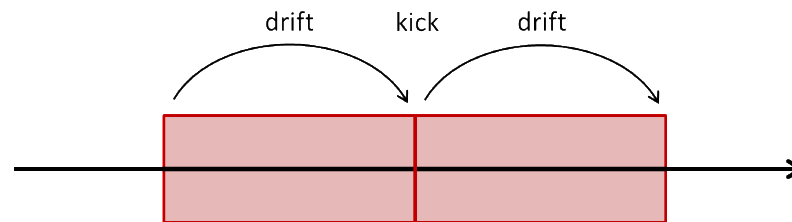
- Exact solutions using some elementary functions do not exist
- By splitting integration into three steps, it is possible to write an **explicit** and **symplectic** approximate solution

$$0 \leq s < L/2 : \quad x_1 = x_0 + \frac{L}{2}p_{x0}, \quad p_{x1} = p_{x0},$$

$$s = L/2 : \quad x_2 = x_1, \quad p_{x2} = p_{x1} - \frac{1}{2}k_2Lx_1^2,$$

$$L/2 < s \leq L : \quad x_3 = x_2 + \frac{L}{2}p_{x2}, \quad p_{x3} = p_{x2}$$

- This is an example of a symplectic integrator known as a “**drift–kick–drift**” approximation



Analytical methods for nonlinear dynamics

- There are two approaches widely used in accelerator physics: **perturbation theory** and **normal form analysis**
 - In both these techniques, the goal is to construct a **quantity** that is **invariant** under application of the single-turn transfer map. In both cases the mathematics is unfortunately complicated and fairly cumbersome ...

- In the case of a **single sextupole** in a storage ring, we find from **normal form analysis** the following expression for the betatron action as a function of the betatron phase (angle variable):

$$J_x \approx I_0 - \frac{k2L}{8} (2\beta_x I_0)^{3/2} \frac{\cos(3\mu_x/2 + 2\phi_x) + \cos(\mu_x/2)}{\sin(3\mu_x/2)} + O(I_0^2)$$

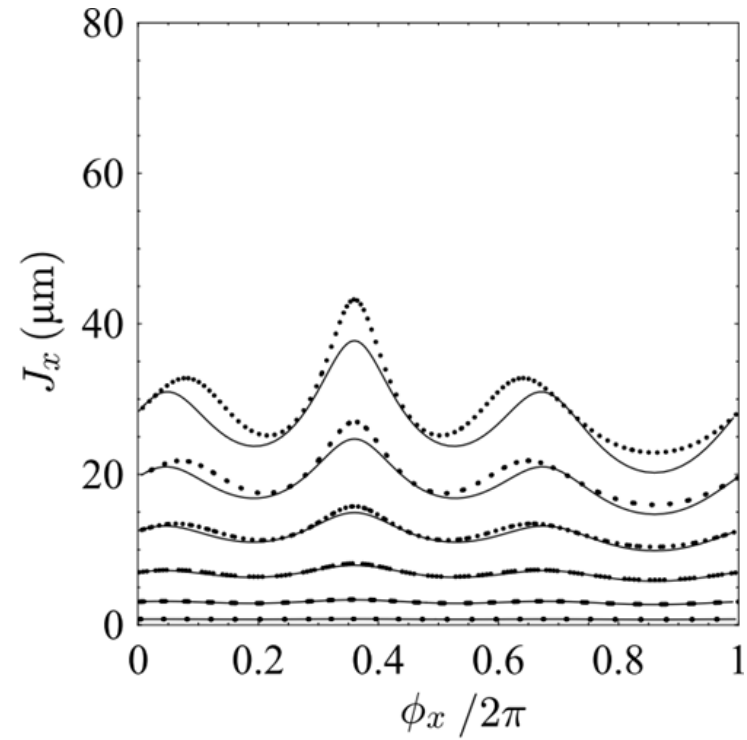
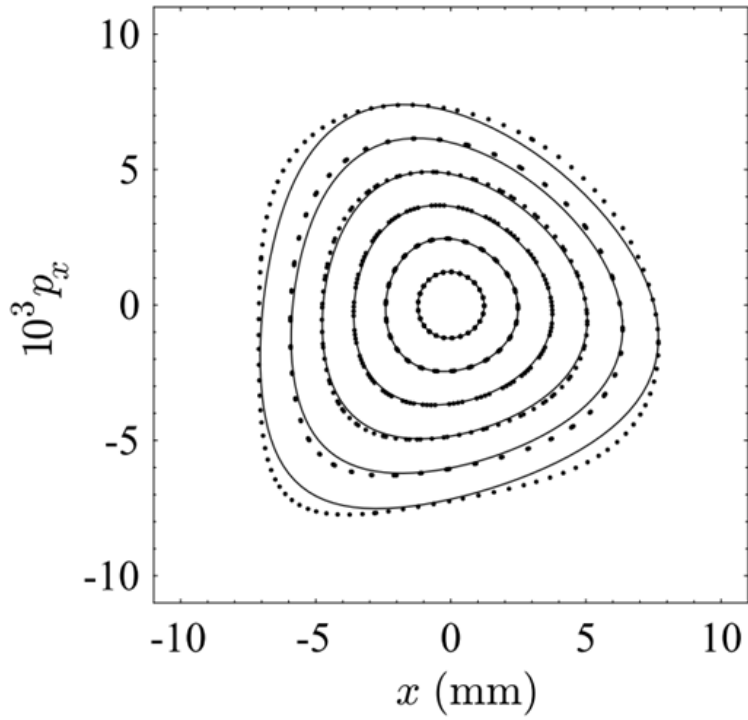
where I_0 is a constant (an invariant of the motion), ϕ_x is the angle variable, and μ_x is the phase advance per cell. The second term becomes **very large** when μ_x is close to **third integer**

- The cartesian variables can be expressed in terms of the action–angle variables:

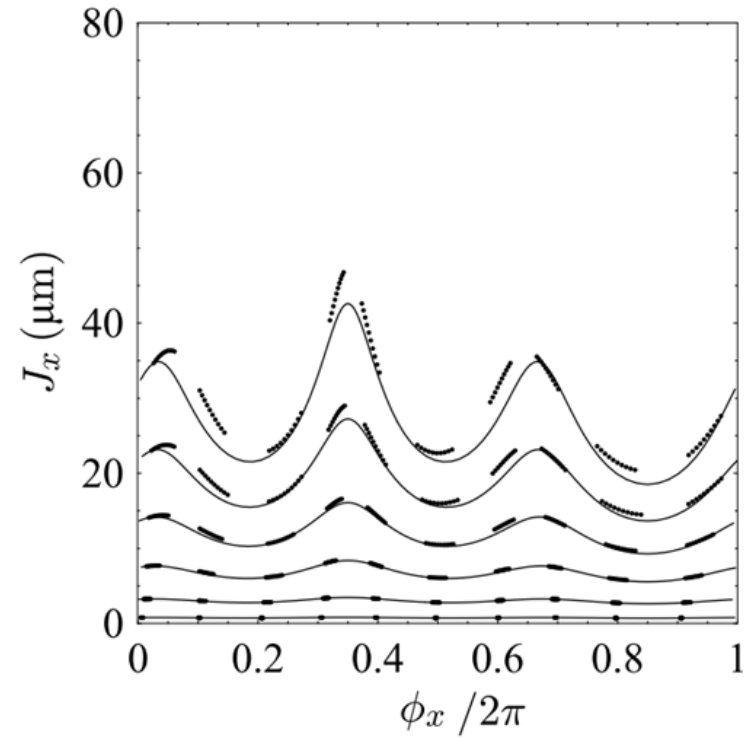
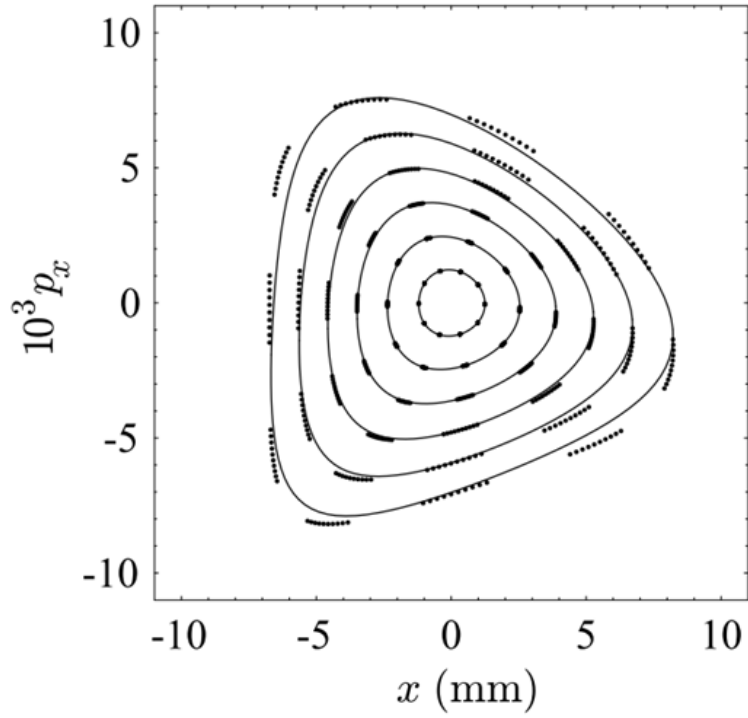
$$x = \sqrt{2\beta_x J_x} \cos \phi_x,$$

$$p_x = -\sqrt{2J_x/\beta_x} (\sin \phi_x + \alpha_x \cos \phi_x)$$

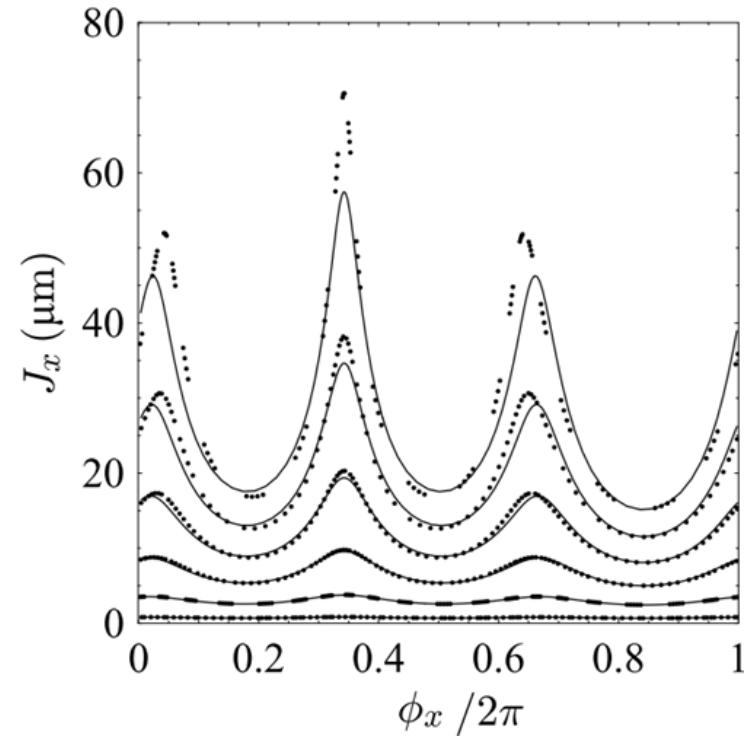
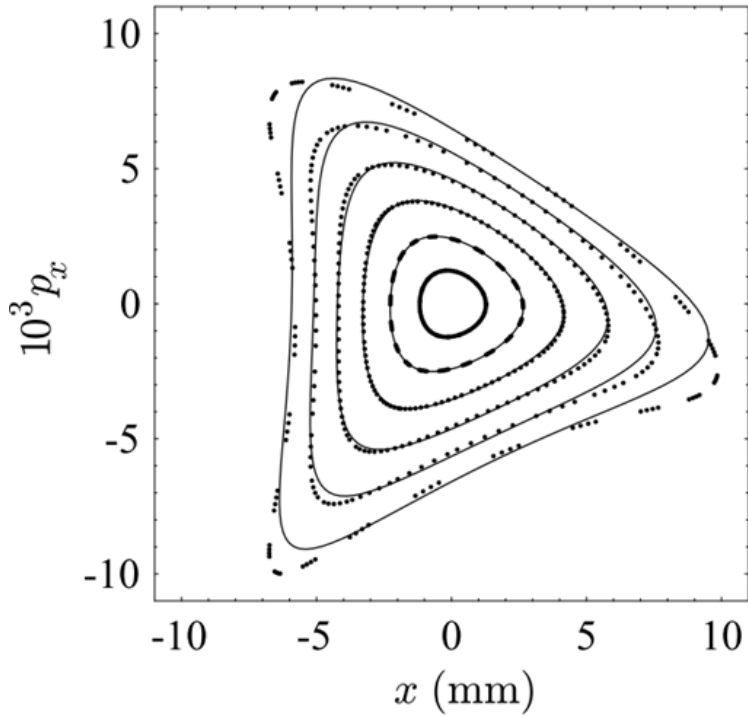
$$\mu_x = 0.28 \times 2\pi$$



$$\mu_x = 0.30 \times 2\pi$$



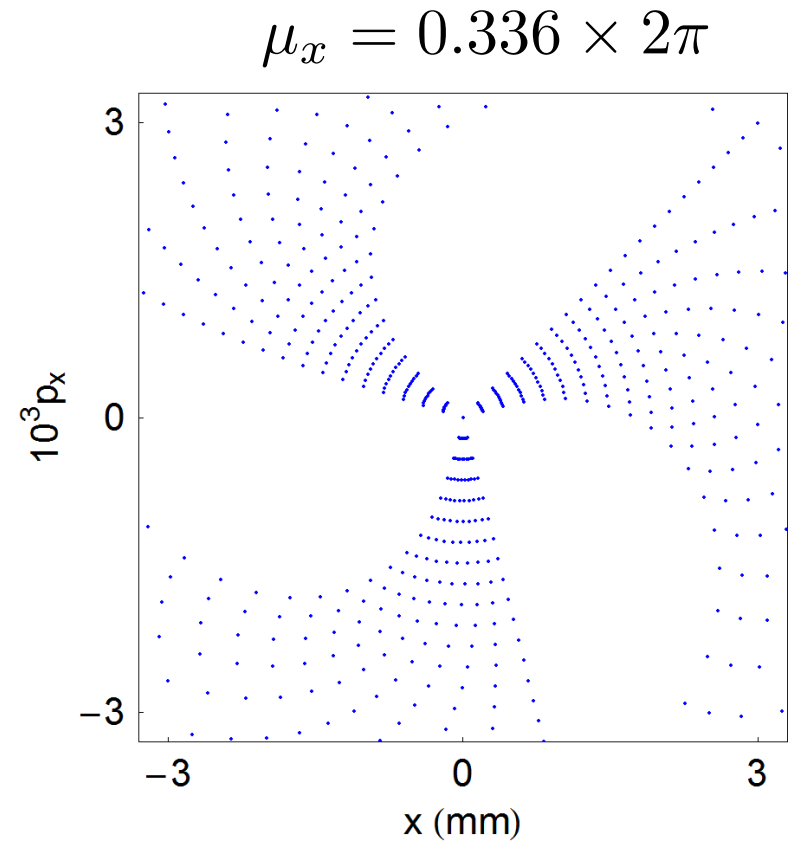
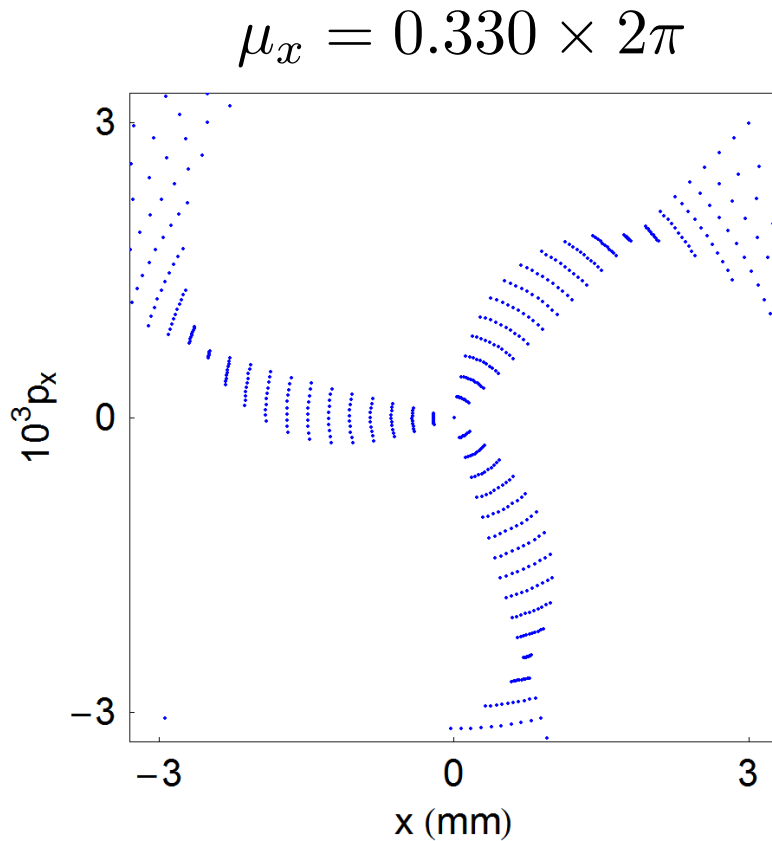
$$\mu_x = 0.315 \times 2\pi$$



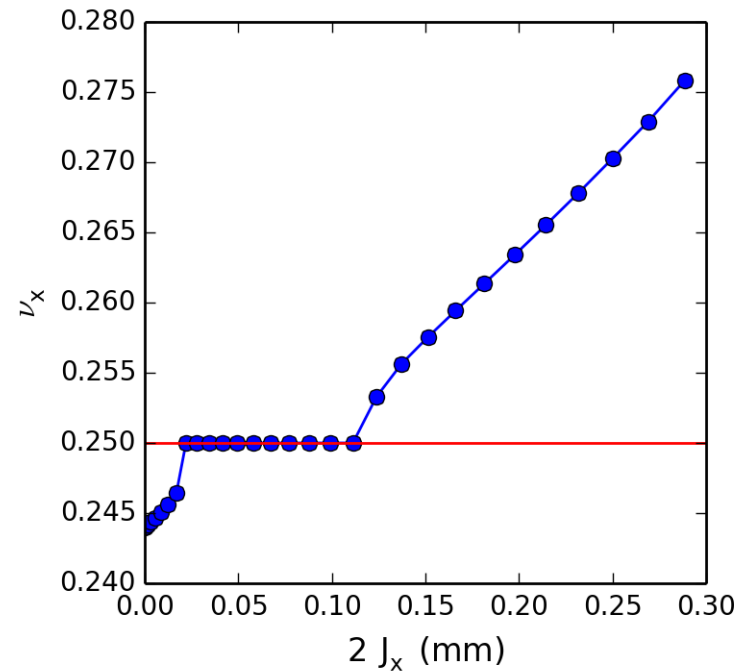
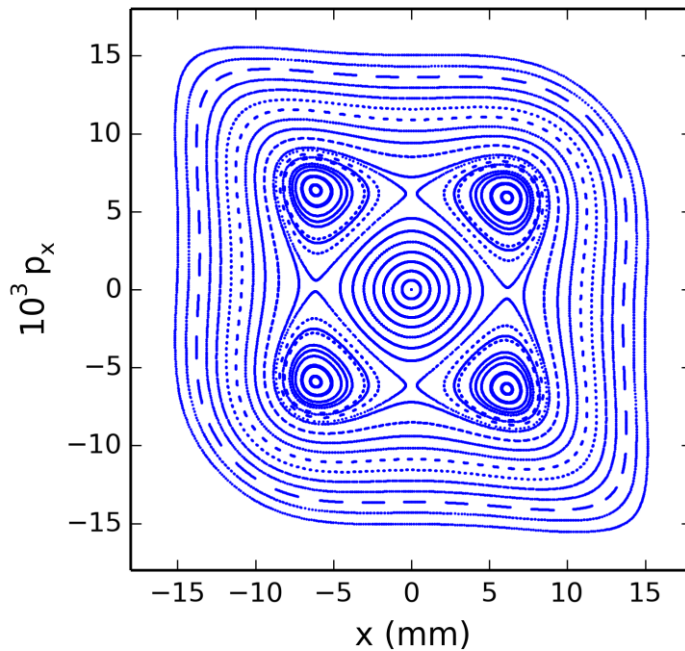
- Close inspection of the plots on the previous slides reveals another effect, in addition to the obvious distortion of the phase space ellipses: the **phase advance** per turn (i.e. the tune) **varies** with increasing **betatron amplitude**
- Normal form analysis (and perturbation theory) can be used to obtain **estimates** for the **tune shift** with amplitude
- In the case of a sextupole, the **tune shift** is **higher-order** in the **sextupole strength**
- An octupole, however, does have a **tune shift** with amplitude in **first-order** of the octupole strength, given by:

$$\nu_x = \nu_{x0} + \frac{k_3 L \beta_x^2}{16\pi} J_x + O(J_x^2)$$

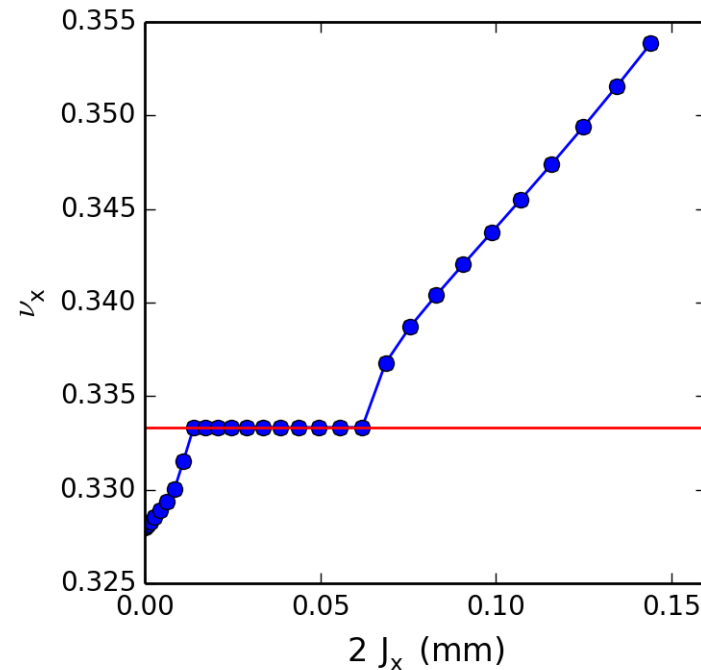
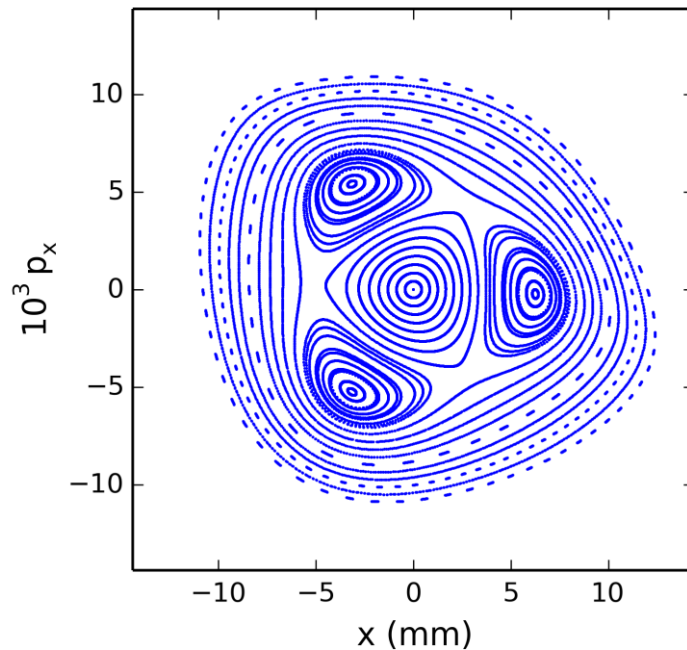
- The tune shift with amplitude becomes obvious if we track a small number of turns (30) in a lattice with a **single octupole**.



- Simulation of simple storage ring with a **single octupole** close to 4th order resonance
- **Detuning with amplitude** (linear in action)
- Particles in the stable islands have a tune **locked** to the **resonance**

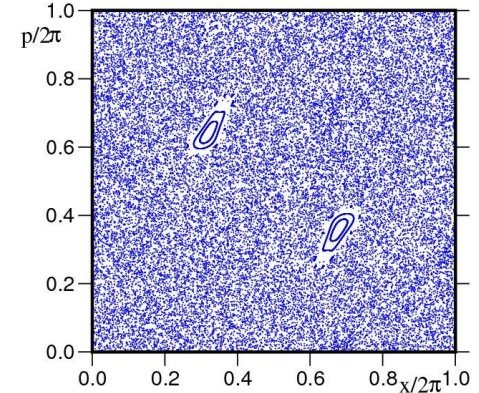
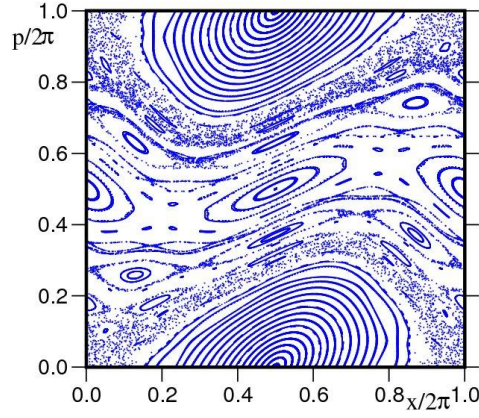
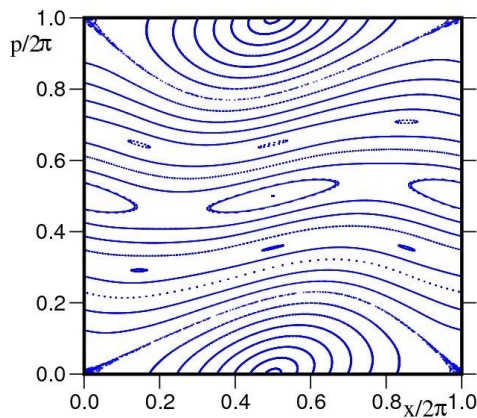


- Simulation of simple storage ring with a **sextupole** and an **octupole** close to **3rd order resonance**
- The **amplitude detuning** induced by the octupole can create **stable islands** even for the **3rd order resonance** (recall the phase-space plot for the case of a single sextupole)

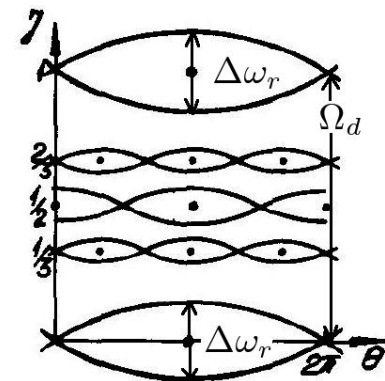


- Perturbation theory and normal form analysis depend on the existence of **constants of motion in the presence of nonlinear perturbations**
- The fact that **constants of motion can exist in the presence of nonlinear perturbations is a consequence of the Kolmogorov–Arnold–Moser (KAM) theorem.**
 - The KAM theorem expresses the general conditions for the existence of constants of motion in nonlinear Hamiltonian systems. Under small perturbations, quasi-periodic phase space trajectories exist.
- **Resonances do not necessarily result in loss of stable motion**
 - **Resonances will typically tend to drive the amplitudes of particles with a particular tune to large amplitudes**
 - **For sufficiently large tune-shift with amplitude, there can be a stable region at amplitudes larger than that at which the resonance occurs (see the example of the octupole in the previous slide)**

- The **overlap of two resonances** is associated with a transition from regular to **chaotic motion**



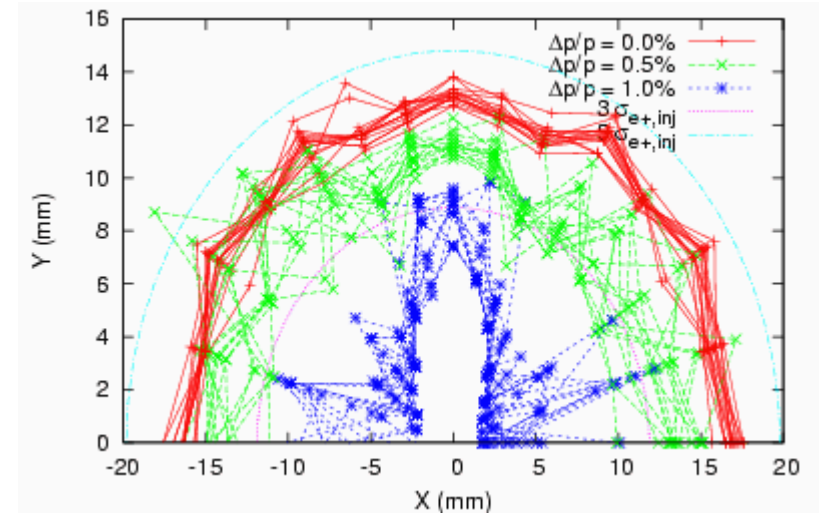
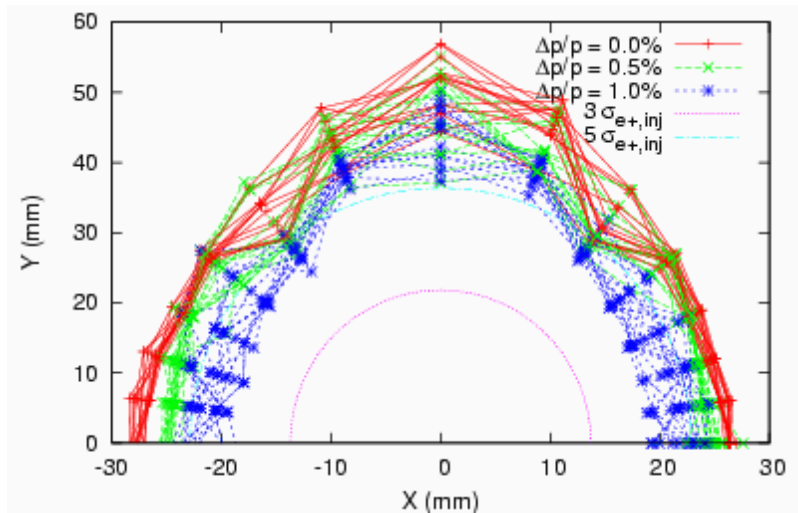
- The **Chirikov criterion** describes the parameter range over which the particle motion becomes chaotic
 - It is a geometric criterion of the resonance overlap based on the size of resonance islands



- We have shown phase space portraits for motion in one degree of freedom. In those cases, instability occurred when the oscillation amplitude exceeded a certain value. Chaotic motion can occur also for smaller amplitudes.
- **In multiple degrees of freedom**, a new phenomenon occurs: **Arnold diffusion**. There can be **regions of phase space where invariant tori exist at large amplitudes compared to regions of chaotic motion**.
- For storage rings, this means that trajectories with (initially) small amplitudes can be unstable, even if trajectories with much larger amplitudes are stable.

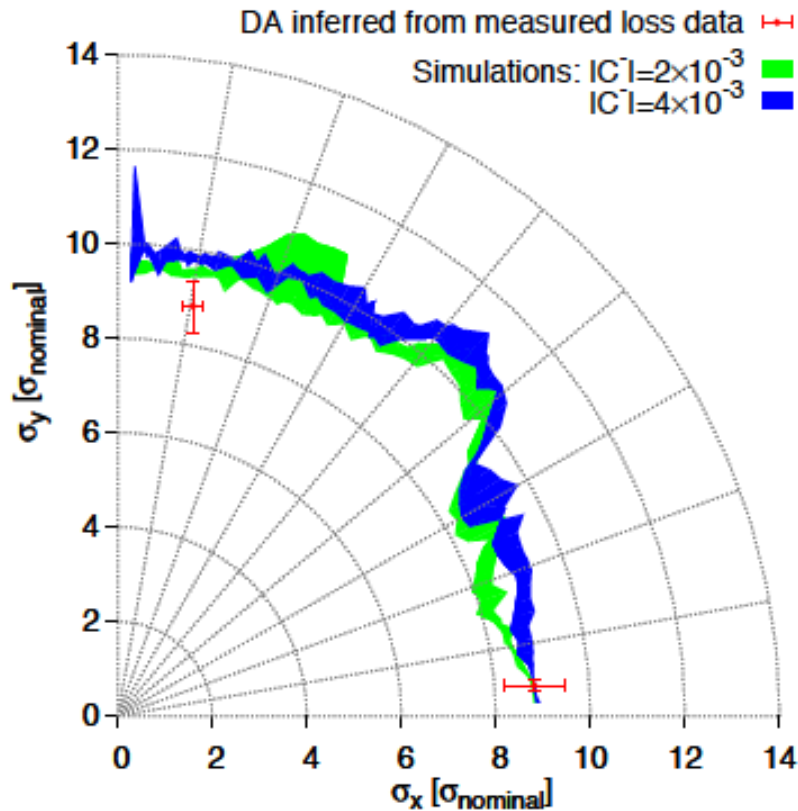
Numerical methods: Dynamic aperture (DA) and Frequency map analysis (FMA)

- The most direct way to evaluate the nonlinear dynamics performance of a ring is the computation of **Dynamic Aperture** (short: DA), which is the **boundary of the stable region in co-ordinate space**
- Need a **symplectic tracking code** to follow particle trajectories (a lot of initial conditions) for a number of turns until particles start getting lost → this boundary defines the **Dynamic aperture**
- Dynamic aperture plots show the maximum initial values of stable trajectories in x-y coordinate space



DA simulations for CLIC damping rings

- LHC design was based on a large campaign of systematic DA simulations (including margin for stability)
 - The goal is to allow significant margin in the design – the measured dynamic aperture is often smaller than the predicted dynamic aperture



- A few years after LHC started operating, a measurement of the DA was performed (kicking the beam to large amplitudes)
- Very good agreement between tracking simulations and measurements in the machine

E.Mclean, PhD thesis, 2014

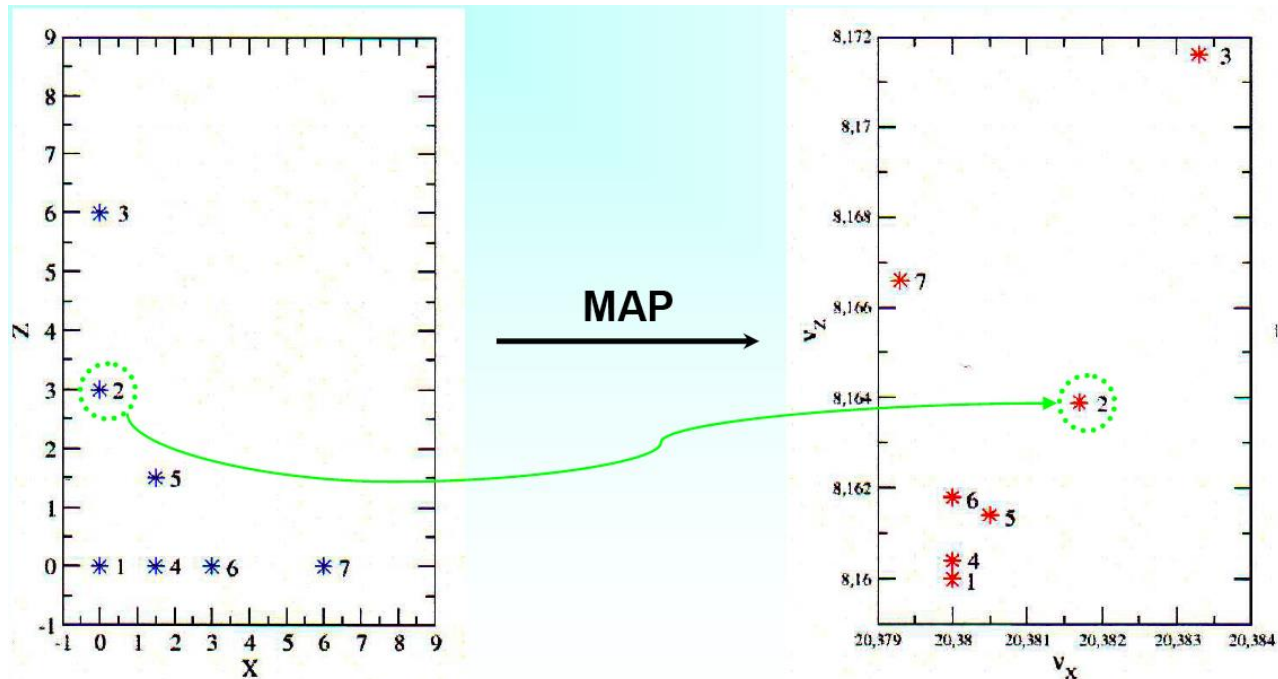
- Frequency Map Analysis (FMA) is a numerical method pioneered by J. Laskar for putting in evidence the chaotic motion in Solar Systems
- FMA relies on precise tune determination using the **Numerical Analysis of Fundamental Frequencies (NAFF) algorithm**
 - When a quasi-periodic function $f(t) = q(t) + ip(t)$ in the complex domain is given numerically, it is possible to recover a quasi-periodic approximation

$$f'(t) = \sum_{k=1}^N a'_k e^{i\omega'_k t}$$

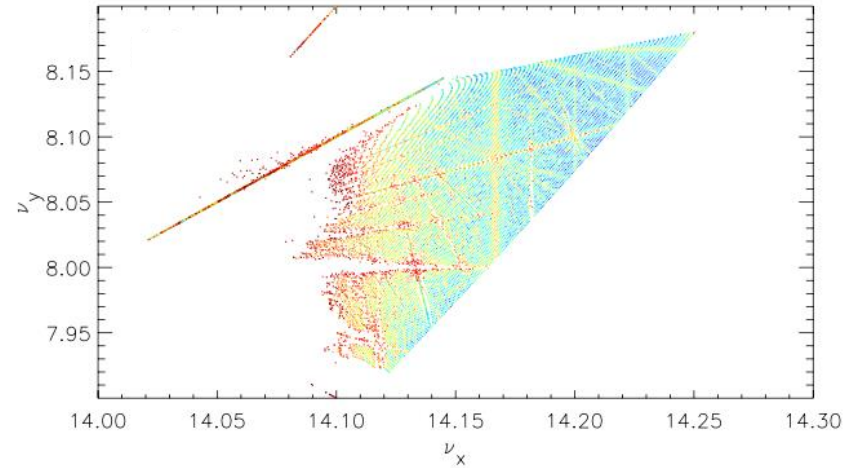
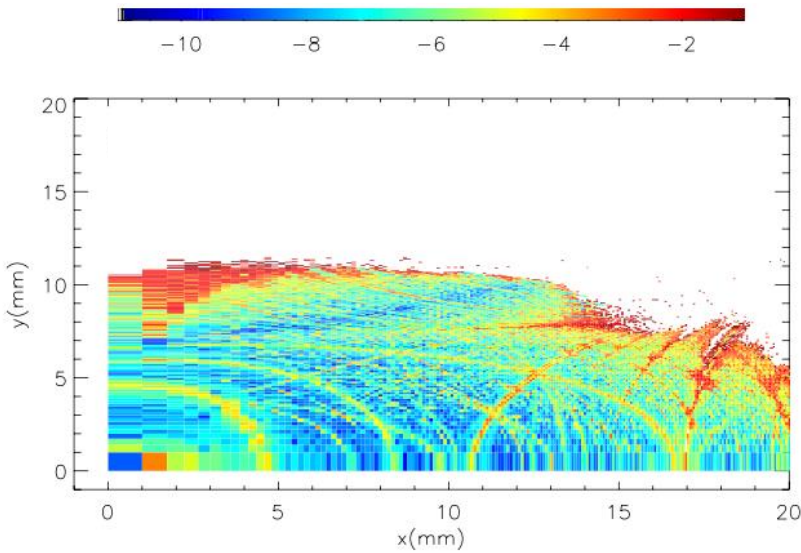
in a very precise way over a finite time span, **several orders of magnitude more precisely than simple Fourier techniques!**

■ Building the FMA from simulations

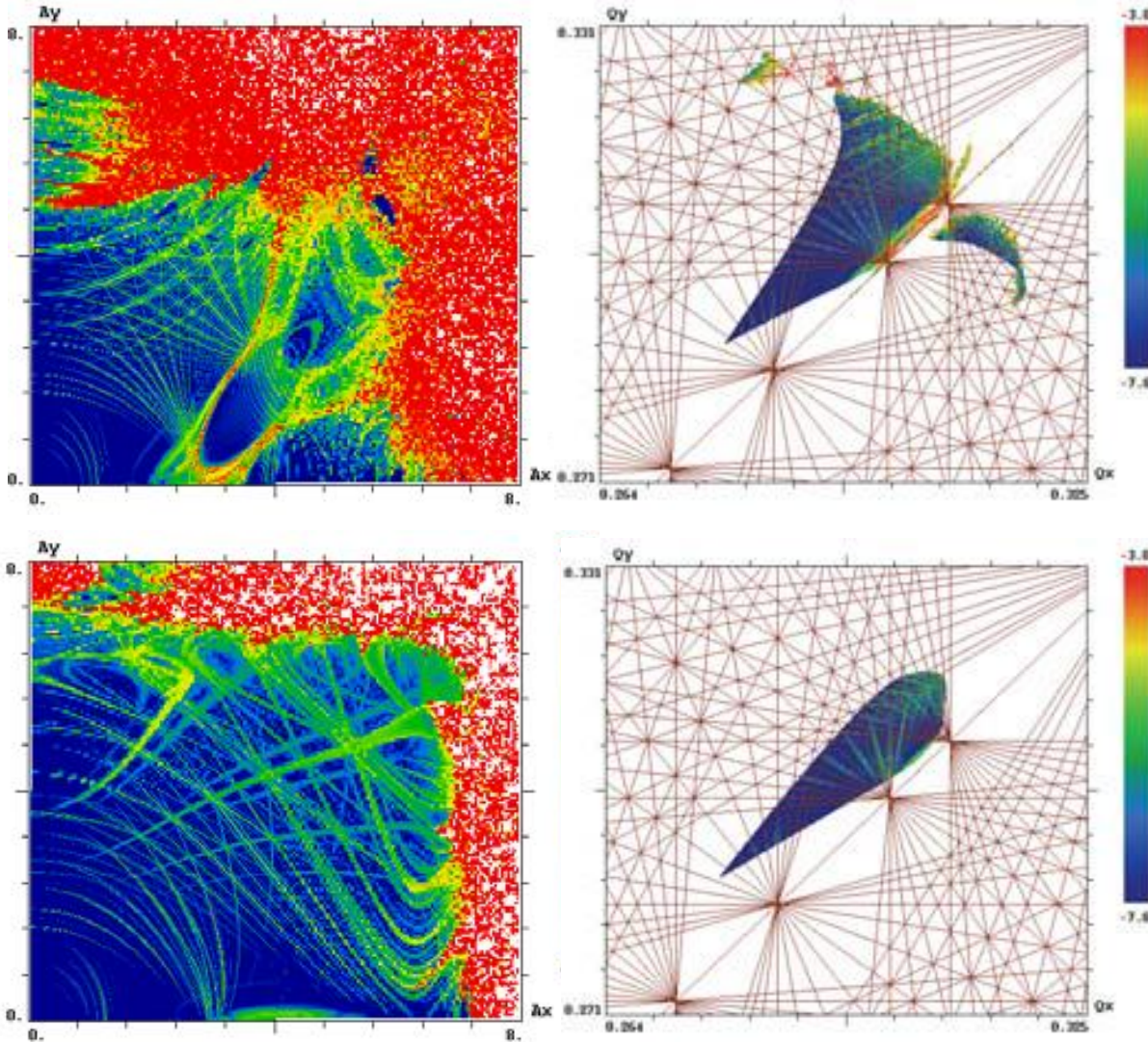
- ❑ Choose coordinates (x_i, y_i) with $p_x=p_y=0$
- ❑ Numerically integrate the particle trajectories through the lattice for sufficient number of turns (“tracking”)
- ❑ Precisely compute Q_x and Q_y after a sufficient number of turns
- ❑ Plot them in the tune diagram



J. Laskar, "Frequency map analysis and particle accelerators", PAC 2003



- Computing the tunes ν_x and ν_y for 2 consecutive intervals through NAFF, allows plotting the **tune variation with a color code in configuration space x-y (left) and tune space ν_x - ν_y (right)**
 - ❑ **small tune diffusion** corresponds to **regular motion**
 - ❑ **Large tune diffusion** is associated with **chaotic motion**, it occurs at the dynamic aperture of the machine
 - ❑ **Resonances appear as distorted lines in frequency space** (or curves in initial configuration space) → allows to determine which part of the configuration space is affected by which resonances



Large tune footprint and DA reduction due to “long range beam-beam” forces

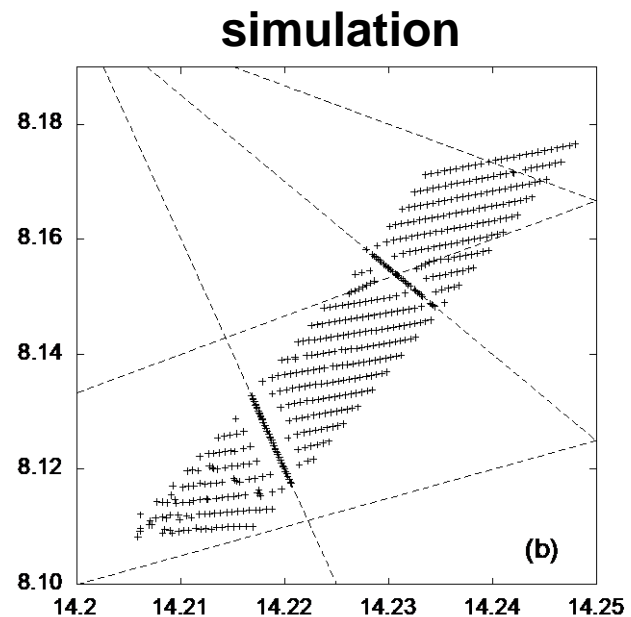
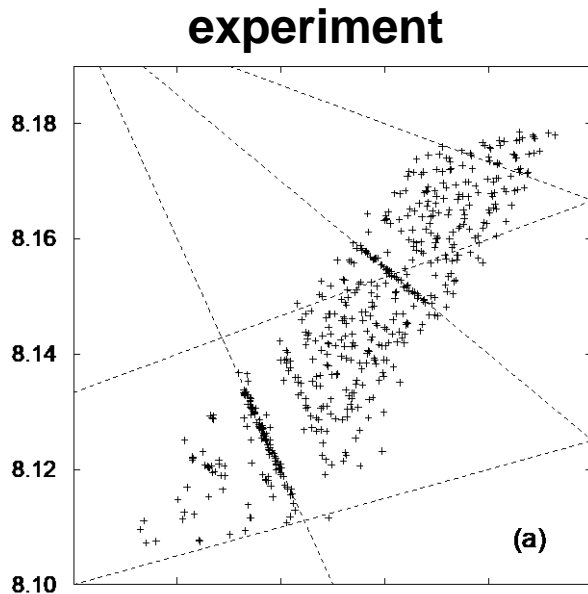
(electromagnetic field from the other beam in the interaction region)

DA clearly improved when compensating long range beam-beam with a wire

S. Fartoukh et al.,
PRSTAB, 2015

- **Frequency analysis of turn-by-turn data** of beam oscillations produced by a fast kicker magnet and recorded on a Beam Position Monitor
- **Reproduced in tracking simulations using the non-linear model of the ALS storage ring**

D. Robin, C. Steier, J. Laskar, and L. Nadolski, PRL 2000



Conclusions and Summary

- **Nonlinear dynamics** appear in a wide variety of accelerator systems, including **single-pass** systems (such as bunch compressors) and **multi-turn** systems (such as storage rings)
- It is possible to model nonlinear dynamics in a given component or section of beamline by representing the **transfer map** as a **power series**
- **Conservation of phase space volumes** is an important feature of the beam dynamics in many systems. To conserve phase space volumes, transfer maps must be **symplectic**
- In general, (**truncated**) **power series** maps are **not symplectic**
- To construct a **symplectic transfer map**, the equations of motion in a given accelerator component must be solved using a **symplectic integrator** (e.g. the “drift–kick–drift” approximation for a multipole magnet)

- **Common features** of nonlinear dynamics in accelerators include **phase space distortion**, **tune shifts with amplitude**, **resonances**, and **chaotic** particle trajectories at large amplitudes (**dynamic aperture** limits)
- Analytical methods such as **perturbation theory** and **normal form** analysis can be used to estimate the impact of nonlinear perturbations in terms of quantities such as **resonance strengths** and **tune shifts with amplitude**
- **Frequency map analysis (FMA)** provides a useful numerical tool for characterising tune shifts and resonance strengths from tracking data
- This can give some **insight** into **limitations** and ways for optimizing the **dynamic aperture (DA)**